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Non-linear mirror instability

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ABSTRACT

Slow dynamical changes in magnetic-field strength and invariance of the particles' magnetic moments generate ubiquitous pressure anisotropies in weakly collisional, magnetized astrophysical plasmas. This renders them unstable to fast, small-scale mirror and firehose instabilities, which are capable of exerting feedback on the macroscale dynamics of the system. By way of a new asymptotic theory of the early non-linear evolution of the mirror instability in a plasma subject to slow shearing or compression, we show that the instability does not saturate quasi-linearly at a steady, low-amplitude level. Instead, the trapping of particles in small-scale mirrors leads to non-linear secular growth of magnetic perturbations, $\delta B/B \propto t^{2/3}$. Our theory explains recent collisionless simulation results, provides a prediction of the mirror evolution in weakly collisional plasmas and establishes a foundation for a theory of non-linear mirror dynamics with trapping, valid up to $\delta B/B = O(1)$.

Key words: instabilities – magnetic fields – MHD – plasmas – turbulence – solar wind – galaxies: clusters: intracluster medium.

1 INTRODUCTION

Dynamical, weakly collisional high-β plasmas develop pressure anisotropies with respect to the magnetic field as a result of the combination of slow changes in magnetic-field strength B and conservation of the first adiabatic invariant of particles $\mu = v_\perp^2/2B$. This renders them unstable to fast (ion cyclotron time-scale Ω_i^{-1}), small-scale (ion gyroscale ρ_i) firehose, mirror and ion cyclotron instabilities (Rosenbluth 1956; Chandrasekhar, Kaufman & Watson 1958; Parker 1958; Vedenov & Sagdeev 1958; Rudakov & Sagdeev 1961; Gary 1992), whose observational signatures have been reported in the solar wind (Hellinger et al. 2006; Bale et al. 2009) and planetary magnetosheaths (Kaufmann, Horng & Wolfe 1970; Erdös & Balogh 1996; André, Erdös & Dougherty 2002; Joy et al. 2006; Génot et al. 2009; Horbury & Lucek 2009; Soucek & Escoubet 2011). These instabilities are also thought to be excited in energetic astrophysical environments such as the intracluster medium (ICM; Fabian 1994; Carilli & Taylor 2002; Govoni & Feretti 2004; Schekochihin et al. 2005; Peterson & Fabian 2006), the vicinity of accreting black holes (Quataert 2001; Blaes 2014), or the warm ionized interstellar medium (Hall 1980), producing strong dynamical feedback at macroscales, with critical astrophysical implications (Chandran & Cowley 1998; Schekochihin & Cowley 2006; Sharma et al. 2006, 2007; Kunz et al. 2011; Mogavero & Schekochihin 2014). A self-consistent description of the multiscale physics of such plasmas requires understanding how these instabilities saturate non-linearly.

Let us consider a typical situation in which slow changes in B due to shearing, compression or expansion of the plasma at large ('fluid') scales build up ion pressure anisotropy $\Delta_i \equiv (p_i^\perp - p_i^\parallel)/p_i^\perp$, driving the plasma through either the ion firehose instability boundary ($\Delta_i < -2/\beta_i$, with $\beta_i = 8\pi p_i/B^2$) in regions of decreasing field, or the mirror instability boundary ($\Delta_i \gtrsim 1/\beta_i$) in regions of increasing field (the ion cyclotron instability appears to be subdominant, see Hellinger et al. 2006; Bale et al. 2009; Riquelme, Quataert & Verscharen 2014). This triggers exponential growth on time-scales up to Ω_i^{-1}, much faster than the shearing time-scale S^{-1}. The separation between these time-scales implies that the instabilities always operate close to threshold and regulate the levels of pressure anisotropy in the plasma non-linearly. However, how they achieve that in the face of the slowly changing B constantly generating more pressure anisotropy, remains an open question.

In the simplest case of the parallel firehose instability, the growth of magnetic perturbations $\delta\boldsymbol{B}$ leads to an increase of the average (rms) field strength and perpendicular pressure, which drives the anisotropy back to marginality, $\Delta_i(t) \to -2/\beta_i$. If a weakly unstable initial state $\Delta_{io} + 2/\beta_i < 0$ is postulated with no further driving of Δ_i, quasi-linear theory (Shapiro & Shevchenko 1964) predicts saturation at a steady, low amplitude $\delta B/B \sim |\Delta_{io} + 2/\beta_i|^{1/2} \ll 1$. However, the shearing or expansion process that drove the plasma through the instability boundary in the first place, must ultimately

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become important again once quasi-linear relaxation has pushed the system sufficiently close back to marginality. When such continued driving is accounted for, asymptotic theory (Schekochihin et al. 2008; Rosin et al. 2011) predicts secular growth of perturbations as $\delta B/B \propto t^{1/2}$ up to $\delta B/B = O(1)$ (cf. Matteini et al. 2006), a very different outcome from steady-state, low-amplitude saturation.

