Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET I

These problems will be discussed in the 1st Examples Class.

1. Magnetic-field spectra in clusters of galaxies. Randomly tangled magnetic fields in galaxy clusters can be measured because the polarisation angle of an electromagnetic wave propagating from an extended radio source in a cluster rotates as it passes though magnetised intracluster medium. Let z be the line-of-sight direction and (x, y) the coordinates in the plane perpendicular to it. The rotation angle is $\Delta\phi = \lambda^2\sigma(x, y)$, where λ is the wavelength and

$$\sigma(x, y) = a_0 \int_{z_{\text{source}}}^{z_{\text{observer}}} dz\, n_e B_z \tag{1}$$

is called the *rotation measure*. Here a_0 is a constant, n_e is the electron density in the intracluster medium and B_z is the projection of the magnetic field in the medium onto the line of sight. While B_z is a (random) function of x, y, and z, you may assume that n_e is constant. Let us assume that we have a two-dimensional data set with values of $\sigma(x, y)$ for all x and y. Your task is to determine the spectrum of magnetic energy based on this information and some statistical assumptions.

1. Assuming spatial homogeneity of the field, the two-point correlation function of the magnetic field depends only on the distance between points: $\langle B_i(\mathbf{r})B_j(\mathbf{r}')\rangle = C_{ij}(\mathbf{r} - \mathbf{r}')$. Define the Fourier transform

$$\hat{\mathbf{B}}(\mathbf{k}) = \int d^3r\, \mathbf{B}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{r} = (x, y, z). \tag{2}$$

Show that $\langle \hat{B}_i(\mathbf{k})\hat{B}_j(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k} + \mathbf{k}')\hat{C}_{ij}(\mathbf{k})$, where

$$\hat{C}_{ij}(\mathbf{k}) = \int d^3r\, C_{ij}(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{3}$$

2. Now assume spatial isotropy and parity (mirror symmetry) of the field. Then the tensor $\hat{C}_{ij}(\mathbf{k})$ can be written in terms of one scalar function of $k = |\mathbf{k}|$. If $\hat{C}_{ii}(\mathbf{k}) = 2H(k)$, what is the expression for $\hat{C}_{ij}(\mathbf{k})$ in terms of $H(k)$ and $\mathbf{k}$?

3. Suppose we have constructed from our data set the correlation function of the rotation measure, i.e., $C_{\text{RM}} = \langle\sigma(\mathbf{r}_{\perp 1})\sigma(\mathbf{r}_{\perp 2})\rangle$ is known for any two points $\mathbf{r}_{\perp 1} = (x_1, y_1)$ and $\mathbf{r}_{\perp 2} = (x_2, y_2)$. Show that C_{RM} and $H(k)$ are related as follows

$$C_{\text{RM}}(|\mathbf{r}_{\perp 1} - \mathbf{r}_{\perp 2}|) = a_0^2 n_e^2 L \int \frac{d^2k_\perp}{(2\pi)^2} H(|\mathbf{k}_\perp|)\, e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_{\perp 1}-\mathbf{r}_{\perp 2})}, \quad \mathbf{k}_\perp = (k_x, k_y), \tag{4}$$

where L is the distance from the source to the observer. You are allowed to take the integration limits in z to $\pm\infty$ wherever you need to and use the formula $\int_{-\infty}^{+\infty} dz\, e^{ik_z z} = 2\pi\delta(k_z)$.

4. Now show that the spectrum of the magnetic field can be recovered from the observed rotation-measure correlation function as follows:

$$H(k) = \frac{2\pi}{a_0^2 n_e^2 L} \int_0^\infty dr r J_0(kr) C_{\text{RM}}(r). \tag{5}$$

You may use the formula $\int_0^{2\pi} d\theta\, e^{\pm ia\cos\theta} = 2\pi J_0(a)$.

2. Scalar Turbulence: Obukhov-Corrsin Theory. Consider the evolution of *passive scalar* $\theta(t, \mathbf{x})$ (this can be temperature, or concentration of an admixture like a dye or salt, or, in 2D hydrodynamics, the vorticity field, or, in RMHD, the magnetic flux function, etc.):

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + S, \tag{6}$$

where $\mathbf{u}$ is the turbulent velocity field, κ is the scalar diffusivity, and S is the source function (scalar "forcing"). We will assume that S varies at some (large) scale $L_\theta < L$, where L is the outer scale of the turbulence.

You are going to develop a dimensional theory of scalar turbulence *à la* the K41 theory I described in my lectures.

