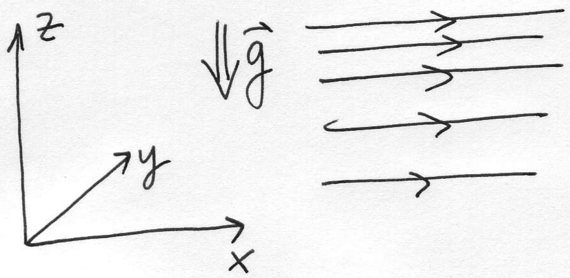
ES2. Problem 1

Instabilities with gravity



$$\vec B_0 = B_0(z)\hat x\ ,\quad p_0 = p_0(z)\ ,\quad \rho_0 = \rho_0(z)$$

$$\frac{\partial}{\partial z}\left(p_0 + \frac{B_0^2}{8\pi}\right) = -\rho_0 g$$

Consider displacement $\vec\xi \sim \vec\xi(z)\, e^{ik_x x + ik_y y}$

We have, for each (k_x, k_y),

$$\delta W_2 = \frac{1}{2}\int dz \left[\gamma p_0 |\nabla\cdot\vec\xi|^2 + \frac{|\vec Q|^2}{4\pi} + (\nabla\cdot\vec\xi^*)\underbrace{\vec\xi\cdot\nabla p_0}_{=\xi_z p_0'} + \frac{\vec J_0\cdot(\vec\xi^*\times\vec Q)}{c} + \underbrace{(\vec\xi^*\cdot\vec g)}_{=-\xi_z^* g}\overbrace{\nabla\cdot(\rho_0\vec\xi)}^{=\rho_0\nabla\cdot\vec\xi + \xi_z\rho_0'}\right]$$

Consider several special cases

(a) <u>No magnetic field</u> $B_0 = 0$: $\frac{\partial p_0}{\partial z} = -\rho_0 g$

$$\delta W_2 = \frac{1}{2}\int dz \left[\gamma p_0 |\nabla\cdot\vec\xi|^2 + (\nabla\cdot\vec\xi^*)\,\vec\xi\cdot\nabla p_0 + (\vec\xi^*\cdot\vec g)\,\nabla\cdot(\rho_0\vec\xi)\right] =$$

$$= \frac{1}{2}\int dz \left[\gamma p_0 |\nabla\cdot\vec\xi|^2 + (\nabla\cdot\vec\xi^*)\,\xi_z p_0' - \xi_z^* \overbrace{\rho_0 g}^{=-p_0'}\,\nabla\cdot\vec\xi - |\xi_z|^2 g \rho_0'\right]$$

We see that δW_2 depends only on ξ_z and $\nabla\cdot\vec\xi$.

Let us consider these to be independent variables and minimize with respect to them.

$$\delta W_2 = \frac{1}{2}\int dz \left[\gamma p_0 |\nabla\cdot\vec\xi|^2 + p_0'\,\xi_z(\nabla\cdot\vec\xi^*) + p_0'\,\xi_z^*(\nabla\cdot\vec\xi) - \rho_0' g\,|\xi_z|^2\right]$$

$$\frac{\partial}{\partial(\nabla\cdot\vec\xi)^*}\left[\dots\right] = \gamma p_0 \nabla\cdot\vec\xi + p_0'\xi_z = 0 \implies \nabla\cdot\vec\xi = -\frac{p_0'}{\gamma p_0}\xi_z$$

$$\delta W_2 = \frac{1}{2}\int dz \left[\gamma p_0\left(\frac{p_0'}{\gamma p_0}\right)^2 - 2p_0'\frac{p_0'}{\gamma p_0} - \rho_0' g\right]|\xi_z|^2$$

$$[\ldots] = \frac{\cancel{(p_0')^2}}{\cancel{\gamma} p_0} - \cancel{2}\frac{(p_0')^2}{\gamma p_0} - \frac{\rho_0'}{\rho_0}\underbrace{\rho_0 g}_{=-p_0'} = -\frac{p_0'}{\gamma}\left(\frac{p_0'}{p_0} - \gamma\frac{\rho_0'}{\rho_0}\right) =$$

$$= -\frac{p_0'}{\gamma}\frac{d}{dz}\ln\frac{p_0}{\rho_0^\gamma} = \frac{\rho_0 g}{\gamma}\frac{d}{dz}\ln\frac{p_0}{\rho_0^\gamma}$$