The non-linear dynamics of the mirror instability (Tajiri 1967; Hasegawa 1969; Southwood & Kivelson 1993; Hellinger 2007) in a weakly collisional shearing (or compressing) plasma driven through its instability boundary is more involved and has only recently been explored numerically (Kunz, Schekochihin & Stone 2014; Riquelme et al. 2014). In this Letter, we show that weakly non-linear mirror modes in such conditions (rather than in a scenario of free relaxation from initial anisotropy, see e.g. Shapiro & Shevchenko 1964; McKean et al. 1993; Califano et al. 2008; Pokhotelov et al. 2008; Hellinger et al. 2009) do not saturate quasi-linearly at a steady, low amplitude either, but continue to grow secularly as $\delta B/B \propto t^{2/3}$. To do this, we introduce a new asymptotic theory in the spirit of earlier work (Califano et al. 2008; Schekochihin et al. 2008; Rosin et al. 2011), in which the combined effects of weak collisionality, large-scale shearing, quasi-linear relaxation (Shapiro & Shevchenko 1964), particle trapping (Kivelson & Southwood 1996; Pantellini 1998; Istomin, Pokhotelov & Balikhin 2009; Pokhotelov et al. 2010) and finite ion Larmor radius (FLR) are consistently retained.

2 ASYMPTOTIC THEORY

We consider the simplest case of a plasma consisting of cold electrons [1] and hot ions of mass m_i, charge $q_i = Ze$, and thermal velocity $v_{\mathrm{th}i} = \sqrt{2\,T_i/m_i}$, coupled to the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. The dynamics is governed by the non-relativistic Vlasov–Maxwell system,

$$\frac{\partial f_i}{\partial t} + \boldsymbol{v}\cdot\nabla f_i + \frac{q_i}{m_i}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_i}{\partial \boldsymbol{v}} = C\left[f_i\right], \tag{1}$$

$$\nabla\cdot\boldsymbol{B} = 0, \quad \frac{\partial \boldsymbol{B}}{\partial t} = -c\,\nabla\times\boldsymbol{E}, \quad \boldsymbol{j} = \frac{c}{4\pi}\nabla\times\boldsymbol{B}, \tag{2}$$

and Ohm's law describing the force balance for electrons,

$$\boldsymbol{E} + \frac{\boldsymbol{u}_i\times\boldsymbol{B}}{c} = \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi Z e n_i}. \tag{3}$$

Here, $f_s(t,\boldsymbol{r},\boldsymbol{v})$, $n_s(t,\boldsymbol{r}) = \int f_s\,\mathrm{d}^3\boldsymbol{v}$ and $\boldsymbol{u}_s(t,\boldsymbol{r}) = \int \boldsymbol{v} f_s\,\mathrm{d}^3\boldsymbol{v}$ are, respectively, the distribution function, number density and mean velocity of species $s = (i, e)$, $\boldsymbol{j} = e\,n_e\,(\boldsymbol{u}_i - \boldsymbol{u}_e)$ is the total current density given the quasi-neutrality condition $n_e = Zn_i$, and $C[f_i]$ is a collision operator. In the following, we use the ion peculiar velocity $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u}_i$ as the velocity-space variable and will henceforth drop the primes. Taking the first moment of equation (1) and using equations (2) and (3), we obtain the ion momentum equation

$$\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = -\frac{\nabla\cdot\boldsymbol{P}_i}{m_i n_i} + \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi m_i n_i}, \tag{4}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{u}_i\cdot\nabla$ and $\boldsymbol{P}_i = m_i\int \boldsymbol{v}\boldsymbol{v} f_i\,\mathrm{d}^3\boldsymbol{v}$ is the ion pressure tensor. Introducing $\hat{\boldsymbol{b}} \equiv \boldsymbol{B}/B$, using equations (2) and (3), we obtain the evolution equation for the field strength

$$\frac{\mathrm{d}\ln B}{\mathrm{d}t} = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla\boldsymbol{u}_i - \nabla\cdot\boldsymbol{u}_i - \frac{\hat{\boldsymbol{b}}}{B}\cdot\nabla\times\left(\frac{\boldsymbol{j}\times\boldsymbol{B}}{Zen_i}\right). \tag{5}$$

