Paper 75. Magnetohydrodynamics & Turbulence

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Problem 1

Kinetic Alfvén Waves

In Electron (Hall) MHD, one assumes that the magnetic-field lines are frozen into the electron flow:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u}_e \times \vec{B}) \qquad \text{(we neglect Ohmic diffusion here)} \qquad (1)$$

The current density is

$$\vec{J} = en(\vec{u} - \vec{u}_e) \qquad \text{(assuming hydrogen plasma)} \qquad (2)$$

n: particle density; $\vec{u}$: ions; $\vec{u}_e$: electrons

Ion velocity satisfies

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \frac{1}{c}\vec{J}\times\vec{B} \qquad (3)$$

The electron velocity is, from (2),

$$\vec{u}_e = \vec{u} - \frac{1}{en}\vec{J} \qquad (4)$$

and the current density

$$\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B} \qquad \text{(Ampère's law)} \qquad (5)$$

Assume incompressibility: $\nabla\cdot\vec{u}=0$ $\left[\Rightarrow \nabla\cdot\vec{u}_e=0 \text{ because } \nabla\cdot\vec{j}=0\right]$

Then our equations are

$$\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u} = -\frac{\nabla\tilde{p}}{\rho}+\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi\rho}\,,\quad \nabla\cdot\vec{u}=0 \qquad (6)$$

$$\frac{\partial\vec{B}}{\partial t}+\vec{u}_e\cdot\nabla\vec{B}=\vec{B}\cdot\nabla\vec{u}_e \qquad (7)$$

$$\text{where}\quad \vec{u}_e=\vec{u}-\frac{c}{4\pi e n}\nabla\times\vec{B} \qquad (8)$$

Equilibrium: $\vec{u}_0=0$, $\vec{B}_0=B_0\hat{z}=\text{const}$

Displacements: ion $\frac{\partial\vec{\xi}_i}{\partial t}=\vec{u}$, electron $\frac{\partial\vec{\xi}_e}{\partial t}=\vec{u}_e$

Linearize eq. (7):

$$\frac{\partial\delta\vec{B}}{\partial t}=\vec{B}_0\cdot\nabla\frac{\partial\vec{\xi}_e}{\partial t} \Rightarrow \delta\vec{B}=B_0\nabla_\parallel\vec{\xi}_e \qquad (9)$$

From eq. (8):
$$\frac{\partial\vec{\xi}_e}{\partial t}=\frac{\partial\vec{\xi}_i}{\partial t}-\frac{c}{4\pi e n}\nabla\times\delta\vec{B} \qquad (10)$$

Linearize eq. (6):

$$\frac{\partial^2\vec{\xi}_i}{\partial t^2}=\frac{B_0\nabla_\parallel\delta\vec{B}}{4\pi\rho}\overset{(9)}{=}\frac{B_0^2}{4\pi\rho}\nabla_\parallel^2\vec{\xi}_e=V_A^2\nabla_\parallel^2\vec{\xi}_e \qquad (11)$$

From (10),
$$\frac{\partial^2\vec{\xi}_i}{\partial t^2}=\frac{\partial^2\vec{\xi}_e}{\partial t^2}+\underbrace{\frac{cB_0}{4\pi e n}\nabla\times\left(\nabla_\parallel\frac{\partial\vec{\xi}_e}{\partial t}\right)}_{=\frac{cV_A}{\omega_{pi}}\nabla\times\left(\nabla_\parallel\frac{\partial\vec{\xi}_e}{\partial t}\right)} \qquad (12)$$

plasma frequency

$$\omega_{pi}=\left(\frac{4\pi e^2 n}{m_i}\right)^{1/2}$$

Combining (11) and (12), we get

$$\frac{\partial^2 \vec{\xi}_e}{\partial t^2} = V_A^2 \nabla_\parallel^2 \vec{\xi}_e - \frac{c V_A}{\omega_{pi}} \nabla \times \left(\nabla_\parallel \frac{\partial \vec{\xi}_e}{\partial t} \right) \qquad (13)$$

Now $\vec{\xi}_e \sim e^{-i\omega t + i\vec{k}\cdot\vec{x}}$ (Fourier transform):

$$-\omega^2 \vec{\xi}_e = -k_\parallel^2 V_A^2 \vec{\xi}_e + i\omega \frac{c V_A}{\omega_{pi}} \vec{k} \times \left(k_\parallel \vec{\xi}_e \right) \qquad (14)$$

