EX. SHEET II
Problem 6 ~~ES2. Problem 2~~

Ambipolar and Viscous Damping

Momentum equations:

ions: $$\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot \nabla \vec{u}_i = -\frac{\nabla p_i}{\rho_i} + \cancel{\nu_i \nabla^2 \vec{u}_i} - \mu_{in}(\vec{u}_i - \vec{u}_n) + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi\rho_i}$$

neutrals: $$\frac{\partial \vec{u}_n}{\partial t} + \vec{u}_n \cdot \nabla \vec{u}_n = -\frac{\nabla p_n}{\rho_n} + \nu_n \nabla^2 \vec{u}_n - \mu_{ni}(\vec{u}_n - \vec{u}_i)$$

incompressibility: $\nabla\cdot\vec{u}_i = \nabla\cdot\vec{u}_n = 0$

m. field: $$\frac{\partial \vec{B}}{\partial t} + \vec{u}_i \cdot \nabla \vec{B} = \vec{B}\cdot\nabla\vec{u}_i + \cancel{\eta \nabla^2 \vec{B}}$$

Ignore ion viscosity: formally, this assumes

$$\nu_i \ll \nu_n \quad \text{and} \quad \nu_i k^2 \ll \mu_{in}$$

(but also, ion viscosity isn't really isotropic and is heavily suppressed in the directions $\perp \vec{B}$, so we can't really keep it and expect correct physics for shear Alfvén waves)

Ignore magnetic diffusivity as well: again,

$$\eta \ll \nu_n \quad \text{and} \quad \eta k^2 \ll \mu_{in}$$

Note that, assuming elastic ion-neutral collisions,

$$\rho_i \mu_{in} = \rho_n \mu_{ni}.$$

Define <u>degree of ionisation</u>: $\chi = \dfrac{\rho_i}{\rho_i + \rho_n}$. Then

$$\mu_{ni} = \frac{\rho_i}{\rho_n}\mu_{in} = \frac{\chi}{1-\chi}\mu_{in}$$

Consider straight-field static equilibrium:

$$\vec{B}_0 = B_0 \hat{z}, \quad \vec{u}_i = \vec{u}_n = 0$$

Perturbed velocities in terms of displacements:

$$\vec{u}_i = \frac{\partial \vec{\xi}_i}{\partial t}, \quad \vec{u}_n = \frac{\partial \vec{\xi}_n}{\partial t}.$$

Then $$\frac{\partial \delta\vec{B}}{\partial t} = \vec{B}_0\cdot\nabla \frac{\partial \vec{\xi}_i}{\partial t} \Rightarrow \delta\vec{B} = B_0 \nabla_\parallel \vec{\xi}_i$$ magnetic field frozen into ions.

$$\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi\rho_i} = \frac{\vec{B}_0 \nabla_\parallel \delta\vec{B}}{4\pi\rho_i} = \frac{B_0^2}{4\pi\rho_i}\nabla_\parallel \vec{\xi}_i = V_A^2\nabla_\parallel^2\vec{\xi}_i$$

So, we get

$$\begin{cases} \dfrac{\partial^2 \vec{\xi}_i}{\partial t^2} = \overbrace{V_A^2 \nabla_\parallel^2 \vec{\xi}_i}^{\text{Alfvén waves}} - \overbrace{\mu_{in}\left(\dfrac{\partial \vec{\xi}_i}{\partial t} - \dfrac{\partial \vec{\xi}_n}{\partial t}\right)}^{\text{ion-neutral collisions}} \\[2ex] \dfrac{\partial^2 \vec{\xi}_n}{\partial t^2} = \underbrace{\nu_n \nabla^2}_{\text{neutral viscosity}} \dfrac{\partial \vec{\xi}_n}{\partial t} - \dfrac{\chi}{1-\chi}\mu_{in}\left(\dfrac{\partial \vec{\xi}_n}{\partial t} - \dfrac{\partial \vec{\xi}_i}{\partial t}\right) \end{cases}$$

