~~E Class. 9.03.05~~
MOJ: 8.11.05

ES ~~III~~ I – Problem 1

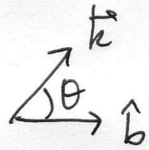
Anisotropic Correlation Functions in k Space

$$\langle u_i(\vec{k})\, u_j(\vec{k}')\rangle = (2\pi)^d\, \delta(\vec{k}+\vec{k}')\, C_{ij}(\vec{k})$$

Suppose we have a preferred direction in space defined by a unit vector $\hat{b}_i$. Then

$$C_{ij}(\vec{k}) = C_1\, \delta_{ij} + C_2\, \hat{k}_i \hat{k}_j + C_3\, \hat{b}_i \hat{b}_j + C_4\, \hat{b}_i \hat{k}_j + C_5\, \hat{k}_i \hat{b}_j\,,$$

where $\hat{k}_i = \frac{k_i}{k}$ unit vector along $\vec{k}$,



and $C_n = C_n(k, \xi)$, where $\xi = \hat{k}\cdot\hat{b} = \cos\theta$
$n = \overline{1,5}$

Symmetry: $C_{ij} = C_{ji}$ gives [This is true if $C_{ij}(\vec{k}) = C_{ij}(-\vec{k})$, i.e. system is mirror-symmetric]

$$C_4\, \hat{b}_i \hat{k}_j + C_5\, \hat{k}_i \hat{b}_j = C_4\, \hat{b}_j \hat{k}_i + C_5\, \hat{k}_j \hat{b}_i$$

$$\hat{b}_i \hat{k}_j (C_4 - C_5) + \hat{k}_i \hat{b}_j (C_5 - C_4) = 0 \quad \forall \hat{k}_i$$

So $\boxed{C_4 = C_5}$

Incompressibility: $\hat{k}_i C_{ij} = 0$

$$C_1 \hat{k}_j + C_2 \hat{k}_j + C_3\, \xi\, \hat{b}_j + C_4\, \xi\, \hat{k}_j + C_4\, \hat{b}_j = 0$$

$$\hat{k}_j (C_1 + C_2 + \xi C_4) + \hat{b}_j (\xi C_3 + C_4) = 0 \quad \forall \hat{k}_j$$

$$\boxed{C_1 + C_2 + \xi C_4 = 0} \Rightarrow C_1 + C_2 - \xi^2 C_3 = 0 \Rightarrow C_2 = -C_1 + \xi^2 C_3$$

$$\boxed{\xi C_3 + C_4 = 0} \Rightarrow C_4 = -\xi C_3$$

So we have

$$C_{ij}(\vec{k}) = C_1 \delta_{ij} + (-C_1 + \xi^2 C_3)\hat{k}_i \hat{k}_j + C_3 \hat{b}_i \hat{b}_j - \xi C_3 (\hat{b}_i \hat{k}_j + \hat{b}_j \hat{k}_i)$$

$$= \underbrace{C_1(k,\xi)}_{\equiv\, C^{(i)}(k,\xi)} (\delta_{ij} - \hat{k}_i \hat{k}_j) + \underbrace{C_3(k,\xi)}_{=\, C^{(a)}(k,\xi)} \left[\hat{b}_i \hat{b}_j + \xi^2 \hat{k}_i \hat{k}_j - \xi (\hat{b}_i \hat{k}_j + \hat{b}_j \hat{k}_i)\right]$$

isotropic part   anisotropic part

We can also work in terms of $\parallel$ and $\perp$ correlation functions (wrt $\hat{b}$):

$$C_\parallel = C_{ij} \hat{b}_i \hat{b}_j = C_1 (1 - \xi^2) + C_3 \left[1 + \xi^4 - 2\xi^2\right] =$$

$$= C_1 (1 - \xi^2) + (1 - \xi^2)^2 C_3$$

$$C_\perp = C_{ij} (\delta_{ij} - b_i b_j) = C_{ii} - C_\parallel = C_1 (d-1) + C_3 \overbrace{(1 + \xi^2 - 2\xi^2)}^{1-\xi^2} -$$

$$- C_1 (1 - \xi^2) - (1 - \xi^2)^2 C_3 =$$

$$= C_1 (\underbrace{d-2}_{=\,1 \text{ for } d=3} + \xi^2) + \xi^2 (1 - \xi^2) C_3$$

<u>2D flow</u>: no spatial correlations in the direction of $\hat{b}$, so $\xi = 0$ or, formally, $C_1, C_3 \propto \delta(\xi)$. Then

$$C_{ij}(\vec{k}) = C_1 (\delta_{ij} - \hat{k}_i \hat{k}_j) + C_3 \hat{b}_i \hat{b}_j$$

and $C_\parallel = C_1 + C_3$, $C_\perp = C_1$