ES III. Problems 2-4 Scalar Turbulence

Yaglom's $\frac{4}{3}$ law for Passive Scalar Fields

$$\boxed{\frac{\partial\theta}{\partial t} + \vec{u}\cdot\nabla\theta = \varkappa\nabla^2\theta + f} \qquad (1)$$

f ← source

Define the correlation function for the scalar:

$$C(\vec{y}) = \langle \theta(\vec{x}_1)\,\theta(\vec{x}_2)\rangle \quad , \; \vec{y} = \vec{x}_2 - \vec{x}_1$$

and the structure function

$$S(\vec{y}) = \langle \delta\theta^2\rangle = \langle [\theta(\vec{x}_2) - \theta(\vec{x}_1)]^2\rangle =$$

$$= \langle\theta_2^2\rangle + \langle\theta_1^2\rangle - 2\langle\theta_1\theta_2\rangle = \underbrace{2\langle\theta^2\rangle}_{*} - 2C(y)$$

We have than

$$\partial_t C(y) = \partial_t\langle\theta_1\theta_2\rangle = \langle\theta_1\partial_t\theta_2\rangle + \langle\theta_2\partial_t\theta_1\rangle =$$

$$= \Big\langle -\theta_1 u_{2l}\frac{\partial\theta_2}{\partial x_{2l}} + \varkappa\,\theta_1\nabla_2^2\theta_2 + \theta_1 f_2$$

$$- \theta_2 u_{1l}\frac{\partial\theta_1}{\partial x_{1l}} + \varkappa\,\theta_2\nabla_1^2\theta_1 + \theta_2 f_1\Big\rangle =$$

$$= -\frac{\partial}{\partial x_{1l}}\langle\theta_1\theta_2 u_{1l}\rangle - \frac{\partial}{\partial x_{2l}}\langle\theta_1\theta_2 u_{2l}\rangle +$$

$$+ \underbrace{\varkappa\nabla_1^2\langle\theta_1\theta_2\rangle + \varkappa\nabla_2^2\langle\theta_1\theta_2\rangle}_{=\,2\varkappa\nabla^2 C(y)} + \underbrace{\langle\theta_1 f_2\rangle + \langle\theta_2 f_1\rangle}_{\equiv\, 2\epsilon_\theta(y)}$$

$$\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_i}C(y) = \frac{\partial}{\partial y_i}\frac{y_i}{y}C'(y) = \frac{d}{y}C' - \frac{y_i}{y^2}\frac{y_i}{y}C' + \frac{y_i}{y}\frac{y_i}{y}C'' =$$

$$= \frac{d-1}{y}C' + C'' = \frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}\frac{\partial C}{\partial y}$$

(for convenience could again assume white forcing with
$\langle f(t,\vec{x}_1)\,f(t',\vec{x}_2)\rangle = \delta(t-t')\,2\epsilon_\theta(y)$)

Introduce the following 3-order mixed correlation function:

$$F_\ell(\vec y) = \langle u_\ell(\vec x_1)\,\theta(\vec x_1)\,\theta(\vec x_2)\rangle = F(y)\,\hat y_\ell \quad \text{by isotropy and homogeneity}$$

and a corresponding structure function:

$$G_\ell(\vec y) = \langle \delta u_\ell\, \delta\theta^2\rangle = \langle (u_{2\ell}-u_{1\ell})(\theta_2-\theta_1)^2\rangle =$$

$$= \langle u_{2\ell}(\theta_2^2 - 2\theta_1\theta_2 + \theta_1^2)\rangle - \langle u_{1\ell}(\theta_2^2 - 2\theta_2\theta_1 + \theta_1^2)\rangle =$$

$$= \underbrace{\langle u_{2\ell}\theta_2^2\rangle}_{0} - 2\underbrace{\langle u_{2\ell}\theta_1\theta_2\rangle}_{= F_\ell(-\vec y) = -F_\ell(\vec y)} + \underbrace{\langle u_{2\ell}\theta_1^2\rangle}_{0} - \underbrace{\langle u_{1\ell}\theta_2^2\rangle}_{0} + 2\underbrace{\langle u_{1\ell}\theta_2\theta_1\rangle}_{= F_\ell(\vec y)} - \underbrace{\langle u_{1\ell}\theta_1^2\rangle}_{0} =$$