Fourier transform:

$$\begin{cases} -\omega^2 \vec{\xi}_i = -k_\parallel^2 V_A^2 \vec{\xi}_i + i\omega\mu_{in}(\vec{\xi}_i - \vec{\xi}_n) \\ -\omega^2 \vec{\xi}_n = i\omega \nu_n k^2 \vec{\xi}_n + i\omega\mu_{in}\dfrac{\chi}{1-\chi}(\vec{\xi}_n - \vec{\xi}_i) \end{cases}$$

$$\begin{cases} (\omega^2 - k_\parallel^2 V_A^2 + i\omega\mu_{in})\vec{\xi}_i = i\omega\mu_{in}\vec{\xi}_n & (1) \\ \left(\omega^2 + i\omega\nu_n k^2 + i\omega\mu_{in}\dfrac{\chi}{1-\chi}\right)\vec{\xi}_n = i\omega\mu_{in}\dfrac{\chi}{1-\chi}\vec{\xi}_i & (2) \end{cases}$$

$$(\omega^2 - k_\parallel^2 V_A^2 + i\omega\mu_{in})\left(\omega^2 + i\omega\nu_n k^2 + i\omega\mu_{in}\frac{\chi}{1-\chi}\right) = -\omega^2\mu_{in}^2\frac{\chi}{1-\chi}$$

$$\omega^4 - \underline{\underline{\omega^2 k_\parallel^2 V_A^2}} + \underline{i\omega^3\mu_{in}} + \underline{i\omega^3\nu_n k^2} - i\omega\nu_n k^2 k_\parallel^2 V_A^2 - \omega^2\underline{\underline{\mu_{in}\nu_n k^2}}$$

$$+\underline{i\omega^3\mu_{in}\frac{\chi}{1-\chi}} - i\omega k_\parallel^2 V_A^2\mu_{in}\frac{\chi}{1-\chi} - \cancel{\omega^2\mu_{in}^2\frac{\chi}{1-\chi}} = \cancel{-\omega^2\mu_{in}^2\frac{\chi}{1-\chi}}$$

$$\omega^3 + i\omega^2\left(\nu_n k^2 + \mu_{in} + \mu_{in}\frac{\chi}{1-\chi}\right) - \omega\left(k_\parallel^2 V_A^2 + \mu_{in}\nu_n k^2\right) -$$

$$- i k_\parallel^2 V_A^2\left(\nu_n k^2 + \mu_{in}\frac{\chi}{1-\chi}\right) = 0$$

$$\boxed{i\omega^3 - \omega^2\left(\nu_n k^2 + \frac{\mu_{in}}{1-\chi}\right) - i\omega\left(k_\parallel^2 V_A^2 + \mu_{in}\nu_n k^2\right) + k_\parallel^2 V_A^2\left(\nu_n k^2 + \mu_{in}\frac{\chi}{1-\chi}\right) = 0}$$

Assume $k_\parallel V_A \ll \mu_{in}$ and write our disp. relation so:

$$i\underbrace{\frac{k_\parallel V_A}{\mu_{in}}}_{\text{small}}\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \left(\frac{\omega}{k_\parallel V_A}\right)^2\left(\frac{\nu_n k^2}{\mu_{in}} + \frac{1}{1-\chi}\right) - i\frac{\omega}{k_\parallel V_A}\left(\underbrace{\frac{k_\parallel V_A}{\mu_{in}}}_{\text{small}} + \frac{\nu_n k^2}{k_\parallel V_A}\right) + \frac{\nu_n k^2}{\mu_{in}} + \frac{\chi}{1-\chi} = 0 \quad (3)$$

In principle, we could solve this using the Cardano formula, but that is extremely ~~unplea~~ unpleasant. It is much more interesting to consider various limiting cases.

For the purposes of estimates, recall

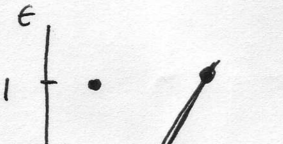
$$\nu_n \sim v_{th}\lambda_{mfp} \quad \text{and} \quad \mu_{in} \sim \frac{v_{th}}{\lambda_{mfp}}$$

(we assume $\chi \sim 1$, i.e. not too small)

1) $\nu_n k^2 \ll \dfrac{k_\parallel^2 V_A^2}{\mu_{in}} \ll k_\parallel V_A \ll \mu_{in}$

Eq. (3) becomes:

$$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \frac{1}{1-\chi}\left(\frac{\omega}{k_\parallel V_A}\right)^2 - i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega}{k_\parallel V_A}\right) + \frac{\chi}{1-\chi} = 0$$



