Paper 69. Question 3

$$\begin{cases} \dfrac{d\tilde{B}}{dt} = \tilde{\sigma}\tilde{B} \\[2mm] \dfrac{d\tilde{\sigma}}{dt} = -\dfrac{1}{\tau}\tilde{\sigma} + f \end{cases}$$

where f is a gaussian white noise:

$$\langle f(t) f(t')\rangle = \varkappa\,\delta(t-t')$$

1) Solution of the second equation with $\sigma(0)=0$:

$$\tilde{\sigma}(t) = \int_0^t dt' e^{-(t-t')/\tau} f(t')$$

Correlation function:

$$\langle \tilde{\sigma}(t_1)\tilde{\sigma}(t_2)\rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\frac{t_1-t'+t_2-t''}{\tau}} \overbrace{\langle f(t') f(t'')\rangle}^{=\varkappa\delta(t'-t'')} =$$

Let $t_1 \geq t_2$

$$= \int_0^{t_2} dt'' e^{+\frac{2t''}{\tau}} \varkappa e^{-\frac{t_1+t_2}{\tau}} = \frac{\varkappa\tau}{2}\left[-1 + e^{+\frac{2t_2}{\tau}}\right] e^{-\frac{t_1+t_2}{\tau}} =$$

$$= \frac{\varkappa\tau}{2} e^{-\frac{t_1-t_2}{\tau}}\left[1 - e^{-\frac{2t_2}{\tau}}\right] \approx \frac{\varkappa\tau}{2} e^{-\frac{t_1-t_2}{\tau}} \quad \text{as } t_2 \to \infty$$

So τ is the correlation time of $\tilde{\sigma}$.

2) Joint pdf of $\tilde{B}$ and $\tilde{\sigma}$:

$$P(B,\sigma) = \langle \delta(B-\tilde{B}(t))\,\delta(\sigma - \tilde{\sigma}(t))\rangle \equiv \langle \tilde{P}\rangle$$

$$\frac{\partial\tilde{P}}{\partial t} = -\frac{\partial}{\partial B}\tilde{P}\frac{d\tilde{B}}{dt} - \frac{\partial}{\partial\sigma}\tilde{P}\frac{d\tilde{\sigma}}{dt} =$$

$$= -\left[\frac{\partial}{\partial B}\sigma B + \frac{\partial}{\partial\sigma}\left(-\frac{\sigma}{\tau} + f\right)\right]\tilde{P}$$

Average:

$$\frac{\partial P}{\partial t} = -\sigma\frac{\partial}{\partial B}BP + \frac{1}{\tau}\frac{\partial}{\partial\sigma}\sigma P - \frac{\partial}{\partial\sigma}\langle f\tilde{P}\rangle$$

Now $\langle f\tilde{P}\rangle = \int dt' \langle f(t)f(t')\rangle \langle \frac{\delta\tilde{P}(t)}{\delta f(t')}\rangle =$

$$= \varkappa \langle \frac{\delta\tilde{P}(t)}{\delta f(t)}\rangle = -\frac{1}{2}\varkappa\frac{\partial}{\partial\sigma}P \quad \text{because}$$

$$\tilde{P}(t) = \int^t dt' \left[-\frac{\partial}{\partial B}\sigma B + \frac{\partial}{\partial\sigma}\frac{\sigma}{\tau} - \frac{\partial}{\partial\sigma}f(t')\right]\tilde{P}(t')$$

$$\frac{\delta\tilde{P}(t')}{\delta f(t)} = -\frac{\partial}{\partial\sigma}\int^t dt'\,\delta(t'-t)\tilde{P}(t') = -\frac{1}{2}\frac{\partial}{\partial\sigma}\tilde{P}(t)$$

(all other terms vanish by causality)

So $$\boxed{\frac{\partial P}{\partial t} = \frac{1}{2}\varkappa\frac{\partial^2}{\partial\sigma^2}P + \frac{1}{\tau}\frac{\partial}{\partial\sigma}\sigma P - \sigma\frac{\partial}{\partial B}BP}$$

3) We need $\langle B^n\rangle = \int dB\, B^n \int d\sigma\, P$

but we can't integrate the P equation over σ because this gives a closure problem.

