ES4. -1-

Spectrum of the passive scalar field at $Sc \gg 1$.

Consider

$$(1) \qquad \partial_t \theta + u^l \frac{\partial \theta}{\partial x^l} = \eta \nabla^2 \theta$$

sorry, I'll call scalar diffusivity η because $\varkappa^{ij}$ will velocity correlator.

– passive-scalar decay in a linear velocity field:

$$u^i(t,\vec{x}) = \sigma^i_m(t) x^m$$

This model is reasonable for scales $l \ll l_\nu$ and $Sc \gg 1$, so there is a non-empty viscous-conv. interval.

1) Seek the solution in the form

$$\theta(t,\vec{x}) = \int \frac{d^d k_0}{(2\pi)^d} \tilde{\theta}(t,\vec{k}_0) e^{i \tilde{k}_m(t,\vec{k}_0) \cdot x^m}, \qquad (2)$$

where $\tilde{k}_m(0,\vec{k}_0) = k_{0m}$. Substituting into (1), we get

$$\boxed{\begin{aligned} \partial_t \tilde{\theta} &= -\eta \tilde{k}^2 \tilde{\theta} && (3) \\ \partial_t \tilde{k}_m &= -\sigma^l_m \tilde{k}_l && (4) \end{aligned}}$$

2) Introduce the joint pdf of $\tilde{\theta}$ and $\tilde{k}$:

$$P(\theta,\vec{k}) = \langle \underbrace{\delta(\theta - \tilde{\theta}(t)) \delta(k_m - \tilde{k}_m(t))}_{\equiv \tilde{P}} \rangle$$

Take velocity to be Gaussian and δ-correlated with

$$\langle \sigma^i_m(t) \sigma^j_n(t') \rangle = \delta(t-t') \varkappa_2 T^{ij}_{mn},$$

$$T^{ij}_{mn} = \delta^{ij}\delta_{mn} - \frac{1}{d+1}\left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m\right)$$

$$\partial_t \tilde{P} = \partial_t \left[\delta(\theta - \tilde{\theta}(t))\, \delta(k_m - \tilde{k}_m(t)) \right] =$$

$$= \frac{\partial}{\partial \theta} \left\{ \tilde{P} \left[\eta \tilde{k}^2(t) \tilde{\theta}(t) \right] \right\} + \frac{\partial}{\partial k_m} \left\{ \tilde{P} \left[\sigma^i_m \tilde{k}_i(t) \right] \right\} =$$

$$= \frac{\partial}{\partial k_m} k_i \sigma^i_m \tilde{P} + \eta k^2 \frac{\partial}{\partial \theta} \theta \tilde{P}$$

Average:

$$\partial_t P = \frac{\partial}{\partial k_m} k_i \langle \sigma^i_m \tilde{P} \rangle + \eta k^2 \frac{\partial}{\partial \theta} \theta P \quad (5)$$

$$\langle \sigma^i_m(t) \tilde{P}(t) \rangle = \int dt' \langle \sigma^i_m(t) \sigma^j_n(t') \rangle \langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t')} \rangle =$$

$$= \varkappa_2 T^{ij}_{mn} \langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t)} \rangle$$

$$\frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t)} = \frac{\delta}{\delta \sigma^j_n(t)} \int^t dt' \left[\frac{\partial}{\partial k_m} k_i \sigma^i_m(t') \tilde{P}(t') + \eta k^2(t') \frac{\partial}{\partial \theta} \theta \tilde{P}(t') \right] =$$

$$= \frac{1}{2} \frac{\partial}{\partial k_n} k_j \tilde{P}(t)$$ same way as in my notes on the dynamo.

So, we get from (5):

$$\partial_t P = \frac{\varkappa_2}{2} \underbrace{T^{ij}_{mn} \frac{\partial}{\partial k_m} k_i \frac{\partial}{\partial k_n} k_j P}_{\equiv \circledast} + \eta k^2 \frac{\partial}{\partial \theta} \theta P \qquad (6)$$

By isotropy, $P(\vec{k}, \theta) = P(k, \theta)$ — only depends on $k = |\vec{k}|$.

