Paper 69. Question 2

EMHD Turbulence

Magnetic field frozen into the electron flow:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u}_e \times \vec{B})$$

Current: $\vec{j} = en(\vec{u}_i - \vec{u}_e) \Rightarrow \vec{u}_e = \underbrace{\cancel{\vec{u}_i}}_{\text{neglect}} - \frac{\vec{j}}{en}$

But $\vec{j} = \frac{c}{4\pi} \nabla \times \vec{B}$ (Ampère's law), so

$$\vec{u}_e = -\frac{c}{4\pi e n} \nabla \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \frac{c}{4\pi e n} \nabla \times \left[\vec{B} \times (\nabla \times \vec{B})\right] = \frac{v_A d_i}{B_0} \nabla \times \left[\vec{B} \times (\nabla \times \vec{B})\right]$$

1) $\vec{B} = B_0 \hat{z} + \delta\vec{B}$. Linear dispersion relation:

$$\frac{\partial \delta\vec{B}}{\partial t} = \frac{c B_0}{4\pi e n} \nabla \times \left[\hat{z} \times (\nabla \times \delta\vec{B})\right]$$

$$-i\omega\,\delta\vec{B} = -\frac{cB_0}{4\pi e n} \begin{bmatrix} k_\perp \\ 0 \\ k_\parallel \end{bmatrix} \times \underbrace{\left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \underbrace{\left(\begin{bmatrix} k_\perp \\ 0 \\ k_\parallel \end{bmatrix} \times \begin{bmatrix} \delta B_x \\ \delta B_y \\ \delta B_z \end{bmatrix} \right)}_{= \begin{bmatrix} -k_\parallel \delta B_y \\ -k_\perp \delta B_z + k_\parallel \delta B_x \\ k_\perp \delta B_y \end{bmatrix}} \right]}_{= \begin{bmatrix} k_\perp \delta B_z - k_\parallel \delta B_x \\ -k_\parallel \delta B_y \\ 0 \end{bmatrix}} =$$

$$= -\frac{cB_0}{4\pi e n} \begin{bmatrix} k_\parallel^2 \delta B_y \\ k_\perp k_\parallel \delta B_z - k_\parallel^2 \delta B_x \\ -k_\perp k_\parallel \delta B_y \end{bmatrix}$$

$$\begin{bmatrix} -i\omega & \frac{cB_0}{4\pi e n} k_\parallel^2 & 0 \\ -\frac{cB_0}{4\pi e n} k_\parallel^2 & -i\omega & \frac{cB_0}{4\pi e n} k_\perp k_\parallel \\ 0 & -k_\perp k_\parallel \frac{cB_0}{4\pi e n} & -i\omega \end{bmatrix} \cdot \delta\vec{B} = 0$$

$$i\omega^3 + i\omega k_\perp^2 k_\parallel^2 \left(\frac{cB_0}{4\pi e n}\right)^2 - i\omega \left(\frac{cB_0}{4\pi e n}\right)^2 k_\parallel^4 = 0$$

$$\omega^2 = \left(\frac{cB_0}{4\pi e n}\right)^2 k_\parallel^2 k^2 = (k_\parallel v_A)^2 \frac{(k_\perp \rho_i)^2}{\beta_i} = (k_\parallel v_A)^2 (k_\perp d_i)^2$$

$$\frac{cB_0}{4\pi e n} = \frac{c m_i}{e B_0} \frac{B_0^2}{4\pi m_i n} = \frac{1}{\Omega_i} v_A^2 = \frac{v_{thi}}{\Omega_i} \frac{v_A}{v_{thi}} v_A = \rho_i \frac{1}{\sqrt{\beta_i}} v_A$$

$$\text{or } \frac{c}{\sqrt{4\pi e^2 n/m_i}} \frac{B_0}{\sqrt{4\pi n m_i}} = \frac{c}{\omega_{pi}} v_A = d_i v_A$$

ion skin depth.