Our derivation is based on an asymptotic expansion of these equations. The separation between the slow magnetic-field-stretching time-scale and the fast instability time-scale implies that the distance to instability threshold $\Gamma \sim \Delta_i - 1/\beta_i$ must remain small, which provides us with a natural expansion parameter. In order to study the dynamics in this regime, we start from an already weakly unstable situation and order $\Gamma = O(\varepsilon^2)$, with $\varepsilon \ll 1$. We then construct a 'maximal' ordering (summarized in equations 8–12) retaining ion FLR, collisional, quasi-linear and trapping effects, as well as the effect of continued slow shearing. Following Hellinger (2007), we order the time and spatial scales of mirror modes as $\gamma \sim \varepsilon^2 k_\parallel v_{\mathrm{th}i}$, $k_\perp \sim \varepsilon^{-1}k_\parallel$, $\rho_i^{-1} \sim \varepsilon^{-2}k_\parallel$, where γ is the instability growth rate, $(\boldsymbol{k}_\perp, k_\parallel)$ the typical perturbation wavenumbers (defined with respect to the unperturbed field), $\rho_i = v_{\mathrm{th}i\perp}/\Omega_i$, and $\Omega_i^{-1} = (m_i c)/q_i B \sim \varepsilon^{-2}k_\parallel v_{\mathrm{th}i}$. The ion distribution function is expanded as $f_i = f_{0i} + f_{2i} + \delta f$, where f_{0i} provides the required pressure anisotropy to pin the system at the threshold, f_{2i} provides an extra $O(\varepsilon^2)$ anisotropy to drive the system away from it, and δf contains mirror perturbations. We also expand $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$ and $\boldsymbol{u}_i = \boldsymbol{u}_{0i} + \delta\boldsymbol{u}_i$, where $\boldsymbol{B}_0$ and $\boldsymbol{u}_{0i}$ have no instability-scale variations, $\boldsymbol{u}_{0i}$ is the slow, large-scale shearing/compressive motion, and $\delta\boldsymbol{B}$ and $\delta\boldsymbol{u}_i$ are the mirror perturbations.

The ordering of δf, $\delta\boldsymbol{B}$, $\delta\boldsymbol{u}_i$ and of the remaining time-scales is guided by physical considerations. The critical pitch-angle $\xi = v_\parallel/v$ below which particles get trapped by magnetic fluctuations is $\xi_{\mathrm{tr}} = (\delta B/B_0)^{1/2}$ and the corresponding bounce frequency is $\omega_B \sim k_\parallel v_{\mathrm{th}i}\,\xi_{\mathrm{tr}}$. To retain their contribution in our calculation, we order $\omega_B \sim \gamma \sim \varepsilon^2 k_\parallel v_{\mathrm{th}i}$, which provides us with the ordering $\delta B/B_0 = O(\varepsilon^4)$ (plus higher order terms). For consistency of the ε expansion, we must order $\delta f = O(\varepsilon^4)$, $\delta\boldsymbol{u}_i = O(\varepsilon^5)$ and higher. We further order $S \equiv \mathrm{d}/\mathrm{d}t\,(\ln B_0)$, the shearing/compression rate, the same size as $\mathrm{d}/\mathrm{d}t\,(\delta B/B_0) \sim \varepsilon^6 k_\parallel v_{\mathrm{th}i}$ to be able to investigate the effect of a slowly changing field strength. The exact $O(\varepsilon^6)$ evolution equation for $\ln B_0$ is obtained by averaging equation (5) over instability scales,

$$\frac{\mathrm{d}\ln B_0}{\mathrm{d}t} = \hat{\boldsymbol{b}}_0\hat{\boldsymbol{b}}_0 : \nabla\boldsymbol{u}_{0i} - \nabla\cdot\boldsymbol{u}_{0i} \;\equiv\; S. \tag{6}$$

Subtracting equation (6) from equation (5), we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B}{B_0} = \hat{\boldsymbol{b}}_0\hat{\boldsymbol{b}}_0 : \nabla\delta\boldsymbol{u}_i, \tag{7}$$

to all orders relevant to the calculation.[2] The aforementioned time-scale separation S/Ω_i is now related to ε through $\varepsilon \sim (S/\Omega_i)^{1/8}$ ($\varepsilon \sim 0.01$ for the ICM, see Rosin et al. 2011).

A separate asymptotic treatment is required for low-pitch-angle resonant particles, which develop a velocity-space boundary layer and evolve into a separate population of trapped particles on the instability time-scale (the process is similar to 'non-linear Landau damping', see Dawson 1961; O'neil 1965; Istomin et al.

[1] Formally, we first expand the electron Vlasov equation in $\sqrt{m_e/m_i}$ and then take the limit $T_e \ll T_i$, so electrons still stream along the field very fast. This is purely a matter of analytical convenience: our results also hold for hot, isothermal electrons.

[2] A Hall term $-\left(\rho_i v_{\mathrm{th}i\perp}/\beta_{0i}^{\perp}\right)\hat{\boldsymbol{b}}_0\cdot\left[\nabla_\perp\times\left(\nabla_\parallel\,\delta\boldsymbol{B}_5^{\perp}/B_0\right)\right]$ is formally present in the induction equation at the maximum order $O(\varepsilon^6)$ needed here. However, it can be proven to be zero because of the particular polarization of mirror modes (Califano et al. 2008) and has therefore been summarily discarded in equation (7) to simplify the algebra. Non-linearities quadratic in perturbations are at most $O(\varepsilon^8)$ and can therefore also be neglected.

