ES2. Problem 3

X-point Collapse

1) Syrovatskii solution

Start from Lagrangian MHD:

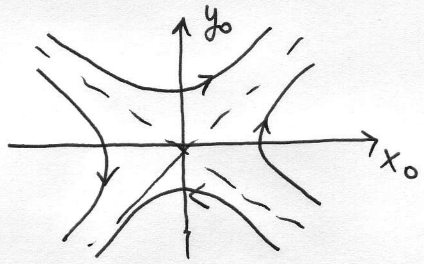
$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = -J(\nabla_0\vec{x})^{-1}\cdot\nabla_0\left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\vec{B}_0\cdot\nabla_0\left(\frac{\vec{B}_0}{J}\cdot\nabla_0\vec{x}\right)$$

Consider the following initial condition:

$$\vec{B}_0 = B_0\hat{z} + \hat{z}\times\nabla_0\psi(x_0,y_0)$$

and set up an X point : $\psi(x_0,y_0) = \frac{1}{2}(a x_0^2 - b y_0^2)$

so $B_{0x} = b y_0$, $B_{0y} = a x_0$



Seek solutions in the form

$$\left.\begin{aligned} x(t,\vec{x}_0) &= \xi(t)\,x_0 \\ y(t,\vec{x}_0) &= \eta(t)\,y_0 \\ z(t,\vec{x}_0) &= z_0 \end{aligned}\right| \Rightarrow J = |\det\nabla\vec{x}_0| = \left|\det\begin{bmatrix}\xi & 0 & 0\\ 0 & \eta & 0\\ 0 & 0 & 1\end{bmatrix}\right| = \xi\eta$$

$$\rho_0 x_0\ddot{\xi} = -\xi\eta\frac{1}{\xi}\frac{\partial}{\partial x_0}\left(\frac{p_0}{(\xi\eta)^\gamma} + \frac{|B_0\hat{z} + b y_0\xi\hat{x} + a x_0\eta\hat{y}|^2}{8\pi(\xi\eta)^2}\right) +$$

$$+\frac{1}{4\pi}\underbrace{\left(B_0\frac{\partial}{\partial z_0} + b y_0\frac{\partial}{\partial x_0} + a x_0\frac{\partial}{\partial y_0}\right)^2\frac{1}{\cancel{\xi}\eta}\cancel{\xi}x_0}_{=} =$$

$$\left(B_0\frac{\partial}{\partial z_0} + b y_0\frac{\partial}{\partial x_0} + a x_0\frac{\partial}{\partial y_0}\right)\frac{1}{\eta}b y_0 = \frac{ab x_0}{\eta}$$

$$= -\eta\frac{\partial}{\partial x_0}\left[\frac{\cancel{p_0}}{(\xi\eta)^\gamma} + \frac{\cancel{B_0^2} + b^2\xi^2 y_0^2 + a^2\eta^2 x_0^2}{8\pi\xi^2\eta^2}\right] + \frac{1}{4\pi}\frac{ab x_0}{\eta}$$

$$= -\eta\frac{a^2\cancel{\eta^2}}{8\pi\xi^2\cancel{\eta^2}}2x_0 + \frac{1}{4\pi}\frac{ab x_0}{\eta}$$

$$\ddot{\xi} = +\frac{1}{4\pi\rho_0}\left[-\frac{\eta}{\xi^2}a^2 + \frac{ab}{\eta}\right] = -\frac{ab}{4\pi\rho_0}\eta\left(\frac{a}{b}\frac{1}{\xi^2} - \frac{1}{\eta^2}\right)$$