Dominant balances:

(i) $i\dfrac{\cancel{k_\parallel V_A}}{\mu_{in}}\left(\dfrac{\omega}{\cancel{k_\parallel V_A}}\right)^{3} \approx \dfrac{1}{1-\chi}\cancel{\left(\dfrac{\omega}{k_\parallel V_A}\right)^2}$

$$\boxed{\omega = -\frac{i\mu_{in}}{1-\chi}} \quad (4)$$

(ii) $\frac{1}{1-\chi}\left(\frac{\omega}{k_\parallel V_A}\right)^2 \approx \frac{\chi}{1-\chi} \Rightarrow \omega_0 = \pm\sqrt{\chi}\, k_\parallel V_A$ to zeroth order

Next order: $\omega = \omega_0 + \delta\omega$

$$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega_0}{k_\parallel V_A}\right)^3 - \frac{1}{1-\chi}\left[\cancel{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2} + 2\frac{\omega_0}{k_\parallel V_A}\frac{\delta\omega}{k_\parallel V_A}\right] - i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega_0}{k_\parallel V_A}\right) + \cancel{\frac{\chi}{1-\chi}} = 0$$

$$\frac{2}{1-\chi}\cancel{\frac{\omega_0}{k_\parallel V_A}}\frac{\delta\omega}{k_\parallel V_A} = i\frac{k_\parallel V_A}{\mu_{in}}\cancel{\frac{\omega_0}{k_\parallel V_A}}\left[\underbrace{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2 - 1}_{=\chi-1}\right]$$

$$\delta\omega = -\frac{i}{2}(1-\chi)^2\frac{k_\parallel^2 V_A^2}{\mu_{in}}$$

So
$$\boxed{\omega = \underbrace{\pm\sqrt{\chi}\, k_\parallel V_A}_{\text{Alfvén waves}} \underbrace{- \frac{i}{2}(1-\chi)^2\frac{k_\parallel^2 V_A^2}{\mu_{in}}}_{\text{Ambipolar damping}}} \qquad (5)$$

NB: (4) and (5) are 3 solutions.
(4) is much more strongly damped than (5), so (5) is the interesting mode.

Note that $k_\parallel V_A \ll \underset{\sim v_{th}/\lambda_{mfp}}{\mu_{in}}$ means $k\lambda_{mfp} \ll \frac{v_{th}}{V_A} \sim \left(\frac{P_n}{\rho_n}\frac{4\pi\rho_i}{B^2}\right)^{1/2} \sim \beta^{1/2}$

$k\lambda_{mfp} \ll \beta^{1/2}$

$\nu_n k^2 \ll \frac{k_\parallel^2 V_A^2}{\mu_{in}}$ means $v_{th}\cancel{\lambda_{mfp}} \ll \frac{V_A^2}{v_{th}}\cancel{\lambda_{mfp}} \Rightarrow \beta \ll 1$

So this is a regime with heavy collisions and strong magnetic field.

In the case of non-small β, we have to consider other limits.

2) $\dfrac{k_\parallel^2 V_A^2}{\mu_{in}} \ll \nu_n k^2$ [redacted] $\ll k_\parallel V_A \ll \mu_{in}$

Eq. (3) becomes:

$$i\underbrace{\frac{k_\parallel V_A}{\mu_{in}}}_{\text{small}}\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \frac{1}{1-\chi}\left(\frac{\omega}{k_\parallel V_A}\right)^2 - i\underbrace{\frac{\nu_n k^2}{k_\parallel V_A}}_{\text{small}}\left(\frac{\omega}{k_\parallel V_A}\right) + \frac{\chi}{1-\chi} = 0 \qquad (6)$$

Dominant balances:

(i) Same as (4)

