# Models of magnetic field evolution and effective viscosity in weakly collisional extragalactic plasmas

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# ABSTRACT

In weakly collisional plasmas such as the intracluster medium (ICM), the viscous stress and the rate of change of the magnetic energy are proportional to the local pressure anisotropy, so subject to constraints imposed by the pressure-anisotropy-driven microinstabilities (mirror and firehose) and controlled by the local instantaneous plasma  $\beta$ . The dynamics of such plasmas can be dramatically different from a conventional magnetohydrodynamic fluid. The plasma is expected to stay locally marginal with respect to the instabilities, but how it does this remains an open question. Two models of magnetic field evolution are investigated. In the first, marginality is achieved via suppression of the rate of change of the field. In the second, the instabilities give rise to anomalous collisionality, reducing pressure anisotropy to marginal – at the same time decreasing viscosity and so increasing the turbulent rate of strain. Implications of these two models are studied in a simplified zero-dimensional setting. In the first model, the field grows explosively but on a time-scale that scales with the initial  $\beta$ , while in the second, dynamical field strength can be reached in one large-scale turbulence turnover time regardless of the initial seed. Both models produce very intermittent fields. Both also suffer from fairly strong constraints on their applicability: for typical cluster-core conditions, scale separation between the fluid motions (with account of suppressed viscous stress) and the miscoscale fluctuations break down at  $\beta \sim 10^4 - 10^5$ . At larger  $\beta$  (weaker fields), a fully collisionless plasma dynamo theory is needed to justify field growth from a tiny primordial seed. However, the models discussed here are appropriate for studying the structure of the currently observed field as well as large-scale dynamics and thermodynamics of the magnetized ICM or similarly dilute astrophysical plasmas.

**Key words:** dynamo – magnetic fields – plasmas – turbulence – galaxies: clusters: intracluster medium.

# **1 INTRODUCTION**

Both analytical theory and numerical modelling of the large-scale dynamics of extragalactic plasmas present conceptual challenges that are more serious than merely constraints of numerical resolution or analytical tractability. One of the most intriguing of these challenges is understanding what happens in a weakly collisional plasma when a dynamically small magnetic field is stretched and tangled by the plasma flows – a process that is both interesting in itself, in the context of the genesis of the magnetic fields ubiquitously observed in the Universe, and integral to any large-scale fluid dynamics of astrophysical plasmas.

We call a plasma *weakly collisional* and *magnetized* when, on the one hand, the collision frequency in it exceeds the typical frequencies associated with fluid motions, waves or instabilities, but, on the other hand, it is much smaller than the Larmor frequency of the plasma's constituent ions and electrons gyrating around the magnetic field (Balbus 2004 calls such plasmas *dilute*). While this regime requires some magnetic field to be present, this field by no means needs to be dynamically significant. As an example, consider the intracluster medium (ICM) in the cores of galaxy clusters. The ratio of the ion collision frequency  $v_{ii}$  to the ion Larmor frequency  $\Omega_i$ , when referred to the conditions typical of this environment (taken from Rosin et al. 2011) turns out to be

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$$\frac{\nu_{ii}}{\Omega_i} \sim \left(\frac{B}{10^{-17} \,\mathrm{G}}\right)^{-1},\tag{1}$$

where *B* is the magnetic field strength. Thus,  $B \sim 10^{-17}$  G (corresponding to the ratio of the thermal to magnetic energy  $\beta = 8\pi p/B^2 \sim 10^{24}$ ) is sufficient for the plasma to be magnetized, but magnetic field is not strong enough to affect plasma motions via the Lorentz force until  $B \sim 10^{-6}$  G ( $\beta \sim 10^2$ ), a value at which the magnetic-energy density is comparable to the kinetic-energy density of the turbulent plasma flows at the viscous scale (Schekochihin & Cowley 2006). Note that this is close to the values of *B* measured in cluster cores (e.g., Carilli & Taylor 2002; Govoni & Feretti 2004; Vogt & Enßlin 2005).

In a weakly collisional magnetized plasma, the magnetic moment of each particle,  $\mu = v_{\perp}^2/B$  (where  $v_{\perp}$  is the peculiar velocity of the particle's Larmor motion), is conserved on the time-scales shorter than the collision time. Therefore, as the magnetic field, which is frozen into the turbulent flow and, being dynamically weak, might appear to be entirely at its mercy, is stretched and tangled, its strength changes (the field gets larger in some places, weaker in others) and pressure anisotropies develop. If we ignore heat fluxes and assume that flows are subsonic (therefore, incompressible), the local pressure anisotropy is<sup>1</sup>

$$\Delta \equiv \frac{p_{\perp} - p_{\parallel}}{p} \approx \frac{1}{\nu_{ii}} \frac{1}{B} \frac{\mathrm{d}B}{\mathrm{d}t},\tag{2}$$

where  $p_{\perp}$  ( $p_{\parallel}$ ) is the perpendicular (parallel) pressure with respect to the local direction of the field. The rate of change of *B* can be related to the local plasma flow velocity *u* according to

$$\frac{\mathrm{d}B}{\mathrm{d}t} = (\hat{b}\hat{b}: \nabla u)B \equiv \gamma B \tag{3}$$

(this follows from the magnetohydrodynamic, MHD, induction equation). Here  $d/dt = \partial_t + \boldsymbol{u} \cdot \nabla$  and  $\hat{\boldsymbol{b}} = \boldsymbol{B}/\boldsymbol{B}$ , so  $\gamma = \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} : \nabla \boldsymbol{u}$  is the local field-stretching rate. All fluid frequencies,  $\gamma$  amongst them, are taken to be small compared to  $v_{ii}$ , so  $\Delta \ll 1$ . For typical core ICM parameters,  $\Delta \sim 0.01$  (Kunz et al. 2011; Rosin et al. 2011).

Even though  $\Delta$  is small, it turns out to be sufficient to render the ICM violently unstable to the firehose and mirror instabilities (see Schekochihin et al. 2005, and references therein), which have growth rates closer to  $\Omega_i$  than to  $v_{ii}$  (see Davidson & Völk 1968; Yoon, Wu & de Assis 1993; Hellinger & Matsumoto 2000; Hellinger 2007; Schekochihin et al. 2010 and Section 3.5) and thus can be viewed as instantaneous from the point of view of all collisional and fluid processes (i.e. all macroscale dynamics). The instabilities are quenched when

$$\Delta = \frac{\gamma}{\nu_{ii}} \in \left[-\frac{2}{\beta}, \frac{1}{\beta}\right],\tag{4}$$

where the lower threshold is for the firehose and the upper threshold for the mirror.<sup>2</sup> Thus, at large enough  $\beta$ , any change in the magnetic field leads to instabilities and in order to understand whether and how magnetic field can continue changing, we must account for the effect these instabilities have on the dynamics of the ICM. It is not currently clear precisely how these instabilities saturate, but it is clear that the result of their saturation will be that the pressure anisotropy averaged over scales smaller than those of the fluid motions will not stray beyond the marginal-stability boundaries (4) (the clearest evidence for this is found in the solar wind; see Kasper, Lazarus & Gary 2002; Hellinger et al. 2006; Bale et al. 2009). This suggests that large-scale dynamics of the ICM might perhaps be modelled in total ignorance of the microphysical complexities associated with the firehose and mirror saturation, simply by assuming that  $\Delta$  stays at most marginal, as per equation (4). There are, unfortunately, two very different ways in which this can be accomplished and which of them is correct depends on how the instabilities saturate.