Similarly,

$$\rho_0 y_0 \ddot{\eta} = -\xi\eta\frac{1}{\eta}\frac{\partial}{\partial y_0}\left(\cancel{\frac{p_0}{(\xi\eta)^\gamma}} + \frac{\cancel{B_0^2} + b^2\xi^2 y_0^2 + a^2\eta^2 x_0^2}{8\pi\xi^2\eta^2}\right) +$$

$$+\frac{1}{4\pi}\underbrace{\left(B_0\frac{\partial}{\partial z_0} + by_0\frac{\partial}{\partial x_0} + ax_0\frac{\partial}{\partial y_0}\right)\frac{1}{\xi\cancel{\eta}}\underbrace{\left(B_0\frac{\partial}{\partial z_0} + by_0\frac{\partial}{\partial x_0} + ax_0\frac{\partial}{\partial y_0}\right)\cancel{\eta}y_0}_{= ax_0}}_{= \frac{aby_0}{\xi}} =$$

$$= -\xi\frac{b^2\cancel{\xi^2}}{8\pi\cancel{\xi^2}\eta^2}2y_0 + \frac{1}{4\pi}\frac{aby_0}{\xi}$$

$$\ddot{\eta} = \frac{ab}{4\pi\rho_0}\xi\left(\frac{1}{\xi^2} - \frac{b}{a}\frac{1}{\eta^2}\right)$$

Now let $b = \frac{B_\perp}{L_y}$, $a = \frac{B_\perp}{L_x}$, so $\frac{ab}{4\pi\rho_0} = \frac{V_A^2}{L_x L_y} \equiv \frac{1}{\tau_A^2}$, $\frac{a}{b} = \frac{L_y}{L_x} \equiv \alpha$

We can rescale time $\frac{t}{\tau_A} \to t$, so

$$\begin{cases} \ddot{\xi} = -\eta\left(\frac{\alpha}{\xi^2} - \frac{1}{\eta^2}\right) \\ \ddot{\eta} = \xi\left(\frac{1}{\xi^2} - \frac{1}{\alpha\eta^2}\right) \end{cases}$$

Let $\xi = \alpha^{1/4}\tilde{\xi}$, $\eta = \alpha^{-1/4}\tilde{\eta}$. Then

$$\boxed{\begin{aligned} \ddot{\tilde{\xi}} &= -\tilde{\eta}\left(\frac{1}{\tilde{\xi}^2} - \frac{1}{\tilde{\eta}^2}\right) \quad (1) \\ \ddot{\tilde{\eta}} &= \tilde{\xi}\left(\frac{1}{\tilde{\xi}^2} - \frac{1}{\tilde{\eta}^2}\right) \quad (2) \end{aligned}}$$

This means we never need have bothered to have $a \neq b$.
So let $a = b$ and drop tildes from now on.

Note that ξ and η are symmetric: we get the same equations if we swap $\xi \leftrightarrow \eta$.

Consider the possibility that one of these quantities becomes small: say, $\eta \to 0$, while the other tends to a constant: $\xi \to \xi_c$ (this is a guess that's about to prove consistent with the equations).

Then we have:

$$\text{Eq. (2)} \Rightarrow \quad \ddot{\eta} \approx -\frac{\xi_c}{\eta^2} \quad \Big| \cdot \dot{\eta}$$

$$\dot{\eta}\ddot{\eta} \approx -\xi_c \frac{\dot{\eta}}{\eta^2}$$

$$\frac{d}{dt}\frac{\dot{\eta}^2}{2} = +\xi_c \frac{d}{dt}\frac{1}{\eta} \quad \Rightarrow \quad \frac{\dot{\eta}^2}{2} = \frac{\xi_c}{\eta} + \cancel{\text{const}}$$

neglect because $\frac{1}{\eta} \to \infty$

Then

$$\dot{\eta} = \pm\sqrt{2\xi_c}\,\frac{1}{\sqrt{\eta}}$$

Some const

$$\sqrt{\eta}\, d\eta = \pm\sqrt{2\xi_c}\, dt \quad \Rightarrow \quad \frac{2}{3}\eta^{3/2} = \pm\sqrt{2\xi_c}\,(t - t_c)$$