2009). As can be seen by considering a simple Lorentz pitch-angle scattering operator (Helander & Sigmar 2002), $C[f_i] = (\nu_{ii}/2)\,\partial_\xi[(1-\xi^2)\partial_\xi f_i]$, this results in a boost of their effective collisionality, $\nu_{ii,\,\mathrm{eff}} \sim \nu_{ii}/\xi_{\mathrm{tr}}{}^2 \gg \nu_{ii}$ for $\xi < \xi_{\mathrm{tr}} \ll 1$. To retain this effect, we order $\nu_{ii,\,\mathrm{eff}} \sim \gamma \sim \omega_B$, or $\nu_{ii} \sim k_\parallel v_{\mathrm{th}i}(\delta B/B_0)^{3/2} \sim \varepsilon^6 k_\parallel v_{\mathrm{th}i}$ (this preserves the low-collisionality assumption, i.e. the rest of the distribution relaxes on a time-scale $1/\nu_{ii} \gg 1/\gamma$). The maximal mirror ordering is summarized as follows:

$$f_i = f_{0i} + f_{2i} + \delta f_{4i} + \cdots, \tag{8}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 + \delta B_4^\parallel \hat{\boldsymbol{b}}_0 + \delta \boldsymbol{B}_5^\perp + \cdots, \quad \boldsymbol{u}_i = \boldsymbol{u}_{0i} + \delta \boldsymbol{u}_{5i}^\perp + \cdots, \tag{9}$$

$$\rho_i^{-1} \sim \varepsilon^{-2} k_\parallel, \quad k_\perp \sim \varepsilon^{-1} k_\parallel, \tag{10}$$

$$\gamma \sim \varepsilon^2 k_\parallel v_{\mathrm{th}i}, \quad \Omega_i \sim \varepsilon^{-2} k_\parallel v_{\mathrm{th}i}, \quad S \sim \nu_{ii} \sim \varepsilon^6 k_\parallel v_{\mathrm{th}i}, \tag{11}$$

$$\xi_{\mathrm{tr}} \sim \left(\delta B_4^\parallel / B_0\right)^{1/2} \sim \varepsilon^2, \quad \omega_B \sim \nu_{ii,\mathrm{eff}} \sim \gamma \sim \varepsilon^2 k_\parallel v_{\mathrm{th}i}. \tag{12}$$

Taking the three lowest non-trivial orders of equation (1), we first find that $f_{0i}, f_{2i}, \delta f_{4i}$ are gyrotropic. Expanding and gyroaveraging equation (1) up to $O(\varepsilon^4)$ then gives δf_{4i} in terms of the mirror perturbation $\delta B_4^\parallel/B_0$, from which the perturbed scalar pressures $\delta p_{4i}^\perp$ and $\delta p_{4i}^\parallel$ are derived (note that resonant/trapped particles are not involved at this stage). Taking the perpendicular projection of equation (4) at the lowest order $O(\varepsilon^3)$, we obtain the threshold condition for the mirror instability (Hellinger 2007):

$$\Gamma_0 = -\frac{2\,m_i}{p_{0i}^\perp} \int \frac{v_\perp^4}{4} \frac{\partial f_{0i}}{\partial v_\parallel^2}\bigg|_{v_\perp} \mathrm{d}^3\boldsymbol{v} - \frac{2}{\beta_{0i}^\perp} - 2 = 0. \tag{13}$$

Next, we expand and gyroaverage equation (1) to three further orders, up to $O(\varepsilon^7)$. This tedious calculation yields FLR corrections and resonant effects (not shown, see Califano et al. 2008 for an almost identical procedure) and provides us with explicit expressions for δf_{5i} and δf_{6i}, from which we obtain higher order elements of $\boldsymbol{P}_i$, $\delta \boldsymbol{p}_{5i}^{\perp\parallel} \equiv m_i \int \boldsymbol{v}_\perp v_\parallel\, \delta f_{5i}\, \mathrm{d}^3\boldsymbol{v}$ and $\delta \boldsymbol{P}_{6i}^{\perp\perp}$, in terms of $\delta B_4^\parallel$ and $\delta B_6^\parallel$. No new information arises from equation (4) at $O(\varepsilon^4)$. Using these results and equation (13) in the perpendicular projection of equation (4) at $O(\varepsilon^5)$, we derive the pressure balance condition:

$$\left[\Gamma_2 + \frac{3}{2}\rho_*^2 \nabla_\perp^2 - \left(\frac{p_{0i}^\perp - p_{0i}^\parallel}{p_{0i}^\perp} + \frac{2}{\beta_{0i}^\perp}\right)\frac{\nabla_\parallel^2}{\nabla_\perp^2}\right]\nabla_\perp \frac{\delta \tilde{B}_4^\parallel}{B_0} = \nabla_\perp \frac{\delta \tilde{p}_{6i}^{\perp(\mathrm{res})}}{p_{0i}^\perp}, \tag{14}$$

where the resonant/trapped particle pressure is

$$\delta \tilde{p}_{6i}^{\perp(\mathrm{res})} = m_i \int_{|\xi|<\xi_{\mathrm{tr}}} \frac{v_\perp^2}{2}\, \delta \tilde{f}_{4i}^{(\mathrm{res})}\, \mathrm{d}^3\boldsymbol{v}, \tag{15}$$

