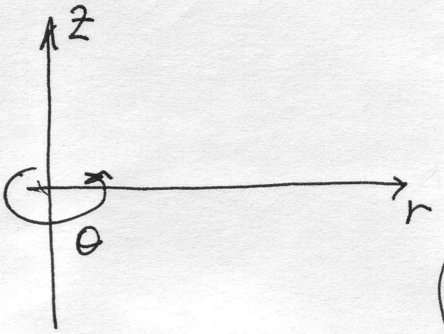
ES1. Problem 9

The Grad-Shafranov Equ in Cylindrical Coordinates



Let $\frac{\partial}{\partial\theta} = 0$

1) $\nabla\cdot\vec{B} = 0$

$$\frac{1}{r}\frac{\partial}{\partial r} r B_r + \frac{\partial B_z}{\partial z} = 0 \qquad (1)$$

Then we can write

$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad B_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad (2)$$

which solves (1).

ψ is the poloidal flux function. We can write the field so:

$$\vec{B} = \underset{\substack{\text{toroidal}\\ \text{field}}}{B_T\hat{\theta}} + \underbrace{\nabla\psi\times\nabla\theta}_{\substack{\vec{B}_p\\ \text{poloidal field}}} \quad , \quad \nabla\theta = \frac{\hat{\theta}}{r}$$

[this is because the ~~pure~~ flux through the surface with $z=0$ and $r_1<r<r_2$ is

$$\Psi_{12} = \int_{r_1}^{r_2} B_z\, 2\pi r\, dr = 2\pi\int_{r_1}^{r_2}\frac{\partial\psi}{\partial r}\,dr = 2\pi\left[\psi(r_2)-\psi(r_1)\right]$$]

2) Ampère's law: $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}$. Let $F = rB_\theta$ by def.

$$j_r = -\frac{c}{4\pi}\frac{\partial B_\theta}{\partial z} = -\frac{c}{4\pi r}\frac{\partial F}{\partial z} \qquad (3)$$

$$j_\theta = \frac{c}{4\pi}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) \qquad (4)$$

$$j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \frac{c}{4\pi r}\frac{\partial F}{\partial r} \qquad (5)$$

Substitute (2) into (4):

$$j_\theta = \frac{c}{4\pi}\left(-\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = -\frac{c}{4\pi r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) \quad (6)$$

3) Force balance: $\frac{1}{c}\vec{J}\times\vec{B} = \nabla p \qquad (7)$

(7)$\cdot\hat{\theta}$:

$$0 = \underset{(5)}{j_z}\,\underset{(2)}{B_r} - \underset{(3)}{j_r}\,\underset{(2)}{B_z} = \frac{c}{4\pi r}\frac{\partial F}{\partial r}\left(-\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + \frac{c}{4\pi r}\frac{\partial F}{\partial z}\frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$-\frac{\partial F}{\partial r}\frac{\partial\psi}{\partial z} + \frac{\partial F}{\partial z}\frac{\partial\psi}{\partial r} = 0$$

This is $\nabla F\times\nabla\psi = 0 \implies \nabla F \parallel \nabla\psi \implies F = F(\psi)$

F is called the <u>poloidal current function</u>

[poloidal current through surface with $z=0$ and $r_1<r<r_2$ is

$$I_{12} = \int_{r_1}^{r_2} j_z\, 2\pi r\, dr = \frac{c}{2}\int_{r_1}^{r_2} dr\,\frac{\partial F}{\partial r} = \frac{c}{2}\left(F(r_2)-F(r_1)\right) = \frac{c}{2}\left(F(\psi_2)-F(\psi_1)\right)]$$

(7)$\cdot\hat{r}$:

$$c\frac{\partial p}{\partial r} = \underset{(6)}{j_\theta}\,\underset{(2)}{B_z} - \underset{(5)}{j_z}\,\underbrace{B_\theta}_{F/r} = -\frac{c}{4\pi r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right)\frac{1}{r}\frac{\partial\psi}{\partial r} -$$

$$-\frac{c}{4\pi r}\underbrace{\frac{\partial F}{\partial r}}_{=\frac{dF}{d\psi}\frac{\partial\psi}{\partial r}}\frac{F}{r} = -\frac{c}{4\pi r^2}\frac{\partial\psi}{\partial r}\left[\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) + F\frac{dF}{d\psi}\right] \quad (8)$$

(7)$\cdot\hat{z}$:

$$c\frac{\partial p}{\partial z} = j_r B_\theta - j_\theta B_r = -\frac{c}{4\pi r}\frac{dF}{d\psi}\frac{\partial\psi}{\partial z}\frac{F}{r} -$$

$$-\frac{c}{4\pi r}\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right)\frac{1}{r}\frac{\partial\psi}{\partial z} =$$

$$= -\frac{c}{4\pi r^2}\frac{\partial\psi}{\partial z}\left[\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) + F\frac{dF}{d\psi}\right] \quad (9)$$

Now take $(8)\cdot\dfrac{\partial\psi}{\partial z} - (9)\cdot\dfrac{\partial\psi}{\partial r}$:

$$\frac{\partial p}{\partial r}\frac{\partial\psi}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial\psi}{\partial r} = 0$$

This is $\nabla p \times \nabla\psi = 0 \;\Rightarrow\; \nabla p \parallel \nabla\psi \;\Rightarrow\; p = p(\psi)$

In (8), write $\dfrac{\partial p}{\partial r} = \dfrac{dp}{d\psi}\dfrac{\partial\psi}{\partial r}$. This gives

$$\boxed{-\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) = 4\pi r^2\frac{dp}{d\psi} + F\frac{dF}{d\psi}} \qquad (10)$$

— this is the Grad-Shafranov eqn.

Here
$$\vec{B} = \frac{F}{r}\hat\theta + \nabla\psi \times \frac{\hat\theta}{r}$$
$$\vec{J} = \frac{c}{4\pi r}\left[-\hat\theta\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{dF}{d\psi}\nabla\psi \times \frac{\hat\theta}{r}\right]$$

$p(\psi)$ and $F(\psi)$ are free functions.

Given their forms, get ψ and then $\vec{B}$ and $\vec{J}$.

Note that if $\dfrac{\partial}{\partial z} = 0$, we get $B_r = 0$, $\dfrac{1}{r}\dfrac{\partial\psi}{\partial r} = B_z$

$$\frac{dp}{d\psi} = \frac{\partial p}{\partial r}\left(\frac{\partial\psi}{\partial r}\right)^{-1}, \quad F\frac{dF}{d\psi} = \underbrace{F\frac{\partial F}{\partial r}}_{''}\left(\frac{\partial\psi}{\partial r}\right)^{-1}$$

So, from (10)

$$r B_\theta \frac{\partial}{\partial r} r B_\theta = r B_\theta^2 + r^2\frac{\partial}{\partial r}\frac{B_\theta^2}{2}$$

$$4\pi r^2\left[\frac{\partial p}{\partial r} + \frac{B_\theta^2}{4\pi r} + \frac{\partial}{\partial r}\frac{B_\theta^2}{8\pi}\right] = -\frac{\partial\psi}{\partial r}\, r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}$$

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{8\pi}\right) + \frac{B_\theta^2}{4\pi r} = -\frac{\partial}{\partial r}\frac{1}{8\pi}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = -\frac{\partial}{\partial r}\frac{B_z^2}{8\pi}$$

$$\boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}}$$