Collisionless Relaxation in Systems with Coulomb Interactions

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We give an analytical consideration of the relaxation of the distribution function for systems of particles with Coulomb interaction. It takes into account two-particle correlations which correspond to formation of some macroparticles, i.e., coherently moving regions. It is argued that such relaxation leads to the Lynden-Bell distribution with an additional high-energy tail.

Recently Lynden-Bell[1] has argued that relaxation of collisionless systems of charged particles or stellar systems should be some kind of chaotic interchange of elements in phase space. The elements in phase space cannot overlap and therefore follow an exclusion principle. This leads to statistics of the Fermi-Dirac type. For the special case where the initial distribution function f is equal to unity over certain regions of phase space and is zero outside these regions, the equilibrium distribution function is exactly equal to the Fermi-Dirac function.

To check the Lynden-Bell theory, numerical calculations were carried out[2] with a one-dimensional model. These calculations have shown that for simple initial conditions the quasiequilibrium state which is reached after several plasma periods is close to the Fermi distribution with an additional high-energy tail. For more complicated initial conditions the final distribution deviates from the Fermi distribution.

We shall discuss this problem analytically from the point of view suggested recently by Dupree's[3] and the authors'[4] idea on the formation of macroparticles in a plasma. The relationship between this approach and quasilinear theory will also be discussed.

For simplicity we consider an electron plasma which is homogeneous on the average except for some small initial perturbations. The evolution of these perturbations is described by the Vlasov equation

$$\frac{\partial F}{\partial t}+\vec{v}\cdot\nabla F=\frac{e}{m}\vec{E}\cdot\frac{\partial F}{\partial\vec{v}} \tag{1}$$

with self-consistent electric field $\vec{E}=-\nabla\varphi$,

$$\operatorname{div}\vec{E}=-4\pi e\left(\int F d^3v-n_0\right), \tag{2}$$

where $n_0=\text{const}$ is the density of heavy ions and F is the distribution function.

It is natural to suppose that after several plasma periods the evolution of electrons may be considered to be stochastic. We introduce the average value $f_0=\langle F\rangle$ so that $F=f_0+f$, where $\langle f\rangle=0$. We find by averaging Eq. (1)

$$\frac{\partial f_0}{\partial t}=\frac{e}{m}\left\langle\vec{E}\cdot\frac{\partial f}{\partial\vec{v}}\right\rangle\equiv St(f_0), \tag{3}$$

where $St(f_0)$ denotes the collision term.

Assuming that perturbation f is not very large, we use for f an equation of the quasilinear type:

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\nabla f=\frac{e}{m}\vec{E}\cdot\frac{\partial f_0}{\partial\vec{v}}. \tag{4}$$

In Fourier representation, neglecting slow variation of f_0 with time, we can write (4) in the form

$$(\omega-\vec{k}\cdot\vec{v})f=\frac{e}{m}\varphi\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}, \tag{5}$$

where φ is the potential. We find from (5)

$$f=\frac{e}{m}\frac{1}{\omega-\vec{k}\cdot\vec{v}}\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}\varphi+f_2, \tag{6}$$

where f_2 is an arbitrary solution of Eq. (5) with zero on the right-hand side. In fact, f_2 corresponds to some initial perturbation of the electron distribution function.

The substitution of (6) into (2) gives us

$$\epsilon\varphi=-(4\pi e/k^2)\int f_2\,d^3v, \tag{7}$$

where ϵ is the well-known dielectric constant:

$$\epsilon=1+\frac{4\pi e^2}{mk^2}\int\frac{\vec{k}}{\omega-\vec{k}\cdot\vec{v}}\cdot\frac{\partial f_0}{\partial\vec{v}}d^3v=1+\frac{4\pi e^2}{mk^2}\int\left(\frac{\mathrm{P}}{\omega-\vec{k}\cdot\vec{v}}-i\pi\delta(\omega-\vec{k}\cdot\vec{v})\right)\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}d^3v. \tag{8}$$

