Problem 1. Ambipolar Damping

Momentum equations:

ions + electrons: $\frac{\partial \vec{u}_i}{\partial t} + \vec{u}_i \cdot \nabla \vec{u}_i = -\frac{\nabla P}{\rho_i} - \mu_{in} (\vec{u}_i - \vec{u}_n) + \frac{\vec{B} \cdot \nabla \vec{B}}{4\pi \rho_i}$

neutrals: $\frac{\partial \vec{u}_n}{\partial t} + \vec{u}_n \cdot \nabla \vec{u}_n = -\frac{\nabla p_n}{\rho_n} - \mu_{ni} (\vec{u}_n - \vec{u}_i)$

Assume incompressibility: $\nabla \cdot \vec{u}_i = \nabla \cdot \vec{u}_n = 0$

m. field: $\frac{\partial \vec{B}}{\partial t} + \vec{u}_i \cdot \nabla \vec{B} = \vec{B} \cdot \nabla u_i$

Assuming elastic i-n collisions,

$$\rho_i \mu_{in} = \rho_n \mu_{ni} \Rightarrow \mu_{ni} = \frac{\rho_i}{\rho_n} \mu_{in}$$

For simplicity, assume $\rho_i = \rho_n$, so $\mu_{in} = \mu_{ni}$

Linearise around static straight-field equilibrium:

$$\vec{B}_0 = B_0 \hat{z}, \quad \vec{u}_{i0} = \vec{u}_{n0} = 0.$$

Perturbed velocities in terms of displacement:

$$\vec{u}_i = \frac{\partial \vec{\xi}_i}{\partial t} \qquad \vec{u}_n = \frac{\partial \vec{\xi}_n}{\partial t}$$

Then $\frac{\partial \delta \vec{B}}{\partial t} = B_0 \frac{\partial}{\partial z} \frac{\partial \vec{\xi}_i}{\partial t} \Rightarrow \delta \vec{B} = B_0 \frac{\partial \vec{\xi}_i}{\partial z}$

$$\frac{\vec{B} \cdot \nabla \vec{B}}{4\pi \rho_i} = \frac{B_0}{4\pi \rho_i} \frac{\partial}{\partial z} \delta \vec{B} = \frac{B_0^2}{4\pi \rho_i} \frac{\partial^2}{\partial z^2} \vec{\xi}_i = V_A^2 \frac{\partial^2 \vec{\xi}_i}{\partial z^2}$$

Momentum equations become:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\vec{\xi}_i = V_A^2 \dfrac{\partial^2}{\partial z^2}\vec{\xi}_i - \mu_{in}\left(\dfrac{\partial \vec{\xi}_i}{\partial t} - \dfrac{\partial \vec{\xi}_n}{\partial t}\right) \\ \dfrac{\partial^2}{\partial t^2}\vec{\xi}_n = -\mu_{in}\left(\dfrac{\partial \vec{\xi}_n}{\partial t} - \dfrac{\partial \vec{\xi}_i}{\partial t}\right) \end{cases}$$

$$\begin{cases} \left(\omega^2 - k_\parallel^2 V_A^2 + i\omega\mu_{in}\right)\vec{\xi}_i = i\omega\mu_{in}\vec{\xi}_n \\ \left(\omega^2 + i\omega\mu_{in}\right)\vec{\xi}_n = i\omega\mu_{in}\vec{\xi}_i \end{cases}$$

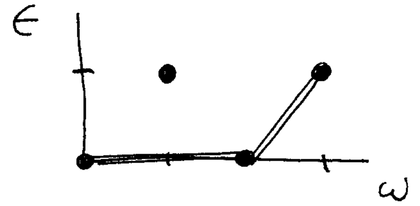
$$\left(\omega^2 - k_\parallel^2 V_A^2 + i\omega\mu_{in}\right)\left(\omega^2 + i\omega\mu_{in}\right) = -\omega^2\mu_{in}^2$$

$$\omega^4 - \omega^2 k_\parallel^2 V_A^2 + i\omega^3\mu_{in} - i\omega k_\parallel^2 V_A^2 \nu_n k^2$$
$$+ i\omega^3\mu_{in} - i\omega\mu_{in} k_\parallel^2 V_A^2 - \omega^2\mu_{in}^2 = -\omega^2\mu_{in}^2$$

$$\omega^3 + i\omega^2\left(2\mu_{in}\right) + \omega\left(-k_\parallel^2 V_A^2\right) - i k_\parallel^2 V_A^2\left(\mu_{in}\right) = 0$$

$$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega}{k_\parallel V_A}\right)^3 - 2\left(\frac{\omega}{k_\parallel V_A}\right)^2 - i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega}{k_\parallel V_A}\right) + 1 = 0$$