In equation (4), by  $\Delta$  let us understand the *mean* pressure anisotropy averaged over the microscales at which the unstable fluctuations appear; similarly,  $\gamma = \hat{b}\hat{b} : \nabla u$  is the *mean* field-stretching rate. Whenever  $\Delta$  attempts to cross either of the boundaries (4), it can be reined in *either* via  $\gamma$  being effectively suppressed by the instabilities (Model I) *or* via the effective collision rate  $v_{ii}$  being enhanced by, say, anomalous particle scattering off the firehose or mirror fluctuations (Model II). The two models amount to two very different closures for the fluid equations and lead to very different physical consequences (see below).

After providing, in the remainder of this section (Section 1.1), some pointers to relevant previous literature pertaining to the justification and/or consequences of these two models, we will, in the rest of this paper, study, using very drastically simplified equations for the evolution of the magnetic field and the local rate of its stretching, what implications they might have for the field growth and its spatial distribution. A 'zero-dimensional-dynamo' paradigm that will be the basis for our investigation of the two models will be introduced in Section 2. Model I will be studied in Section 3 and Model II in Section 4. The results for each of these will be summarized at the end of the section devoted to it and a more global discussion given in Section 5.

#### 1.1 Current status of the two models

The suppressed-stretching-rate model (Model I) was used by Kunz et al. (2011) in a theory of ICM thermal stability (this will be further discussed in Section 5). Various versions of the anomalousscattering model (Model II) have been used in the theory of explosive dynamo (Schekochihin & Cowley 2006), simulations of accretion flows (Sharma et al. 2006, 2007), anisotropic heat-conduction instabilities (Kunz et al. 2012) and of turbulent dynamo (Santos-Lima et al. 2014) as well as a number of simulations and models of space plasmas (Samsonov et al. 2001, 2007; Chandran et al. 2011; Meng et al. 2012). In Section 5.4, we will explain why, in light of the analysis of Section 4, adequate numerical implementation of Model II may be harder than it appears.

We would like to avoid a lengthy discussion of the relative microphysical merits of the two models but it is perhaps worthwhile to outline the state of play as we see it. Traditionally, modelling results and observational evidence (in the solar wind) have been interpreted in terms of anomalous scattering of particles pinning the pressure anisotropy at the instability thresholds (e.g. Gary et al. 1997, 1998; Gary, Yin & Winske 2000; Bale et al. 2009). Some doubt is, however, cast on the possibility of enhanced scattering at the mirror threshold by the fact that near marginal stability, the mirror fluctuations have scales much larger than the Larmor radius and growth rates much smaller than the Larmor frequency (Davidson

<sup>&</sup>lt;sup>1</sup> This follows from the so-called CGL equations (Chew, Goldberger & Low 1956), with collisions retained (see, e.g. Schekochihin et al. 2010). Besides  $\mu$  conservation, these equations are also an expression of the conservation of the so-called longitudinal invariant (e.g. Kulsrud 1983; Quest & Shapiro 1996), which is related to the bounce invariant of particles trapped in the local inhomogeneities of the magnetic field strength.

 $<sup>^2</sup>$  For the mirror, this is only an approximate bound assuming cold electrons and bi-Maxwellian ions (Hellinger 2007) – approximations that are generally incorrect quantitatively but give a threshold that is simple enough and close enough to the truth to be useful in a qualitative discussion.

& Völk 1968; Hellinger 2007) and so should not be able to break the conservation of the first adiabatic invariant. Similar reasoning (fluctuation scales too large to break adiabatic invariance) holds for parallel firehose fluctuations – but the oblique firehose does produce Larmor-scale fluctuations (Yoon et al. 1993; Hellinger & Matsumoto 2000).

A weakly non-linear theory of the parallel firehose (Schekochihin et al. 2008; Rosin et al. 2011) finds a mechanism for pinning  $\Delta$  at marginal stability that relies on effective cancellation of the mean rate of change of the magnetic field strength rather than on anomalous scattering (see Section 3.1). An example of magnetic field evolution in a Braginskii plasma where marginality with respect to the firehose is maintained via modification of the flow field was recently found by Melville & Schekochihin (2014). A no-scattering mechanism for maintaining marginal  $\Delta$  has also been proposed for the mirror, with a special role assigned to trapped particles in setting up magnetic 'holes' that compensate for the mean field growth (Pantellini, Burgess & Schwartz 1995; Kivelson & Southwood 1996; Pantellini 1998; Schekochihin et al. 2008; Rincon, Schekochihin & Cowley 2014; cf. Califano et al. 2008; Pokhotelov et al. 2010).

Recent PIC simulations of a shearing, high- $\beta$  plasma by Kunz, Schekochihin & Stone (2014) appear to support this line of thinking for the mirror threshold (no scattering, non-linear evolution broadly in agreement with Schekochihin et al. 2008; Rincon et al. 2014) – except possibly in cases of very strong drive (see also Riquelme, Quataert & Verscharen 2014) or at late stages of the non-linear evolution, which may or may not be relevant to a situation where the turbulent velocity field that drives the pressure anisotropy decorrelates every turnover time. Incidentally, both Kunz et al. (2014) and Riquelme et al. (2014) confirm that it is the mirror instability that dominates in high- $\beta$  regimes with positive pressure anisotropy, not the ion-cyclotron instability (cf. Gary et al. 1997, 2000). Note that solar-wind measurements also appear to support mirror over the ioncyclotron instability as controlling the positive-pressure-anisotropy threshold (Hellinger et al. 2006).

At the firehose threshold, Kunz et al. (2014) find evidence both of transient evolution reminiscent of the no-scattering theory (Schekochihin et al. 2008; Rosin et al. 2011) and of oblique fluctuations at Larmor scales vigorously breaking adiabatic invariance in their saturated non-linear state.

Thus, the state of play appears to be in flux and, with the complete theory or complete understanding lacking, we take the view that neither of our two models can as yet be ruled out – and that possible implications of either are worth investigating.

# 2 ZERO-DIMENSIONAL DYNAMO

We model the magnetic field evolution by a 'zero-dimensional' equation,

$$\partial_t B = \gamma B,$$
 (5)

where  $\gamma(t)$  is a random time-dependent stretching rate and we will interpret different realizations of  $\gamma$  as corresponding to different (strictly speaking, Lagrangian) spatial positions. In the absence of either dynamical back reaction by the field on the flow or of any plasma microphysical effects, the stretching rate is of order of the local rate of strain in the turbulent flow advecting the field. So we set  $\gamma = \sigma(t)$ , where  $\sigma$  is a scalar quantity representing this local rate of strain and modelled as an Ornstein–Uhlenbeck process:

$$\partial_t \sigma = -2\sigma_0 \sigma + 2\sigma_0^{3/2} \chi(t), \tag{6}$$

where  $\chi(t)$  is a unit Gaussian white noise,  $\langle \chi(t)\chi(t')\rangle = \delta(t - t')$ , and  $\sigma_0$  is the rms value of  $\sigma$  (in statistically steady state), which is also the decorrelation rate (see Appendix A; the factors of 2 are for future normalization convenience). In Kolmogorov turbulence, the largest rate of strain is associated with the motions *at the viscous scale* (because smaller scale 'eddies' have shorter turnover times) and so  $\sigma_0 \sim (\varepsilon/\mu)^{1/2}$ , where  $\varepsilon$  is the mean power injected into the turbulence (originating from large-scale driving mechanisms; in clusters, merger-excited instabilities, ejecta from active galactic nuclei, or galaxy wakes; see, e.g. Norman & Bryan 1999; Enßlin & Vogt 2006; Subramanian, Shukurov & Haugen 2006; Ruszkowski & Oh 2010) and  $\mu \sim p/v_{ii}$  is the dynamical viscosity of the ICM.<sup>3</sup> Thus, the first term on the right-hand side of equation (6) stands for viscous damping of the velocity and the second for the fresh input of turbulent power into the viscous-scale motions coming from the inertial range.