(ii) Again $\omega_0 = \pm\sqrt{\chi}\, k_\parallel V_A$. Next order:

$$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega_0}{k_\parallel V_A}\right)^3 - \frac{1}{1-\chi}\left[\cancel{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2} + 2\frac{\omega_0}{k_\parallel V_A}\frac{\delta\omega}{k_\parallel V_A}\right] - i\frac{\nu_n k^2}{k_\parallel V_A}\left(\frac{\omega_0}{k_\parallel V_A}\right) + \cancel{\frac{\chi}{1-\chi}} = 0$$

$$\frac{2}{1-\chi}\cancel{\frac{\omega_0}{k_\parallel V_A}}\frac{\delta\omega}{k_\parallel V_A} = i\cancel{\frac{\omega_0}{k_\parallel V_A}}\left[\frac{k_\parallel V_A}{\mu_{in}}\underbrace{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2}_{=\chi} - \frac{\nu_n k^2}{k_\parallel V_A}\right]$$

$$\delta\omega = -\frac{i}{2}(1-\chi)\left(\frac{\nu k^2}{[\text{redacted}]} - \underbrace{\cancel{\chi\frac{k_\parallel V_A^2}{\mu_{in}}}}_{\text{small}}\right)$$

So,

$$\boxed{\omega \approx \pm\underbrace{\sqrt{\chi}\, k_\parallel V_A}_{\text{Alfrén waves}} - \underbrace{\frac{i}{2}(1-\chi)\,\nu_n k^2}_{\text{Viscous damping}}} \qquad (7)$$

(4) and (7) are the 3 solutions, (7) is the slower-damped one.

Conditions for this to work: $k_\parallel V_A \ll \mu_n$ is $k\lambda_{mfp} \ll \beta^{1/2}$ (automatically satisfied)

$\dfrac{k_\parallel^2 V_A^2}{\mu_{in}} \ll \nu_n k^2$ is $\beta \gg 1$ → automatically satisfied

and $\nu_n k^2 \ll k_\parallel V_A$ means $V_{th}\lambda_{mfp} k \ll V_A$ $\quad k\lambda_{mfp} \ll \dfrac{1}{\beta^{1/2}}$

So weak magnetic field and heavy collisions

3) $k_\parallel V_A \ll \nu_n k^2 \ll \mu_{in}$

Eq. (6) still valid, but $\dfrac{\nu_n k^2}{k_\parallel V_A}$ is large.

Dominant balances:

(i) Same as (4)

(ii) $$\frac{1}{1-\chi}\left(\frac{\omega}{\cancel{k_\parallel V_A}}\right)^{\cancel{2}} \approx -i\frac{\nu_n k^2}{\cancel{k_\parallel V_A}}\cancel{\left(\frac{\omega}{k_\parallel V_A}\right)}$$

$$\boxed{\omega \approx -i(1-\chi)\,\nu_n k^2} \qquad (8)$$

(iii) $$i\frac{\nu_n k^2}{k_\parallel V_A}\frac{\omega}{k_\parallel V_A} \approx \frac{\chi}{1-\chi}$$

$$\boxed{\omega \approx -i\frac{\chi}{1-\chi}\frac{k_\parallel^2 V_A^2}{\nu_n k^2}} \qquad (9)$$

This is called <u>viscous relaxation</u> (no waves!)

(4) and (8) are much faster damping rates, so (9) is the most interesting.

Conditions for it to work: $\beta \gg 1$, $k\lambda_{mfp} \ll \beta^{1/2}$ as before

$\nu_n k^2 \gg k_\parallel V_A$ is $k\lambda_{mfp} \gg \dfrac{1}{\beta^{1/2}}$

and $\nu_n k^2 \ll \mu_n$ means $V_{th}\lambda_{mfp}k^2 \ll \dfrac{V_{th}}{\lambda_{mfp}} \Rightarrow k\lambda_{mfp} \ll 1$

Thus, $$\frac{1}{\beta^{1/2}} \ll k\lambda_{mfp} \ll 1$$ collisional regime, high β.

4) $\nu_n k^2 \gg \mu_{in} \gg k_\parallel V_A$.