Assume $k_\parallel V_A \ll \mu_{in}$



Dominant balances:

(i) $$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{\omega}{k_\parallel V_A}\right)^3 \approx 2\left(\frac{\omega}{k_\parallel V_A}\right)^2$$

$$\boxed{\omega = -2i\mu_{in}}$$ damping due to collisions with neutrals

Eigenvector: $(-4\mu_{in}^2 + 2\mu_{in}^2)\vec{\xi}_i = 2\mu_{in}^2 \vec{\xi}_n$

$$\vec{\xi}_i = -\vec{\xi}_n$$

(ii) $$2\left(\frac{\omega}{k_\parallel V_A}\right)^2 = 1 \Rightarrow \omega_0 = \pm\frac{k_\parallel V_A}{\sqrt{2}}$$ zeroth-order

Now $\omega = \omega_0 + \delta\omega$

$$i\frac{k_\parallel V_A}{\mu_{in}}\left(\frac{1}{\sqrt{2}}\right)^3 - 2\left(\frac{1}{2} + 2\frac{1}{\sqrt{2}}\frac{\delta\omega}{k_\parallel V_A}\right) - i\frac{k_\parallel V_A}{\mu_{in}}\frac{1}{\sqrt{2}} + 1 = 0$$

$$i\frac{k_\parallel V_A}{\mu_{in}}\frac{1}{2} - 4\frac{\delta\omega}{k_\parallel V_A} - i\frac{k_\parallel V_A}{\mu_{in}} = 0$$

$$\delta\omega = -\frac{i}{8}\frac{k_\parallel^2 V_A^2}{\mu_{in}}$$

So, $$\omega = \pm\frac{k_\parallel V_A}{\sqrt{2}} - \frac{i}{8}\frac{k_\parallel^2 V_A^2}{\mu_{in}}$$ ← Ambipolar damping

Eigenvector: to lowest order, $\vec{\xi}_n = \vec{\xi}_i$

Next order: $\vec{\xi}_i = \vec{\xi}_n + \delta\vec{\xi}$

$$\begin{cases} (\omega^2 - k_\parallel^2 V_A^2 + i\omega\mu_{in})(\vec{\xi}_n + \delta\vec{\xi}) = i\omega\mu_{in}\vec{\xi}_n \\ (\omega^2 + i\omega\mu_{in})\vec{\xi}_n = i\omega\mu_{in}(\vec{\xi}_n + \delta\vec{\xi}) \end{cases}$$

$$\begin{cases} \left(\omega^2 - k_\parallel^2 V_A^2\right) \vec{\xi}_n + i\omega\mu_{in}\,\delta\vec{\xi} = 0 \\ \omega^2 \vec{\xi}_n = i\omega\mu_{in}\,\delta\vec{\xi} \end{cases}$$

$$\pm i \frac{k_\parallel V_A}{\sqrt{2}} \mu_{in}\,\delta\vec{\xi} = \frac{k_\parallel^2 V_A^2}{2} \vec{\xi}_n$$

$$\delta\vec{\xi} = \mp i \frac{k_\parallel V_A}{\sqrt{2}\,\mu_{in}} \vec{\xi}_n$$

Thus, $$\vec{\xi}_i = \left(1 \mp i \frac{k_\parallel V_A}{\sqrt{2}\,\mu_{in}}\right) \vec{\xi}_n$$

↑ slippage between ions and neutrals.