1. Let us figure out when the diffusive term in Eq. (6) is negligible. Assume that the convective term in Eq. (6) is dominated by interactions between velocity fluctuations δu_l and scalar fluctuations $\delta\theta_l$ on comparable scales. Compare it with the diffisive term and show that the latter is negligible if

$$\frac{\kappa}{\delta u_l l} \ll 1. \tag{7}$$

2. Show that, for δu_l satisfying the K41 scaling, Eq. (7) reduces to $l \gg l_\kappa = \mathrm{Sc}^{-3/4} l_\nu$, where $l_\nu = (\nu^3/\epsilon)^{1/4}$ is the viscous scale, ϵ is the Kolmogorov flux, and $\mathrm{Sc} = \nu/\kappa$ is called the *Schmidt number*.

   Note that, since you have used K41 inertial-range scaling for the cascade time, your estimates are only correct for $\mathrm{Sc} \ll 1$ (do you understand why that is?).

3. Show that an equivalent expression for the diffusive scale is $l_\kappa \sim \mathrm{Pe}^{-3/4} L_\theta$ (provided the characteristic scale of the scalar source is $L_\theta < L$), where $\mathrm{Pe} = \delta u_{L_\theta} L_\theta / \kappa$ is called the *Péclet number* (analog of the Reynolds number for scalars).

   The scale range of l such that $L > L_\theta \gg l \gg l_\theta \gg l_\nu$ is called *the inertial-convective range.* It is non-empty if $\mathrm{Re} \gg \mathrm{Pe} \gg 1$.

4. Define the *scalar variance* $\mathcal{E}_\theta = \langle \theta^2 \rangle / 2$ ("energy" of the scalar field). Derive an evolution equation for it and show that in the statistically stationary state, when $\mathcal{E}_\theta = \mathrm{const}$, the average injected scalar "power" is

$$\epsilon_\theta = \langle S\theta \rangle = \kappa \langle |\nabla \theta|^2 \rangle. \tag{8}$$

5. Find an argument (similar to K41) that leads to the relation, valid in the inertial-convective range,

$$\frac{\delta \theta_l^2}{\tau_l} \sim \epsilon_\theta = \mathrm{const}. \tag{9}$$

   What is the cascade time? Use the above relation to prove that

$$\delta \theta_l \sim \left(\frac{\epsilon_\theta}{\epsilon} \right)^{1/2} \delta u_l. \tag{10}$$

6. So what is the scaling of $\delta\theta_l$ if $\mathbf{u}$ is the Kolmogorov turbulent velocity field? Show that the spectrum of scalar variance, defined analogously to the spectrum of the velocity field, is

$$E_\theta(k) \sim \frac{\epsilon_\theta}{\epsilon^{1/3}} k^{-5/3} \tag{11}$$

(the Obukhov-Corrsin spectrum). Sketch the spectra of the kinetic energy and of the scalar variance, indicating all relevant wavenumbers (scales) and slopes.

3. Scalar Turbulence: Batchelor Theory. What if $\mathrm{Sc} \gg 1$? Then l_κ we calculated in Question 2 is smaller than l_ν. Our dimensional theory only applies to $l \gg l_\nu$. Let us figure out what the scalar does at $l \ll l_\nu$.

1. Use the scaling of δu_l in the viscous range ($l \ll l_\nu$) derived in my lectures to show that Eq. (7) reduces to $l \gg l_\kappa = \mathrm{Sc}^{-1/2} l_\nu$ — the new expression for the diffusive scale in the limit $\mathrm{Sc} \gg 1$.

   The scale range $l_\nu \gg l \gg l_\kappa$ is called *the viscous-convective range* (or *subviscous range*).

2. In a manner analogous to what you did in Question 2, show that, for l in the viscous-convective range,

$$\delta\theta \sim {\epsilon_\theta}^{1/2} \epsilon^{-1/4} \nu^{1/4}, \tag{12}$$

   (independent of scale!) or, for the spectrum of scalar variance,

$$E_\theta(k) \sim \epsilon_\theta \epsilon^{-1/2} \nu^{1/2} k^{-1} \tag{13}$$

   (the Batchelor spectrum). This spectrum is the result of these two properties of the viscous-convective range: (i) flux of scalar variance is independent of l, (ii) cascade time is independent of l (and equal to the turnover time of the viscous-scale eddies — confirm this is so!).

3. Thus, in the inertial-convective range, we have the Oboukhov-Corrsin spectrum, in the viscous-convective range, we have the Batchelor spectrum. Sketch the spectra of the kinetic energy and of the scalar variance in the case $\mathrm{Sc} \gg 1$, indicating all relevant wavenumbers $k \sim 1/l$ and slopes.