In the absence of further constraints, equations (5) and (6) lead to an ensemble of realizations of *B* with exponentially growing moments,  $\langle B^n \rangle \propto \exp(\sigma_0 n^2 t/2)$ , and a lognormal probability density,  $P(B) = B^{-1} \exp\left[-(\ln B)^2/2\sigma_0 t\right]/\sqrt{2\pi\sigma_0 t}$  (see Appendix A). This is similar to the standard properties of a kinematic dynamo in a one-scale stochastic velocity field (see Schekochihin et al. 2002b, 2004, and references therein). Thus, the magnetic energy grows exponentially but the field is quite intermittent (note that, unlike the magnetic energy, a typical realization of the field grows subexponentially,  $\ln B \sim \sqrt{\sigma_0 t}$ , so only a small fraction of the realizations contribute to the exponential growth of  $\langle B^2 \rangle$ ).

The field will grow until the Lorentz force is strong enough to affect the rate of strain. This happens when the magnetic-energy density becomes comparable to the kinetic-energy density of the motions at the viscous scale

$$\frac{B^2}{8\pi} \sim (\varepsilon\mu)^{1/2} \sim \frac{\sigma_0 p}{\nu_{ii}}.$$
(7)

Since magnetic energy grows exponentially at the rate  $\gamma = \sigma \sim \sigma_0$ , the dynamical strength (7) is achieved after the time

$$t_{\rm dyn} \sim \frac{1}{\sigma_0} \ln\left(\frac{\sigma_0 \beta_0}{\nu_{ii}}\right),$$
 (8)

where  $\beta_0 = 8\pi p/B_0^2$  is plasma  $\beta$  associated with the initial (seed) field.

Further growth of the field does occur after that, but requires modelling of its dynamical effect on the flow (see Schekochihin et al. 2002a, 2004; Cho et al. 2009; Beresnyak 2012). The field can typically grow in this non-linear regime by a factor of  $Re^{1/2}$ , where  $Re = \rho UL/\mu$  is the Reynolds number of the intracluster turbulence ( $\rho$  is the mass density and U the typical velocity at the outer scale L). With the viscosity based on the Coulomb collisionality  $v_{ii}$ , Re is not very much larger than unity in the ICM (Schekochihin & Cowley 2006; Kunz et al. 2011; Rosin et al. 2011), so the difference between the field given by equation (7) and the final saturated level is not very large (we will come back to these non-linear issues in Sections 3.4 and 4.2). Observationally, the  $B \sim 1-10 \,\mu\text{G}$  fields found ubiquitously in clusters (Carilli & Taylor 2002; Govoni & Feretti 2004; Vogt & Enßlin 2005) are quite close to the magnitude given by equation (7); this is also the field magnitude that gives marginal values of the pressure anisotropy (see equation 4).

The 'zero-dimensional-dynamo' paradigm is, of course, a gross simplification, not least because it says nothing of the spatial

<sup>&</sup>lt;sup>3</sup> In a magnetized plasma, this is the parallel Braginskii (1965) viscosity. While only the parallel component of the rate-of-strain tensor,  $\hat{b}\hat{b} : \nabla u$ , is damped by this viscosity and the perpendicular viscosity is much smaller, the parallel viscosity is the relevant one for the magnetic field strength evolution because only motions with  $\hat{b}\hat{b} : \nabla u \neq 0$  can change *B*.

structure of the field or of its direction relative to the flow and also ignores the effect of resistivity (or whatever other flux unfreezing mechanism turns out to be important in a weakly collisional plasma, another area of very poor current knowledge). However, it is a useful tool to explore what possible effect our two microphysical closure models might have on the evolution of the magnetic field.

# **3 MODEL I: SUPPRESSED STRETCHING RATE**

#### 3.1 Microphysical closure

In this approach, we assume that the effect of pressure-anisotropydriven instabilities is to suppress the stretching rate  $\gamma$  in equation (5) whenever the rate of strain  $\sigma$  becomes large enough to violate the mirror or firehose stability conditions. In order to model this, we stipulate that in equation (5),

if 
$$\sigma \ge 0$$
,  $\gamma = \min\left\{\sigma, \frac{\nu_{ii}}{\beta}\right\}$ , (9)

if 
$$\sigma < 0$$
,  $\gamma = \max\left\{\sigma, -\frac{2\nu_{ii}}{\beta}\right\}$ , (10)

i.e. while the collision rate  $v_{ii}$  is always the Coulomb collision rate and the rate of strain  $\sigma$  ranges freely, the pressure anisotropy  $\Delta$  and, therefore, the field-stretching rate  $\gamma$  (see equation 4) cannot cross the stability boundaries.

An example of how such a closure might be achieved microphysically is provided by the calculation by Rosin et al. (2011) of the non-linear evolution of the parallel firehose instability (the only analytical example we are aware of from which any inferences about an effective closure can be drawn). In that calculation, both the largescale magnetic field and the large-scale velocity field are perturbed by fast-growing (and oscillating), small-scale firehose fluctuations:  $B = B_0 + \delta B_{\perp}$ ,  $u = u_0 + \delta u_{\perp}$ . The mean rate of change of the magnetic field is

$$\frac{\frac{1}{B}\overline{\frac{\mathrm{d}B}{\mathrm{d}t}}}{=\overline{\hat{b}\hat{b}}:\nabla u} = \hat{b}_0\hat{b}_0:\nabla u_0 + \overline{\hat{b}_0\cdot(\nabla\delta u_\perp)\cdot\frac{\delta B_\perp}{B_0}}$$
$$= \frac{1}{B_0}\frac{\mathrm{d}B_0}{\mathrm{d}t} + \frac{1}{2}\frac{\partial}{\partial t}\frac{\overline{|\delta B_\perp|^2}}{B_0^2} \approx -\frac{2\nu_{ii}}{\beta}, \tag{11}$$

so the (fast) growth of the firehose fluctuations largely cancels the (slow) decrease of the large-scale field and keeps the mean rate of change marginal. This is achieved by perturbations both to the field and to the flow. However, while their combined effect on  $\gamma$  is dramatic, the effect of the perturbations just on the rate-of-strain tensor averages out:  $\overline{\nabla u} = \nabla u_0 + \overline{\nabla \delta u_\perp} = \nabla u_0$ . The rate of strain may be large, but the mean magnetic field does not feel it.

PIC simulations by Kunz et al. (2014) of shearing a mean magnetic field in a high- $\beta$  plasma provide a numerical example of how the mean rate of change of the magnetic field strength might be reduced (at least transiently) by both firehose and mirror fluctuations. Another such example, for a collisional magnetized (Braginskii) plasma at the firehose threshold, is reported by Melville & Schekochihin (2014).