$\delta f_{4i}^{(\mathrm{res})}$ is the resonant part of the perturbed distribution function, the distance to instability threshold is

$$\Gamma_2 = -\frac{2\,m_i}{p_{0i}^\perp}\left(\int \frac{v_\perp^4}{4}\frac{\partial f_{2i}}{\partial v_\parallel^2}\bigg|_{v_\perp} \mathrm{d}^3\boldsymbol{v} + \int_{|\xi|<\xi_{\mathrm{tr}}} \frac{\partial \overline{\delta f_{4i}^{(\mathrm{res})}}}{\partial v_\parallel^2}\frac{v_\perp^4}{4}\bigg|_{v_\perp} \mathrm{d}^3\boldsymbol{v}\right) - \frac{2\,p_{2i}^\perp}{p_{0i}^\perp}, \tag{16}$$

and the effective Larmor radius is

$$\rho_*^2 = \frac{\rho_i^2}{12}\frac{m_i}{p_{0i}^\perp\, v_{\mathrm{th}i\perp}^2} \int \left(-v_\perp^6 \frac{\partial f_{0i}}{\partial v_\parallel^2}\bigg|_{v_\perp} - 3\,v_\perp^4\, f_{0i}\right) \mathrm{d}^3\boldsymbol{v}. \tag{17}$$

Tildes denote fluctuating (zero field-line average) parts of the perturbed fields and overlines denote line averages. The l.h.s. of equation (14) describes the non-resonant response.[3] Γ_2 and $\delta \tilde{p}_{6i}^{\perp(\mathrm{res})}$ depend on the regime considered. However, both only involve $\delta f_{4i}^{(\mathrm{res})}$ because restricting the integration to ξ_{tr} brings in an extra $O(\varepsilon^2)$ smallness, whereas FLR corrections only start to affect the distribution function at $O(\varepsilon^6)$ within our expansion. Thus, $\delta \tilde{f}_{4i}^{(\mathrm{res})}$ and $\delta \tilde{p}_{6i}^{\perp(\mathrm{res})}$ can be directly calculated from the much simpler drift-kinetic equation which, in $(\mu, v_\parallel)$ variables, reads (Kulsrud 1983):

$$\frac{\mathrm{d} f_i}{\mathrm{d}t} + v_\parallel \nabla_\parallel f_i = -\mu B\,(\nabla\cdot\hat{\boldsymbol{b}})\frac{\partial f_i}{\partial v_\parallel} + C[f_i] \tag{18}$$

to all orders relevant to our calculation (here $E_\parallel = 0$ because the electrons are cold). For resonant particles, $v_\parallel \sim \varepsilon^2 v_{\mathrm{th}i}$ and $\partial \delta f_{4i}^{(\mathrm{res})}/\partial v_\parallel \sim (\varepsilon^{-2}/v_{\mathrm{th}i})\,\delta f_{4i}^{(\mathrm{res})}$, so the expansion of equation (18) at the first non-trivial order $O(\varepsilon^6)$ is

$$\frac{\mathrm{d}\delta f_{4i}^{(\mathrm{res})}}{\mathrm{d}t} + v_\parallel \nabla_\parallel \delta f_{4i}^{(\mathrm{res})} = \mu B_0 \left(\frac{\nabla_\parallel \delta B_4^\parallel}{B_0}\right)\left(\frac{\partial f_{0i}}{\partial v_\parallel} + \frac{\partial \delta f_{4i}^{(\mathrm{res})}}{\partial v_\parallel}\right) + C\left[\delta f_{4i}^{(\mathrm{res})}\right]. \tag{19}$$

We have omitted the collision term $C[f_{0i}]$: it gives a $O(\varepsilon^4)$ correction to the line-averaged pressure on the instability time-scale that can be absorbed into Γ_0.

3 LINEAR AND QUASI-LINEAR REGIMES

Neglecting the non-linear and collision terms in equation (19) and taking its space–time Fourier transform, we obtain

$$\delta \hat{f}_{4i\boldsymbol{k}}^{(\mathrm{res})} = \frac{\mu\, i k_\parallel\, \delta \hat{B}_{4\boldsymbol{k}}^\parallel}{\gamma_{\mathrm{L}} + i\, k_\parallel v_\parallel}\frac{\partial f_{0i}}{\partial v_\parallel}, \tag{20}$$

where γ_{L} is the linear instability growth rate. Using equation (15) to calculate $\delta \tilde{p}_{6i}^{\perp(\mathrm{res})}$ and substituting it into equation (14), we recover the linear mirror dispersion relation (Hellinger 2007):

$$\gamma_{\mathrm{L}} = \sqrt{\frac{2}{\pi}}|k_\parallel| v_* \left[\Gamma_2 - \frac{3}{2}\rho_*^2 k_\perp^2 - \left(\frac{p_{0i}^\perp - p_{0i}^\parallel}{p_{0i}^\perp} + \frac{2}{\beta_{0i}^\perp}\right)\frac{k_\parallel^2}{k_\perp^2}\right], \tag{21}$$

with the effective thermal speed

$$v_*^{-1} = -\sqrt{2\pi}\,\frac{2m_i}{p_{0i}^\perp}\int \frac{v_\perp^4}{4}\frac{\partial f_{0i}}{\partial v_\parallel^2}\bigg|_{v_\perp} \delta(v_\parallel)\,\mathrm{d}^3\boldsymbol{v}. \tag{22}$$