Now we can substitute the value $\vec{E}=-i\vec{k}\varphi$ from (7) in Eq. (3). It is convenient to separate the wave [$k<\omega_0/v_{\text{th}}$, where v_{th} is some average "thermal" velocity and $\omega_0=(4\pi e^2n_0/m)^{1/2}$] and nonwave ($k>\omega_0/v_{\text{th}}$) regions. In the wave region, neglecting the small term f_2, we get the usual quasilinear approximation.[5,6] In the nonwave region the electric-field fluctuations are produced by f_2 only. Replacing approximately the real part of ϵ by unity we obtain the nonwave collision term in a form similar to the Balescu-Lenard type:

$$St_{\mathrm{nw}}=\pi\left(\frac{e}{m}\right)^2\int\left(\frac{4\pi e}{k^2}\right)^2\vec{k}\cdot\frac{\partial}{\partial\vec{v}}\left\{\langle f_2'f_2''^*\rangle_{\vec{k}\omega}\delta(\omega-\vec{k}\cdot\vec{v})\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}-\langle f_2''f_2^*\rangle_{\vec{k}\omega}\delta(\omega-\vec{k}\cdot\vec{v}')\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}'}\right\}d^3v'd^3v''d^3kd\omega,\quad(9)$$

where $f_2'=f_2(\vec{v}')$, $f_2''=f_2(\vec{v}'')$, and $\langle f_2'f_2''^*\rangle_{\vec{k}\omega}$ is the Fourier component of the two-particle correlation function. We define the integral over $\vec{k}$ and ω so that

$$\int\langle f_2f_2'^*\rangle_{\vec{k}\omega}\exp(-i\omega\tau+i\vec{k}\cdot\vec{\xi})d^3kd\omega=\langle f_2(\vec{r}+\vec{\xi},\vec{v},t+\tau)f_2(\vec{r},\vec{v}',t)\rangle.\quad(10)$$

We see that $\langle f_2f_2'^*\rangle_{\vec{k}\omega}$ corresponds to a two-time correlation function. Since the function f_2 is a solution of the equation $(\omega-\vec{k}\cdot\vec{v})f_2=0$, the correlation function $\langle f_2f_2'^*\rangle$ should be proportional to $\delta(\omega-\vec{k}\cdot\vec{v})$. It is natural to believe that correlations of particles with quite different velocities and positions should quickly decrease so that the correlation function is not zero only when particles are close together, say in the interval $\Delta v\ll v_{\mathrm{th}}$, $\Delta x\ll v_{\mathrm{th}}/\omega_0$. This means that we can write

$$\langle f_2f_2'^*\rangle_{\vec{k}\omega}=A\delta(\omega-\vec{k}\cdot\vec{v})\delta(\vec{v}-\vec{v}'),\quad(11)$$

where $A\cong$const at $k<1/\Delta x$ and is equal to zero at $k>1/\Delta x$. To find A we should solve the equation for the two-particle correlation function. But even without such a solution we can estimate the correlation function in the following way.

As we see from (10), the integral from (11) over ω is proportional to the one-time correlation function

$$\langle f_2(\vec{r}+\vec{\xi},\vec{v},t)f_2(\vec{r},\vec{v}',t)\rangle=(2\pi)^3A\delta(\vec{v}-\vec{v}')\delta(\vec{\xi}).\quad(12)$$

Let us consider a special case when in the initial state the function F was equal to unity over certain regions of the phase space and was zero outside these regions. This condition is conserved in time so that

$$\langle F(\vec{v},\vec{r})F(\vec{v},\vec{r})\rangle=\langle 1\times F(\vec{v})\rangle=f_0,\ \text{i.e.,}\ \langle f_2(\vec{r},\vec{v},t)f_2(\vec{r},\vec{v},t)\rangle=\langle(F-f_0)(F-f_0)\rangle=f_0-f_0^2.$$

If the points $\vec{r},\vec{v}$ and $\vec{r}',\vec{v}'$ do not coincide, the correlation function will decrease. We can approximate it by its value $f_0-f_0^2$ inside some regions $(\Delta r)^3$, $(\Delta v)^3$ and zero outside so that

$$(2\pi)^3A=(\Delta r)^3(\Delta v)^3f_0(1-f_0)\equiv q^2f_0(1-f_0),\quad(13)$$

where we have used the notation q^2 for the region of correlation. The quantity qe plays the role of the effective charge of the macroparticles, i.e., correlated regions.