#### 3.2 One-scale flow

Let us first consider a rather artificial situation in which the rate of strain  $\sigma$  is set without regard to microphysics by the model equation (6). This amounts to assuming a fixed viscosity determining

a definite cutoff scale for the turbulence and, therefore, a definite decorrelation rate,  $\sigma_0$ , for the rate of strain. In reality, the pressure anisotropy will have a dramatic effect on the viscosity of the ICM – we will take up this further complication in Section 3.4.

#### 3.2.1 Qualitative discussion

In any given realization, how large the field has managed to grow determines how strongly it can be further stretched. This means that in the realizations where the field is particularly strong, its growth (or decay) can also be faster. Conversely, in realizations where the field is weak, it can neither grow nor decay very fast because higher values of  $\beta$  constrain the rate of change of the field more stringently. This suggests a positive feedback mechanism: consider for a moment a realization where  $\sigma$  has managed to stay at the mirror threshold at all times:  $\sigma = B^2 v_{ii}/8\pi p$ . Then  $\partial_t B = (v_{ii}/8\pi p)B^3$  and the field growth is explosive:

$$B(t) = \frac{B_0}{\sqrt{1 - t/t_c}}, \qquad t_c = \frac{\beta_0}{2\nu_{ii}}.$$
 (12)

Thus, arbitrary field strength can be achieved in finite time. The condition (7) for the field to become dynamically important is reached at

$$t_{\rm dyn} = \frac{\beta_0}{2\nu_{ii}} - \frac{1}{2\sigma_0}.$$
 (13)

The first term here will typically be much larger than the second and so, in comparison with equation (8), this is a rather sluggish field-growth mechanism - not a surprising outcome as the field amplification rate is capped by the mirror threshold.

A further setback for the field growth in this scenario arises from the fact that not all realizations of the random stretching rate  $\gamma$  will manage to keep close to the mirror threshold for a time of order  $t_{\rm dyn}$  necessary to consummate the explosive growth. Every time  $\gamma$  strays into negative values and towards the firehose threshold, the field gets weaker, the maximum value of  $|\gamma|$  decreases and so recovery and growth become less likely. Furthermore, because of the asymmetry of the stable interval of stretching rates,  $\left[-2/\beta\right]$ ,  $1/\beta v_{ii}$ , the decrements in the magnetic energy produced by the negative values of the fluctuating rate of strain are on average twice as large as the increments produced by its positive values. Thus, there is a net tendency for field realizations to decay and any growth will have to come from the rare realizations at the mirror end of the distribution (see Appendix B for a quantitative discussion of this point; this tendency is in fact largely an artefact of the zerodimensional dynamo model and will be cured in Section 3.3).

The result of all these effects is that only a small fraction f of the realizations of the field will manage to grow to the dynamical level (i.e. the field will be very poorly space filling). To obtain a crude estimate of f, let us estimate the probability for a realization to stay at positive values of  $\sigma$  (and therefore with  $\gamma$  at or just below the mirror threshold) throughout the evolution from  $B_0$  to the dynamical strength (7). Since the decorrelation rate of  $\sigma$  is  $\sigma_0$ , the rate of strain has an opportunity to change sign roughly  $t_{dyn}\sigma_0$  times during the lifetime of such a successful realization and the probability for it to stay positive each time is 1/2. Therefore, the fraction of such realizations is

$$f \sim \left(\frac{1}{2}\right)^{t_{\rm dyn}\sigma_0} \sim \exp\left(-\frac{\sigma_0\beta_0}{\nu_{ii}}\right) \tag{14}$$

(this is derived in a more quantitative fashion in Appendix B2).

# 3.2.2 Numerical results

We non-dimensionalize our equations by letting  $M = B^2 v_{ii}/8\pi p\sigma_0 = v_{ii}/\sigma_0\beta$ ,  $\tau = 2\sigma_0 t$  and  $\tilde{\sigma} = \sigma/\sigma_0$ . Then the equations are

$$\partial_{\tau} M = \begin{cases} M^2, & \tilde{\sigma} > M, \\ \tilde{\sigma} M, & \tilde{\sigma} \in [-2M, M], \\ -2M^2, & \tilde{\sigma} < -2M, \end{cases}$$
(15)

$$\partial_{\tau}\tilde{\sigma} = -\tilde{\sigma} + \sqrt{2}\,\chi(\tau). \tag{16}$$

The dynamically-strong-field condition (7) converts into M = 1. This system depends on no parameters except the initial normalized magnetic energy  $M_0 = v_{ii}/\sigma_0\beta_0$ .

Equations (15) and (16) are quite straightforward to solve numerically. To obtain good statistics, a large number of realizations (typically  $N = 10^7$  to  $10^9$ ) were used and a parameter scan in the initial magnetic energy  $M_0$  was carried out; for each value of  $M_0$ , all realizations had  $M(\tau = 0) = M_0$ , i.e. the initial distribution of the magnetic energy was  $\delta(M - M_0)$ .

In Fig. 1 (left-hand panel), we show the time  $\tau_f$  it takes for a given fraction f of the realizations to achieve the dynamical level (7), M = 1. At  $M_0$  that is not too small, all  $\tau_f$  follow equation (13), which in dimensionless terms is simply  $\tau_{dyn} = 1/M_0 - 1$ . However, for each  $M_0$ , there is a maximum fraction of realizations  $f_{max}(M_0)$  that will ever reach M = 1, while the rest will decay, and so  $\tau_f \to \infty$  as  $f \to f_{max}$ . The inset in the left-hand panel of Fig. 1 shows that  $f_{max}$  follows the estimate (14) quite dutifully.

Examples of realizations that grow or decay are shown in Fig. 2. Note that the typical behaviour of a successful dynamo realization is explosive growth (on the time-scale  $\sim \tau_{dyn}$ ), preceded perhaps by a period of hesitation, in line with the qualitative discussion in Section 3.2.1. Fig. 1 (right-hand panel) quantifies these periods of hesitation in terms of the probability density function (PDF)  $P_1(\tau|M_0)$  of the time  $\tau$  it takes a realization to get from  $M = M_0$  to M = 1. The PDF has a peak at  $\tau \sim \tau_{dyn}$  and an exponential tail for  $\tau \gg \tau_{dyn}$  (see Appendix B2 for the derivation of this result).

The conclusion from the above is that getting to dynamically significant fields from small initial seeds is both a very slow and a very rare event in the scenario we have investigated. If a 'dynamo' is defined by the requirement of growth of mean magnetic energy



**Figure 2.** Examples of time evolution of growing and decaying realizations in Model I, equations (15) and (16) with  $M_0 = 0.1$ . Note the explosive episodes that take the field to dynamical level (M = 1).

 $\langle M \rangle$ , then this is clearly *not* a dynamo (in Appendix B, it is shown that both the typical realizations and all moments of *M* decay). However, there are several ways in which our treatment is in fact overly pessimistic and reasonable amendments to the model render it much more germane to magnetic field growth. These are discussed in the next two subsections.

