Problem 2.

Stability of the θ Pinch.

Consider a cylindrically symmetric equilibrium

$$\vec{B}_0 = B_0(r)\,\hat{z}\,, \quad \frac{\partial}{\partial\theta} = 0\,, \quad \frac{\partial}{\partial z} = 0$$

$$\Downarrow$$

Ampère's law: $\vec{J}_0 = \frac{c}{4\pi}\nabla\times\vec{B}_0 \qquad j_{0r} = 0\,,\; j_{0\theta} = -\frac{c}{4\pi}\frac{\partial B_0}{\partial r}\,,\; j_{0z} = 0$

Force balance: $\nabla p = \frac{1}{c}\vec{J}_0\times\vec{B}_0$

$$\frac{\partial p_0}{\partial r} = -\frac{\partial}{\partial r}\frac{B_0^2}{8\pi} \qquad \text{(pressure balance)}$$

Consider general displacement:

$$\vec{\xi} = \hat{\xi}(r)\,e^{im\theta + ikz}$$

Then

$$\delta W = 2\pi L_z \int_0^\infty dr\,r\left[\gamma p_0\,|\nabla\cdot\vec{\xi}|^2 + \frac{|\vec{Q}|^2}{4\pi} + (\nabla\cdot\vec{\xi}^*)\,\vec{\xi}\cdot\nabla p_0 + \frac{\vec{J}_0\cdot(\vec{\xi}^*\times\vec{Q})}{c}\right]$$

domain size in z

where $\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}_0)$.

Calculate everything:

- $$\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}_0) = \nabla\times\left(\begin{bmatrix}\xi_r\\ \xi_\theta\\ \xi_z\end{bmatrix}\times\begin{bmatrix}0\\ 0\\ B_0\end{bmatrix}\right) = \nabla\times\begin{bmatrix}\xi_\theta B_0\\ -\xi_r B_0\\ 0\end{bmatrix} =$$

$$= \hat{r}\,ik\xi_r B_0 + \hat{\theta}\,ik\xi_\theta B_0 + \hat{z}\left[-\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r B_0) - \frac{im}{r}\xi_\theta B_0\right]$$

• $$\frac{1}{c}\vec{j}_0\cdot\left(\vec{\xi}^*\times\vec{Q}\right) = -\frac{1}{4\pi}\frac{\partial B_0}{\partial r}\left(-\xi_r^* Q_z + \xi_z^* Q_r\right) =$$

$$= \frac{1}{4\pi}\frac{\partial B_0}{\partial r}\left(-\frac{\xi_r^*}{r}\underbrace{\frac{\partial}{\partial r} r\xi_r B_0}_{=\,\xi_r B_0 + B_0 r\frac{\partial \xi_r}{\partial r} + r\xi_r\frac{\partial B_0}{\partial r}} - \frac{im}{r}\xi_\theta\xi_r^* B_0 - ik\xi_r\xi_z^* B_0\right) =$$

$$-\frac{|\xi_r|^2}{r}B_0 - B_0\xi_r^*\frac{\partial \xi_r}{\partial r} - |\xi_r|^2\frac{\partial B_0}{\partial r}$$

$$= -\frac{1}{4\pi}|\xi_r|^2\left(\frac{\partial B_0}{\partial r}\right)^2 + \frac{\partial p_0}{\partial r}\left(\frac{|\xi_r|^2}{r} + \xi_r^*\frac{\partial \xi_r}{\partial r} + \frac{im}{r}\xi_\theta\xi_r^* + ik\xi_r\xi_z^*\right)$$

• $$(\nabla\cdot\vec{\xi}^*)\,\vec{\xi}\cdot\nabla p_0 = \left(\frac{1}{r}\frac{\partial}{\partial r} r\xi_r^* - \frac{im}{r}\xi_\theta^* - ik\xi_z^*\right)\xi_r\frac{\partial p_0}{\partial r} =$$

$$= \frac{\partial p_0}{\partial r}\left(\frac{|\xi_r|^2}{r} + \xi_r\frac{\partial \xi_r^*}{\partial r} - \frac{im}{r}\xi_\theta^*\xi_r - ik\xi_z^*\xi_r\right)$$

• $$\frac{|\vec{Q}|^2}{4\pi} = \frac{1}{4\pi}\Big[B_0^2 k^2|\xi_r|^2 + B_0^2 k^2|\xi_\theta|^2 +$$

$$+ \underbrace{\left|\frac{1}{r}\xi_r B_0 + B_0\frac{\partial \xi_r}{\partial r} + \xi_r\frac{\partial B_0}{\partial r} + \frac{im}{r}\xi_\theta B_0\right|^2}\Big] =$$

$$\left|B_0\left(\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right) + \xi_r\frac{\partial B_0}{\partial r}\right|^2 =$$

$$= B_0^2\left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right|^2 + |\xi_r|^2\left(\frac{\partial B_0}{\partial r}\right)^2 +$$

$$+ \left(2\frac{|\xi_r|^2}{r} + \xi_r\frac{\partial \xi_r^*}{\partial r} + \xi_r^*\frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta\xi_r^*}{r} - \frac{im\xi_\theta^*\xi_r}{r}\right)\frac{\partial}{\partial r}\frac{B_0^2}{2}$$

$$= \frac{B_0^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right|^2\right] + \cancel{\frac{|\xi_r|^2}{4\pi}\left(\frac{\partial B_0}{\partial r}\right)^2} -$$

$$- \frac{\partial p_0}{\partial r}\left[\cancel{2\frac{|\xi_r|^2}{r}} + \cancel{\xi_r \frac{\partial \xi_r^*}{\partial r}} + \cancel{\xi_r^* \frac{\partial \xi_r}{\partial r}} + \frac{im}{r}\left(\cancel{\xi_r^* \xi_\theta} - \cancel{\xi_\theta^* \xi_r}\right)\right]$$

So, finally,

$$\delta W = 2\pi L_z \int_0^\infty dr\, r \left\{ \gamma p_0 |\nabla\cdot\vec{\xi}|^2 + (\dots) + \right.$$

$$+ \frac{\partial p_0}{\partial r}\left(\cancel{\frac{|\xi_r|^2}{r}} + \cancel{\xi_r \frac{\partial \xi_r^*}{\partial r}} - \cancel{\frac{im}{r}\xi_\theta^* \xi_r} - \cancel{ik\xi_z^* \xi_r}\right) -$$

$$\left. - \cancel{\frac{1}{4\pi}|\xi_r|^2\left(\frac{\partial B_0}{\partial r}\right)^2} + \frac{\partial p_0}{\partial r}\left(\cancel{\frac{|\xi_r|^2}{r}} + \cancel{\xi_r^* \frac{\partial \xi_r}{\partial r}} + \cancel{\frac{im}{r}\xi_\theta \xi_r^*} + \cancel{ik\xi_r \xi_z^*}\right)\right\}$$

$$= 2\pi L_z \int_0^\infty dr\, r \left\{ \gamma p_0 |\nabla\cdot\vec{\xi}|^2 + \frac{B_0^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right|^2\right]\right\}$$

$\delta W > 0$ always, so θ pinch is always stable.