because 1-rank tensors that depend on nothing

the same way as showed in the lecture that $\langle p_1 u_{2k}\rangle = \langle p_2 u_{1k}\rangle = 0$

$$= 4F_\ell(\vec y) \equiv G(y)\,\hat y_\ell, \quad \text{where } G(y) = 4F(y)$$

So we get

$$\partial_t C(y) - 2\varkappa \frac{1}{y^{d-1}}\frac{\partial}{\partial y} y^{d-1}\frac{\partial C}{\partial y} - 2\epsilon_\theta(y) = \frac{\partial}{\partial y_\ell} 2F_\ell(y) =$$

$$= \frac{\partial}{\partial y_\ell} 2F(y)\frac{y_\ell}{y} = 2\left[\frac{d}{y}F - \frac{y_\ell}{y^2}\frac{y_\ell}{y}F + F'\frac{y_\ell}{y}\frac{y_\ell}{y}\right] =$$

$$= 2\left[\frac{d-1}{y}F + F'\right] = 2\frac{1}{y^{d-1}}\frac{\partial}{\partial y} y^{d-1}F(y) = \frac{1}{2}\frac{1}{y^{d-1}}\frac{\partial}{\partial y} y^{d-1}G(y)$$

Recall that

$$C(y) = -\frac{1}{2}S(y) + 2\mathcal{E}_\theta$$

where $\mathcal{E}_\theta = \frac{1}{2}\langle\theta^2\rangle$ scalar variance (like energy)

Then we can write

$$2\frac{d\mathcal{E}_\theta}{dt}-\frac{1}{2}\partial_t S(y)-2\epsilon_\theta(y)+\chi\frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}\frac{\partial S(y)}{\partial y}=\frac{1}{2}\frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}G(y)$$

$$\boxed{\frac{\partial S(y)}{\partial t}=4\frac{d\mathcal{E}_\theta}{dt}-4\epsilon_\theta(y)-\frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}G(y)+2\chi\frac{1}{y^{d-1}}\frac{\partial}{\partial y}y^{d-1}\frac{\partial S(y)}{\partial y}}\quad(2)$$

– analog of the von Kármán–Howarth equation for the scalar.

Steady state: $\frac{\partial}{\partial t}=0$. Then

Denote $\bar{\epsilon}_\theta=\epsilon_\theta(0)$

$$\frac{\partial}{\partial y}y^{d-1}\left[G(y)-2\chi S'(y)\right]=-4\underbrace{\epsilon_\theta(y)}_{\simeq\,\epsilon_\theta(0)+\dots\text{ for } y\ll L_\theta}y^{d-1}\approx-4\bar{\epsilon}_\theta\,y^{d-1}$$

L_θ – the scale of the scalar forcing

Integrate:

$$G(y)-2\chi S'(y)=\Big[-\frac{4}{d}\bar{\epsilon}_\theta\,y^d+\underbrace{\text{const}}_{=0}\Big]\frac{1}{y^{d-1}}$$

0 so there is no singularity at $y=0$

Finally,

$$\boxed{G(y)=-\frac{4}{d}\bar{\epsilon}_\theta\,y+\underbrace{2\chi S'(y)}_{(*)}}\quad(3)$$

$\frac{4}{3}$ for $d=3$

Now let us figure out when the diffusive term (*) is smaller than the linear ($\propto y$) term.

$$\frac{|2\chi S'(y)|}{|\frac{4}{3}\bar{\epsilon}_\theta y|}\sim\frac{\chi\,\delta\theta_\ell^2/\ell}{\bar{\epsilon}_\theta\,\ell}\sim\frac{\chi\tau_\ell}{\ell^2}\ll 1$$

$y\sim\ell$ scale
$\delta\theta_\ell$ – variation of θ over scale ℓ

flux of scalar variance: $\bar{\epsilon}_\theta\sim\delta\theta_\ell^2\cdot\tau_\ell^{-1}$
"cascade time"

Now what is the cascade time?