Let $\vec{k} = \begin{pmatrix} k_\perp \\ 0 \\ k_\parallel \end{pmatrix}$ (without loss of generality)

Then

$$\vec{k} \times \vec{\xi}_e = \begin{pmatrix} k_\perp \\ 0 \\ k_\parallel \end{pmatrix} \times \begin{pmatrix} \xi_{ex} \\ \xi_{ey} \\ \xi_{ez} \end{pmatrix} = \begin{pmatrix} -k_\parallel \xi_{ey} \\ -k_\perp \xi_{ez} + k_\parallel \xi_{ex} \\ k_\perp \xi_{ey} \end{pmatrix}$$

So, (14) becomes:

$$\begin{cases} \left(\omega^2 - k_\parallel^2 V_A^2\right) \xi_{ex} = i \dfrac{\omega}{\omega_{pi}} c V_A k_\parallel^2 \xi_{ey} & (15) \\ \left(\omega^2 - k_\parallel^2 V_A^2\right) \xi_{ey} = i \dfrac{\omega}{\omega_{pi}} c V_A \left(k_\parallel k_\perp \xi_{ez} - k_\parallel^2 \xi_{ex} \right) & (16) \\ \left(\omega^2 - k_\parallel^2 V_A^2\right) \xi_{ez} = -i \dfrac{\omega}{\omega_{pi}} c V_A k_\parallel k_\perp \xi_{ey} & (17) \end{cases}$$

Take $(17) \cdot k_\parallel k_\perp - (15) \cdot k_\parallel^2$:

$$\left(\omega^2 - k_\parallel^2 V_A^2\right)\left(k_\parallel k_\perp \xi_{ez} - k_\parallel^2 \xi_{ex}\right) = -i \frac{\omega}{\omega_{pi}} c V_A \left(k_\parallel^2 k_\perp^2 + k_\parallel^4 \right) \xi_{ey} =$$

$$= -i \frac{\omega}{\omega_{pi}} c V_A k_\parallel^2 k^2 \xi_{ey} \qquad (18)$$

where $k^2 = k_\parallel^2 + k_\perp^2 = |\vec{k}|^2$

Combine (18) and (16):

$$(\omega^2 - k_\parallel^2 V_A^2)^2 \xi_{ey} = i\frac{\omega}{\omega_{pi}} c V_A \left(-i\frac{\omega}{\omega_{pi}} c V_A k_\parallel^2 k^2\right)\xi_{ey}$$

$$\left(\omega^2 - k_\parallel^2 V_A^2\right)^2 = \frac{\omega^2}{\omega_{pi}^2} c^2 k^2 k_\parallel^2 V_A^2$$

$$\boxed{\omega^4 - \omega^2\left(2k_\parallel^2 V_A^2 + \frac{c^2k^2}{\omega_{pi}^2} k_\parallel^2 V_A^2\right) + k_\parallel^4 V_A^4 = 0} \qquad (19)$$

$$\left(\frac{\omega}{k_\parallel V_A}\right)^4 - 2\left(\frac{\omega}{k_\parallel V_A}\right)^2\left(1 + \frac{c^2k^2}{2\omega_{pi}^2}\right) + 1 = 0$$

$$\frac{\omega^2}{k_\parallel^2 V_A^2} = \frac{1}{2}\left[2\left(1 + \frac{c^2k^2}{2\omega_{pi}^2}\right) \pm \sqrt{4\left(1+\frac{c^2k^2}{2\omega_{pi}^2}\right)^2 - 4}\,\right] =$$

$$= 1 + \frac{c^2k^2}{2\omega_{pi}^2} \pm \sqrt{\left(1+\frac{c^2k^2}{2\omega_{pi}^2}\right)^2 - 1} \qquad (20)$$

Note that $\frac{c}{\omega_{pi}}$ is called plasma skin depth.

Also note that

$$\frac{c^2}{2\omega_{pi}^2} = \frac{c^2 m_i}{8\pi e^2 n} = \frac{B_0^2}{8\pi}\frac{c^2 m_i}{ne^2B_0^2} = \frac{B_0^2}{8\pi}\overbrace{\frac{c^2 m_i^2}{e^2 B_0^2}}^{=\frac{1}{\Omega_i^2}}\overbrace{\frac{1}{m_i n}}^{=\frac{1}{\rho}} =$$

$$= \frac{B_0^2}{8\pi}\frac{V_{th}^2}{\Omega_i^2}\frac{1}{\rho V_{th}^2} = \frac{1}{\beta}\rho_i^2, \text{ so } \frac{c}{\omega_{pi}} = \sqrt{\frac{2}{\beta}}\,\rho_i$$

← ion Larmor radius.

