

Intro to Pressure-Anisotropy - Driven Instabilities

Alex Schekochihin (Oxford)

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[see http://www-thphys.physics.ox.ac.uk/people/Alexander.Schekochihin/notes/LES_HOUCHES_15.pdf]

S1. A Super-Brief Review of the kMHD (Part I)

I want to start by showing how for magnetized, weakly collisional plasmas, low-frequency, degenerate long-wavelength

$$\begin{aligned} \nu_{\text{coll}} &\ll \Omega_s \\ \text{coll.} & \quad \text{Larmor} \\ \text{frequencies} & \quad \text{frequencies} \\ p_s &\ll \lambda_{\text{Lfp}} \\ \text{Larmor} & \quad \text{mean free} \\ \text{radii} & \quad \text{path} \end{aligned}$$

$$\omega \ll \Omega_s$$

$$k_p s \ll 1$$

~~the~~ dynamics can be described by a set of equations that look almost like the familiar MHD.

We will see later on that the ways in which they are not MHD will profoundly affect the ~~same~~ dynamics — indeed we do not really fully understand the full implications of this in high-β plasmas.

This is consequently one of the ~~frontiers~~ frontier topics in theoretical plasma astrophysics.

Let's start from first principles.

Any plasma that is going to be of interest to us is described by the Maxwell - Landau system of equations:

these notes +
supp. materials

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \nabla f_s + \frac{e_s}{m_s} \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) \cdot \frac{\partial \vec{f}_s}{\partial \vec{v}} = C[f_s] \quad (1)$$

distr. fn of species -2-

Vlasov

collisions
Landau

$$\text{Maxwell: } \cancel{\nabla \cdot \vec{E}} = 4\pi \sum_s e_s n_s, \quad n_s = \int d^3 v f_s \text{ density} \quad (2)$$

small when $k^2 \lambda_{De}^2 \ll 1$

||

O quasineutrality

Debye length

$$\nabla \cdot \vec{B} = 0 \quad (3)$$

$$\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} \quad \text{Faraday} \quad (4)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \cancel{\frac{\partial \vec{E}}{\partial t}} \quad \begin{array}{l} \text{small when} \\ \omega \ll k c \end{array} \quad (5)$$

(low-freq. waves,
non-relat. motions)

$$\vec{j} = \sum_s e_s n_s \vec{u}_s, \quad \vec{u}_s = \frac{1}{n_s} \int d^3 v \vec{v} f_s$$

~~This is quite difficult because each of solutes is different~~
 Intuitively we tend to want to think of the plasma as a fluid (or a multi-fluid of several species) with some density n_s , velocity \vec{u}_s and perhaps pressure, temperature, etc. This is rooted in our experience with collisional gases ($v \gg \omega$), which are in local Maxwellian equilibrium:

$$f_s = \frac{n_s}{(\pi v_{th,s}^2)^{3/2}} e^{-\frac{(v-\vec{u}_s)^2}{4v_{th,s}^2}}, \quad v_{th,s} = \sqrt{\frac{2T_s}{m_s}} \quad (6)$$

where n_s , \vec{u}_s and T_s are governed by fluid equations.

With this desire to think of the plasma as a fluid, let us break the motion of the particles into two parts:

$$\vec{v} = \vec{u}_s(t, \vec{r}) + \vec{w}$$

↑ ↑
 mean velocity of "peculiar" velocity
 species s - "internal" motion
 ↑ ↑
 "fluid" "kinetic"

This amounts to a transformation of variables

$$(t, \vec{r}, \vec{v}) \rightarrow (t, \vec{r}, \vec{w}), \quad \vec{w} = \vec{v} - \vec{u}_s(t, \vec{r})$$

under which

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \frac{\partial \vec{u}_s}{\partial t} \cdot \frac{\partial}{\partial \vec{w}}, \quad \nabla \rightarrow \nabla - (\nabla \vec{u}_s) \cdot \frac{\partial}{\partial \vec{w}}, \quad \frac{\partial}{\partial \vec{v}} \rightarrow \frac{\partial}{\partial \vec{w}}$$

and eq. (1) becomes

$$\underbrace{\left(\frac{\partial}{\partial t} + \vec{u}_s \cdot \nabla \right) f_s + \vec{w} \cdot \nabla f_s}_{\substack{\text{III} \\ \frac{d}{dt} \text{ convective derivative}}} + \left(\frac{e_s}{m_s} \frac{\vec{w} \times \vec{B}}{c} + \vec{a}_s - \vec{w} \cdot \nabla \vec{u}_s \right) \cdot \frac{\partial f_s}{\partial \vec{w}} = C[f_s]$$

(7)

$\frac{e_s}{m_s} \left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) - \frac{d \vec{u}_s}{dt}$

and now we have always $\int d^3 \vec{w} \vec{w} f_s = 0$ by definition.

The strategy now is to take moments of this equation.

$$\int d^3 \vec{w} (7) : \quad \frac{d n_s}{dt} + (\nabla \cdot \vec{u}_s) n_s = 0$$

↑ $\int d^3 \vec{w} (-\vec{w} \cdot \nabla \vec{u}_s) \cdot \frac{\partial f_s}{\partial \vec{w}}$ by parts

↓ All other terms = 0.

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \vec{u}_s) = 0 \quad \begin{matrix} \text{continuity} \\ \text{equation} \end{matrix} \quad (8)$$

$$\int d^3\vec{w} m_s \vec{w} (f) : \underbrace{\nabla \cdot \int d^3\vec{w} n_s \vec{w} \vec{w} f_s - m_s n_s \vec{a}_s}_{\text{All other terms = 0 because}} = \int d^3\vec{w} n_s \vec{w} C[f] = \vec{R}_s \text{ collisional friction}$$

\hat{P}_s pressure factor

Unpacking \vec{a}_s , we get the momentum equation

$$m_s n_s \frac{d\vec{u}_s}{dt} = -\nabla \cdot \hat{P}_s + e_s n_s \left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) + \vec{R}_s \quad (9)$$

$$\sum_s m_s n_s \frac{d\vec{u}_s}{dt} = -\nabla \cdot \sum_s \hat{P}_s + \sum_s e_s n_s \vec{E} + \left(\frac{\sum_s e_s n_s \vec{u}_s \times \vec{B}}{c} \right) + \sum_s \vec{R}_s$$

mass flow

quasineutrality

$\int = \frac{c}{4\pi} \nabla \times \vec{B}$ Ampère's law.

collision conserve momentum

$$\rho = \sum_s m_s n_s, \quad \vec{u} = \frac{1}{\rho} \sum_s m_s n_s \vec{u}_s \quad \text{c. of u. velocity}$$