Recall $\partial_t \theta + \underline{\vec{u}\cdot\nabla\theta} = \varkappa \nabla^2\theta + f$

$\uparrow$ $\underline{\underline{\text{cascade term}}} \Rightarrow \tau_\ell \sim \frac{\ell}{\delta u_\ell}$ just like in turbulence

So $$\frac{\varkappa \tau_\ell}{\ell^2} \sim \frac{\varkappa \ell}{\ell^2 \delta u_\ell} \sim \frac{\varkappa}{\delta u_\ell \ell} \sim \frac{\varkappa}{\epsilon^{1/3}\ell^{4/3}} \ll 1 \quad (4)$$

$\delta u_\ell \sim (\epsilon \ell)^{1/3}$ for inertial range of turbulence

So, we must have

$$\ell \gg \underbrace{\varkappa^{3/4}\epsilon^{-1/4}}_{} \equiv \ell_\varkappa = \left(\frac{\varkappa}{\nu}\right)^{3/4} \underbrace{\frac{\nu^{3/4}}{\epsilon^{1/4}}}_{=\ell_\nu} \equiv Sc^{-3/4}\,\ell_\nu$$

$\ell_\varkappa$: diffusive scale

ℓ_ν: viscous scale

where $Sc = \frac{\nu}{\varkappa}$ is the $\underline{\text{Schmidt number}}$

NB: we used the inertial-range scaling for the velocity field, so the diffusive scale we have estimated must lie inside the inertial range:

$$L \gg \ell_\varkappa \gg \ell_\nu \Rightarrow \underline{\underline{Sc \ll 1}}$$

NB: We can rewrite $\ell_\varkappa$ as follows:

$$\ell_\varkappa = \varkappa^{3/4}\epsilon^{-1/4} = \left(\frac{\varkappa}{\delta u_{L_\theta} L_\theta}\right)^{3/4} \frac{L_\theta^{1/4}}{(\delta u_{L_\theta}^3)^{1/4}} (\delta u_{L_\theta} L_\theta)^{3/4}$$

$\epsilon = \frac{\delta u_{L_\theta}^3}{L_\theta}$ where L_θ is the scale of the scalar forcing (source)

$= Pe^{-3/4} L_\theta$, where $Pe = \frac{\delta u_{L_\theta} L_\theta}{\varkappa}$ is the $\underline{\text{Péclet number}}$

Note that we assumed $L_\theta < L$ (turbulence outer scale)

Thus, we have found that if

$$L > L_\theta \gg \underset{\substack{\wr \\ l}}{y} \gg l_\chi \gg l_\nu = Re^{-3/4} L,$$

$$l_\chi = Pe^{-3/4} L_\theta = Sc^{-3/4} l_\nu$$

This is called <u>inertial-convective range</u>

$$\boxed{G(y) = -\frac{4}{3}\bar{\epsilon}_\theta\, y}$$

3D

, where $\bar{\epsilon}_\theta$ is the flux of scalar variance.

This is called <u>Yaglom's $\frac{4}{3}$ law</u>.

What is the <u>scaling / spectrum of the passive scalar in the inertial-convective range</u>?

Recall $G(y) = \langle (\delta\theta)^2 \delta u_L \rangle \sim \delta\theta_l^2\, \delta u_l$, where $l \sim y$

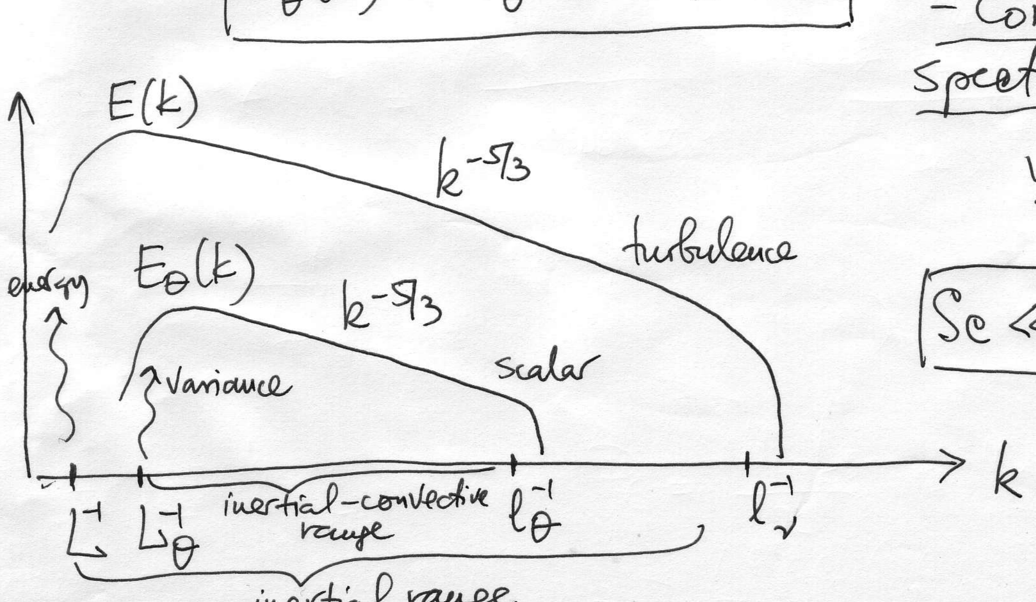
longitudinal
$\delta u_L = \delta\vec{u}\cdot\hat{y}$