So $\boxed{\omega = \pm k_\parallel v_A k_\perp d_i}$

2)

Assume weak interaction:

$$\omega \sim \frac{v_A}{l_\parallel} \frac{d_i}{l_\perp} \gg \underbrace{\frac{\delta B_l}{l_\perp^2} \frac{v_A d_i}{B_0}}_{\text{nonlinear time}^{-1}} \Leftrightarrow \frac{l_\perp}{l_\parallel} \gg \frac{\delta B_l}{B_0}$$

— interaction time: $\Delta t \sim \dfrac{l_\parallel l_\perp}{V_A d_i}$

— amplitude kick in one interaction:

$$\Delta \delta B_l \sim \frac{V_A d_i}{B_0} \frac{\delta B_l^2}{l_\perp^2} \Delta t \sim \frac{\delta B_l^2}{B_0} \frac{l_\parallel}{l_\perp} \quad (\text{NB: } \ll \delta B_l)$$

— kicks accumulate as random walk over time:

$$\sum^t \Delta \delta B_l \sim \frac{\delta B_l^2}{B_0} \frac{l_\parallel}{l_\perp} \cdot \underbrace{\sqrt{\frac{t}{\Delta t}}}_{\#\text{ of kicks}} \sim \delta B_l \Rightarrow t \sim \tau_{cascade}$$

$$\frac{\delta B_l}{B_0} \frac{l_\parallel}{l_\perp} \sqrt{\frac{\tau_{cascade}}{\Delta t}} \sim 1$$

$$\tau_{cascade} \sim \Delta t \frac{l_\perp^2}{l_\parallel^2} \frac{B_0^2}{\delta B_l^2} \sim \frac{l_\perp^3}{l_\parallel} \frac{B_0^2}{\delta B_l^2} \frac{1}{V_A d_i}$$

— energy flux:

$$\epsilon \sim \left(\frac{\delta B_l}{B_0} V_A\right)^2 \frac{1}{\tau_{cascade}} \sim \left(\frac{\delta B_l}{B_0}\right)^4 V_A^2 \frac{l_\parallel}{l_\perp^3} V_A d_i$$

So $$\frac{\delta B_l}{B_0} \sim \left(\frac{\epsilon}{V_A^3 d_i} \frac{l_\perp^3}{l_\parallel}\right)^{1/4}$$

3) Suppose $l_\parallel \sim l_\perp$ (isotropy). Then

$$\delta B_l \sim l^{1/2} \Rightarrow \text{spectrum } E(k) \sim k^{-2}$$

[Suppose $l_\parallel \sim$ const (no cascade in $k_\parallel$). Then

$$\delta B_l \sim l_\perp^{3/4} \Rightarrow \text{spectrum } E(k_\perp) \sim k_\perp^{-5/2}$$

But in the latter case,

$$\frac{\delta B_\ell}{B_0} \ll \frac{l_\perp}{l_\parallel} \quad \text{is only satisfied for} \quad l_\perp \gg \frac{\epsilon}{V_A^3 d_i} l_\parallel^3$$

(weak interactions)]

4) Now assume the cascade is critically balanced:

$$\omega \sim \frac{V_A}{l_\parallel}\frac{d_i}{l_\perp} \sim \frac{\delta B_\ell}{l_\perp^2}\frac{V_A d_i}{B_0} \Rightarrow \frac{\delta B_\ell}{B_0} \sim \frac{l_\perp}{l_\parallel} \ll 1$$

anisotropy

Kolmogorov argument for cascade:

$$\epsilon \sim \left(\frac{\delta B_\ell}{B_0} V_A\right)^2 \underbrace{\frac{\delta B_\ell}{B_0}\frac{V_A d_i}{l_\perp^2}}_{\text{nonlinear/cascade time}^{-1}} \sim \left(\frac{\delta B_\ell}{B_0}\right)^3 \frac{V_A^3 d_i}{l_\perp^2}$$

$$\frac{\delta B_\ell}{B_0} \sim \left(\frac{\epsilon}{V_A^3 d_i}\right)^{1/3} l_\perp^{2/3} \Rightarrow \boxed{E(k_\perp) \sim k_\perp^{-7/3}}$$

$$\frac{\delta B_\ell}{B_0} \sim \frac{l_\perp}{l_\parallel} \Rightarrow \boxed{l_\parallel \sim \left(\frac{V_A^3 d_i}{\epsilon}\right)^{1/3} l_\perp^{1/3}}$$