Therefore

$$\circledast = T^{ij}_{mn} \frac{\partial}{\partial k_m} k_i \frac{\partial}{\partial k_n} k_j P = T^{ij}_{mn} \left(\cancel{\delta^m_i} + k_i \frac{\partial}{\partial k_m} \right) \left(\cancel{\delta^n_j} + k_j \frac{\partial}{\partial k_n} \right) P =$$

vanish because $T^{ij}_{in} = T^{ij}_{mj} = 0$

$$= T^{ij}_{mn} k_i \frac{\partial}{\partial k_m} k_j \frac{k_n}{k} \frac{\partial P}{\partial k} = T^{ij}_{mn} \left[\frac{k_i k_j k_n k_m}{k} \frac{\partial}{\partial k} \frac{1}{k} \frac{\partial P}{\partial k} + \right.$$

$$\left. + \cancel{\delta^m_j} \frac{k_i k_n}{k} \frac{\partial P}{\partial k} + \frac{k_i k_j}{k} \delta^{mn} \frac{\partial P}{\partial k} \right] =$$

vanishes because $T^{ij}_{in} = 0$

$$= \frac{d-1}{d+1}\, k^3 \frac{\partial}{\partial k} \frac{1}{k} \frac{\partial P}{\partial k} + \frac{(d-1)(d+2)}{d+1}\, k \frac{\partial P}{\partial k} =$$

$$T^{ij}_{mn} \frac{k_i k_j k_m k_n}{k^4} \qquad\qquad T^{ij}_{mn}\, \delta^{mn} \frac{k_i k_j}{k^2}$$

$$= \frac{d-1}{d+1} \left[k^2 \frac{\partial^2 P}{\partial k^2} + (d+1)\, k \frac{\partial P}{\partial k} \right]$$

Eq. (6) becomes:

$$\partial_t P = \frac{1}{2} \frac{d-1}{d+1}\, \varkappa_2 \left[k^2 \frac{\partial^2 P}{\partial k^2} + (d+1)\, k \frac{\partial P}{\partial k} \right] + \eta k^2 \frac{\partial}{\partial \theta} \theta P \qquad (7)$$

Normalization:

$$1 = \int d^d k \int_{-\infty}^{+\infty} d\theta\, P = S_d \int_0^\infty dk\, k^{d-1} \int_{-\infty}^{+\infty} d\theta\, P$$

Define $F(k,\theta) = S_d\, k^{d-1} P$, so Eq. (7) gives [see p. 69 of my notes]

$$\partial_t F = \frac{1}{2} \frac{d-1}{d+1}\, \varkappa_2 \left[\frac{\partial}{\partial k} k^2 \frac{\partial F}{\partial k} - 2d \frac{\partial}{\partial k} kF + \cancel{d(d+1)F} + \right.$$

$$\left. + (d+1) \frac{\partial}{\partial k} kF - \cancel{(d+1)d\, F} \right] + \eta k^2 \frac{\partial}{\partial \theta} \theta F =$$

$$= \frac{1}{2} \frac{d-1}{d+1}\, \varkappa_2 \frac{\partial}{\partial k} \left[k^2 \frac{\partial F}{\partial k} - (d-1)\, kF \right] + \eta k^2 \frac{\partial}{\partial \theta} \theta F \qquad (8)$$

Spectrum of the passive scalar is, by the same derivation as on p. 71 of my notes,

$$\frac{1}{2} \int d\Omega_d\, k^{d-1} \langle |\vec{B}(t,\vec{k})|^2 \rangle = \int d^d k_0\, T(t,k), \quad \text{where}$$

$$T(t,k) = \frac{1}{2} \int_{-\infty}^{\infty} d\theta\, \theta^2 F(\theta,k)$$

Now $\int_{-\infty}^{+\infty} d\Theta\,\Theta^2$ [Eq. (8)]:

$$\boxed{\partial_t T = \frac{1}{2}\frac{d-1}{d+1}\varkappa_2 \frac{\partial}{\partial k}\left[k^2\frac{\partial T}{\partial k} - (d-1)kT\right] - 2\eta k^2 T} \qquad (9)$$

Denote $\frac{1}{2}\frac{d-1}{d+1}\varkappa_2 \equiv D$. This is an equation first derived by Kraichnan in 1968.