Yaglom's law says $G(y) \propto \bar{\epsilon}_\theta y$, so $\delta\theta_l^2\, \delta u_l \sim \bar{\epsilon}_\theta\, l$

$$\delta\theta_l \sim \left(\frac{\bar{\epsilon}_\theta\, l}{\delta u_l}\right)^{1/2} \sim \left(\frac{\bar{\epsilon}_\theta\, l}{(\epsilon l)^{1/3}}\right)^{1/2} = \bar{\epsilon}_\theta^{1/2}\, \epsilon^{-1/6}\, l^{1/3} \quad (5)$$

whence $\boxed{E_\theta(k) \sim \bar{\epsilon}_\theta\, \epsilon^{-1/3}\, k^{-5/3}}$

<u>Obukhov – Corrsin spectrum of scalar variance</u>



$\boxed{Sc \ll 1}$

This result could also be derived analogously to the K41 dimensional theory by assuming

$$\bar{\epsilon}_\theta \sim \delta\theta_\ell^2 \cdot \tau_\ell^{-1} \sim \delta u_\ell \delta\theta_\theta^2 / \ell = \text{const}$$

$$\Downarrow$$

$$\delta\theta_\ell \sim \left(\frac{\bar{\epsilon}_\theta \ell}{\delta u_\ell}\right)^{1/2} \text{ etc...}$$

What if $Sc \gg 1$? Then $\ell_\varkappa$ calculated above is $\ll \ell_\nu$. So for all $\ell > \ell_\nu$, we have the same theory, but for $\ell < \ell_\nu$, we have to reassess the situation. In eq. (4), we had:

$$\frac{\varkappa \tau_\ell}{\ell^2} \sim \frac{\varkappa}{\delta u_\ell \ell} \sim \frac{\varkappa}{(\epsilon/\nu)^{1/2} \ell^2} \ll 1 \Rightarrow \ell \gg \left(\frac{\epsilon}{\nu}\right)^{-\frac{1}{4}} \varkappa^{1/2}$$

$\ell < \ell_\nu$ (dissipation range)
$\delta u_\ell \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell$ – see lecture on K41 dim. theory

So, the new estimate for the diffusive scale is

$$\ell_\varkappa \sim \left(\frac{\epsilon}{\nu}\right)^{-1/4} \varkappa^{1/2} \sim \left(\frac{\varkappa}{\nu}\right)^{1/2} \underbrace{\frac{\nu^{3/4}}{\epsilon^{1/4}}}_{= \ell_\nu} \sim Sc^{-1/2} \ell_\nu \ll \ell_\nu$$

because $Sc \gg 1$

~~What is the spectrum in~~

We have another scale range:

$$\ell_\nu \gg \ell \gg \ell_\varkappa \sim Sc^{-1/2} \ell_\nu$$

– viscous-convective range.

What is the passive scalar scaling/spectrum in the viscous-convective range?

From Yaglom's law or by assuming $\bar{\epsilon}_\theta = \text{const}$, we get [as in Eq. (5)]:

$$\delta\theta_\ell \sim \left(\frac{\bar{\epsilon}_\theta \, \ell}{\delta u_\ell}\right)^{1/2} \sim \bar{\epsilon}_\theta^{1/2} \, \epsilon^{-\frac{1}{4}} \, \nu^{1/4}$$

$$\hookrightarrow \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \ell$$

or $\boxed{E_\theta(k) \sim \bar{\epsilon}_\theta \, \epsilon^{-1/2} \, \nu^{1/2} \, k^{-1}}$ Batchelor spectrum.

Note that this spectrum is a result of

1) const. flux $\bar{\epsilon}_\theta$
2) const. cascade time $\tau_\ell \sim \frac{\ell}{\delta u_\ell} \sim \left(\frac{\nu}{\epsilon}\right)^{1/2}$ in the visc. range.