#### 3.3 One-scale flow with a mean stretching rate

The main reason for the decay of the field in the above treatment is that the decay rate at the firehose threshold ( $\tilde{\sigma} = -2M$ ) is larger than the growth rate at the mirror threshold ( $\tilde{\sigma} = M$ ) while the probability for the rate of strain to stray beyond either of these thresholds is approximately the same when  $M \ll 1$ . This is in fact an artefact of the zero-dimensional dynamo model we have adopted (Section 2), in which it is hard-wired that  $\langle \sigma \rangle = 0$ . Therefore, in the absence of instabilities,  $\langle \gamma \rangle = 0$  and typical realizations neither grow nor decay on average,  $\langle \ln B \rangle = 0$ ; with the instabilities, this gives the field decay at firehose threshold a competitive edge. In 3D dynamo, this is certainly not the case: the growing field configures itself in such a way with respect to the rate-of-strain tensor that





**Figure 1.** Left-hand panel: times  $\tau_f$  for the fractions f = 10, 1, 0.1 and 0.01 per cent of all realizations to achieve dynamical field strength (M = 1; equation 7) in Model I, equations (15) and (16), plotted versus initial normalized field energy  $M_0 = v_{ii}/\sigma_0\beta_0$ . The dotted line is  $\tau_{dyn} = 1/M_0 - 1$  (equation 13). Inset: total fraction of realizations that ever make it to M = 1. The dotted line is the theoretical curve (B9). Right-hand panel: PDF of the time  $\tau$  it took those realizations to grow to M = 1. The case of  $M_0 = 0.05$  is shown together with the theoretical curve (B8).

 $\langle \gamma \rangle = \langle \hat{b}\hat{b} : \nabla u \rangle > 0$  and  $\langle \ln B \rangle = \langle \gamma \rangle t$  (see, e.g. Schekochihin et al. 2004; the value of  $\langle \gamma \rangle$  is related to the mean Lyapunov exponents associated with the rate-of-strain tensor; see Zeldovich et al. 1984; Chertkov et al. 1999; Schekochihin & Cowley 2007). Spatially this means that regions where magnetic field decays occupy smaller area on average than those where it grows, i.e. the rate of strain is more likely to cross the mirror threshold than the firehose one (this was quantitatively confirmed in the recent study by Santos-Lima et al. 2014).

To incorporate this feature into Model I, we may stipulate that, in the absence of thresholds,  $\gamma$  (or, equivalently,  $\sigma$ ) has a non-zero prescribed mean  $\gamma_0$ . Then in equations (9) and (10), we replace  $\sigma \rightarrow \sigma + \gamma_0$ . Equation (15) is then replaced by

$$\partial_{\tau} M = \begin{cases} M^2, & \tilde{\sigma} > M - \bar{\gamma}, \\ (\tilde{\sigma} + \bar{\gamma}) M, & \tilde{\sigma} \in [-2M - \bar{\gamma}, M - \bar{\gamma}], \\ -2M^2, & \tilde{\sigma} < -2M - \bar{\gamma}, \end{cases}$$
(17)

where  $\bar{\gamma} = \gamma_0 / \sigma_0 > 0$ .

For  $M \ll \bar{\gamma}$ , the system is at the mirror threshold with probability

$$p = \frac{1}{\sqrt{2\pi}} \int_{-\bar{\gamma}}^{\infty} d\tilde{\sigma} \ e^{-\bar{\sigma}^2/2} > \frac{1}{2}$$
(18)

and at the firehose threshold with probability 1 - p < 1/2. Then

$$\left\langle \frac{\partial_{\tau} M}{M^2} \right\rangle = 3p - 2. \tag{19}$$

The tendency will be for *M* to decay if p < 2/3 and to grow if p > 2/3. Using equation (18), we find that the threshold value corresponds to  $\bar{\gamma} \approx 0.4$ , but one should not regard this as a quantitative prediction for real 3D turbulence because the distribution of the rate of strain is in reality very far from being Gaussian. The salient point is that the spatial structure of the field and the consequent relative space-filling properties of the regions of growth and decay will matter. They cannot unfortunately be captured adequately in the zero-dimensional modelling framework adopted in this paper and require direct numerical simulations of the dynamo saturation in 3D.

For completeness, an analytical treatment of the Model I with one-scale flow and non-zero mean stretching rate is given in Appendix B3. When p > 2/3, the mean magnetic energy is

$$\langle M \rangle = \frac{1}{1/M_0 - (3p-2)\tau},$$
 (20)

which explodes at  $\tau = 1/(3p - 2)M_0 \sim \tau_{dyn}$  (as do all higher moments of *M*); all realizations eventually reach the dynamical level M = 1,  $f_{max} = 1$ . The dynamo is explosive and 100 per cent 'efficient', although the growth time is still given by equation (13) – long if the initial field is small. Fig. 3 confirms this result by showing the time it takes for a given fraction of realizations to reach M = 1in a numerical solution of equations (17) and (16) with  $\bar{\gamma} = 0.75$ (so  $p \approx 0.77$ ).

#### 3.4 Plasma dynamo and the ICM viscosity

The one-scale-flow dynamo model ignores an essential effect. The suppression of the mean field-stretching rate by the microinstabilities affects not just the rate of growth of the field but also the effective viscosity of the plasma and, therefore, changes the spatial scale of the maximum turbulent rate of strain, which is, in Kolmogorov turbulence, the viscous cutoff scale. This in turn sets the magnitude of the rate of strain and so also its decorrelation rate ( $\sigma_0$ , thus far assumed fixed).



**Figure 3.** Same as Fig. 1 (left-hand panel), but for the modified Model I, equations (17) and (16) with mean stretching rate  $\bar{\gamma} = 0.75$ . This time all realizations eventually reach M = 1 on a time-scale consistent with equation (13); times  $\tau_f$  for f = 90, 50, 10, 1, 0.1 and 0.01 per cent are shown.

To be more quantitative, consider the momentum equation in a magnetized plasma, valid at time and spatial scales longer than the Larmor period and radius, respectively:

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla \left( p_{\perp} + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[ \hat{\boldsymbol{b}} \hat{\boldsymbol{b}} p \left( \Delta + \frac{2}{\beta} \right) \right]. \tag{21}$$

The mean pressure anisotropy is  $p\Delta = \mu \, \hat{b}\hat{b} : \nabla u$ , where  $\mu = p/v_{ii}$ is the (parallel) viscosity of the plasma (see equations 2 and 3). In the absence of the microinstabilities, it is this term that provides the viscous damping of the component of the rate of strain that can change the magnetic field strength (the stretching rate), while the  $2/\beta$  term is the tension force (Maxwell stress), responsible for back reaction of the field on the flow - this back reaction was assumed small for weak fields being amplified by a dynamo. Since, under our modelling assumptions,  $\hat{b}\hat{b}: \nabla u$  is suppressed by the instabilities and so  $|\Delta|$  is never larger than  $2/\beta$ , there is no mechanism left to enforce the viscous cutoff on any part of the rate-of-strain tensor. Thus, as the system crosses the stability thresholds (4), the turbulent cascade is free to extend to very small scales, probably into the microscale range where the finite-Larmor-radius (FLR) effects (omitted in equation 21) determine the shape of the velocity spectrum and where also the firehose and mirror instabilities operate, so scale separation is lost between mascroscopic motions and the microphysics.