If we look for (decaying) eigenfunctions $T(t,k) \propto e^{-\lambda D t}$, we get

$$\boxed{T(t,k) = T_0\, e^{-\lambda D t}\left(\frac{k}{k_\eta}\right)^{-1+\frac{d}{2}} K_{\sqrt{\frac{d^2}{4}-\lambda}}\left(\frac{k}{k_\eta}\right)} \qquad (10)$$

where $k_\eta \doteq \left(\frac{D}{2\eta}\right)^{1/2}$, K is the Macdonald function (modified Bessel function that decays at $\frac{k}{k_\eta}\to\infty$); T_0 is some constant.

If we define $\mathcal{F}(k) = -D\left[k^2 T' - (d-1)kT\right]$, eq. (9) is

$$\frac{\partial T}{\partial t} = -\frac{\partial}{\partial k}\mathcal{F}(k) - 2\eta k^2 T$$

Total scalar variance at $k>k_*$ $\mathcal{E}_\theta(t) = \int_{k_*}^{\infty} dk\, T(t,k)$ satisfies

$$\frac{\partial \mathcal{E}_\theta}{\partial t} = -\cancelto{0}{\mathcal{F}(\infty)} + \underbrace{\mathcal{F}(k_*)}_{\text{flux from } k<k_*} - \underbrace{2\eta\int dk\, k^2 T}_{=\frac{1}{2}\langle|\nabla\theta|^2\rangle}$$

~~But $\partial_t\theta + \mathbf{u}\cdot\nabla\theta = \eta\nabla^2\theta + f$ — large-scale source f does not appear at $k>k_*$~~

~~So for scalar variance at $k<k_*$, we have~~

$$\cancel{\frac{\partial \mathcal{E}^*_\theta}{\partial t} = \frac{1}{2}\langle\theta f\rangle - \mathcal{F}(k_*)}$$ ~~flux into $k>k_*$~~

Clearly, we can write

$$\mathcal{F}(k_*) = \epsilon_\theta \leftarrow \text{flux of scalar variance arriving from scales } k < k_*$$

So the boundary condition (that determines λ) is

$$\boxed{D\left[k_*^2 T'(k_*) - (d-1) k_* T'(k_*)\right] = -\epsilon_\theta} \qquad (11)$$

1) Forced scalar : $\epsilon_\theta = \text{const}$ (source at large scales)

Then $\lambda = \overset{(\text{st-state})}{0}$ and

$$T(k) = T_0 \left(\frac{k}{k_\eta}\right)^{-1+\frac{d}{2}} K_{\frac{d}{2}}\left(\frac{k}{k_\eta}\right)$$

(d=3) : useful to know that $K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2x}}\left(1+\frac{1}{x}\right)e^{-x}$,

so, for $k \ll k_\eta$,

$$T(k) \simeq T_0 \sqrt{\frac{\pi}{2}} \left(\frac{k}{k_\eta}\right)^{-1}$$ and at $k = k_*$, eq. (11) gives

$$D k_* T_0 \sqrt{\frac{\pi}{2}} \left[\cancel{-\left(\frac{k_*}{k_\eta}\right)^{-1}} - (d-\cancel{1}) \left(\frac{k_*}{k_\eta}\right)^{-1} \right] = -\epsilon_\theta$$

$$D T_0 \sqrt{\frac{\pi}{2}}\, d\, k_\eta = \epsilon_\theta \Rightarrow T_0 = \sqrt{\frac{2}{\pi}} \frac{\epsilon_\theta}{D k_\eta \underbrace{d}_{=3}}$$

(d=2) : $K_1\left(\frac{k}{k_\eta}\right) \approx \left(\frac{k}{k_\eta}\right)^{-1}$ for $k \ll k_\eta$, so,

by an analogous calculation, $T_0 = \dfrac{\epsilon_\theta}{D k_\eta \cdot 2}$

In either case, we have the Batchelor k^{-1} spectrum.