Because of the resonance, $\delta \tilde{f}_{4i}^{(\mathrm{res})}$ develops a velocity-space boundary layer on the time-scale $1/\gamma_{\mathrm{L}}$, resulting in a correction to the line-averaged distribution function that satisfies

$$\frac{\mathrm{d}\overline{\delta f_{4i}^{(\mathrm{res})}}}{\mathrm{d}t} = -\mu B_0 \overline{\left(\frac{\nabla_\parallel \delta \tilde{B}_4^\parallel}{B_0}\right)\frac{\partial \delta f_{4i}^{(\mathrm{res})}}{\partial v_\parallel}} + C[\overline{\delta f_{4i}^{(\mathrm{res})}}]. \tag{23}$$

Assuming a monochromatic perturbation for simplicity and using equation (20) to calculate the line-averaged non-linear term on the

[3] Non-linearities quadratic in $\delta B/B$ are negligible here because of the weakly non-linear ordering $\delta B/B \sim \Gamma^2$, which differs from $\delta B/B \sim \Gamma$ used in Califano et al. (2008).

r.h.s. of equation (23), we recover the resonant quasi-linear diffusion equation (Shapiro & Shevchenko 1964):

$$\frac{\partial \overline{\delta f_{4i}^{(\mathrm{res})}}}{\partial t} = \frac{\partial}{\partial v_\parallel}\left[\frac{2\,(\mu B_0)^2 k_\parallel^2 \gamma_\mathrm{L}}{\gamma_\mathrm{L}^2 + \left(k_\parallel v_\parallel\right)^2}\overline{\left(\frac{\delta \tilde{B}_4^\parallel}{B_0}\right)^2}\frac{\partial f_{0i}}{\partial v_\parallel}\right] + C[\overline{\delta f_{4i}^{(\mathrm{res})}}]. \qquad (24)$$

The effect of the first term on the r.h.s. of equation (24) is to relax Γ_2 (see equation 16) by flattening the total averaged distribution function at low ξ, thereby decreasing the growth rate (Pokhotelov et al. 2008; Hellinger et al. 2009).

4 TRAPPING REGIME

Quasi-linear relaxation ceases to be the dominant saturation mechanism once particle trapping becomes dynamically significant ($\omega_B \sim k_\parallel v_{\mathrm{th}i}\,(\delta B/B)^{1/2} \sim \gamma_\mathrm{L}$). Indeed, due to the growth of $\delta B/B$ and quasi-linear reduction of the growth rate $\partial/\partial t \ll \gamma_\mathrm{L}$ for $t \gg 1/\gamma_\mathrm{L}$, (i) the system eventually reaches a bounce-dominated regime, $\omega_B \gg \partial/\partial t$, and (ii) collisional and shearing effects, however slow their time-scales are, compared to the initial linear instability time-scale, inevitably become important after a few instability times (hence the maximal ordering equations 8–12). To elicit these effects, we rewrite equation (18) in (μ, $E = v^2/2$) variables:

$$\frac{\mathrm{d} f_i}{\mathrm{d}t} \pm \sqrt{2(E-\mu B)}\,\frac{\partial f_i}{\partial \ell} = -\mu\frac{\mathrm{d}B}{\mathrm{d}t}\frac{\partial f_i}{\partial E} + C[f_i], \qquad (25)$$

where ℓ is the distance along the perturbed field line. Expanding equation (5) and equation (25) at $O(\varepsilon^6)$, we obtain

$$\frac{\mathrm{d}\delta f_{4i}^{(\mathrm{res})}}{\mathrm{d}t} \pm \sqrt{2(E-\mu B)}\,\frac{\partial \delta f_{4i}^{(\mathrm{res})}}{\partial \ell} = -\mu B_0 \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_4^\parallel}{B_0}\frac{\partial f_{0i}}{\partial E}$$
$$-\mu B_0\left(\frac{\mathrm{d}\ln B_0}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_4^\parallel}{B_0}\right)\frac{\partial \delta f_{4i}^{(\mathrm{res})}}{\partial E} + C[\delta f_{4i}]. \qquad (26)$$

Here $C[f_{0i}]$ and $-\mu B_0\,(\mathrm{d}\ln B_0/\mathrm{d}t)(\partial f_{0i}/\partial E)$ have been discarded for the same reason as in equation (19). Note that both $\partial \delta f_{4i}^{(\mathrm{res})}/\partial E$ and $\partial \delta f_{4i}^{(\mathrm{res})}/\partial \mu$ are $O(1)$ because of the velocity-space boundary layer in $|\xi| < \xi_\mathrm{tr} = O(\varepsilon^2)$.