As this happens, the maximum rate of strain  $\sigma$  becomes very large even as the velocity of the motions becomes very small ( $\sigma \propto l^{-2/3}$  but  $u \propto l^{1/3}$  in Kolmogorov turbulence; *l* is scale). Therefore, already very weak magnetic fields can exert dynamical back reaction on these motions and, if the field is sufficient to do that, the smallest scale at which the motions are capable of amplifying the field will be the scale where the kinetic-energy density of the motions is comparable to the total magnetic-energy density,  $\rho u_l^2/2 \sim B^2/8\pi$ . Since  $u_l \sim (\varepsilon l/\rho)^{1/3}$ , this gives

$$l \sim \frac{p v_{\text{thi}}}{\varepsilon} \beta^{-3/2}, \qquad (22)$$

where  $v_{\text{th}i} = (p/\rho)^{1/2}$  is the ion thermal speed. The corresponding rate of strain is

$$\sigma \sim \frac{u_l}{l} \sim \frac{\varepsilon}{p} \beta \sim \frac{\sigma_0}{M}.$$
(23)

As the magnetic field grows, the scale (22) gets larger and the rate of strain (23) smaller. Note, however, that the rate at which the field is amplified is at all times limited by the instabilities to be  $\leq v_{ii}/\beta \ll \sigma$ , as given by equation (9), so equation (23) just gives the decorrelation rate of the stretching, not its actual magnitude, i.e. the inverse of the typical time that the system might spend at the mirror threshold before flipping the sign of  $\sigma$  and ending up at the firehose threshold.

The argument involving the tension force limiting the motions does not apply at the firehose threshold: indeed, there  $\Delta + 2/\beta = 0$  and so the viscous damping and the non-linear back reaction in equation (21) exactly cancel each other: field lines lose tension and the motions no longer feel them at all. This means that in the regions and instances where the field weakens, the cutoff scale for the motions is microphysical (FLR determined), giving the rate of strain – and, therefore, the decorrelation rate – much larger than in the growing-field regions.

Thus, as the field grows, the typical time the system spends at the mirror threshold becomes much larger than the time it spends at the firehose threshold. This resembles the dynamo discussed in Section 3.3:  $\partial_{\tau} M = M$  with probability  $p \to 1$  and  $\partial_{\tau} M = -2M$  with probability  $1 - p \to 0$ . The result will, therefore, be a robust explosive dynamo with the typical growth time again given by equation (13).

One can construct model dynamical equations for  $\sigma$  (to replace equation 16) that would include all of the above effects and then solve them numerically together with equation (15).<sup>4</sup> However, the level of uncertainty about the way in which the rates of strain at the mirror and, especially, firehose thresholds are determined is such that a zero-dimensional modelling exercise is unlikely to teach us much more than the above qualitative discussion has done – and any further conclusions will be sensitive to a large number of modelling choices. A more promising course of action here would be direct 3D numerical simulations using Braginskii MHD equations with a suitable implementation of the microphysical closure corresponding to Model I – such a study, although highly desirable, is outside the scope of this paper.

#### 3.5 Constraints on the seed field

In order for the above considerations to be applicable, time- and spatial-scale separation between macro- and microphysics must be present. This imposes a number of constraints all of which can be expressed as lower bounds on the magnitude of the magnetic field – and, therefore, on the initial seed field from which any of the dynamo models considered so far is allowed to start.

These constraints can be expressed in terms of  $\beta$  (or *M*) and the three relevant time-scales in the problem, for which we will adopt reference core ICM values (cf. Enßlin & Vogt 2006; Schekochihin & Cowley 2006; Kunz et al. 2011; Rosin et al. 2011):

$$\sigma_0 \sim 10^{-14} \, \mathrm{s}^{-1},\tag{24}$$

$$v_{ii} \sim 10^{-12} \,\mathrm{s}^{-1},$$
 (25)

$$\Omega_1 \sim 1 \text{ s}^{-1},\tag{26}$$

where  $\Omega_1 = e\sqrt{8\pi p}/mc$  is the Larmor frequency corresponding to  $\beta = 1$ . Let us itemize the constraints.

<sup>4</sup> The treatment of the enhanced-collisionality case in Section 4 is an example of how non-linear equations for  $\sigma$  incorporating dependence on *M* might be constructed.

(i) Plasma must be magnetized:  $v_{ii} \ll \Omega_i$ , provided

$$\frac{1}{\beta} \gg \left(\frac{\nu_{ii}}{\Omega_1}\right)^2 \sim 10^{-24}, \text{ or } M \gg \frac{\nu_{ii}^3}{\sigma_0 \Omega_1^2} \sim 10^{-22},$$
(27)

a constraint overriden by the more stringent ones to come.

(ii) The typical growth rates of the mirror and firehose instabilities must be larger than the turbulent rate of strain in order for an 'instantaneous' suppression of the latter to be a sensible assumption. The peak growth rates of the parallel firehose (e.g. Davidson & Völk 1968; Schekochihin et al. 2010) and mirror (Hellinger 2007) are

$$\gamma_{\rm f} \sim \Omega_i \left| \Delta + \frac{2}{\beta} \right| \sim \frac{\Omega_1}{\beta^{3/2}},$$
(28)

$$\gamma_{\rm m} \sim \Omega_i \beta \left( \Delta - \frac{1}{\beta} \right)^2 \sim \frac{\Omega_1}{\beta^{3/2}}.$$
 (29)

For the purposes of these estimates, we take  $\Delta \sim 1/\beta$  and assume that the distance to threshold is also of order  $1/\beta$  (assuming  $\beta \gg 1$ ). This is reasonable (e.g. Rosin et al. 2011), but not necessarily obvious (especially for the mirror; see Hellinger 2007 – but we do not have a better a priori estimate). If the microinstabilities, in their non-linear state, manage to keep the pressure anisotropy even more tightly pinned to the threshold, their effective growth rates become smaller, scales larger and the resulting lower bounds on the seed field even more stringent. Note that the oblique firehose grows faster than the parallel one (Yoon et al. 1993; Hellinger & Matsumoto 2000), so if  $\gamma_f$  and  $\gamma_m$  given by equations (28) and (29) are large enough, so will be the growth rate of the oblique firehose.

Thus, the reference instability growth scale we will use is  $\gamma_m\sim\Omega_1/\beta^{3/2}.$  Then  $\sigma_0\ll\gamma_m$  if

$$\frac{1}{\beta} \gg \left(\frac{\sigma_0}{\Omega_1}\right)^{2/3} \sim 10^{-10}, \text{ or } M \gg \frac{\nu_{ii}}{\sigma_0^{1/3}\Omega_1^{2/3}} \sim 10^{-8}.$$
 (30)

With the heuristic model of the ICM viscosity proposed in Section 3.4, the effective rate of strain at the mirror threshold is given by equation (23) and so  $\sigma \ll \gamma_m$  if

$$\frac{1}{\beta} \gg \left(\frac{\varepsilon}{p\Omega_1}\right)^{2/5} \sim 10^{-7}, \text{ or } M \gg \frac{\nu_{ii}^{3/5}}{\sigma_0^{1/5}\Omega_1^{2/5}} \sim 10^{-5}$$
(31)

(we have used  $\varepsilon/p = \sigma_0^2/v_{ii}$ ). Note that  $\gamma_f$  might be a sensible estimate for the maximum rate of strain accessible at the firehose threshold, in which case the condition for the system to be at the mirror threshold with larger probability  $(p \sim 1 - \sigma/\gamma_f)$  than at the firehose threshold is  $\sigma \ll \gamma_f$ , which is automatically ensured by equation (31).

