ES2. Problem 4

Tearing Mode with Viscosity

We now restore the viscous term in eq. (1). $\underbrace{}_{\nu \nabla_\perp^4 \phi}$

Equation numbers refer to my lecture notes

$$\nu \nabla_\perp^4 \phi = \nu \left(\frac{\partial^2}{\partial x^2} - k^2 \right)^2 \phi_1$$ – this is added to the rhs of eq. (4)

This brings in another time scale:

$\tau_{visc} = (\nu k^2)^{-1}$. Assume $\frac{1}{\gamma} \ll \tau_{visc}$

and another spatial scale

$$\delta_{visc} \sim \sqrt{\frac{\nu}{\gamma}} \sim \frac{1}{k} \frac{1}{\sqrt{\gamma \tau_{visc}}} \ll \frac{1}{k}$$

The outer solution is not affected by viscosity.

For the inner solution, we have to add

$$\nu \left(\frac{\partial^2}{\partial x^2} - k^2 \right)^2 \phi_1 \simeq \nu \frac{\partial^4 \phi_1}{\partial x^4} \qquad \left(\text{NB: } \frac{\partial}{\partial x} \gg k \right)$$

to the rhs of eq. (9):

$$\gamma \phi_1'' = -k \psi_0''(0) x \psi_1'' + \nu \phi_1^{IV} \qquad (23)$$

whence $$\psi_1'' = -\frac{\gamma \phi_1''}{k x \psi_0''(0)} + \frac{\nu \phi_1^{IV}}{k x \psi_0''(0)} =$$

$$= -\frac{\gamma}{k x \psi_0''(0)} \frac{\partial^2}{\partial x^2} \left[1 - \frac{\nu}{\gamma} \frac{\partial^2}{\partial x^2} \right] \phi_1$$

So we simply need to replace ϕ_1'' by $\frac{\partial^2}{\partial x^2} \left(1 - \frac{\nu}{\gamma} \frac{\partial^2}{\partial x^2} \right) \phi_1$

δx ‖ defined by (16)

Go again to the X, $\chi(X)$ variables.

In place of eq. (18), we get

$$\underbrace{\frac{\eta\gamma}{\delta^4}\left[kB'_{0y}\right]^{-2}}\left(1-\frac{\nu}{\gamma\delta^2}\frac{\partial^2}{\partial X^2}\right)\frac{\partial^2\chi}{\partial X^2} = X(1+\chi X) \qquad (24)$$

Last time we defined δ so that this is 1 [eq. (17)]
Now we keep δ undefined

This is $\frac{\delta_{visc}^2}{\delta^2}$

If $\delta_{visc}^2/\delta^2 \ll 1$, then we simply get another boundary layer inside the resistive layer. The solution we obtained previously is the outer solution and we have to calculate the inner solution as well. This will produce a correction to the previous value of γ, but the correction is small, so it is not particularly interesting.

What if, however, $\frac{\delta_{visc}^2}{\delta} \gg 1$?

Then Eq. (24) becomes

$$\underbrace{-\frac{\eta\nu}{\delta^6}\left[kB'_{0y}\right]^{-2}}_{\text{define } \delta \text{ to set this to } 1}\frac{\partial^4\chi}{\partial X^4} = X(1+\chi X)$$

define δ to set this to 1: $\boxed{\delta^6 = \eta\nu\left[kB'_{0y}\right]^{-2}}$ (25)

$$\boxed{-\frac{\partial^4\chi}{\partial X^4} = X(1+\chi X)} \qquad (26)$$

Again we have equation with no parameters.

The matching condition (8) now becomes [using (23)]

$$\Delta' = \frac{\gamma^2}{\delta^3}\left[kB'_{0y}\right]^{-2}\int_{-\infty}^{+\infty} d\mathrm{X}\,\frac{1}{\mathrm{X}}\Big(1-\underbrace{\frac{\nu}{\gamma\delta^2}}_{\text{large}}\frac{\partial^2}{\partial \mathrm{X}^2}\Big)\frac{\partial^2 \chi}{\partial \mathrm{X}^2} =$$

$$= + \frac{\gamma\nu}{\delta^5}\left[kB'_{0y}\right]^{-2}\underbrace{\int_{-\infty}^{+\infty} d\mathrm{X}\,(1+\chi\,\mathrm{X})}_{\text{again this is just a number, } I_{visc}}$$

use (26)