Problem 2

Reduced EMHD

$$\frac{\partial \vec{B}}{\partial t} = -\frac{c}{4\pi e n} \nabla \times \left[(\nabla \times \vec{B}) \times \vec{B} \right]$$

Let $\vec{B} = B_0 \hat{z} + \delta \vec{B}$. Then

$$\frac{\partial}{\partial t} \frac{\delta \vec{B}}{B_0} = -\frac{c B_0}{4\pi e n} \nabla \times \left[\left(\nabla \times \frac{\delta \vec{B}}{B_0}\right) \times \left(\hat{z} + \frac{\delta \vec{B}}{B_0}\right) \right]$$

$$\frac{c B_0}{4\pi e n} = \frac{B_0}{\sqrt{4\pi n m_i}} \, c \sqrt{\frac{m_i}{4\pi e^2 n}} = v_A d_i$$

Dispersion relation: $\omega = \pm k_\parallel v_A k d_i$

1) Ordering: critical balance (strong turbulence)

$$\omega \sim k_\parallel v_A k_\perp d_i \sim v_A d_i \frac{\delta B}{B_0} k_\perp^2 \quad \text{(nonlinear time)}$$

$$\frac{\delta B}{B_0} \sim \frac{k_\parallel}{k_\perp}$$

2) $\nabla \cdot \delta \vec{B} = 0 = \nabla_\perp \cdot \delta \vec{B} + \frac{\partial \delta B_\parallel}{\partial z} = 0 \Rightarrow \nabla_\perp \cdot \delta B_\perp = 0$

$\qquad\qquad\qquad\quad \epsilon \qquad\qquad \epsilon^2$

$$\frac{\delta \vec{B}_\perp}{\sqrt{4\pi n m_i}} = \hat{z} \times \nabla_\perp \psi$$

So,

$$\frac{\delta \vec{B}}{B_0} = \frac{1}{v_A} \hat{z} \times \nabla_\perp \psi + \hat{z} \frac{\delta B_\parallel}{B_0}$$

3) Now derive evolution equations.

$$\nabla \times \frac{\delta \vec{B}}{B_0} = \frac{1}{v_A} \nabla \times (\hat{z} \times \nabla_\perp \psi) - \hat{z} \times \nabla_\perp \frac{\delta B_\parallel}{B_0} =$$

$$= \frac{1}{V_A}\left[\hat{z}\,\nabla_\perp^2\psi - \frac{\partial}{\partial z}\nabla_\perp\psi\right] - \hat{z}\times\nabla_\perp\frac{\delta B_\parallel}{B_0}$$

$$\nabla\times\left[\left(\nabla\times\frac{\delta\vec{B}}{B_0}\right)\times\left(\hat{z}+\frac{\delta\vec{B}}{B_0}\right)\right] = \left(\hat{z}+\frac{\delta\vec{B}}{B_0}\right)\cdot\nabla\left(\nabla\times\frac{\delta\vec{B}}{B_0}\right) -$$

$$- \left(\nabla\times\frac{\delta\vec{B}}{B_0}\right)\cdot\nabla\frac{\delta\vec{B}}{B_0}$$

$$\hat{z}\cdot\frac{\partial}{\partial t}\frac{\delta\vec{B}}{B_0} = \frac{\partial}{\partial t}\frac{\delta B_\parallel}{B_0} = -V_A d_i\,\hat{z}\cdot\left\{\nabla\times\left[\dots\right]\right\} =$$

$$= -V_A d_i\left[\left(\hat{z}+\frac{\delta\vec{B}}{B_0}\right)\cdot\underset{\epsilon^2}{\nabla}\frac{1}{V_A}\nabla_\perp^2\psi - \underset{\epsilon^3}{\cancel{\frac{1}{V_A}\nabla_\perp^2\psi\frac{\partial}{\partial z}\frac{\delta B_\parallel}{B_0}}} + \right.$$

$$\left. + \underset{\epsilon^3}{\cancel{\frac{1}{V_A}\left(\nabla_\perp\frac{\partial\psi}{\partial z}\right)\cdot\nabla\frac{\delta B_\parallel}{B_0}}} + \underset{0}{\cancel{\left(\hat{z}\times\nabla_\perp\frac{\delta B_\parallel}{B_0}\right)\cdot\nabla\frac{\delta B_\parallel}{B_0}}}\right]$$

$$\approx -d_i\frac{\vec{B}}{B_0}\cdot\nabla\nabla_\perp^2\psi \qquad \text{q.e.d.}$$