(iii) The typical scales of the mirror and firehose instabilities must be shorter than the scale of the motions that stretch the magnetic field. The (parallel) scales at which the peak growth rates (28) and (29) are achieved are

$$l_{\rm f} \sim \frac{\rho_i}{\sqrt{|\Delta + 2/\beta|}} \sim \frac{v_{\rm thi}}{\Omega_1} \beta,$$
 (32)

$$l_{\rm m} \sim \frac{\rho_i}{\beta(\Delta - 1/\beta)} \sim \rho_i \sim \frac{v_{\rm thi}}{\Omega_1} \beta^{1/2},$$
 (33)

where  $\rho_i = v_{\text{th}i}/\Omega_i$  is the Larmor radius. For the oblique firehose,  $l_{\text{f}} \sim \rho_i$ . The constraints that are obtained by requiring  $l \gg l_{\text{m}}$ ,  $l_{\text{f}}$  are less stringent than equation (31).

(iv) The use of equation (9) for the effective rate of amplification of the magnetic field is predicated on the calculation of the pressure anisotropy from the balance of the rate of change of the field and the collisional isotropization (see equation 2), which requires the collision time-scale to be shorter than any fluid time-scales. On the other hand, the collision rate must be smaller than the growth rates of the instabilities. Requiring  $v_{ii} \ll \gamma_m$  gives

$$\frac{1}{\beta} \gg \left(\frac{\nu_{ii}}{\Omega_1}\right)^{2/3} \sim 10^{-8}, \text{ or } M \gg \frac{\nu_{ii}^{5/3}}{\sigma_0 \Omega_1^{2/3}} \sim 10^{-6}.$$
 (34)

In Section 3.4, the rate of strain becomes large for small M (see equation 23) and so the requirement  $\sigma \ll v_{ii}$  imposes what turns out to be the most stringent of the lower bounds on the magnetic energy:

$$\frac{1}{\beta} \gg \frac{\varepsilon}{p\nu_{ii}} \sim 10^{-4}, \text{ or } M \gg \frac{\sigma_0}{\nu_{ii}} \sim 10^{-2}.$$
(35)

If this is violated, further modelling choices have to be made regarding the handling of the collisionless mechanism for setting the relationship between the mean pressure anisotropy and the rate of change of the magnetic field,<sup>5</sup> currently a poorly understood issue, which we leave outside the scope of this paper and which is likely to require some form of collisionless Landau-fluid closure (e.g. Snyder, Hammett & Dorland 1997; Passot & Sulem 2007; Passot, Sulem & Hunana 2012).

#### 3.6 Summary for Model I

In this model, the pressure anisotropy is kept from crossing the firehose and mirror thresholds by the effective suppression of the mean rate of change of the magnetic field. This means that the field growth is generally less efficient than it would have been without the microinstabilities. While the growth is, mathematically speaking, explosive, the time for it to happen is  $t_{dyn} \sim \beta_0 / v_{ii}$  (equation 13), which can be very long if the initial seed is small and collisions are rare.

If we accept this scenario, the implication is that it is virtually impossible to generate fields of observed strength ( $B \sim 10^{-6}$  G, or  $\beta \sim 10^2$ , or magnetic energy  $M \sim 1$  in our dimensionless terms) from purely primordial seeds ( $B \sim 10^{-21} - 10^{-9}$  G; see review by Durrer & Neronov 2013). If, on the other hand, there is (or was) a sustained source of field only one or two orders of magnitude below the dynamical strength, then the dynamical field level can perhaps be effectively maintained by turbulence, possibly in very intermittent patches. To study this, one would need to combine a dynamo saturation model (e.g. Boldyrev 2001; Schekochihin et al. 2002a) with the model of field evolution in the presence of pressure anisotropies. Given the large amount of theoretical uncertainty in the understanding of both, we leave this outside the scope of this paper and note that significant progress could be made via direct numerical simulations incorporating a microphysical closure represented by Model I.

An essential caveat to the above conclusion is that all versions of Model I that we have discussed are valid only for relatively large magnetic fields, owing to a number of lower bounds on the magnetic energy imposed by the requirement of scale separation between fluid, collisional and microinstability time and spatial scales (Section 3.5). The most pessimistic of these bounds (equation 35) constrains *B* to values only an order of magnitude below the target dynamical strength and thus means that these ideas are primarily useful for the study of how the field is structured and maintained in the currently observed turbulent ICM rather than how it grew to its present level from a tiny seed. The question of magnetogenesis starting from a tiny seed may have to wait for a better understanding of field growth in a collisionless (as opposed to weakly collisional) plasma (see further discussion in Section 5.1).

#### **4 MODEL II: ENHANCED COLLISIONALITY**

#### 4.1 Microphysical closure

Now we examine the possibility that it is not the stretching rate of the field but the effective collisionality of the plasma that keeps the pressure anisotropy from crossing the instability thresholds. The stretching rate is always  $\gamma = \sigma$  and the rate of strain  $\sigma$  continues to obey equation (6) as long as  $\sigma \in [-2, 1]v_{ii}/\beta$ , but whenever it falls outside this interval, we postulate an instantaneous adjustment of the collision rate:

$$\nu_{\rm eff} = \xi \,|\sigma|\beta,\tag{36}$$

where  $\xi = 1/2$  and 1 at the firehose and mirror thresholds, respectively. Larger collision frequency means smaller effective viscosity,  $\mu_{\text{eff}} = p/v_{\text{eff}} = \mu v_{ii}/v_{\text{eff}}$ , where  $\mu$  and  $v_{ii}$  continue to denote the 'bare' viscosity and collisionality associated with Coulomb collisions. Therefore, locally, the Kolmogorov cascade will extend to smaller scales and larger rates of strain  $\sim (\varepsilon/\mu_{\text{eff}})^{1/2}$  (Schekochihin & Cowley 2006). We can model this by replacing in equation (6)

$$\sigma_0 \to \sigma_0 \left(\frac{\nu_{\text{eff}}}{\nu_{ii}}\right)^{1/2} = \sigma_0 \left(\frac{\xi |\sigma|\beta}{\nu_{ii}}\right)^{1/2},\tag{37}$$

where  $\sigma_0$  will continue to denote the 'bare' rms rate of strain. Adopting again the non-dimensionalization introduced in Section 3.2.2, we replace equations (15) and (16) with<sup>6</sup>

$$\partial_{\tau} M = \tilde{\sigma} M, \tag{38}$$

$$\partial_{\tau}\tilde{\sigma} = -\tilde{\sigma} + \sqrt{2}\,\chi(\tau), \qquad \qquad \xi |\tilde{\sigma}| \le M, \quad (39)$$

$$\partial_{\tau}\tilde{\sigma} = -\left(\frac{\xi|\tilde{\sigma}|}{M}\right)^{1/2} \tilde{\sigma} + \sqrt{2} \left(\frac{\xi|\tilde{\sigma}|}{M}\right)^{3/4} \chi(\tau), \ \xi|\tilde{\sigma}| > M.$$
(40)

Note that our approach here differs from the earlier work by Schekochihin & Cowley (2006; where the possibility of an explosive dynamo was first explored) in that there a specific microphysically inspired formula for the effective collisionality was postulated, while here we take a more agnostic attitude and simply assume that the anisotropy will always be effectively pinned at the marginal level. In this sense, the model proposed here is a more adequate reflection of what ought to happen in numerical simulations that adopt the prescription of sharply increased local collisionality to prevent firehose and mirror instabilities from developing (Sharma et al. 2006; Kunz et al. 2012; Meng et al. 2012; Santos-Lima et al. 2014). However, in none of these simulations has it so far been possible to accommodate numerically the dramatic local refinement of

<sup>&</sup>lt;sup>5</sup> We stress that it is the relationship between  $\Delta$  and  $\gamma$  that is unclear; the pressure anisotropy itself is likely always to be at the mirror or firehose thresholds (Kasper et al. 2002; Hellinger et al. 2006; Bale et al. 2009; Laveder et al. 2011).