Eq.(3) becomes

$$i\underbrace{\frac{k_\parallel V_A}{\mu_{in}}}_{\text{small}}\left(\frac{\omega}{k_\parallel V_A}\right)^3-\underbrace{\frac{\nu_n k^2}{\mu_{in}}}_{\text{large}}\left(\frac{\omega}{k_\parallel V_A}\right)^2-i\underbrace{\frac{\nu_n k^2}{k_\parallel V_A}}_{\text{large}}\left(\frac{\omega}{k_\parallel V_A}\right)+\underbrace{\frac{\nu_n k^2}{\mu_{in}}}_{\text{large}}=0$$

Dominant balances:

(i) $$i\frac{\cancel{k_\parallel V_A}}{\cancel{\mu_{in}}}\left(\frac{\omega}{\cancel{k_\parallel V_A}}\right)^{\cancel{3}}\approx\frac{\nu_n k^2}{\cancel{\mu_{in}}}\left(\frac{\omega}{\cancel{k_\parallel V_A}}\right)^{\cancel{2}} \Rightarrow \boxed{\omega=-i\nu_n k^2} \quad (10)$$

(ii) $$-\frac{\cancel{\nu_n k^2}}{\mu_{in}}\left(\frac{\omega}{\cancel{k_\parallel V_A}}\right)^{\cancel{2}}\approx i\frac{\cancel{\nu_n k^2}}{\cancel{k_\parallel V_A}}\left(\frac{\cancel{\omega}}{\cancel{k_\parallel V_A}}\right) \Rightarrow \boxed{\omega=-i\mu_{in}} \quad (11)$$

(iii) $$-i\frac{\cancel{\nu_n k^2}}{k_\parallel V_A}\frac{\omega}{k_\parallel V_A}\approx\frac{\cancel{\nu_n k^2}}{\mu_{in}} \Rightarrow \boxed{\omega\simeq-i\frac{k_\parallel^2 V_A^2}{\mu_{in}}} \quad (12)$$

This is the slowest damping rate. It's called <u>ambipolar relaxation</u> (no waves!)

$\nu_n k^2 \gg \mu_{in}$ means $k\lambda_{mfp}\gg 1$, also $\beta \gg 1$, $k\lambda_{mfp} \ll \beta^{1/2}$ as before

Thus, $1 \ll k\lambda_{mfp} \ll \beta^{1/2}$ weakly collisional regime, high β.

Note that the fluid treatment is not really valid in this regime, so the results are suspect.

(collisionless $k\lambda_{mfp}\gg 1$)

Now let us take $k_\parallel V_A \gg \mu_{in}$ or $k\lambda_{mfp} \gg \beta^{1/2}$

In order for a fluid treatment to make sense, we need to have $\beta \ll 1$ (strong m. field.).

Rewrite eq. (3) so:

$$i\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \left(\frac{\omega}{k_\parallel V_A}\right)^2\left(\frac{\nu_n k^2}{k_\parallel V_A} + \frac{1}{1-\chi}\underbrace{\frac{\mu_{in}}{k_\parallel V_A}}_{\text{small}}\right) - i\frac{\omega}{k_\parallel V_A}\left(1 + \underbrace{\cancel{\frac{\nu_n k^2}{k_\parallel V_A}\frac{\mu_{in}}{k_\parallel V_A}}}_{\text{small}}\right) + \frac{\nu_n k^2}{k_\parallel V_A} + \overbrace{1}^{\left(\frac{\chi}{1-\chi}\frac{\mu_{in}}{k_\parallel V_A}\right)} = 0 \qquad (13)$$

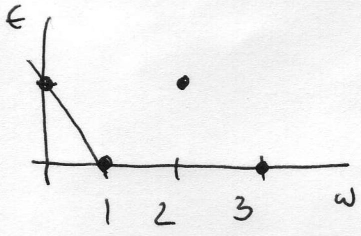
We already know that

$\nu_n k^2 \ll \dfrac{k_\parallel^2 V_A^2}{\mu_{in}}$ means $\beta \ll 1$, so this term is small.

1) $\nu_n k^2 \ll \mu_{in} \ll k_\parallel V_A$.

Eq. (13) becomes

$$i\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \frac{1}{1-\chi}\underbrace{\frac{\mu_{in}}{k_\parallel V_A}}_{\text{small}}\left(\frac{\omega}{k_\parallel V_A}\right)^2 - i\frac{\omega}{k_\parallel V_A} + \frac{\chi}{1-\chi}\underbrace{\frac{\mu_{in}}{k_\parallel V_A}}_{\text{small}} = 0$$



Dominant balances:

(i) $+i\dfrac{\omega}{\cancel{k_\parallel V_A}} \approx \dfrac{\chi}{1-\chi}\dfrac{\mu_{in}}{\cancel{k_\parallel V_A}} \Rightarrow \boxed{\omega \approx -i\dfrac{\chi}{1-\chi}\mu_{in}}$ (14)

(ii) $i\left(\dfrac{\omega}{k_\parallel V_A}\right)^3 \approx i\dfrac{\omega}{k_\parallel V_A} \Rightarrow \omega_0 = \pm k_\parallel V_A$ to zeroth order.