We'll take $s = i, e$, $m_e \ll m_i$, so $\vec{u} \approx \vec{u}_i$, $\rho \approx m_i n_i$

$$\begin{aligned} \rho \frac{d\vec{u}}{dt} &= -\nabla \cdot \hat{P} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi} \\ &= -\nabla \cdot \left[\hat{P} + \frac{B^2}{8\pi} \mathbb{I} - \frac{\vec{B} \vec{B}}{4\pi} \right] \end{aligned} \quad (14)$$

We also need an equation for the magnetic field:

It's Faraday's law:

$$\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E}$$

$$\text{From eq. (9), } \vec{E} = -\frac{\vec{u}_s \times \vec{B}}{c} + \cancel{\frac{\nabla \cdot \hat{P}_s}{e_s n_s}} - \cancel{\frac{\vec{R}_s}{e_s n_s}} + \cancel{\frac{m_s}{e_s} \frac{d\vec{u}_s}{dt}}$$

$$k p_i / M_a \ll 1$$

$$\gamma_s / S_s \ll 1$$

$$\omega / S_s \ll 1$$

This means that $\vec{u}_{\perp s} = c \frac{\vec{E} \times \vec{B}}{B^2} = \vec{u}_\perp$, the \perp part of the velocity is the same for all species
 and $\nabla \times \frac{d\vec{B}}{dt} = \vec{B} \cdot \nabla \vec{u} - \vec{B} \nabla \cdot \vec{u}$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}) \quad (11) \text{ induction equation.}$$

Thus, we have eqns (10-12), which look just like MHD, except for the pressure tensor in (11).

Obviously, all the kinetic nastiness is hidden in \hat{P} .

Going back to eq. (7), it is key to notice that

$$\frac{e_s}{m_s} \frac{\vec{w} \times \vec{B}}{c} \cdot \frac{\partial f_s}{\partial \vec{w}} = - S_s \left(\frac{\partial f_s}{\partial \vec{v}} \right)_{w_\perp, w_\parallel} \downarrow$$

$\frac{e_s B}{m_s c}$ gyroangle
particle

eq. (7) is

$$S_s \left(\frac{\partial f_s}{\partial \vec{v}} \right)_{w_\perp, w_\parallel} = \cancel{\frac{df_s}{dt}} + \cancel{\vec{w} \cdot \nabla f_s} + + (\vec{a}_s - \vec{w} \cdot \nabla \vec{u}_s) \cdot \frac{\partial f_s}{\partial \vec{w}} - c [f_s] = 0$$

$$\begin{cases} \omega \ll S_s \\ k_p s \frac{u_s}{V_{th s}} \ll 1 \end{cases}$$

prove this by
 changing ~~variables~~
~~variables~~ to
 cylindrical
 coords in \vec{w} space:

$$\vec{w} = \begin{pmatrix} w_\perp \cos \vartheta \\ w_\perp \sin \vartheta \\ w_\parallel \end{pmatrix}$$

To lowest order
 in $k_p s$, $\frac{\omega}{S_s}$

$$f_s = f_s (w_\perp, w_\parallel)$$

gyrotropic!

We can use this fact to simplify the pressure tensor:

$$\hat{P}_S = \int d^3\vec{w} m_S \underbrace{\langle \vec{w} \vec{w} \rangle}_{\parallel} f_S(t, \vec{r}, w_{\perp}, w_{\parallel}) =$$

$$\frac{w_{\perp}^2}{2} (\mathbb{I} - \vec{B}\vec{B}) + w_{\parallel}^2 \vec{B}\vec{B}$$

Diagonal wrt
local direction of \vec{B}
↓

$$= (\mathbb{I} - \vec{B}\vec{B}) \underbrace{\int d^3\vec{w} \frac{m_S w_{\perp}^2}{2} f_S}_{\parallel P_{\perp S}} + \vec{B}\vec{B} \underbrace{\int d^3\vec{w} m_S w_{\parallel}^2 f_S}_{\parallel P_{\parallel S}} = \begin{pmatrix} P_{\perp S} & 0 & 0 \\ 0 & P_{\perp S} & 0 \\ 0 & 0 & P_{\parallel S} \end{pmatrix}$$

Eq. (11) becomes

$$\boxed{\rho \frac{d\vec{u}}{dt} = -\nabla(p_{\perp} + \frac{B^2}{8\pi}) + \nabla \cdot [\vec{B}\vec{B}(p_{\perp} - p_{\parallel} + \frac{B^2}{4\pi})]} \quad (13)$$

↑ total scalar pressure ↑ pressure anisotropy stress ↑ Maxwell stress

This is the key new feature compared to usual MHD.

Should be important provided

$$p_{\perp} - p_{\parallel} \gtrsim \frac{B^2}{4\pi}, \text{ or } \frac{p_{\perp} - p_{\parallel}}{p} \gtrsim \frac{2}{\beta}$$

} so more likely to matter in high- β plasmas

To summarize where we have got so far:

to work out motions and m-fields in a plasma, solve eqns. (13) for \vec{u} and (12) for \vec{B} ,

where $p_{\perp} = \sum_s \int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} f_s$, $p_{\parallel} = \sum_s \int d^3 \vec{w} m_s w_{\parallel}^2 f_s$

and $\rho = \sum_s m_s \int d^3 \vec{w} f_s$ [or solve eq. (10)]

We still need the kinetic equation to calculate f_s
 — this kinetic equation will need to be somewhat
 reduced to solve for the lowest-order, gyrotopic
 pdf $f_s(w_{\perp}, w_{\parallel})$.

In pursuit of instant gratification, I will postpone
 doing this and first derive some results that
 do not ~~depend on~~ need the f_s equation.

S2 Firehole Instability : Linear Theory

Suppose we have some "macroscopic" solution
 of our (yet to be fully derived!) epus.

Allow high-frequency, short-wavelength
 perturbations of this solution

Seek total soln in the form

$$\vec{u} + \delta \vec{u}, \vec{B} + \delta \vec{B}, \text{etc.}$$

$$\omega \ll \frac{u}{l}$$

$$kl \gg 1$$

$$\text{infinitesimal perturbations} \propto e^{-i\omega t + ik \cdot r}$$

Fourier-transform and linearize our epus.

in practice, this amounts to
 perturbing about some
 constant homogeneous equilibrium (or non-equilibrium.)