We anticipate that magnetic fluctuations will grow secularly as $\delta B_4^\parallel \sim \delta B_4^\parallel(t_\mathrm{L})(t/t_\mathrm{L})^s$, with $s > 0$, for $t \gg t_\mathrm{L} \sim 1/\gamma_\mathrm{L}\,(\sim 1/\omega_B)$, and introduce a secondary ordering parameter $\chi = (t_\mathrm{L}/t)^{s/2} \ll 1$, so now $\delta B_4^\parallel/B_0 = O(\varepsilon^4/\chi^2)$ and $\xi_\mathrm{tr} \sim (\delta B_4^\parallel/B_0)^{1/2} = O(\varepsilon^2/\chi) \gg \varepsilon^2$. The instantaneous non-linear growth rate is $\gamma_\mathrm{NL} \sim \partial/\partial t \sim 1/t = O(\varepsilon^2\chi^{2/s})$, so the new ordering guarantees $\omega_B \sim k_\parallel v_{\mathrm{th}i}\,\xi_\mathrm{tr} \gg \gamma_\mathrm{NL}$. For trapped particles to play a role in the non-linear evolution, their pressure in equation (14) must be taken to be of the same order as the instability-driving term, $\Gamma_2\,(\delta B_4^\parallel/B_0) \sim \delta p_{6i}^{\perp(\mathrm{res})}/p_{0i}^\perp = O(\varepsilon^2/\chi^2)$. Given that $\delta p_{6i}^{\perp(\mathrm{res})} \sim \xi_\mathrm{tr}\delta f_{4i}^{(\mathrm{res})}$, we must therefore order $\delta f_{4i}^{(\mathrm{res})} = O(\varepsilon^4/\chi)$. Expanding equation (26) to lowest order $O(\varepsilon^6/\chi^2)$, we find that the distribution function of the trapped particles is homogenized along the field lines within the traps: $\partial \delta f_{4i}^{(\mathrm{res})}/\partial \ell = 0$. Therefore, $\delta f_{4i}^{(\mathrm{res})} = \langle \delta f_{4i}^{(\mathrm{res})}\rangle + \delta f_{4i}^{(\mathrm{res})'}$, where $\delta f_{4i}^{(\mathrm{res})'} \ll \langle \delta f_{4i}^{(\mathrm{res})}\rangle$ and $\langle \bullet \rangle = \oint \bullet\,\mathrm{d}\ell$ denotes a bounce average between bounce points ℓ_1 and ℓ_2 defined by the relation $E = \mu B(\ell_1) = \mu B(\ell_2)$. Looking at the next orders of equation (26), we find that the first term on the r.h.s. (the betatron term linear in perturbations) is $O(\varepsilon^6\chi^{2/s-2})$, while the time derivative on the l.h.s. and the r.h.s. term quadratic in perturbations are $O(\varepsilon^6\chi^{2/s-1})$, so quasi-linear effects are subdominant. The terms involving $\mathrm{d}\ln B_0/\mathrm{d}t$ and collisions are $O(\varepsilon^6\chi)$. For equation (26) to have a solution at $O(\varepsilon^6\chi)$, we see that $s = 2/3$ is required, so $\delta B_4^\parallel/B_0 \propto t^{2/3}$. The resulting equation for $\delta f_{4i}^{(\mathrm{res})'}$ is

$$\frac{\partial \delta f_{4i}^{(\mathrm{res})'}}{\partial \ell} = \mp\frac{\mu B_0}{\sqrt{2(E-\mu B)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_4^\parallel}{B_0}\frac{\partial f_{0i}}{\partial E} + \frac{\mathrm{d}\ln B_0}{\mathrm{d}t}\frac{\partial\left\langle \delta f_{4i}^{(\mathrm{res})}\right\rangle}{\partial E}\right)$$
$$\pm C\left[\left\langle \delta f_{4i}^{(\mathrm{res})}\right\rangle\right]. \qquad (27)$$

Using a Lorentz operator and bounce averaging, we obtain

$$\left\langle \frac{\mu B_0}{\sqrt{2(E-\mu B)}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_4^\parallel}{B_0}\right\rangle \frac{\partial f_{0i}}{\partial E} =$$
$$-\frac{\mathrm{d}\ln B_0}{\mathrm{d}t}\left\langle \frac{\mu B_0}{\sqrt{2(E-\mu B)}}\right\rangle \frac{\partial\left\langle \delta f_{4i}^{(\mathrm{res})}\right\rangle}{\partial E}$$
$$+\frac{\nu_{ii}}{B_0}\frac{\partial}{\partial \mu}\left(\mu\left\langle\sqrt{2(E-\mu B)}\right\rangle\frac{\partial\left\langle \delta f_{4i}^{(\mathrm{res})}\right\rangle}{\partial \mu}\right). \qquad (28)$$