Next order:

$$i\left[\cancel{\left(\frac{\omega_0}{k_\parallel V_A}\right)^3} + 3\overbrace{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2}^{=1}\frac{\delta\omega}{k_\parallel V_A}\right] - \frac{1}{1-\chi}\frac{\mu_{in}}{k_\parallel V_A}\overbrace{\left(\frac{\omega_0}{k_\parallel V_A}\right)^2}^{=1} -$$

$$-i\left[\cancel{\frac{\omega_0}{k_\parallel V_A}} + \frac{\delta\omega}{k_\parallel V_A}\right] + \frac{\chi}{1-\chi}\frac{\mu_{in}}{k_\parallel V_A} = 0$$

$$2i\frac{\delta\omega}{k_\parallel V_A} = \frac{\mu_{in}}{k_\parallel V_A} \Rightarrow \delta\omega = -\frac{i}{2}\mu_{in}$$

So $$\boxed{\omega = \pm k_\parallel V_A - \frac{i}{2}\mu_{in}} \qquad (15)$$

Alfvén wave

weakly damped by collisions

$\nu_n k^2 \ll \mu_{in}$ means $k\lambda_{mfp} \ll 1$, so $\beta^{1/2} \ll k\lambda_{mfp} \ll 1$

collisional, low β.

2) $\mu_{in} \ll \nu_n k^2$ Eq. (13) becomes

$$i\left(\frac{\omega}{k_\parallel V_A}\right)^3 - \frac{\nu_n k^2}{k_\parallel V_A}\left(\frac{\omega}{k_\parallel V_A}\right)^2 - i\frac{\omega}{k_\parallel V_A} + \frac{\nu_n k^2}{k_\parallel V_A} = 0$$

$$i\frac{\omega}{k_\parallel V_A}\left[\left(\frac{\omega}{k_\parallel V_A}\right)^2 - 1\right] - \frac{\nu_n k^2}{k_\parallel V_A}\left[\left(\frac{\omega}{k_\parallel V_A}\right)^2 - 1\right] = 0$$

$$\left[\left(\frac{\omega}{k_\parallel V_A}\right)^2 - 1\right]\left(i\omega - \nu_n k^2\right) = 0$$

Solutions: $$\boxed{\omega = -i\nu_n k^2} \qquad (16)$$

and $$\boxed{\omega = \pm k_\parallel V_A} \qquad (17)$$ undamped Alfvén wave

$\mu_{in} \ll \nu_n k^2$ means $k\lambda_{mfp} \gg 1$ collisionless regime, fluid results suspect...

Summary

$\boxed{\beta \ll 1}$ $\quad k\lambda_{mfp} \ll \beta^{1/2} \qquad \omega = \pm\sqrt{\chi}\, k_\parallel v_A - \frac{i}{2}(1-\chi)^2 \frac{k_\parallel^2 v_A^2}{\mu_{in}} \quad (5)$

$\beta^{1/2} \ll k\lambda_{mfp} \ll 1 \qquad \omega = \pm k_\parallel v_A - \frac{i}{2}\mu_{in} \quad (15)$

$k\lambda_{mfp} \gg 1 \qquad \omega = \pm k_\parallel v_A \quad (17)$

$\boxed{\beta \gg 1}$ $\quad k\lambda_{mfp} \ll \frac{1}{\beta^{1/2}} \qquad \omega = \pm\sqrt{\chi}\, k_\parallel v_A - \frac{i}{2}(1-\chi)\, \nu_n k^2 \quad (7)$

$\frac{1}{\beta^{1/2}} \ll k\lambda_{mfp} \ll 1 \qquad \omega = -i\frac{\chi}{1-\chi}\frac{k_\parallel^2 v_A^2}{\nu_n k^2} \quad (9)$

$1 \ll k\lambda_{mfp} \ll \beta^{1/2} \qquad \omega = -i\frac{k_\parallel^2 v_A^2}{\mu_{in}} \quad (12)$