Eq. (12):

$$-\omega \vec{S} \vec{B} = \cancel{\text{Term}} \vec{B} \cdot \vec{k} \delta \vec{u} - \vec{B} \vec{k} \cdot \delta \vec{u}$$

Eq. (13):

$$-\omega p \delta \vec{u} = B (k_{\parallel} \delta \vec{u}_{\perp} - \vec{k}_{\perp} \cdot \delta \vec{u}_{\perp}) \quad (14)$$

$$-\omega p \delta \vec{u} = -\vec{k} (\delta p_{\perp} + \frac{B \delta B}{4\pi}) +$$

N.B.: $\vec{k} \cdot \vec{\delta b} = -k_{\parallel} \frac{\delta B}{B}$ because $\nabla \cdot \vec{B} = 0$

$$\begin{aligned} & \vec{k} \cdot \left[(\delta \vec{b} \cdot \vec{b} + \vec{b} \cdot \delta \vec{b}) (p_{\perp} - p_{\parallel} + \frac{B^2}{4\pi}) \right. \\ & \left. + \vec{b} \cdot \vec{b} (\delta p_{\perp} - \delta p_{\parallel} + \frac{B \delta B}{2\pi}) \right] = \\ & = -\vec{k}_{\perp} \left(\delta p_{\perp} + \frac{B \delta B}{4\pi} \right) \end{aligned}$$

$$\begin{aligned} & -\vec{b} k_{\parallel} \left[\delta p_{\parallel} + (p_{\perp} - p_{\parallel}) \frac{\delta B}{B} \right] \\ & + \delta \vec{b} k_{\parallel} \left(p_{\perp} - p_{\parallel} + \frac{B^2}{4\pi} \right) \quad (15) \end{aligned}$$

$$\delta \vec{b} = \frac{\delta \vec{B}_{\perp}}{B} = -\frac{k_{\parallel}}{\omega} \delta \vec{u}_{\perp} \text{ from eq. (14)} \quad \text{"Alfvénic" perturbation}$$

Isolate the Alfvénic response in eq. (15) by

$\vec{k}_{\perp} \times [Eq.(15)]_{\perp}$:

$$-\omega p \vec{k}_{\perp} \times \delta \vec{u}_{\perp} = k_{\parallel} \left(p_{\perp} - p_{\parallel} + \frac{B^2}{4\pi} \right) \vec{k}_{\perp} \times \delta \vec{b}$$

$$\omega^2 = k_{\parallel}^2 \left(\frac{B^2}{4\pi p} + \frac{p_{\perp} - p_{\parallel}}{S_c} \right) = k_{\parallel}^2 V_{th\parallel}^2 \left(\frac{p_{\perp} - p_{\parallel}}{p_{\parallel}} + \frac{2}{\beta_{\parallel}} \right)$$

V_A^2 Alfvén wave!

will be unstable ($\omega^2 < 0$)

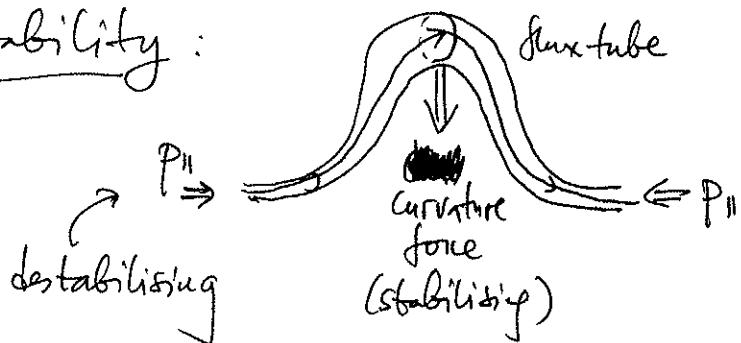
if $\Delta < -\frac{2}{\beta_{\parallel}}$!

$$\boxed{\gamma = k_{\parallel} V_{th\parallel} \sqrt{|\Delta + \frac{2}{\beta}|}}$$

growth rate.

Thus, negative Δ ($p_{\parallel} > p_{\perp}$) locally weakens
tension (= slows down Alfvén waves)

and makes it energetically easier to bend
w. field lines. For $\Delta < -2/\beta_{\parallel}$, the elasticity of
field lines is lost and we have the firehose
instability:



Rosenbluth 1956

Chandrasekhar
et al. 1958

Parker 1958

Vedenov
& Sydnev 1958

Key point:

— nothing surprising that $p_{\parallel} > p_{\perp}$ leads
to an instability: it's a non-equilibrium situation,
so a source of free energy

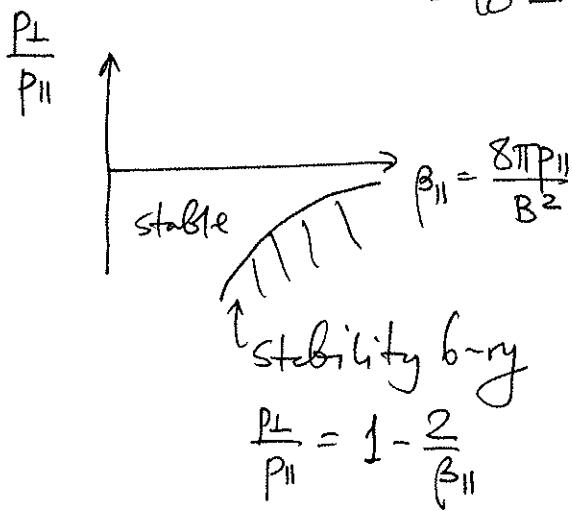
— $\propto k_{\parallel}$ UV catastrophe: so, within KMFV
($\omega \ll \Omega_i$, $k p_i \ll 1$), the wavenumber of peak γ
is not captured. Including FLR gives

$$\gamma_{\text{peak}} \sim |\Delta + \frac{2}{\beta_{\parallel}}| \Omega_i, k_{\parallel \text{peak}} p_i \sim |\Delta + \frac{2}{\beta_{\parallel}}|^{1/2}$$

So the instability is v. fast at microscale

Any ^{high- β} macroscopic solution with $p_{\parallel} > p_{\perp}$ will blow
up instantly — what happens next is decided
by the nonlinear saturation of the firehose (later!)

Simplest calculation: MNRAS 405, 291 (2010)



Showed that there is an increased fluctuation level at the boundary.

It was a transformative moment when J. Kasper (2002) discovered that the firehose stability boundary contains most observed solar wind states

[followed by Hellinger et al 2006, Bale et al. 2009]

§3 Mirror Instability : Linear Theory

Let us go back to eqs. (14-15) and see what other (apart from Alfvénic) perturbations there are and whether they are stable.

We have already looked at $\delta \vec{B} = \frac{\delta \vec{B}_\perp}{B}$.