$$\hat{z}\times\frac{\partial}{\partial t}\frac{\delta\vec{B}}{B_0} = \frac{\partial}{\partial t}\frac{1}{V_A}\hat{z}\times\left(\hat{z}\times\nabla_\perp\psi\right) = -\frac{1}{V_A}\nabla_\perp\frac{\partial\psi}{\partial t} =$$

$$= -V_A d_i\,\hat{z}\times\left\{\nabla\times\left[\dots\right]\right\} =$$

$$= -V_A d_i\left\{\frac{\vec{B}}{B_0}\cdot\nabla\left[\hat{z}\times\left(\nabla\times\frac{\delta\vec{B}}{B_0}\right)\right] - \left(\nabla\times\frac{\delta\vec{B}}{B_0}\right)\cdot\nabla\left(\hat{z}\times\frac{\delta\vec{B}}{B_0}\right)\right\}$$

$$= -V_A d_i\left\{\underbrace{\frac{\vec{B}}{B_0}\cdot\nabla\left[\underset{\epsilon}{\nabla\frac{\delta B_\parallel}{B_0}}\right.}_{\epsilon^2} - \left.\underset{\epsilon^2}{\cancel{\frac{\partial}{\partial z}\frac{\delta\vec{B}}{B_0}}}\right] - \underset{\epsilon^3}{\cancel{\frac{1}{V_A}\nabla_\perp^2\psi\frac{\partial}{\partial z}\left(\hat{z}\times\frac{\delta\vec{B}}{B_0}\right)}} + \right.$$

$$\left. + \underset{\epsilon^3}{\cancel{\frac{1}{V_A}\left(\nabla_\perp\frac{\partial\psi}{\partial z}\right)\cdot\nabla\left(\hat{z}\times\frac{\delta\vec{B}}{B_0}\right)}} + \left(\hat{z}\times\nabla_\perp\frac{\delta B_\parallel}{B_0}\right)\cdot\nabla\underbrace{\left(\hat{z}\times\frac{\delta\vec{B}}{B_0}\right)}_{= -\frac{1}{V_A}\nabla_\perp\psi}\right\} =$$

$$= -v_A d_i \left[\frac{\partial}{\partial z} \nabla \frac{\delta B_\parallel}{B_0} + \frac{1}{v_A} \left\{ \psi, \nabla \frac{\delta B_\parallel}{B_0} \right\} + \frac{1}{v_A} \underbrace{\left(\hat{z} \times \nabla_\perp \frac{\delta B_\parallel}{B_0} \right) \cdot \nabla \nabla_\perp \psi} \right]$$

$$\hat{z} \cdot \left(\nabla_\perp \frac{\delta B_\parallel}{B_0} \times \nabla \nabla_\perp \psi \right) = \left\{ \frac{\delta B_\parallel}{B_0}, \nabla \psi \right\}$$

$$= -v_A d_i \left[\frac{\partial}{\partial z} \nabla \frac{\delta B_\parallel}{B_0} + \frac{1}{v_A} \left\{ \psi, \nabla \frac{\delta B_\parallel}{B_0} \right\} + \frac{1}{v_A} \left\{ \nabla \psi, \frac{\delta B_\parallel}{B_0} \right\} \right] =$$

$$= -v_A d_i \nabla \left[\frac{\partial}{\partial z} \frac{\delta B_\parallel}{B_0} + \frac{1}{v_A} \left\{ \psi, \frac{\delta B_\parallel}{B_0} \right\} \right]$$

Thus, $\frac{\partial \psi}{\partial t} = v_A^2 d_i \frac{\vec{B}}{B_0} \cdot \nabla \frac{\delta B_\parallel}{B_0}$ q.e.d.

4) Linearised eqns:

$$\begin{cases} -i\omega \frac{\delta B_\parallel}{B_0} = i d_i k_\parallel k_\perp^2 \psi \\ -i\omega \psi = i k_\parallel v_A^2 d_i \frac{\delta B_\parallel}{B_0} \end{cases}$$

$$\omega^2 = k_\parallel^2 k_\perp^2 d_i^2 v_A^2 \quad \text{OK}$$

Problem 3

Material covered in the lecture course.

Marking Scheme: 50 pts for each problem (so full score 100)

Problem 1
1. (20)
2. (20)
3. (10)

Problem 2
1. (5)
2. (5)
3. (35)
4. (5)

Problem 3
1. (20)
2. (10)
3. (20)