(you will always get this in systems with const flux and a single timescale)

2) <u>Decaying scalar</u>: $\epsilon_\theta = 0$ (no forcing)

Then eq. (11) reduces to [substitute (10)]:

$$\frac{k_*}{k_\eta} K'_{\sqrt{\frac{d^2}{4}-\lambda}}\left(\frac{k_*}{k_\eta}\right) - \frac{d}{2} K_{\sqrt{\frac{d^2}{4}-\lambda}}\left(\frac{k_*}{k_\eta}\right) = 0$$

$$\underbrace{\left(\sqrt{\frac{d^2}{4}-\lambda} - \frac{d}{2}\right)} K_{\sqrt{\frac{d^2}{4}-\lambda}}\left(\frac{k_*}{k_\eta}\right) - \left(\frac{k_*}{k_\eta}\right) K_{\sqrt{\frac{d^2}{4}-\lambda}+1}\left(\frac{k_*}{k_\eta}\right) = 0$$

cannot be positive, otherwise $\lambda < 0$ (growth -- impossible for unforced scalar)

For $k_* \ll k_\eta$, we get from the above for $\lambda > \frac{d^2}{4}$:

$$K_{i\sqrt{\lambda-\frac{d^2}{4}}}\left(\frac{k_*}{k_\eta}\right) \approx \frac{\sin\left(\sqrt{\lambda-\frac{d^2}{4}}\ln\frac{k_*}{2k_\eta}\right)}{\sqrt{\lambda-\frac{d^2}{4}}} = 0$$

$$\Downarrow$$

$$\sqrt{\lambda-\frac{d^2}{4}}\ln\frac{k_*}{2k_\eta} = \pi$$

$$\Downarrow$$

$$\boxed{\lambda = \frac{d^2}{4} + \frac{\pi^2}{[\ln(k_*/2k_\eta)]^2} = \frac{d^2}{4} + O\left(\frac{1}{(\ln Sc)^2}\right)}$$

The spectrum is $\Gamma(k,t) \approx e^{-\frac{d^2}{4}Dt}\left(\frac{k}{k_\eta}\right)^{-1+\frac{d}{2}} K_0\left(\frac{k}{k_\eta}\right)$

$\sim k^{-1+\frac{d}{2}}$ for $k \ll k_\eta$.

This is the result that has been believed to describe scalar decay until it was shown experimentally that the decay rate is $\ll \frac{d^2}{4}D \sim \varkappa_2 \sim \frac{1}{t_\nu}$ (visc. eddy time).

3) Slow decay.

What actually happens is that the decay rate of the scalar is determine by the decay rate of the slowest-decaying mode, which is a large-scale (system-size) mode — it decays at the rate of turbulent diffusion associated with the energy-containg eddies; so

$$\text{decay rate} \sim \delta u_L L \frac{1}{\underset{\uparrow \text{"box size"}}{L_{box}^2}}.$$

$$\left(\frac{l_\nu}{L}\right)^{2/3}$$

Then
$$\lambda \sim \frac{\delta u_L L / L_{box}^2}{\delta u_{l_\nu}/l_\nu} \sim \left(\frac{L}{L_{box}}\right)^2 \underbrace{\frac{\delta u_L/L}{\delta u_{l_\nu}/l_\nu}}_{=\left(\frac{l_\nu}{L}\right)^{2/3}} \sim \left(\frac{L}{L_{box}}\right)^2 Re^{-\frac{1}{2}} \ll 1$$

(assuming $L_{box} \geq L$).

~~This corresponds to the boundary condition at [illegible] with slowly decaying~~

Solution (10) is still valid, but with λ determined by the box mode. At $k \ll k_\eta$, we have

$$T(t,k) \sim e^{-\lambda D t} \left(\frac{k}{k_\eta}\right)^{\overbrace{-1+\frac{d}{2}-\sqrt{\frac{d^2}{4}-\lambda}}} \sim e^{-\lambda D t}\left(\frac{k}{k_\eta}\right)^{-1+\frac{\lambda}{d}}$$

$$-1+\frac{d}{2}\left(1-\sqrt{1-\frac{4\lambda}{d^2}}\right) \approx -1+\frac{\lambda}{d} \quad \text{for } \lambda \ll 1$$

Thus, the spectrum is only slightly shallower than k^{-1}

[Fereday & Haynes (2004), Schekochihin, Haynes, & Cowley (2004)] } see course log.