Now consider $\frac{\delta B}{B} = \frac{\delta B_\parallel}{B}$: from eq.(14),

$$\omega \frac{\delta B}{B} = k_\perp \cdot \delta \vec{u}_\perp \quad (\perp \text{ compression increases } B)$$

Take $k_\perp \cdot \text{Eq. (15)}$:

$$\omega \rho k_\perp \cdot \delta \vec{u}_\perp = \rho \omega^2 \frac{\delta B}{B} = k_\perp^2 \left(\delta p_\perp + \frac{B \delta B}{4\pi} \right) + k_\parallel^2 \left(p_\perp - p_\parallel + \frac{B^2}{4\pi} \right) \frac{\delta B}{B}$$

so we need kinetic theory to calculate this!

(16)

S1' KMHD Cont'd (Part II)

Let's go back to the form of the kinetic eqn we wrote on p.5:

$$\text{S}_s \left(\frac{\partial f_s}{\partial \theta} \right)_{w_1, w_{\parallel}} = \frac{df_s}{dt} + \vec{w} \cdot \nabla f_s + (\vec{a}_s - \vec{w} \cdot \nabla \vec{u}_s) \cdot \frac{\partial f_s}{\partial \vec{w}} - C[f_s]$$

||

$$\left(\frac{e_s}{m_s} \left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) - \frac{d\vec{u}_s}{dt} \right) \quad (17)$$

To lowest order,

To lowest order,

$$\left(\frac{\partial f_s}{\partial x} \right)_{w_2, w_{11}} = 0$$

To next order, $S_S \left(\frac{\partial f_S^{(1)}}{\partial \theta} \right)_{W_1, W_2} = \text{rhs of Eq. (17) with } f_S = f_S^{(0)}$

eliminate this by integrating over Ω

$$\text{Then } \left\langle \frac{d\vec{f}_S}{dt} + \vec{w} \cdot \nabla \vec{f}_S + (\vec{a}_S - \vec{w} \cdot \nabla \vec{u}_S) \cdot \frac{\partial \vec{f}_S}{\partial \vec{w}} - C[\vec{f}_S] \right\rangle_{\theta} = 0$$

where $f_s = f_s(w_\perp, w_\parallel)$. To do this averaging, transform variables

$$(t, \vec{r}, \vec{w}) \rightarrow (t, \vec{r}, w_{\perp}, w_{\parallel}, \vartheta)$$

$$\text{daere } w_{||} = \vec{w} \cdot \hat{b}(t, \vec{r}), \quad w_{\perp} = |\vec{w} - w_{||} \hat{b}|.$$

I will skip the algebra (exercise! - or look it up
(the more detailed old notes posted on)
in [http://www-thphys.physics.ox.ac.uk/people/Alexander.Schekochihin/](http://www-thphys.physics.ox.ac.uk/people/Alexander.Schekochihin/notes/)
notes/~~homework~~) and write the answer:

$$\frac{Df_s}{Dt} + \frac{1}{B} \frac{DB}{Dt} \frac{w_L}{2} \frac{\partial f_s}{\partial w_L} + \left(\frac{e_s}{m_s} E_{||} - \frac{D \vec{U}_s}{Dt} \cdot \vec{b} - \frac{w_L^2}{2} \frac{\nabla_{||} B}{B} \right) \frac{\partial f_s}{\partial w_{||}} = C[f_s]$$

(B)

This is not terribly transparent and it is perhaps better to write this equation in different, "more physical" variables:

$$\text{let } f_s(w_{\perp}, w_{\parallel}) = F_s(\mu, \varepsilon)$$

where $\mu = \frac{m_s w_{\perp}^2}{2B}$ magnetic moment of a gyrating particle - 1st adiabatic invariant, conserved when $\omega \ll \Omega_s$!

[physically, angular momentum:

$$m_s w_{\perp} r = m_s w_{\perp}^2 \propto \frac{w_{\perp}^2}{B}$$

and $\varepsilon = \frac{m_s w^2}{2} = \frac{m_s}{2} (w_{\perp}^2 + w_{\parallel}^2)$ "internal" energy of the particle

Since μ is conserved, F_s satisfies

$$\boxed{\frac{DF_s}{Dt} + \left[m_s w_{\parallel} \left(\frac{e_s}{m_s} E_{\parallel} - \frac{D\vec{U}_s \cdot \vec{b}}{Dt} \right) + \mu \frac{dB}{dt} \right] \frac{\partial F_s}{\partial \varepsilon} = C[F_s]} \quad (19)$$

$\frac{d}{dt} + \vec{w} \cdot \vec{\nabla}$

acceleration by parallel electric field
 $e_s w_{\parallel} E_{\parallel}$ - work done on the particle

this takes account of the fact that ε does not include the mean part of velocity

"betatron acceleration" due to μ conservation:
 $\varepsilon = \mu B + \frac{m_s w_{\parallel}^2}{2}$
 $\dot{\varepsilon} = \mu \dot{B}$

E_{\parallel} is determined by imposing $\sum_s e_s n_s = 0$

Exercise. Derive this

from Eq. (18) [but I hope eq. (19) is intuitive enough to be believable on its own]

S3'. Mirror Instability Cont'd.

We are now ready to return to eq. (16) and calculate

$$S_{p\perp} = \int d^3 \vec{w} \frac{m_s w_\perp^2}{2} \delta f_s(w_\perp, w_\parallel)$$

get this by calculating
 $F_s(\mu, \epsilon)$ and transforming
to w_\perp, w_\parallel

There is a cute subtlety here: our macroscopic state, around which we are expanding [$\delta(\text{everything})=0$] is

$$F_{os}(\mu, \epsilon) = f_{os}(w_\perp, w_\parallel) = f_{os}\left(\sqrt{\frac{2B\mu}{m_s}}, \sqrt{\frac{2(\epsilon - \mu B_0)}{m_s}}\right)$$

Now perturb everything:

this contains B_0 , the unperturbed field.

$$\begin{aligned} F_s(\mu, \epsilon) &= F_{os}(\mu, \epsilon) + \delta F_s \\ &= f_{os}(w_\perp, w_\parallel) + \delta f_s \end{aligned}$$

at μ now contains $B_0 + \delta B$! and this has to be taken into account when transforming to w_\perp, w_\parallel variables

$$f_{os}\left(\sqrt{\frac{2\mu(B_0 + \delta B)}{m_s}}, \sqrt{\frac{2[\epsilon - \mu(B_0 + \delta B)]}{m_s}}\right)$$

$$= f_{os}\left(\sqrt{\frac{2\mu B_0}{m_s} + \frac{2\mu}{m_s} \delta B}, \sqrt{\frac{2(\epsilon - \mu B_0)}{m_s} - \frac{2\mu}{m_s} \delta B}\right)$$