Thus,
$$\delta W_2 = \frac{1}{2}\int dz\,|\xi_z|^2\,\frac{\rho_0 g}{\gamma}\frac{d}{dz}\ln\frac{p_0}{\rho_0^\gamma}$$

So stable ($\delta W_2 > 0$) if $\boxed{\frac{d}{dz}\ln\frac{p_0}{\rho_0^\gamma} > 0}$ <u>Schwarzschild criterion</u> for convection

Otherwise instability – <u>interchange instability</u> (unmagnetized)

(b) <u>Restore magnetic field</u>: <u>Magnetized interchange instability</u>

$$\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}_0) = -\vec{\xi}\cdot\nabla\vec{B}_0 + \vec{B}_0\cdot\nabla\vec{\xi} - \vec{B}_0\nabla\cdot\vec{\xi} =$$

$$= -\xi_z B_0'\hat{x} + B_0 ik_x\vec{\xi} - B_0\hat{x}\nabla\cdot\vec{\xi} =$$

$$= \underbrace{-(\xi_z B_0' + B_0\nabla\cdot\vec{\xi} - ik_x B_0\xi_x)\hat{x}}_{\substack{\delta B \\ \text{field strength.}}} + B_0\underbrace{ik_x\vec{\xi}_\perp}_{\substack{\delta\hat{b} \\ \text{field direction}}} = \delta\vec{B}$$

$$\vec{j}_0 = \frac{c}{4\pi}B_0'\hat{y}$$

$$\frac{1}{c}\vec{j}_0\cdot(\vec{\xi}^*\times\vec{Q}) = -\frac{B_0'}{4\pi}(\xi_x^* Q_z - \xi_z^* Q_x) =$$

$$= -\frac{B_0'}{4\pi}\left[ik_x\xi_x^*\xi_z B_0 + |\xi_z|^2 B_0' + B_0\xi_z^*\nabla\cdot\vec{\xi} - ik_x\xi_x\xi_z^* B_0\right]$$

Specialise to a class of displacement with $k_x = 0$

$\Downarrow$

$\delta\hat{b} = 0$ no perturbation of field direction (no bending of field lines)

Then $\vec{Q} = -\left(\xi_z B_0' + B_0 \nabla\cdot\vec{\xi}\right)\hat{x}$

$$\frac{1}{c}\vec{J}_0\cdot\left(\vec{\xi}^*\times\vec{Q}\right) = -\frac{B_0'}{4\pi}\left(|\xi_z|^2 B_0' + B_0\,\xi_z^*\,\nabla\cdot\vec{\xi}\right)$$

Substitute this into δW_2:

$$\delta W_2 = \frac{1}{2}\int dz\Big[\gamma p_0|\nabla\cdot\vec{\xi}|^2 + |\xi_z B_0' + B_0\nabla\cdot\vec{\xi}|^2 + (\nabla\cdot\vec{\xi}^*)\,\xi_z p_0' -$$

$$- \frac{(B_0')^2}{4\pi}|\xi_z|^2 \underbrace{- \frac{B_0 B_0'}{4\pi}\xi_z^*\nabla\cdot\vec{\xi} - \rho_0 g\,\xi_z^*\nabla\cdot\vec{\xi}}_{=\,(\nabla\cdot\vec{\xi})\,\xi_z^*\,p_0'} - \rho_0 g\frac{\rho_0'}{\rho_0}|\xi_z|^2\Big]$$

$$= \frac{1}{2}\int dz\Big[\gamma p_0|\nabla\cdot\vec{\xi}|^2 + \frac{1}{4\pi}|\xi_z B_0' + B_0\nabla\cdot\vec{\xi}|^2 - \frac{(B_0')^2}{4\pi}|\xi_z|^2 +$$

$$+ p_0'(\nabla\cdot\vec{\xi})\,\xi_z^* + p_0'(\nabla\cdot\vec{\xi}^*)\,\xi_z - \rho_0 g\frac{\rho_0'}{\rho_0}|\xi_z|^2\Big]$$