5 PHYSICAL BEHAVIOUR AND EVOLUTION

This equation has taken some effort to derive but is fairly transparent physically. It represents a competition between perpendicular betatron cooling of the equilibrium distribution due to the local decrease of the magnetic field in the deepening mirror traps (the l.h.s. of equation 28), the perpendicular betatron heating of the trapped-particle population associated with the increasing mean field B_0 (the first term on the r.h.s.), and their collisional isotropization (the second term on the r.h.s.). In the weakly collisional, unsheared regime ($\nu_{ii} \neq 0$, $\mathrm{d}\ln B_0/\mathrm{d}t \equiv S = 0$), the balance is between betatron cooling and collisions. In the collisionless, shearing regime ($\nu_{ii} = 0$, $S > 0$), it is instead between betatron cooling (of the bulk distribution in mirror perturbations) and heating (of the perturbed distribution in the growing mean field). For the system to stay marginal in the face of continued driving and/or collisional relaxation, perturbations have to continue growing. A physical interpretation of the collisionless case is that trapped particles regulate the evolution so as to 'see' effectively a constant total magnetic field.

Solutions of equations (14) and (15)–(28) can be found in the form $\delta B_4^\parallel/B_0 = \mathcal{A}(t)\mathcal{B}(\boldsymbol{\ell})$. Using equation (28), this implies $\langle \delta f_{4i}^{(\mathrm{res})}\rangle = \alpha\,\mathcal{A}\,(\mathrm{d}\mathcal{A}/\mathrm{d}t)\,\mathcal{H}[(E-\mu B_0)/\mathcal{A}(t)]\,\partial f_{0i}/\partial E$, where $\alpha = 1/S$ if $\nu_{ii} = 0$ and $1/\nu_{ii}$ if $S = 0$ ($\mathcal{H}$ also depends on the functional form of $\mathcal{B}(\boldsymbol{\ell})$). Then, from equation (15), $\delta p_{6i}^{\perp(\mathrm{res})} = \alpha\,\mathcal{A}^{3/2}(\mathrm{d}\mathcal{A}/\mathrm{d}t)\,\mathcal{F}(\boldsymbol{\ell})$. Equation (14) will have solutions if $\alpha\,\mathcal{A}^{1/2}(\mathrm{d}\mathcal{A}/\mathrm{d}t) = \Lambda\Gamma_2$, with Λ a constant of order unity. As anticipated in our discussion of the secondary ordering (in χ), this implies secular growth of perturbations,

$$\mathcal{A}(t) = (\Lambda\Gamma_2 St)^{2/3} \quad \text{and} \quad \mathcal{A}(t) = (\Lambda\Gamma_2\nu_{ii}t)^{2/3} \qquad (29)$$

in the shearing-collisionless regime and unsheared-collisional regime, respectively. This result is formally valid for times St, $\nu_{ii}t = O(\varepsilon^4)$. The $t^{2/3}$ time dependence also holds in mixed regimes[4]

[4] It does not apply to the frequently discussed case $\nu_{ii} = S = 0$, because continued growth of fluctuations was assumed to derive equation (28). In that regime, fluctuations should instead settle in a steady state $\delta B/B \sim \Gamma^2$ through quasi-linear relaxation, possibly after a few transient bounce oscillations (Istomin et al. 2009). However, only a small amount of collisions or continued shearing/compression is required for the present theory to apply: these effects are bound to become dynamically important after a few instability times, once quasi-linear relaxation has reduced the instability drive Γ sufficiently.

($\nu_{ii} \neq 0$, $S \neq 0$). Λ and $\mathcal{B}(\ell)$ (which is not sinusoidal) are obtained by solving a non-linear eigenvalue equation involving trapping integrals. A detailed classification of these solutions lies beyond the scope of this Letter.

6 CONCLUSION

Equation (28) and the secular growth of the non-linear mirror instability, $\delta B/B \propto t^{2/3}$, in both collisional and collisionless regimes, are the main results of this work. Thus, we appear to be approaching a theory in which trapping effects, higher amplitude non-linearities (Kuznetsov, Passot & Sulem 2007; Califano et al. 2008; Pokhotelov et al. 2010) and relaxation of anisotropy through anomalous particle scattering (Kunz et al. 2014) blend together harmoniously. The results make manifest the importance of particle trapping (Kivelson & Southwood 1996; Pantellini 1998). Numerical simulations (Kunz et al. 2014) confirm $t^{2/3}$ secular growth of mirror perturbations in a collisionless, shearing plasma, with saturation amplitudes $\delta B/B = O(1)$ independent of S (cf. Riquelme et al. 2014).

The weakly collisional, weakly shearing regimes studied in this Letter occur in many natural environments (Quataert 2001; Bale et al. 2009) and are increasingly the focus of attention in the context of high-energy astrophysical plasmas (Schekochihin et al. 2008; Rosin et al. 2011; Kunz et al. 2014; Riquelme et al. 2014). The emergence of finite-amplitude magnetic mirrors with scales smaller than the mean free path lends credence to the idea that microscale instabilities regulate heat conduction (Chandran & Cowley 1998), viscosity, heating (Sharma et al. 2006, 2007; Kunz et al. 2011) and dynamo (Schekochihin & Cowley 2006; Mogavero & Schekochihin 2014) processes in such plasmas, and profoundly alter their large-scale energetics and dynamics.

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