$$\approx f_{os}\left(\sqrt{\frac{2\mu B_0}{m_s}}, \sqrt{\frac{2(\epsilon - \mu B_0)}{m_s}}\right) + \underbrace{\frac{2\mu}{m_s} \delta B}_{w_\perp^2 \frac{\delta B}{B}} \left(\frac{\partial f_{os}}{\partial w_\perp^2} - \frac{\partial f_{os}}{\partial w_\parallel^2}\right)$$

$$F_{os}(\mu, \epsilon)$$

$$\text{Thus, } \delta f_s = \delta F_s - w_\perp^2 \frac{\delta B}{B} \left(\frac{\partial f_{os}}{\partial w_\perp^2} - \frac{\partial f_{os}}{\partial w_\parallel^2}\right) \quad (20)$$

if $f_{os} = \frac{n_s}{\pi^{3/2} v_{th, \perp S} v_{th, \parallel S}^{1/2}} e^{-\frac{w_\perp^2}{v_{th, \perp S}^2} - \frac{w_\parallel^2}{v_{th, \parallel S}^2}}$ bi-Maxwellian,

then this is $-\left(\frac{1}{v_{th, \perp S}^2} - \frac{1}{v_{th, \parallel S}^2}\right) f_{os} = -\frac{m_s n_s}{2} \left(\frac{1}{p_{\perp S}} - \frac{1}{p_{\parallel S}}\right) f_{os}$

This gives us

$$\delta P_{\perp S} = \int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} \delta f_S = \int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} \delta F_S$$

$$- \int d^3 \vec{w} \frac{m_s w_{\perp}^4}{2} \left(\frac{\partial f_{OS}}{\partial w_{\perp}^2} - \frac{\partial f_{OS}}{\partial w_{\parallel}^2} \right) \frac{\delta B}{B} =$$

by parts

$$2\pi \int dw_{\perp} w_{\perp} \int dw_{\parallel} \frac{m_s w_{\perp}^4}{2} \left(\frac{\partial f_{OS}}{\partial w_{\perp}^2} - \frac{\partial f_{OS}}{\partial w_{\parallel}^2} \right) =$$

$$= -2 \int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} f_{OS} * - \int d^3 \vec{w} \frac{m_s w_{\perp}^4}{2} \frac{\partial f_{OS}}{\partial w_{\parallel}^2}$$

P_{IS}

$\frac{-2 P_{IS}^2}{P_{IS}}$ if f_{OS} is bi-Maxwellian
(times some coefficient d_s
of order 1 if it's not)

$$= \int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} \delta F_S + \frac{\delta B}{B} \left(2P_{IS} - \frac{2P_{IS}^2}{P_{IS}} d_s \right) \quad (21)$$

Get this by linearising and Fourier-transforming eq.(19) :

$$-i(\omega - k_{\parallel} w_{\parallel}) \delta F_S = - \left[m_s w_{\parallel} \left(\frac{e_s}{m_s} E_{\parallel} - i(\omega - k_{\parallel} w_{\parallel}) \delta U_{\parallel S} * - i \omega \mu \delta B \right) \frac{\partial f_{OS}}{\partial \epsilon} \right] \quad (\text{ignore collisions, } \omega \gg \nu)$$

$$\delta F_S = -i \frac{w_{\parallel} e_s E_{\parallel}}{\omega - k_{\parallel} w_{\parallel}} \frac{\partial f_{OS}}{\partial \epsilon} \quad \text{and so this term will not contribute to } \delta P_{\perp S} \quad \left(\frac{\partial f_{OS}}{\partial w_{\parallel}} \right)'' \text{ (exercise)}$$

This term can be ignored
if $\frac{1}{\beta} \sim \Delta \ll 1$

(otherwise get E_{\parallel} by imposing $\sum_s \delta f_{Ss} = 0$)

$\frac{\partial f_{OS}}{\partial w_{\parallel}}$ and so this term will not contribute to $\delta P_{\perp S}$ (integrates to 0)

$$\frac{w_{\perp}^2}{2} \frac{\delta B}{B} \frac{1}{w_{\parallel}} \frac{\partial f_{OS}}{\partial w_{\parallel}} = w_{\perp}^2 \frac{\delta B}{B} \frac{\partial f_{OS}}{\partial w_{\parallel}^2}$$

Thus, the "relevant" part of δF_S is

$$\delta F_S = - \frac{\omega}{\omega - k_{\parallel} w_{\parallel}} w_{\perp}^2 \frac{\delta B}{B_0} \frac{\partial f_{OS}}{\partial w_{\parallel}^2}$$

and its contribution to δP_{LS} is

$$\int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} \delta F_S = + \frac{\delta B}{B} \frac{\omega}{|k_{\parallel}|} \int \frac{dw_{\parallel}}{w_{\parallel} - \frac{\omega}{|k_{\parallel}|}} \left[\frac{\partial}{\partial w_{\parallel}^2} \int d^2 \vec{w}_{\perp} \frac{m_s w_{\perp}^4}{2} f_{OS} \right]$$

the usual Landau integral (if you don't know about these things, do learn! Here are some notes: <http://www-thphys.physics.ox.ac.uk/people/Alexander.Schekochihin/KT/2014> or read Landau's pages!)

The method of loop it reduces to the following prescription, called the Plemelj formula:

$$\frac{1}{w_{\parallel} - \frac{\omega}{|k_{\parallel}|}} \rightarrow \text{P} \frac{1}{w_{\parallel} - \frac{\omega}{|k_{\parallel}|}} + i\pi \delta(w_{\parallel} - \frac{\omega}{|k_{\parallel}|})$$

pole

principal value integral

$$\rightarrow S_{01} = \frac{\delta B}{B} \left[\frac{\omega}{|k_{\parallel}|} \cancel{\text{P} \int \frac{dw_{\parallel}}{w_{\parallel} - \frac{\omega}{|k_{\parallel}|}}} [\dots] + i\pi \frac{\omega}{|k_{\parallel}|} [\dots]_{w_{\parallel} = \frac{\omega}{|k_{\parallel}|}} \right]$$

assuming $\omega \ll k_{\parallel} v_{thS\parallel}$
can neglect this compared
to other terms in
in eq.(21)

must keep this
because it is the
lowest-order imaginary
bit which will give
us the instability

For a bi-Maxwellian,

$$\left[\frac{2}{\omega^2} \left(d^2 \vec{w}_\perp \frac{m_s w_\perp^4}{2} f_{0s} \right) \right]_{w_\parallel = \frac{\omega}{|k_\parallel|}} = - \frac{2 p_{\perp s}^2}{p_{\parallel s}} \frac{C - \frac{\omega^2}{k_\parallel^2 V_{th s\parallel}^2}}{\sqrt{\pi} V_{th s\parallel}} \approx 1$$

times a coefficient
 $\sigma_s \sim 1$
if not a
bi-Max.