Define $P_n^{(\sigma)} = \int dB\, B^n P(B,\sigma)$. This satisfies

$$\frac{\partial P_n}{\partial t} = \frac{\varkappa}{2}\frac{\partial^2 P_n}{\partial\sigma^2} + \frac{1}{\tau}\frac{\partial}{\partial\sigma}\sigma P_n + n\sigma P_n$$

Look for solutions in the form $P_n = e^{\gamma_n t}\psi(\sigma)e^{S(\sigma)}$

4) Choose $S(\sigma)$ so as to get rid of the first derivative:

$$\frac{\partial P_n}{\partial\sigma} = e^{\gamma_n t}\left[\psi' + \psi S'\right] e^{S}$$

$$\frac{\partial^2 P_n}{\partial\sigma^2} = e^{\gamma_n t}\left[\psi'' + 2\psi' S' + \psi S'' + \psi (S')^2\right] e^{S}$$

The equation becomes:

$$\gamma_n \psi = \frac{\varkappa}{2}\left[\psi'' + \underline{2\psi' S'} + \psi S'' + \psi S'^2\right] + \frac{1}{\tau}\psi + n\sigma\psi +$$

$$+ \frac{\sigma}{\tau}\left[\underline{\psi'} + \psi S'\right] =$$

$$= \frac{\varkappa}{2}\psi'' + \underbrace{\left[\varkappa S' + \frac{\sigma}{\tau}\right]}_{=0}\psi' + \left[\frac{\varkappa}{2}S'' + \frac{\varkappa}{2}S'^2 + \frac{\sigma}{\tau}S' + \frac{1}{\tau} + n\sigma\right]\psi$$

Will give this substitution as a hint.

$$S' = -\frac{\sigma}{\varkappa\tau} \Rightarrow S = -\frac{\sigma^2}{2\varkappa\tau}$$

$$P_n = e^{\gamma_n t}\psi(\sigma)\, e^{-\frac{\sigma^2}{2\varkappa\tau}}, \text{ where}$$

$$\gamma_n \psi = \frac{\varkappa}{2}\psi'' + \underbrace{\left[-\frac{1}{2\tau} + \frac{\sigma^2}{2\varkappa\tau^2} - \frac{\sigma^2}{\varkappa\tau^2} + \frac{1}{\tau} + n\sigma\right]}_{=}\psi$$

$$-\frac{\sigma^2}{2\varkappa\tau^2} + \frac{1}{2\tau} + n\sigma =$$

$$= -\frac{1}{2\varkappa\tau^2}\left(\sigma^2 - 2\varkappa\tau^2 n\sigma + \varkappa^2\tau^4 n^2\right) + \frac{\varkappa\tau^2 n^2}{2} + \frac{1}{2\tau}$$

$$= \frac{\varkappa}{2}\psi'' - \frac{(\sigma - \varkappa\tau^2 n)^2}{2\varkappa\tau^2}\psi + \left(\frac{\varkappa\tau^2 n^3}{2} + \frac{1}{2\tau}\right)\psi$$

Let $x = \dfrac{\sigma - \varkappa\tau^2 n}{\sqrt{\varkappa\tau}}$. Then

$$\gamma_n \psi = \frac{1}{2\tau}\frac{\partial^2\psi}{\partial x^2} - \frac{x^2}{2\tau}\psi + \left(\frac{\varkappa\tau^2 n^2}{2} + \frac{1}{2\tau}\right)\psi$$

$$\frac{\partial^2\psi}{\partial x^2} - x^2\psi = \left(2\tau\gamma_n - 1 - \varkappa\tau^3 n^2\right)\psi$$

This is a harmonic oscillator.

Ground state: $-2E_0 = 2\tau\gamma_n - 1 - \varkappa\tau^3 n^2 = -1$

So $\boxed{\gamma_n = \dfrac{\varkappa\tau^2 n^2}{2}}$

$$\langle B^n \rangle = \int d\sigma\, P_n = e^{\frac{\varkappa\tau^2 n^2}{2} t} \underbrace{\int d\sigma\, \psi(\sigma) e^{S(\sigma)}}_{\text{const.}}$$