Magnetohydrodynamics and Turbulence

Alexander Schekochihin, Part III (CASM) Lent Term 2005

EXAMPLE SHEET I: Problem 10

These problems will be discussed in the 1st Examples Class (9.02.05, 14:30 in MR5).

10. Axially Symetric Force-Free Field. Consider axially symmetric ($\partial/\partial\theta = 0$) magnetic field in the half-space $z > 0$ that satisfies

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \tag{1}$$

with $\alpha = \text{const}$ and $\mathbf{B} \to 0$ as $z \to \infty$. In Problem 9, we saw that for an axially symmetric field solenoidality implies

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad B_z = \frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{2}$$

1. From Eq. (1), derive an equation for ψ and a relation between B_θ and ψ. Observe that your equation is the same as the Grad-Shafranov equation with $p = 0$ and $F(\psi) = \alpha\psi$.

2. Find a solution of this equation that decays expontially when $z \to \infty$. Show that the resulting field is

$$B_r = \frac{k}{\sqrt{\alpha^2 + k^2}} B_0 J_1(\sqrt{\alpha^2 + k^2}\, r)\, e^{-kz}, \tag{3}$$

$$B_\theta = \frac{\alpha}{\sqrt{\alpha^2 + k^2}} B_0 J_1(\sqrt{\alpha^2 + k^2}\, r)\, e^{-kz}, \tag{4}$$

$$B_z = B_0 J_0(\sqrt{\alpha^2 + k^2}\, r)\, e^{-kz}, \tag{5}$$

   where $B_0 = B_z(0, 0)$ and k is an arbitrary positive real number. You will find it useful to look up the formulae for the derivatives of Bessel functions.