Eq. (21) becomes

$$\delta p_{\perp s} = \frac{\delta B}{B} \left[2 p_{\perp s} - \frac{2 p_{\perp s}^2}{p_{\parallel s}} \left(\alpha_s + i \sqrt{\pi} \frac{\omega}{|k_\parallel| V_{th s\parallel}} \sigma_s \right) \right]$$

This goes into eq. (16):

$$\cancel{\rho \omega^2} = \cancel{k_\perp^2 \frac{B^2}{4\pi}} \left[\sum_s \left(1 - \frac{p_{\perp s}}{p_{\parallel s}} \alpha_s \right) \beta_{\perp s} - i \sum_s \frac{p_{\perp s}}{p_{\parallel s}} \beta_{\perp s} \sqrt{\pi} \frac{\omega}{|k_\parallel| V_{th s\parallel}} + 1 \right]$$

neglect because $\omega \ll k_\parallel V_{th s\parallel}$

$$+ k_\perp^2 \frac{B^2}{4\pi} \left[\sum_s \frac{\beta_{\perp s}}{2} \left(1 - \frac{p_{\perp s}}{p_{\parallel s}} \right) + 1 \right]$$

Can discard
 $s = e$ term
because
 $V_{th e\parallel} \gg V_{th i\parallel}$

$$\sigma_i \frac{p_{\perp i}}{p_{\parallel i}} \beta_{\perp i} \sqrt{\pi} \frac{\gamma}{|k_\parallel| V_{th i\parallel}} = \sum_s \left(\frac{p_{\perp s}}{p_{\parallel s}} \alpha_s - 1 \right) \beta_{\perp s} - 1 \equiv \Delta \text{ inst. parameter}$$

$$- \frac{k_\perp^2}{k_\parallel^2} \left[\sum_s \frac{\beta_{\perp s}}{2} \left(1 - \frac{p_{\perp s}}{p_{\parallel s}} \right) + 1 \right] \quad (22)$$

instability if this is positive:

$$\sum_s \left(\frac{p_{\perp s}}{p_{\parallel s}} \alpha_s - 1 \right) \beta_{\perp s} > 1$$

Examining where this came

from, we see that this amounts to δp_\perp

modifying the magnetic pressure force and turn it
from positive to negative:

$$\delta p_\perp + \frac{B \delta B}{4\pi} = \frac{\delta B B}{4\pi} \left[\cancel{1 - \sum_s \left(\frac{p_{\perp s}}{p_{\parallel s}} \alpha_s - 1 \right) \beta_{\perp s} + \dots} \right]$$

mag. pressure non-resonant particle pressure (i.e. resonant particle pressure)

This happens because at $\Delta S > 0$,
there are more $w_{\perp} > w_{\parallel}$ particles
and this gives worse $\delta p_{\perp} \ll \delta p_{\parallel}$ when SB occurs in the troughs)

Thus, fundamentally, pressure anisotropy makes it easier to compress or rarefy magnetic field — and things become unstable when the sign of the pressure flips and it becomes energetically profitable to create compressions and rarefactions:



This is a destabilized slow wave
(but aperiodic, $\omega = 0$)

~~The dispersion relation (22)~~ The dispersion relation (22) is basically a statement of pressure balance.

The balance is between magnetic pressure, the non-res. particle pressure δp_{\perp} and the resonant particle pressure $\propto \gamma$, which came from the "location acceleration" $\mu \frac{dB}{dt}$ in eq. (19).

This term refers to what happens in the stable case.

When magnetic pressure opposes formation of SB perturbations (say, troughs), to compensate it, we must have $\gamma < 0$ and energy goes from ~~SB~~ to ^{resonant} particles, which are accelerated (by the mirror force).

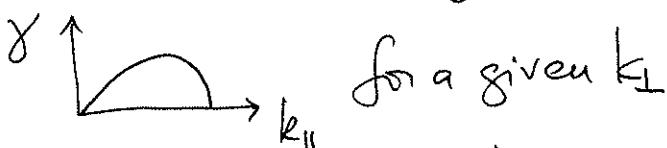
The corresponding damping of SB is the well-known

Barnes damping (Landau damping of "mirror field")

[Barnes 1966 — also known as "transit-time damping", see Stix's book]

Conversely, when mag. pressure is effectively negative, δB grows ($\gamma > 0$) while particles decelerate - energy is transferred from the resonant particles into δB . This is the mirror instability.

OK, to finish the job, note that, from eq.(22),



for a given k_{\perp}

$$\left(\frac{\partial \gamma}{\partial k_{\parallel}}\right)_{k_{\perp}} \propto \Lambda - 3 \frac{k_{\parallel}^2}{k_{\perp}^2} \left[\sum_s \frac{\beta_{1s}}{2} \left(1 - \frac{P_{1s}}{P_{1s}} \right) + 1 \right] = 0$$

and so,

$\frac{1}{3}\Lambda$ at maximum

$$\boxed{\gamma = \frac{|k_{\parallel}| V_{thi} k_{\parallel}}{\sqrt{\pi}} \frac{2}{3} \Lambda \frac{P_{1i}}{P_{1i}} \frac{1}{\sigma_i \beta_{1i}}} \quad (24)$$

NB: We have assumed $\gamma \ll k_{\parallel} V_{thi} k_{\parallel}$, which is indeed true if ~~we are close to marginality~~

$$\text{and so } \Lambda \frac{1}{\beta_{1i}} = \left(\sum_s \Delta_s \beta_{1s} - 1 \right) \frac{1}{\beta_{1i}} \ll 1$$

So our approximations are consistent.

$$\Delta_s \sim \frac{1}{\beta} \ll 1$$

was assumed $\beta \gg 1$

NB: $\frac{k_{\parallel}}{k_{\perp}} \sim \sqrt{\Lambda} \ll 1$ if close to marginal,

so mirror modes are highly oblique near the threshold.

Further key points:

- $\gamma \propto k_{\parallel}$ so UV catastrophe again.

Again we have a fast, microscale instability whose peak growth is outside KMD.

