ESI. Problem 10.

Axisymmetric Force-Free Fields.

We have cylindrical coordinates and $\frac{\partial}{\partial\theta} = 0$.

As we saw in Problem 9, the solenoidality $(\nabla\cdot\vec{B}=0)$ implies

$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad B_z = \frac{1}{r}\frac{\partial\psi}{\partial r}. \qquad (1)$$

and, from $\nabla\times\vec{B} = \alpha\vec{B}$,

$$(\nabla\times\vec{B})_r = -\frac{\partial B_\theta}{\partial z} = \alpha B_r = -\frac{\alpha}{r}\frac{\partial\psi}{\partial z} \qquad (2)$$

$$(\nabla\times\vec{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r} =$$

$$= -\frac{1}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial\psi}{\partial r} = \alpha B_\theta \qquad (3)$$

$$(\nabla\times\vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r} r B_\theta = \alpha B_z = \frac{\alpha}{r}\frac{\partial\psi}{\partial r} \qquad (4)$$

(2) and (4) are solved by $\boxed{B_\theta = \frac{\alpha}{r}\psi}$ (5)

Substitute this into (3):

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \alpha^2\psi = 0 \qquad (6)$$

Let $\psi = r f$. Then $\frac{\partial\psi}{\partial r} = r\frac{\partial f}{\partial r} + f$

$$\frac{\partial^2\psi}{\partial r^2} = r\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r}$$

$$r\frac{\partial^2 f}{\partial z^2} + r\frac{\partial^2 f}{\partial r^2} + 2\frac{\partial f}{\partial r} - \frac{\partial f}{\partial r} - \frac{1}{r}f + \alpha^2 r f = 0$$

$$\frac{\partial^2 f}{\partial z^2} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(\alpha^2 - \frac{1}{r^2}\right) f = 0$$

Now let $f(r,z) = g(r)\, e^{-kz}$ - solution decays at $z \to \infty$ if $k>0$.

Then

$$g'' + \frac{1}{r} g' + \left(\alpha^2 + k^2 - \frac{1}{r^2}\right) g = 0$$

This is a Bessel equation with solution

$$g(r) = \text{const} \cdot J_1\left(\sqrt{\alpha^2+k^2}\, r\right)$$

So $\psi = rf = rg e^{-kz} = \text{const}.\, r J_1\left(\sqrt{\alpha^2+k^2}\, r\right) e^{-kz}$

From (1), $B_z = \frac{1}{r}\frac{\partial}{\partial r}\psi = \text{const}\, \underbrace{\frac{1}{r}\frac{\partial}{\partial r} r J_1\left(\sqrt{\alpha^2+k^2}\, r\right)}_{= \sqrt{\alpha^2+k^2}\, J_0\left(\sqrt{\alpha^2+k^2}\, r\right)} e^{-kz}$

Denote $\text{const}\sqrt{\alpha^2+k^2} = B_0$. Then

$$\begin{cases} B_z = B_0 J_0\left(\sqrt{\alpha^2+k^2}\, r\right) e^{-kz} \\ B_\theta = \dfrac{\alpha}{\sqrt{\alpha^2+k^2}} B_0 J_1\left(\sqrt{\alpha^2+k^2}\, r\right) e^{-kz} \\ B_r = \dfrac{k}{\sqrt{\alpha^2+k^2}} B_0 J_1\left(\sqrt{\alpha^2+k^2}\, r\right) e^{-kz} \end{cases}$$

Note that 1) $k=0$ corresponds to the case of cylindrical symmetry: the above solution reduces to

$B_r = 0$, $B_\theta = B_0 J_1(\alpha r)$, $B_z = B_0 J_0(\alpha r)$ derived in class.

2) Eq. (6) is the ~~same as the~~ Grad-Shafranov equ from Problem 9 with $p=0$ and $F = \alpha\psi$