$\frac{c}{\omega_{pi}} k \ll 1 \Rightarrow$ magnetized ions, (20) becomes

$$\boxed{\omega^2 = k_\parallel^2 V_A^2} \quad \text{— Alfvén waves.}$$

$\frac{c}{\omega_{pi}} k \gg 1 \Rightarrow$ unmagnetized ions, (20) becomes

$$\frac{\omega^2}{k_\parallel^2 V_A^2} = 1 + \frac{c^2k^2}{2\omega_{pi}^2} \pm \frac{c^2k^2}{2\omega_{pi}^2} \sqrt{\underbrace{\left(1+\frac{2\omega_{pi}^2}{c^2k^2}\right)^2 - \frac{4\omega_{pi}^4}{c^4k^4}}_{=\ 1+\frac{4\omega_{pi}^2}{c^2k^2}}} \approx$$

$$\approx 1 + \frac{c^2k^2}{2\omega_{pi}^2} \pm \frac{c^2k^2}{2\omega_{pi}^2}\left(1+\frac{2\omega_{pi}^2}{c^2k^2}\right) =$$

$$= 1 + \frac{c^2k^2}{2\omega_{pi}^2} \pm \left(\frac{c^2k^2}{2\omega_{pi}^2} + 1\right) \cong \begin{cases} \dfrac{c^2k^2}{\omega_{pi}^2} & \text{"+"} \\ 0 & \text{"−"} \end{cases}$$

ignore.

The "+" branch is

$$\boxed{\omega^2 = \frac{c^2k^2}{\omega_{pi}^2} k_\parallel^2 V_A^2 = \frac{2k^2\rho_i^2 k_\parallel^2 V_A^2}{\beta}} \quad \text{— Kinetic Alfvén wave.} \tag{21}$$

Note that the incompressibility assumption works if

not essential.

$$\frac{\omega^2}{k^2 V_{th}^2} = \frac{2}{\beta}\frac{\rho_i^2}{V_{th}^2} k_\parallel^2 V_A^2 = \frac{2}{\beta}\frac{k_\parallel^2 V_A^2}{\Omega_i^2} =$$

$$= \frac{c^2k_\parallel^2}{\omega_{pi}^2}\frac{V_A^2}{V_{th}^2} = \frac{c^2k_\parallel^2}{\omega_{pi}^2}\frac{B_0^2}{4\pi\rho V_{th}^2} = \frac{c^2k_\parallel^2}{\omega_{pi}^2}\frac{2}{\beta} \ll 1$$

so $1 \ll \frac{c^2k_\parallel^2}{2\omega_{pi}^2} \ll \beta$

or $1 \ll k_\parallel^2\rho_i^2 \ll \beta^2$

validity interval non-empty in high-β regime.

Note that, if we assume from the outset

$$\ell \ll \frac{c}{\omega_{pi}} \quad \text{characteristic length}$$

$$\tau \ll \frac{\ell}{v_A} \quad \text{characteristic time}$$

we get: from (6),

$$u \sim \tau \frac{B_0}{4\pi\rho} \frac{\delta B}{\ell} = \tau \frac{B_0}{4\pi n m_i} \frac{\delta B}{\ell}$$

in (8):

$$\frac{\left| \frac{c}{4\pi e n} \nabla \times \vec{B} \right|}{|\vec{u}|} \sim \frac{\frac{c}{4\pi e n} \frac{\delta B}{\ell}}{\tau \frac{B_0}{4\pi n m_i} \frac{\delta B}{\ell}} \sim \frac{\frac{c}{4\pi e n} \sqrt{4\pi n m_i}}{\tau v_A} \sim$$

$$\sim \frac{\frac{c}{\omega_{pi}} \frac{1}{\ell}}{\tau v_A / \ell} \gg 1$$

so the ion velocity is negligible and the induction equation becomes simply

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u}_e \times \vec{B}) = -\frac{c}{4\pi e n} \nabla \times \left[(\nabla \times \vec{B}) \times \vec{B} \right]$$

and Eq. (21) can be derived by linearising just this equation — with no recourse to the momentum equation (and to incompressibility).