Including FLR gives

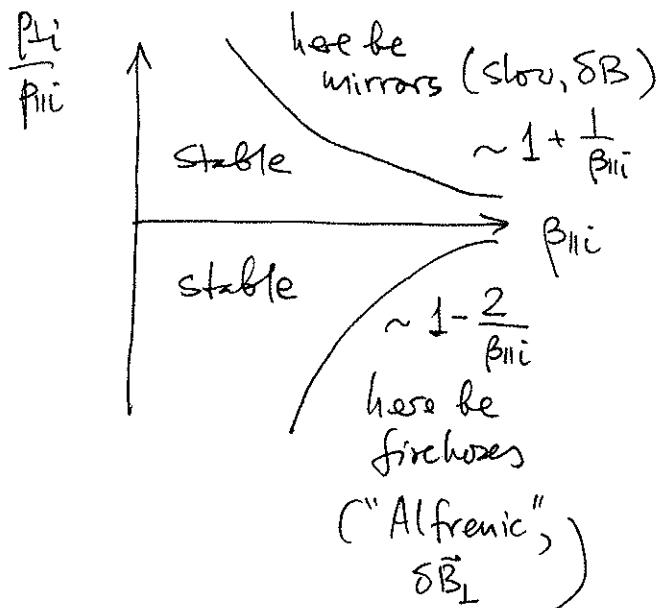
$$\gamma_{\text{peak}} \sim (\Delta - \frac{1}{\beta})^2 \beta S_i, \quad k_{\text{peak}} f_i \sim (\Delta - \frac{1}{\beta}) \beta$$

[See Hellinger 2007 PoP [4, 082105]]

Thus, any ^{high- β} macroscopic solution of KMD with $p_{\perp} > p_{\parallel}$. ~~will develop reconnection~~ with blow up (just like was the case for $p_{\parallel} > p_{\perp}$) — and again, what happens next depends on how mirror saturates!

- Ignoring Δ_e ($\ll \Delta_i$; usually, as we will see), the mirror instability condition is

$$\frac{p_{\perp i}}{p_{\parallel i}} - 1 > \frac{1}{\beta_{\perp i}} = \frac{1}{\beta_{\parallel i}} \frac{p_{\parallel i}}{p_{\perp i}} \Rightarrow \frac{p_{\perp i}}{p_{\parallel i}} \left(\frac{p_{\parallel i}}{p_{\perp i}} - 1 \right) > \frac{1}{\beta_{\parallel i}}$$



The solar wind indeed seems to stay within these boundaries: Hellinger et al 2006, Bale et al 2009

(they see predominantly δB fluctuations at the mirror boundary and δB_{\perp} at the firehose boundary!)

§4. Origin of Pressure Anisotropy

OK, so the bottom line so far is that any macroscopic KMD solution that has $p_{\perp} \neq p_{\parallel}$ (more precisely, $\frac{|p_{\perp} - p_{\parallel}|}{P} \geq \frac{1}{\beta}$) will be violently unstable to either firehose or mirror — both of which are fast and micro-scale modes giving rise to ~~to~~ fluctuations outside the KMD regime (and, by the way, also outside gyrokinetics — too close to cyclotron frequency, ~~and~~ k_{\parallel}/k_{\perp} not small enough, nonlinearly also $\delta B/B$ not small enough).

How worried should this make us about applicability of KMD (and, indeed, MHD ~~or HD~~) to high- β (astrophysical) plasmas that are not ~~fast~~ collisional enough (^{not to be fully fluid} i.e. for which $v \ll \Omega_s$)?

Answer: VERY WORRIED! (Be afraid, Be very afraid...)

As I already indicated (p.12), a key property of low-frequency, weakly collisional dynamics is that

$$\mu = \frac{m_s w_{\perp}^2}{2B} \text{ is conserved by particles}$$

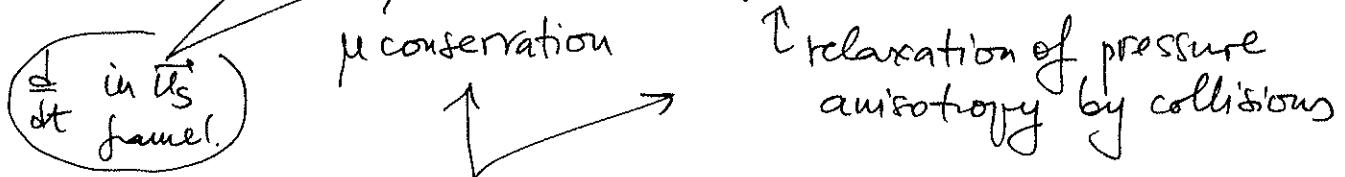
The mean μ of particles of species s is

$$\langle \mu \rangle_w = \frac{1}{n_s} \int d^3 \vec{w} \mu f_s = \frac{P_{\perp s}}{n_s B} = \text{const} \quad (25)$$

For the purposes of a qualitative discussion, let us pretend for a moment that $n_s = \text{const}$ (incompressible plasma, $\beta \gg 1$). Then the above conservation relation says that, locally in a fluid element (\vec{w} is peculiar velocity!), every time you change B , you must change $P_{\perp s}$ proportionally (but not $P_{\parallel s}$)

Thus, we expect

$$\frac{1}{P_{\perp s}} \frac{dP_{\perp s}}{dt} \sim \frac{1}{B} \frac{dB}{dt} - v_s \frac{P_{\perp s} - P_{\parallel s}}{P_{\perp s}} \quad (26)$$



Balancing these effects,

$$\Delta_s = \frac{P_{\perp s} - P_{\parallel s}}{P_{\perp s}} \sim \frac{1}{v_s} \frac{1}{B} \frac{dB}{dt} \quad (27)$$

[this expression is valid
only if $\Delta_s \ll 1$, i.e.
 $v_s \gg \frac{1}{B} \frac{dB}{dt}$, otherwise]

Δ_s will grow with time
as B is changed]

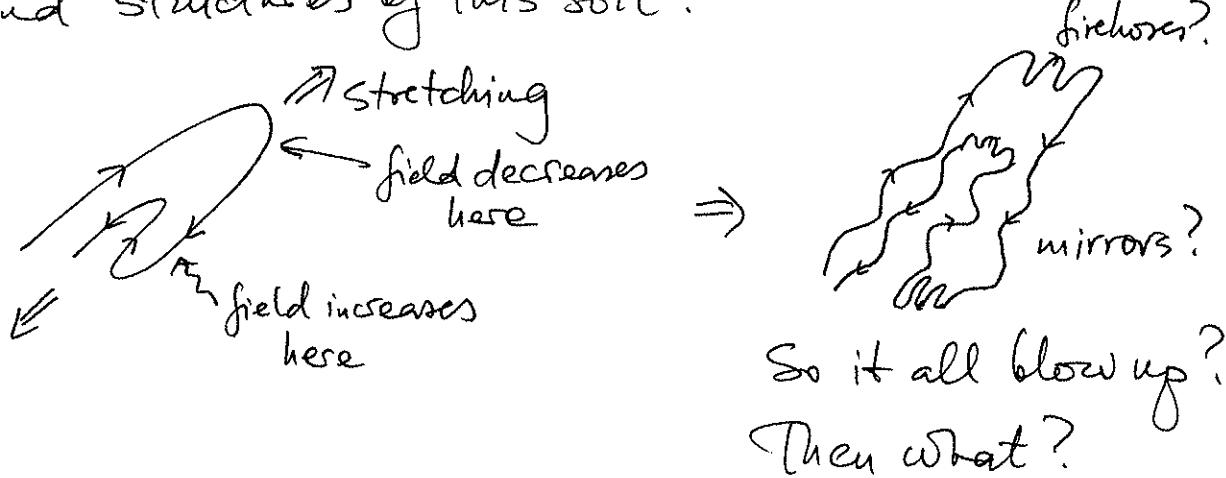
Thus, B increases locally $\Rightarrow \Delta_s > 0 \Rightarrow$ mirror

B decreases locally $\Rightarrow \Delta_s < 0 \Rightarrow$ firehose

If nearly any large-scale dynamics involves local changes in B , this means that nearly any macroscopic solution of KMHD in the high- β regime will be unstable!

