Problem 3

PDF of Magnetic Field in Kinematic Dynamo with Zero Diffusion.

Consider $$\frac{\partial \tilde{B}^i}{\partial t} = \sigma^i_m \tilde{B}^m \tag{1}$$

where $$\langle \sigma^i_m(t)\, \sigma^j_n(t') \rangle = \delta(t-t')\, \varkappa_2\, T^{ij}_{mn}, \tag{2}$$

$$T^{ij}_{mn} = \delta^{ij}\delta_{mn} - \frac{1}{d+1}\left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m\right) \qquad (d=3)$$

PDF: $$P(B^i, t) = \langle \delta(B^i - \tilde{B}^i(t)) \rangle \equiv \langle \tilde{P} \rangle$$

$$\partial_t \tilde{P} = \frac{\partial}{\partial B^i} \tilde{P}\left(-\partial_t \tilde{B}^i\right) = -\frac{\partial}{\partial B^i} \tilde{P}\, \sigma^i_m(t) \tilde{B}^m =$$

$$= -\frac{\partial}{\partial B^i} B^m \sigma^i_m \tilde{P} \tag{3}$$

$$\partial_t P = -\frac{\partial}{\partial B^i} B^m \langle \sigma^i_m \tilde{P} \rangle \tag{4}$$

(2)

$$\langle \sigma^i_m(t) \tilde{P}(t) \rangle = \int dt' \langle \sigma^i_m(t)\, \sigma^j_n(t') \rangle \left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t')} \right\rangle =$$

$$= \varkappa_2 T^{ij}_{mn} \left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t)} \right\rangle$$

From (3), $$\tilde{P}(t) = -\frac{\partial}{\partial B^i} B^m \int^t dt'\, \sigma^i_m(t')\, \tilde{P}(t')$$

$$\frac{\delta \tilde{P}(t)}{\delta \sigma^j_m(t)} = -\frac{\partial}{\partial B^i} B^m \int^t dt' \Bigg[\underbrace{\frac{\delta \sigma^i_m(t')}{\delta \sigma^j_n(t)}}_{\delta(t-t')\delta^i_j \delta^n_m} \tilde{P}(t) + \sigma^i_m(t') \underbrace{\frac{\delta \tilde{P}(t')}{\delta \sigma^j_n(t)}}_{0 \text{ by causality}} \Bigg]$$

$$= -\frac{1}{2}\frac{\partial}{\partial B^j} B^n \tilde{P}$$

Therefore,

$$\partial_t P = +\frac{\partial}{\partial B^i} B^m \frac{1}{2}\varkappa_2 T^{ij}_{mn} \frac{\partial}{\partial B^j} B^n P =$$

$$= \frac{1}{2}\varkappa_2 T^{ij}_{mn}\left(\cancel{\delta^m_i} + B^m\frac{\partial}{\partial B^i}\right)\left(\cancel{\delta^n_j} + B^n\frac{\partial}{\partial B^j}\right) P$$

these vanish because

$T^{ij}_{in} = T^{ij}_{mj} = 0$

$d=3$ everywhere

Now $P(\vec{B}) = P(B)$, so

$$\partial_t P = \frac{1}{2}\varkappa_2 T^{ij}_{mn} B^m \frac{\partial}{\partial B^i} B^n \frac{B^j}{B}\frac{\partial P}{\partial B} =$$

$$= \frac{1}{2}\varkappa_2 T^{ij}_{mn} B^m\left(\cancel{\delta^n_i}\frac{B^j}{B} + \delta_{ij}\frac{B^n}{B} + \frac{B^n B^j B^i}{B}\frac{\partial}{\partial B}\frac{1}{B}\right)\frac{\partial P}{\partial B} =$$

vanishes because

$T^{ij}_{mi} = 0$

$$= \frac{1}{2}\varkappa_2\left(\frac{(d-1)(d+2)}{d+1} B + \frac{d-1}{d+1} B^3 \frac{\partial}{\partial B}\frac{1}{B}\right)\frac{\partial P}{\partial B} =$$

$$= \frac{1}{2}\frac{d-1}{d+1}\varkappa_2\left[B^2\frac{\partial^2 P}{\partial B^2} + (d+1) B\frac{\partial P}{\partial B}\right] \qquad (5)$$

Normalization: $1 = \int d^dB\, P = S_d \int_0^\infty dB\, B^{d-1} P(B)$, $S_d = 4\pi$ in $d=3$

Define $F(B) = S_d B^{d-1} P(B)$. This, substituted in (5), gives

$$\partial_t F = \underbrace{\frac{1}{2}\frac{d-1}{d+1}}\varkappa_2 \frac{\partial}{\partial B}\left[B^2\frac{\partial F}{\partial B} - (d-1) BF\right] \qquad (6)$$

$$\frac{\gamma}{d+2} \Rightarrow \gamma = \frac{(d-1)(d+2)}{2(d+1)}\varkappa_2 \overset{d=3}{=} \frac{5}{4}\varkappa_2$$

So, $$\frac{\partial F}{\partial t} = \frac{\gamma}{5}\frac{\partial}{\partial B}\left[B^2\frac{\partial F}{\partial B} - 2BF\right]$$

1) *Magnetic energy:*

$$\frac{\langle B^2\rangle}{8\pi} = \frac{1}{8\pi}\int_0^\infty dB\, B\, B^2 F(B)$$

Then

$$\partial_t \langle B^2\rangle = \frac{\gamma}{5}\left[2\cdot 3 + 4\right]\langle B^2\rangle = 2\gamma\langle B^2\rangle$$

So $$\langle B^2\rangle(t) = \langle B_0^2\rangle\, e^{2\gamma t}$$

2) Change variables: $z = \ln B$. Then

$$\frac{\partial F}{\partial t} = \frac{\gamma}{5}\left(\frac{\partial}{\partial z}+1\right)\left(\frac{\partial F}{\partial z} - 2F\right) =$$

$$= \frac{\gamma}{5}\left(\frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial z} - 2F\right)$$

Green's function solution:

$$F(z,t) = \int dz'\, F(z',0)\, \frac{e^{-\frac{[z-z'-\frac{\gamma}{5}t]^2}{\frac{4}{5}\gamma t}}}{\sqrt{\frac{4\pi}{5}\gamma t}}\, e^{-\frac{2}{5}\gamma t}$$

$$\downarrow$$

$$F(B,t) = \int \frac{dB'}{B'}\,\delta(B'-B_0)\,\frac{e^{-\frac{2}{5}\gamma t}}{\sqrt{(4/5)\pi\gamma t}}\, e^{-\frac{[\ln\frac{B}{B'}]^2}{\frac{4}{5}\gamma t} + [\ln\frac{B}{B'}]\frac{1}{2} - \frac{1}{20}\gamma t}$$

$$= \frac{e^{-\frac{9}{20}\gamma t}}{B_0\sqrt{\frac{4}{5}\pi\gamma t}}\left(\frac{B}{B_0}\right)^{\frac{1}{2}} e^{-[\ln B/B_0]^2/\frac{4}{5}\gamma t}$$