Substituting (11) and (13) into (9) we obtain the collision term for the nonwave thermal region:

$$\frac{\partial f_0}{\partial t}=St_{\mathrm{nw}}=\frac{2e^4q^2}{m^2}\int\vec{k}\cdot\frac{\partial}{\partial\vec{v}}\delta\frac{(\vec{k}\cdot\vec{v}-\vec{k}\cdot\vec{v}')}{k^4}\left\{\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}f_0'(1-f_0')-\vec{k}\cdot\frac{\partial f_0'}{\partial\vec{v}'}f_0(1-f_0)\right\}d^3v'd^3k,\quad(14)$$

where $f_0'=f_0(\vec{v}')$. The integration over k should be carried out from $k\cong\omega_0/v_{\mathrm{th}}$ to $k\approx1/\Delta x$ and gives, as usual, $\ln(\omega_0\Delta x/v_{\mathrm{th}})$.

We see that (14) has as the stationary solution a Fermi distribution

$$f_0=\frac{1}{1+\exp(mv^2/2T+\mu/T)},\quad(15)$$

as was predicted by Lynden-Bell.

But for this final state to be reached the quantity q^2 should not be very small. In fact q^2 is a function of time, $q^2=q^2(t)$, which decreases with time. For example, in the case of beam-plasma interaction considered in Ref. 4, the quantity q^2 decreases like $t^{-1/2}$. In the three-dimensional case it should decrease even more rapidly. As we see from (14), the possibility of reaching the final state is determined by $\int q^2dt$. If the initial deviation from the final state is large enough, which corresponds to the "simple cases" of Ref. 2, then the final state will be reached. For small perturbations it probably cannot be reached. These arguments are in qualitative agreement with the data on the numerical calculations.[2]

Now we can discuss the wave region. The arguments of Lynden-Bell are not applicable to this region because, in addition to the particles, the collective modes (Langmuir waves) are present. These waves lead, as is well known, to the formation of a high-energy tail. The particles of this tail interact with the thermal part rather weakly so that they represent their own subsystem.

We have considered here the three-dimensional case. Unfortunately, this consideration cannot be applied to the one-dimensional case since the collision term of Balescu-Lenard type goes to zero and the triple interactions have to be taken into account. Therefore we can have only qualitative correspondence between our consideration and the numerical results of the paper.[2]

Our arguments can equally well be applied to stellar systems with some modification of "particle" trajectories. They show that the approach to equilibrium in such systems should be much faster than predicted by binary interactions.

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Brillouin Spectrum of Xenon Near Its Critical Point*

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We have accurately measured the Brillouin spectrum of pure xenon along the critical isochore using two high-resolution spherical Fabry-Perot interferometers in tandem. The spectrum, which contained an extra diffusive mode, is analyzed in terms of a hydrodynamic model employing a relaxing bulk viscosity. We obtain the temperature dependence of the relaxation time, the bulk viscosity, the specific heat ratio C_p/C_v at finite k and ω, the correlation range, and the $k=0$, $\omega=0$ values for the compressibility and C_p-C_v.

This Letter reports accurate measurements of the Brillouin portion of the spectrum of light scattered by a pure fluid, xenon, near its critical point. The measurements were made along the critical isochore at temperatures ranging from 20°C above the critical temperature T_c to within 0.10°C of T_c. The spectral measurements were made using two high-resolution spherical Fabry-Perot interferometers in tandem. This technique enabled us to resolve clearly the weak Brillouin portion of the spectrum despite the presence of the extremely intense Rayleigh component. In addition to the normal Rayleigh and Brillouin components the spectrum contained an additional diffusive mode centered at the frequency of the incident light. The intensity of this extra mode increased as the critical point was approached, and for the lowest temperature studied, $T_c+0.10$°C, its integrated intensity was at least twice the integrated intensity of one Brillouin component. The general appearance of the spectrum as well as its dependence upon temperature is shown in Fig. 1.

The experimental setup consisted of a single-mode, frequency-stabilized, helium-neon laser; a high-pressure cell having two optical-quality glass windows; an axiconical collecting lens; a spectrometer consisting of two high-resolution spherical Fabry-Perot interferometers which were pressure swept in tandem; a photomultiplier tube; and a strip-chart recorder. The cell was carefully cleaned and filled to within 0.1% of the critical density with xenon containing less than 18 ppm of impurities. The cell temperature was controlled to within ±0.001°C, and was measured using a platinum resistance thermometer. The meniscus was observed to disappear at a temperature of (16.597±0.01)°C, which was taken as the critical temperature, in good agreement with the accepted value of 16.590°C. Light scattered at an angle of 170°, corresponding to a scattering wave vector $k=2.25\times10^5$ cm^{-1}, was collected by an axiconical lens and spectrally analyzed using the tandem interferometer. The extremely high contrast of the interferometer, and its narrow instrumental width of 20