A very good example is the dynamo problem:

When magnetic field is randomly stretched by turbulence, leading (in MHD) to exponential growth of magnetic energy (and, eventually to saturated fields we observe), locally one finds structures of this sort:



Generally speaking, in order to understand long-time evolution, we need some sort of mean-field theory for the large-scale effect of the microscale instabilities on the dynamics.

Presumably, this is to keep pressure anisotropy marginal wrt the instabilities (as indeed appears to be confirmed by the solar wind

measurements — see Bde et al (2009 etc.)

There are two ways in which this can happen

- firehose and mirror fluctuations might scatter particles, leading to higher effective collisionality and thus control the pressure anisotropy

$$-\frac{2}{\beta} \lesssim \frac{P_{\perp} - P_{\parallel}}{P} \lesssim \frac{1}{\beta}$$

$\frac{1}{\sqrt{s}} \left(\frac{1}{B} \frac{dB}{dt} \right)$

increase collisions

- they might inhibit ~~a change of~~ of B , which is another way of keeping Δ under control.

Which of these operates matters for dynamics.

{ See ^{speculative} overview of the possible consequences of either mechanism in MNRAS 440, 3226 (2014);

nonlinear firehose: Rosin et al MNRAS 413, 7 (2011)

nonlinear mirror: Rincon et al MNRAS 497, L45 (2015)

PIC simulations : Kunz et al PRL 112, 205003 (2014)

[... and follow the reference trail from there]

I will show these results if there is time.

Some relevant papers and a ppt presentation on nonlinear results are here:

http://www-thphys.physics.ox.ac.uk/people/Alexander.Schekochihin/notes/LES_HOUCHES15

Remark 1.

$$\frac{d\vec{B}}{dt} = \vec{B} \cdot \nabla \vec{u} - \vec{B} \times \vec{u}$$

↓

assume incompressibility

$$\frac{1}{B} \frac{d\vec{B}}{dt} = \vec{B} \vec{B} : \nabla \vec{u}$$

Then, from eq. (27),

$$p_{\perp s} - p_{\parallel s} \sim \left(\frac{p_s}{v_c} \right) \vec{B} \vec{B} : \nabla \vec{u} \quad (28)$$

Putting this back into eq. (B), you will get the lowest order Braginskii MHD equation.

So, from the large-scale point of view, pressure anisotropy is viscous stress — but the resulting equations are ill posed (blow up via instabilities with $\propto k_{\parallel}$).

Remark 2. More rigorously, eq. (26) can be obtained via "CSL equations" — the evolution equations of $p_{\perp s}$ and $p_{\parallel s}$. Namely [exercise]:

$$\int d^3 \vec{w} \frac{m_s w_{\perp}^2}{2} \text{ Eq. (18)} :$$

conservation of μ

↑ see notes cited on p. II for details

$$p_{\perp s} \frac{d}{dt} \ln \left(\frac{p_{\perp s}}{n_s B} \right) = - \nabla \cdot (q_{\perp s} \hat{b}) - q_{\perp s} \nabla \cdot \hat{b} - v_s (p_{\perp s} - p_{\parallel s})$$

$$\int d^3 \vec{w} m_s w_{\parallel}^2 \text{ Eq. (18)} :$$

using simplified coll. operator

(29)

$$p_{\parallel s} \frac{d}{dt} \ln \left(\frac{p_{\parallel s} B^2}{n_s^3} \right) = - \nabla \cdot (q_{\parallel s} \hat{b}) + 2q_{\parallel s} \nabla \cdot \hat{b} - 2v_s (p_{\parallel s} - p_{\perp s})$$

2 conservation of $J = \oint dl w_{\parallel}$ "bounce invariant"

The new feature here is heat fluxes:

$$q_{\perp s} = \int d^3 \vec{w} \frac{m_s w_\perp^2}{2} w_\parallel f_s, \quad q_{\parallel s} = \int d^3 \vec{w} m_s w_\parallel^3 f_s$$

$\propto D_{\parallel\perp} T_s$ in the
(collisional approx.)

They are here because particles can flow in and out of a fluid element and thus affect the conservation (or otherwise) of $\langle \mu \rangle_w$ and $\langle J^2 \rangle_w$ within it.

Finally, from eqs. (29-30),

$$\frac{d}{dt} (p_{\perp s} - p_{\parallel s}) = (p_{\perp s} + 2p_{\parallel s}) \frac{1}{B} \frac{dB}{dt} + (p_{\perp s} - 3p_{\parallel s}) \frac{1}{n_s} \frac{dn_s}{dt}$$

$$- \nabla \cdot [(q_{\perp s} - q_{\parallel s}) \hat{b}] - 3q_{\perp s} \nabla \cdot \hat{b} + 3v_s (p_{\perp s} - p_{\parallel s})$$



cf. eq. (27)

balance this if $v_s \gg \omega$
(and $p_{\perp s} - p_{\parallel s} \ll p_s$)

$$\Delta_s = \frac{p_{\perp s} - p_{\parallel s}}{p_s} \approx \frac{1}{v_s} \left\{ \frac{1}{B} \frac{dB}{dt} - \frac{2}{3} \frac{1}{n_s} \frac{dn_s}{dt} \right\}$$

compressions and
tensions lead
to anisotropy

$$- \nabla \cdot [(q_{\perp s} - q_{\parallel s}) \hat{b}] + \frac{3q_{\perp s} \nabla \cdot \hat{b}}{3p_s}$$

typically,
dominant
here are
electron
heat fluxes

So heat fluxes can also lead to anisotropies and so macroscopic solutions of KMTD involving temperature gradients will also go unstable at microscales!

So here we are, we can't change B at large scales, we can't compress/vary the plasma and we can't have temperature gradients without having to deal everything exploding and needing new equations. Enjoy!