Magnetohydrodynamics and Turbulence

Alexander Schekochihin, Part III (CASM) Lent Term 2005

EXAMPLE SHEET III

These problems will be discussed in the 3d Examples Class (9.03.05, 14:30 in MR5).

1. Anisotropic k-Space Correlation Functions. Consider the correlation function if the velocity field in k space:

$$\langle u_i(\mathbf{k}) u_j(\mathbf{k}')\rangle = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') C_{ij}(\mathbf{k}) \tag{1}$$

Suppose there is one special direction in space, defined by the unit vector $\hat{\mathbf{b}}$ (this can be the direction of an imposed magnetic field or the axis of rotation or the direction of gravity). Then the general form of the tensor C_{ij} is

$$C_{ij}(\mathbf{k}) = C_1\delta_{ij} + C_2\hat{k}_i\hat{k}_j + C_3\hat{b}_i\hat{b}_j + C_4\hat{b}_i\hat{k}_j + C_5\hat{k}_i\hat{b}_j, \tag{2}$$

where $\hat{k}_i = \mathbf{k}/k$ and C_1, ..., C_5 are functions of k and of $\xi = \hat{\mathbf{b}}\cdot\hat{\mathbf{k}} = \cos\theta$ (θ is the angle between $\mathbf{k}$ and $\hat{\mathbf{b}}$, so $k_\parallel = \xi k$).

1. Assuming mirror symmetry, $C_{ij}(\mathbf{k}) = C_{ij}(-\mathbf{k})$, and incompressibility of the velocity field, show that C_{ij} can be written in the form

$$C_{ij}(\mathbf{k}) = C^{\mathrm{iso}}(k,\xi)\left(\delta_{ij} - \hat{k}_i\hat{k}_j\right) + C^{\mathrm{aniso}}(k,\xi)\left[\hat{b}_i\hat{b}_j + \xi^2\hat{k}_i\hat{k}_j - \xi\left(\hat{b}_i\hat{k}_j + \hat{k}_i\hat{b}_j\right)\right]. \tag{3}$$

   Express C^{iso} and C^{aniso} in terms of C_1, ..., C_5. Thus, second-order velocity correlator depends on two scalar functions only. We can get back the isotropic result by setting $C^{\mathrm{aniso}} = 0$.

2. An alternative pair of scalar functions is often useful: the correlation function $C_\parallel(k,\xi)$ of the velocities along $\hat{\mathbf{b}}$ and the correlation function $C_\perp(k,\xi)$ of the velocities in the plane perpendicular to $\hat{\mathbf{b}}$. Give definitions for these functions that you think are appropriate and express them in terms of C^{iso} and C^{aniso}.

3. Suppose all variation of the velocity along $\hat{\mathbf{b}}$ is suppressed. What happens to the tensor C_{ij}?

2. Scalar Turbulence. Part I: Yaglom's $\frac{4}{3}$ Law. Consider the equation for the evolution of *passive scalar* $\theta(t,\mathbf{x})$ (this can be temperature, or concentration of an admixture like a dye or salt, or, in 2D hydrodynamics, the vorticity field, or, in RMHD, the magnetic flux function, etc.):

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \kappa\nabla^2\theta + f, \tag{4}$$

where $\mathbf{u}$ is the (turbulent) velocity field, κ is the scalar diffusivity, and f is the source function (scalar "forcing"). We will assume that f varies at some (large) scale $L_\theta < L$ (L is the outer scale of the turbulence).

1. Define the scalar variance $\mathcal{E}_\theta = \langle\theta^2\rangle/2$ ("energy" of the scalar field), the scalar correlation function $C(y) = \langle\theta(\mathbf{x}_1)\theta(\mathbf{x}_2)\rangle$, and the scalar structure function $S(y) = \langle\delta\theta^2\rangle$, where $\delta\theta = \theta(\mathbf{x}_2) - \theta(\mathbf{x}_1)$ and $\mathbf{y} = \mathbf{x}_2 - \mathbf{x}_1$. Express $S(y)$ in terms of $C(y)$ and $\mathcal{E}$.

2. Define a mixed 3d-order correlation function $F_i(\mathbf{y}) = \langle u_i(\mathbf{x}_1)\theta(\mathbf{x}_1)\theta(\mathbf{x}_2)\rangle = F(y)\hat{\mathbf{y}}_i$ and the corresponding structure function $G_i(\mathbf{y}) = \langle \delta u_i \delta\theta^2\rangle = G(y)\hat{\mathbf{y}}_i$, where $\delta u_i = u_i(\mathbf{x}_2) - u_i(\mathbf{x}_1)$ and $\hat{\mathbf{y}} = \mathbf{y}/y$. Show that $G(y) = 4F(y)$.
   Hint. Any one-point average that is a first-rank tensor (vector) is zero by isotropy (why?). Also, $\langle \mathbf{u}(\mathbf{x}_1)a(\mathbf{x}_2)\rangle = 0$ for any scalar field a (at which point in my lecture on the $\frac{4}{5}$ Law did I prove this?).

3. Now, proceeding analogously to the derivation of the $\frac{4}{5}$ Law in my lectures, derive the analog of the von Kármán–Howarth equation for the passive scalar:
$$\frac{\partial S}{\partial t} = 4\frac{d\mathcal{E}}{dt} - 4\epsilon_\theta(y) - \frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}G(y) + 2\kappa\frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}\frac{\partial S}{\partial y}, \tag{5}$$
   where $\epsilon_\theta(y) = \langle \theta(\mathbf{x}_1)f(\mathbf{x}_2)\rangle$.