"$-$" solution satisfies our assumption: as time approaches t_c, $\eta \to 0$. So we get

$$\boxed{\eta \approx \left(\frac{9\xi_c}{2}\right)^{1/3} (t_c - t)^{2/3}} \qquad t \to t_c \qquad (3)$$

$$\text{Eq. (1)} \Rightarrow \quad \ddot{\xi} \approx \frac{1}{\eta} \approx \left(\frac{2}{9\xi_c}\right)^{1/3} (t_c - t)^{-2/3}$$

$$\dot{\xi} = -\left(\frac{2}{9\xi_c}\right)^{1/3} 3\,(t_c - t)^{1/3} + \text{const}_1$$

$$\xi = \frac{9}{4}\left(\frac{2}{9\xi_c}\right)^{1/3} (t_c - t)^{4/3} + \underbrace{\text{const}_1}_{=0} t + \overbrace{\text{const}_2}^{\xi_c}$$

$=0$, otherwise $\xi \not\to$ const as $t \to t_c$

Thus,

$$\boxed{\xi \approx \xi_c + \frac{9}{4}\left(\frac{2}{9\xi_c}\right)^{1/3} (t_c - t)^{4/3}} \qquad t \to t_c \qquad (4)$$

There is also an analogous solution with $\xi \leftrightarrow \eta$.

(3)-(4) is called the Syrovatskii solution.

Magnetic field in this solution behaves as follows:

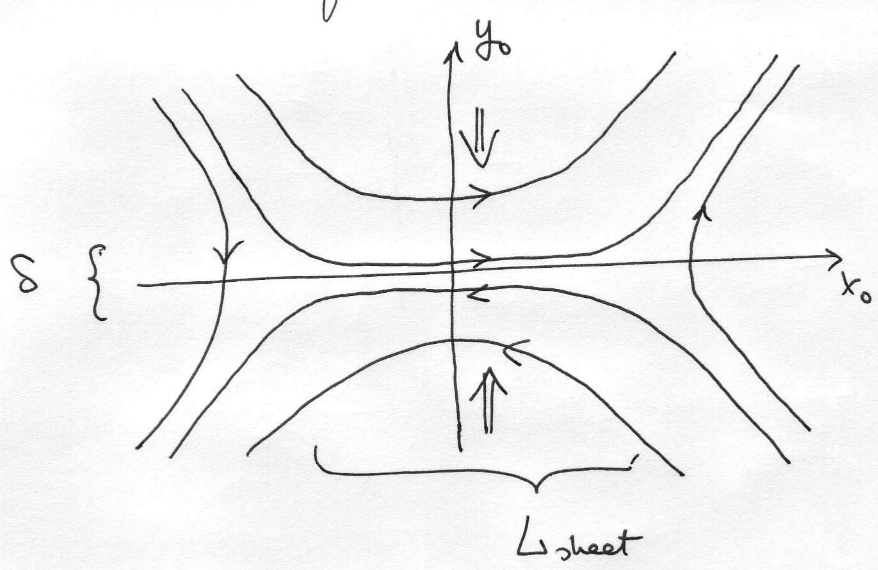
$$B_x = \frac{1}{J}\left(B_0 \frac{\partial}{\partial z_0} + b y_0 \frac{\partial}{\partial x_0} + a x_0 \frac{\partial}{\partial y_0}\right) \xi x_0 = \frac{1}{\xi \eta}\, b y_0 \xi = b \frac{y_0}{\eta(t)}$$

$$B_y = \ldots = a \frac{x_0}{\xi(t)}$$

Thus, $B_y \approx a \left(\frac{a}{b}\right)^{-1/4} \frac{x_0}{\xi_c}$

$$B_x \approx b \left(\frac{a}{b}\right)^{1/4} \left(\frac{2}{9\xi_c}\right)^{1/3} \frac{y_0}{(t_c - t)^{2/3}}$$

Thus, the field, which initially had an x-point configuration is explosively compressed into a sheet along the x axis, with B_x reversing direction on the scale $\sim (t_c - t)^{2/3} \to 0$.
At $t = t_c$, a singularity has formed and it is not meaningful to consider ideal MHD anymore.
Since B_x reverses at a very small scale, resistivity will become important and resolve the singularity:



$$\frac{\delta}{L_y} \sim \left(\frac{9\xi_c}{2}\right)^{1/3} \left(\frac{L_x}{L_y}\right)^{-1/4} (t_c - t)^{2/3} \to 0$$

2) Chapman-Kendall solution

Consider the incompressible case: $J=1$

$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = -(\nabla_0 \vec{x})^{-1} \cdot \nabla_0 \tilde{p} + \frac{1}{4\pi} \vec{B}_0 \cdot \nabla_0 \vec{B}_0 \cdot \nabla_0 \vec{x}$$

and $\tilde{p}$ is determined by $J=1$.