So now we have it in an explicitly self-adjoint form, which is, btw, a good check that no errors crept in. Again δW_2 depends on ξ_z and $\nabla\cdot\vec{\xi}$ only. Treat these as independent and minimize:

$$\frac{\partial}{\partial(\nabla\cdot\vec{\xi}^*)}\big[\ldots\big] = \gamma p_0\nabla\cdot\vec{\xi} + \frac{B_0}{4\pi}\left(\xi_z B_0' + B_0\nabla\cdot\vec{\xi}\right) + p_0'\xi_z = 0$$

$$\left(\gamma p_0 + \frac{B_0^2}{4\pi}\right)\nabla\cdot\vec{\xi} = -\frac{B_0 B_0'}{4\pi}\xi_z - p_0'\xi_z = \rho_0 g\,\xi_z$$

$$\nabla\cdot\vec{\xi} = \frac{\rho_0 g}{\gamma p_0 + B_0^2/4\pi}\,\xi_z$$

Substitute this back into δW_2:

$$\delta W_2 = \frac{1}{2}\int dz \left[\gamma p_0 \left(\frac{\rho_0 g}{\gamma p_0 + \frac{B_0^2}{4\pi}}\right)^2 + \frac{1}{4\pi}\left| B_0' + B_0 \frac{\rho_0 g}{\gamma p_0 + B_0^2/4\pi}\right|^2 - \frac{(B_0')^2}{4\pi} + \right.$$

$$\left. + 2 p_0' \frac{\rho_0 g}{\gamma p_0 + B_0^2/4\pi} - \rho_0 g \frac{\rho_0'}{\rho_0} \right] |\xi_z|^2 =$$

$$= \frac{1}{2}\int dz\, |\xi_z|^2 \left[\frac{\gamma p_0}{\left(\gamma p_0 + \frac{B_0^2}{4\pi}\right)^2}\left(p_0' + \frac{B_0 B_0'}{4\pi}\right)^2 + \cancel{\frac{(B_0')^2}{4\pi}} + 2\frac{B_0 B_0'}{4\pi}\frac{\rho_0 g}{\gamma p_0 + \frac{B_0^2}{4\pi}} + \right.$$

$$\left. + \frac{B_0^2}{4\pi}\frac{\left(p_0' + \frac{B_0 B_0'}{4\pi}\right)^2}{\left(\gamma p_0 + \frac{B_0^2}{4\pi}\right)^2} - \cancel{\frac{(B_0')^2}{4\pi}} + 2 p_0' \frac{\rho_0 g}{\gamma p_0 + \frac{B_0^2}{4\pi}} - \rho_0 g \frac{\rho_0'}{\rho_0} \right] =$$

$$= \frac{1}{2}\int dz\, |\xi_z|^2 \left[\cancel{\frac{1}{\gamma p_0 + \frac{B_0^2}{4\pi}}\left(p_0' + \frac{B_0 B_0'}{4\pi}\right)^2} - \cancel{2}\left(p_0' + \frac{B_0 B_0'}{4\pi}\right)^2 \frac{1}{\gamma p_0 + \frac{B_0^2}{4\pi}} - \rho_0 g \frac{\rho_0'}{\rho_0} \right]$$

$$= \frac{1}{2}\int dz\, |\xi_z|^2 \frac{\rho_0 g}{\gamma p_0 + \frac{B_0^2}{4\pi}} \left[p_0' + \frac{B_0 B_0'}{4\pi} - \gamma p_0 \frac{\rho_0'}{\rho_0} - \frac{B_0^2}{4\pi}\frac{\rho_0'}{\rho_0} \right] =$$

$$= \frac{1}{2}\int dz\, |\xi_z|^2 \frac{\rho_0 g}{\gamma p_0 + \frac{B_0^2}{4\pi}}\, p_0 \left[\underbrace{\frac{p_0'}{p_0} - \gamma \frac{\rho_0'}{\rho_0}}_{= \frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma}} + \underbrace{\frac{B_0^2}{4\pi p_0}}_{= \frac{2}{\beta}} \underbrace{\left(\frac{B_0'}{B_0} - \frac{\rho_0'}{\rho_0}\right)}_{= \frac{d}{dz} \ln \frac{B_0}{\rho_0}} \right] =$$

$$= \frac{1}{2}\int dz\, |\xi_z|^2 \frac{\rho_0 g}{\gamma + 2/\beta} \left[\frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta}\frac{d}{dz} \ln \frac{B_0}{\rho_0} \right]$$

We have generalised the Schwarzschild criterion:

instability if $\boxed{\frac{d}{dz} \ln \frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta}\frac{d}{dz} \ln \frac{B_0}{\rho_0} < 0}$, $\beta^{(z)} = \frac{p_0}{B_0^2/8\pi}$

Note that < 0 would not guarantee stability because we restricted ourselves to perturbations with $k_x = 0$ (no bending of field lines) → if bending is allowed, get Parker instability