4. Consider the statistically steady state and show that for $y \ll L_\theta$,
$$G(y) = -\frac{4}{d}\,\bar{\epsilon}_\theta y + 2\kappa S'(y), \tag{6}$$
   where $\bar{\epsilon}_\theta = \epsilon_\theta(0) = \langle \theta f\rangle$ the input variance per unit time. Show from Eq. (4) that $\bar{\epsilon}_\theta = \kappa\langle|\nabla\theta|^2\rangle$ (scalar dissipation per unit time). Equation (6) for $d = 3$ is the *Yaglom's* $\frac{4}{3}$ *Law.*

5. Show that if $f = 0$ and we consider a self-similar decay of the scalar ($\partial S/\partial t = 0$), Eq. (6) is still satisfied. What is $\bar{\epsilon}_\theta$ in this case?

3. Scalar Turbulence. Part II: The Oboukhov-Corrsin Spectrum. Now you are going to develop a dimensional theory of scalar turbulence *à la* K41 theory I described in my lectures.

1. Let us figure out when the diffusive term in Eq. (6) is negligible. Assume that $S(y) \sim \delta\theta_l^2$ and (dimensionally) $\bar{\epsilon}_\theta \sim \delta\theta_l^2/\tau_l$ (flux of scalar variance), where $\delta\theta$ is the scalar variation across scale $l = y$ and τ_l is some cascade time. Show that the diffusive term is negligible if
$$\frac{\kappa\tau_l}{l^2} \ll 1. \tag{7}$$

2. Assume that $\tau_l \sim l/\delta u_l$ (why?) and show that, for δu_l satisfying the K41 scaling, Eq. (7) reduces to $l \gg l_\kappa = \mathrm{Sc}^{-3/4} l_\nu$, where $l_\nu = (\nu^3/\epsilon)^{1/4}$ is the viscous scale, ϵ is the Kolmogorov flux, and $\mathrm{Sc} = \nu/\kappa$ is called the *Schmidt number.*

   Note that, since you have used K41 inertial-range scaling for the cascade time, your estimates are only correct for $\mathrm{Sc} \ll 1$ (why?).

3. Show that an equivalent expression for the diffusive scale is $l_\kappa \sim \mathrm{Pe}^{-3/4} L_\theta$ (provided the characteristic scale of the scalar source is $L_\theta < L$), where $\mathrm{Pe} = \delta u_{L_\theta} L_\theta/\kappa$ is called the *Péclet number* (analog of the Reynolds number for scalars).

   The scale range of l such that $L > L_\theta \gg l \gg l_\theta \gg l_\nu$ is called *the inertial-convective range.* It is non-empty if $\mathrm{Re} \gg \mathrm{Pe} \gg 1$.

4. Using the Yaglom's law, show that, for l in the inertial-convective range,
$$\delta\theta \sim \bar{\epsilon}_\theta^{1/2}\epsilon^{-1/6}l^{1/3}, \tag{8}$$
   or, for the spectrum of scalar variance,
$$E_\theta(k) \sim \bar{\epsilon}_\theta\epsilon^{-1/3}k^{-5/3} \tag{9}$$
   (the Oboukhov-Corrsin spectrum). Sketch the spectra of the kinetic energy and of the scalar variance, indicating all relevant wavenumbers $k \sim 1/l$ and slopes.

5. Show that Eq. (8) can be derived purely dimensionally (without recourse to Yaglom's law) by assuming that the flux of scalar variance $\bar{\epsilon}_\theta$ is independent of l in the inertial-convective range.

4. Scalar Turbulence. Part III: The Batchelor Spectrum. What if $\mathrm{Sc} \gg 1$? Then l_κ we calculated in Problem 3 is smaller than l_ν. Our dimensional theory only applies to $l \gg l_\nu$. Let us figure out what the scalar does at $l \ll l_\nu$.

1. Use the scaling of $\tau_l \sim l/\delta u_l$ in the viscous range ($l < l_\nu$) derived in my lectures to show that Eq. (7) reduces to $l \gg l_\kappa = \mathrm{Sc}^{-1/2} l_\nu$ — the new expression for the diffusive scale in the limit $\mathrm{Sc} \gg 1$.

   The scale range $l_\nu \gg l \gg l_\kappa$ is called *the viscous-convective range* (or *subviscous range*).

2. In a manner analogous to what you did in Problem 3, use Yaglom's law or the assumption that $\bar{\epsilon}_\theta$ is independent of l to show that, for l in the viscous-convective range,

$$\delta\theta \sim \bar{\epsilon}_\theta^{1/2} \epsilon^{-1/4} \nu^{1/4}, \tag{10}$$

   (independent of scale!) or, for the spectrum of scalar variance,

$$E_\theta(k) \sim \bar{\epsilon}_\theta \epsilon^{-1/2} \nu^{1/2} k^{-1} \tag{11}$$

   (the Batchelor spectrum). This spectrum is the result of these two properties of the viscous-convective range: (i) flux of scalar variance is independent of l, (ii) cascade time is independent of l (and equal to the turnover time of the viscous-scale eddies — confirm this is so!).

3. Thus, in the inertial-convective range, we have the Oboukhov-Corrsin spectrum, in the viscous-convective range, we have the Batchelor spectrum. Sketch the spectra of the kinetic energy and of the scalar variance in the case $\mathrm{Sc} \gg 1$, indicating all relevant wavenumbers $k \sim 1/l$ and slopes.