Consider the same initial configuration as before and seek solutions in the same form.

Incompressibility: $J = \xi\eta = 1 \Rightarrow \eta = \frac{1}{\xi}$

$$\rho_0 x_0 \ddot{\xi} = -\frac{1}{\xi}\frac{\partial \tilde{p}}{\partial x_0} + \frac{1}{4\pi} \underbrace{\left(B_0 \frac{\partial}{\partial z_0} + b y_0 \frac{\partial}{\partial x_0} + a x_0 \frac{\partial}{\partial y_0}\right)^2 \xi x_0}_{(\dots)\, b y_0 \xi = a b x_0 \xi}$$

$$\begin{cases} \ddot{\xi} = -\dfrac{1}{\xi x_0} \dfrac{\partial}{\partial x_0} \dfrac{\tilde{p}}{\rho_0} + \dfrac{ab}{4\pi\rho_0}\xi \\[2ex] \ddot{\eta} = -\dfrac{1}{\eta y_0} \dfrac{\partial}{\partial y_0} \dfrac{\tilde{p}}{\rho_0} + \dfrac{ab}{4\pi\rho_0}\eta \end{cases}$$

Again rescale time $\frac{t}{\tau_A} \to t$, $\tau_A^2 = \left(\frac{ab}{4\pi\rho_0}\right)^{-1}$ and $\frac{\tilde{p}}{\rho_0}\tau_A^2 \equiv p$

$$\begin{cases} \ddot{\xi} = -\dfrac{1}{\xi x_0} \dfrac{\partial p}{\partial x_0} + \xi \\[2ex] \ddot{\eta} = -\dfrac{1}{\eta y_0} \dfrac{\partial p}{\partial y_0} + \eta \end{cases} \quad \text{and} \quad \xi\eta = 1$$

Let $\xi = e^{S(t)}$, $\eta = e^{-S(t)}$, so $\xi\eta = 1$ is satisfied.

$\dot{\xi} = e^S \dot{S}$, $\ddot{\xi} = e^S(\dot{S}^2 + \ddot{S})$,

$\dot{\eta} = -e^{-S}\dot{S}$, $\ddot{\eta} = e^{-S}(\dot{S}^2 - \ddot{S})$

$$\ddot{S} + \dot{S}^2 = -e^{-2S}\frac{1}{x_0}\frac{\partial p}{\partial x_0} + 1 \quad \Rightarrow \quad \frac{\partial p}{\partial x_0} = -x_0 e^{2S}\left(\ddot{S} + \dot{S}^2 - 1\right)$$

$$\ddot{S} - \dot{S}^2 = e^{2S}\frac{1}{y_0}\frac{\partial p}{\partial y_0} - 1 \quad \Rightarrow \quad \frac{\partial p}{\partial y_0} = y_0 e^{-2S}\left(\ddot{S} - \dot{S}^2 + 1\right)$$

So $p = -\frac{1}{2}x_0^2 e^{2S}\left(\ddot{S} + \dot{S}^2 - 1\right) + \frac{1}{2}y_0^2 e^{-2S}\left(\ddot{S} - \dot{S}^2 + 1\right)$

and $\xi = e^{S(t)}$, $\eta = e^{-S(t)}$ with $S(t)$ undetermined.

So, for the magnetic field, we now have

$$\begin{cases} B_y = a x_0 e^{-S(t)} \\ B_x = b y_0 e^{S(t)} \end{cases}$$

For growing functions $S(t)$, e.g. $S(t) = \Lambda t$, we again have the X point collapse.

~~However, this solution does not give a finite length the current sheet~~:

We should expect this solution to break down at distances too far from $x_0 = y_0 = 0$ because the form of $\psi(x_0, y_0)$ was an expansion around the x point. (Same is true about the Syrovatskii solution)