So, using (25) for δ, we get

$$\Delta' = \gamma\nu\left[kB'_{0y}\right]^{-2}\eta^{-5/6}\nu^{-5/6}\left[kB'_{0y}\right]^{5/3} I_{visc}$$

$$= I_{visc}\left[kB'_{0y}\right]^{-1/3}\gamma\,\eta^{-5/6}\nu^{1/6}$$

$$\boxed{\gamma = \eta^{5/6}\nu^{-1/6}\Delta'\left[kB'_{0y}\right]^{1/3}\cdot I_{visc}} \qquad (27)$$

Using (25)

$$\boxed{\delta = \eta^{1/6}\nu^{1/6}\left[kB'_{0y}\right]^{-1/3}} \qquad (28)$$

and $\boxed{\delta_{visc} \sim \nu^{7/12}\eta^{-5/12}\Delta'^{-1/2}\left[kB'_{0y}\right]^{-1/6}}$

Thus, w/o viscosity, $\gamma \sim \eta^{3/5}$

with viscosity, $\gamma \sim \eta^{5/6}\nu^{-1/6}$ different scaling.

~~This is valid if~~ $\cancel{\frac{\nu}{\gamma\delta^2} \gg 1}$. ~~Let's go about this meaning~~

$\cancel{\frac{\nu}{\gamma\delta^2} = \gamma^{-1}\eta^{-5/6}\nu^{1/6}\Delta'^{-1}\left[kB'_{0y}\right]^{-1/3}\eta^{-1/3}\nu^{-1/3}\left[kB'_{0y}\right]^{2/3} =}$

$\cancel{= \eta^{-7/6}\nu^{5/6}\Delta'^{-1}\left[kB'_{0y}\right]^{1/3} \gg 1}$

Now let's write these results in terms of characteristic times:

$$\gamma \sim \tau_R^{-5/6}\, k^{-5/3}\, \tau_{visc}^{1/6}\, k^{1/3}\, \Delta'\, \tau_A^{-1/3} \left[\frac{B_{0y}'}{B_{0y}}\right]^{1/3} =$$

$$= \tau_R^{-5/6}\, \tau_{visc}^{1/6}\, \tau_A^{-1/3} \underbrace{k^{-4/3}\, \Delta' \left[\frac{B_{0y}'}{B_{0y}}\right]^{1/3}}_{\text{order } 1}$$

We need

$$\gamma \tau_A \sim \tau_A^{2/3}\, \tau_R^{-5/6}\, \tau_{visc}^{1/6} \sim \left(\frac{\tau_A}{\tau_R}\right)^{2/3} \left(\frac{\tau_{visc}}{\tau_R}\right)^{1/6} \sim S^{-2/3}\, P_m^{-1/6} \ll 1$$

where $P_m = \frac{\nu}{\eta} = \frac{\tau_R}{\tau_{visc}}$ <u>magnetic Prandtl number</u> $P_m \gg S^{-4/3}$

$$\gamma \tau_R \sim \tau_R^{1/6}\, \tau_{visc}^{1/6}\, \tau_A^{-1/3} \sim S^{1/3}\, P_m^{-1/6} \gg 1$$

$$\gamma \tau_{visc} \sim \tau_R^{-5/6}\, \tau_{visc}^{7/6}\, \tau_A^{-1/3} \sim S^{1/3}\, P_m^{-7/6} \gg 1$$

So we need

$P_m \gg S^{-4/3}$, $P_m \gg S^{-4}$, $P_m \ll S^{2}$, $P_m \ll S^{2/7}$

Assuming $S \gg 1$, this gives $\boxed{\frac{1}{S^{4/3}} \ll P_m \ll S^{2/7}}$

Also $\delta \sim \tau_R^{-1/6}\, \tau_{visc}^{-1/6}\, \tau_A^{1/3}\, L \sim S^{-1/3}\, P_m^{1/6}\, L \ll L$

Finally, we required $\frac{\nu}{\gamma \delta^2} \gg 1$. Check this:

$$\frac{\nu}{\gamma \delta^2} \sim \tau_{visc}^{-1}\, \tau_R^{5/6}\, \tau_{visc}^{-1/6}\, \tau_A^{1/3}\, \tau_R^{1/3}\, \tau_{visc}^{1/3}\, \tau_A^{-2/3} =$$

$$= \tau_R^{7/6}\, \tau_{visc}^{-5/6}\, \tau_A^{-1/3} \sim S^{1/3}\, P_m^{5/6} \gg 1$$

$\boxed{P_m \gg S^{-2/5}}$ overrides

Thus, viscosity is important unless it is $\lll \eta$!