<sup>&</sup>lt;sup>6</sup> We use the Stratonovich stochastic calculus discretization rule for the term containing white noise in equation (40), i.e.  $\partial_{\tau}\tilde{\sigma} = [\sigma(\tau + d\tau) - \sigma(\tau)]/d\tau$  and  $\tilde{\sigma}$  on the right-hand side is  $[\tilde{\sigma}(\tau + d\tau) + \tilde{\sigma}(\tau)]/2$ ;  $\chi(\tau) d\tau$  is the Wiener measure. The numerical results obtained using the Itô calculus instead are not significantly different.

the viscous dissipation scale – the key effect here! – and thus we remain in the realm of largely unexplored physics (see discussion in Section 5.4).

#### 4.2 Qualitative discussion

Ignoring for the purposes of a quick estimate the difference between the rate of strain  $\sigma$  and its rms value, we find from equation (37)

$$\sigma \sim \frac{\sigma_0^2}{\nu_{ii}} \beta \sim \frac{\varepsilon}{p} \beta \sim \frac{\sigma_0}{M} \quad \Rightarrow \quad \mu_{\rm eff} \sim \mu \frac{\nu_{ii}}{\nu_{\rm eff}} \sim \mu M^2, \tag{41}$$

so realizations with weaker magnetic field will have smaller viscosity, faster stretching, smaller viscous scale (Kolmogorov scale  $\propto \mu_{\rm eff}^{3/4}$ ), but also smaller velocities at this scale ( $\propto \mu_{\rm eff}^{1/4}$ ). This means that, with respect to these velocities, the magnetic field will not need to be very strong in order to start having a dynamical effect – and examining the condition (7) with  $\mu$  replaced by  $\mu_{\rm eff}$ , we discover that it is in fact exactly satisfied by the estimates (41). Thus, in this scenario, the dynamo becomes non-linear the moment either of the instability thresholds is crossed.

It might appear that this development requires some amendment to our model reflecting the role of the newly ascendant Lorentz back reaction in moderating the dynamo. In fact, our model already takes care of this effect, on a qualitative level. Indeed, let us consider what happens when the magnetic-energy density becomes comparable to the kinetic-energy density of the viscous-scale turbulent 'eddies'. These eddies (or, to be precise, the stretching rate associated with them,  $\hat{b}\hat{b}: \nabla u$ ) become suppressed and the field is now stretched by the eddies at the next largest scale, grows to have energy density comparable to the eddies at that scale, suppresses them, and so on until it has thus worked its way up the inertial range to be in energy equipartition with the largest turbulent motions. This scenario (Schekochihin et al. 2002a) was recently studied and validated in the MHD numerical simulations of Cho et al. (2009) and Beresnyak (2012; ideologically it goes back to the classic paper by Biermann & Schlüter 1951). The field growth can be modelled by requiring that magnetic energy, as it grows, is always amplified by the turbulent eddies  $u_l$  at scale l that have the same energy density:

$$\frac{\mathrm{d}}{\mathrm{d}t}B^2 \sim \frac{u_l}{l}B^2 \sim \frac{\rho u_l^3}{l} \sim \varepsilon \quad \Rightarrow \quad B^2 \sim \varepsilon t, \tag{42}$$

where  $\varepsilon$  is the constant Kolmogorov cascade rate. This is precisely what would happen in our model except the scale *l* will always be the viscous scale set by the effective viscosity  $\mu_{\text{eff}}$ : indeed, using equation (41),  $\tilde{\sigma} \sim 1/M$ , in equation (38), we get

$$\partial_{\tau} M \sim 1,$$
 (43)

and so the magnetic field reaches M = 1 (which corresponds to dynamical strength with respect to the eddies at the viscous scale set by the 'bare' Coulomb collision rate; see equation 7) at the time

$$\tau_{\rm dyn} \sim 1 - M_0, \quad \text{or} \quad t_{\rm dyn} \sim \frac{1}{\sigma_0} \left( 1 - \frac{\nu_{ii}}{\sigma_0 \beta_0} \right).$$
(44)

This is at most one large-scale stretching time, regardless of the size of the initial field – a much faster dynamo than achieved by Model I (equation 13) or by the conventional MHD dynamo (equation 8).



**Figure 4.** Examples of the time evolution of particular realizations in Model II, equations (38)–(40) with  $M_0 = 0.01$ . Episodes of intense growth are manifest here; this growth is exponential in time with the rate 1/M, where M is the magnetic energy at the beginning of the episode – the corresponding slopes are shown as dotted lines. See discussion in Sections 4.3.1 and 4.3.2.

# **4.3** Numerical results: effects of randomness and finite decorrelation rates

# 4.3.1 Why the field grows

In view of the experience of a dynamo failure in the simplest version of Model I (Section 3.2), we ought to ask why the above scenario should preferentially produce growth of the field rather than decay. The key difference between Models II and I is that in Model II, the modification of the field-stretching rate as a result of crossing the instability thresholds is not instantaneous: it is the effective collisionality  $v_{\text{eff}}$  that is modified on the short time-scales associated with the instabilities, which then leads to the change in the instantaneous decorrelation rate of  $\tilde{\sigma}$  (see equation 40). The actual value of  $\tilde{\sigma}$  then takes the time  $\tau \sim (|\tilde{\sigma}|/M)^{-1/2}$  to change to a new random value, with a modified rms of order 1/M.

Consider the evolution of any particular realization of the field. If it finds itself at the mirror threshold, the field grows, with it increases the effective collisionality, so the instantaneous decorrelation rate of  $\tilde{\sigma}$  decreases and the system can spend a longer time at this threshold before  $\tilde{\sigma}$  flips sign. In contrast, at the firehose threshold, the field drops, the decorrelation rate goes up and so the system can revert to positive growth rate sooner – and when it does, it grows at a relatively higher rate,  $\tilde{\sigma} \sim 1/M$ , because it starts at a lower value of M.<sup>7</sup>

Fig. 4 shows examples of time histories  $M(\tau)$  obtained by numerical solution of equations (38)–(40). A particularly striking feature that is manifest here is that the field growth can happen in short intense bursts during which the rate of increase is exponential and so much faster than suggested by the 'non-linear-dynamo' secular-growth estimate (42). These fast-growth episodes allow the

<sup>&</sup>lt;sup>7</sup> Note that in truth, the system's reluctance to linger at the firehose threshold may be even greater than our model allows because the exact cancellation of the viscous stress will likely drive the effective decorrelation rate even higher. This is reminiscent of the discussion in Section 3.4 except it is the enhanced collisionality that sets  $\Delta + 2/\beta = 0$  this time, so the rate of change of the field is not instantly affected – and by the time it is affected, it may have changed sign. We do not attempt to model the effect of the cancellation of the viscous stress because the enhanced collisionality produced by the firehose fluctuations during an episode of field decay may actually set the expectation value of the growth rate after one decorrelation time.



**Figure 5.** Left-hand panel: times  $\tau_f$  for the fractions f = 90, 50, 10, 1 and 0.1 per cent of all realizations to achieve M = 1 in Model II, equations (38)–(40), plotted versus initial normalized field energy  $M_0 = v_{ii}/\sigma_0\beta_0$ . The two dotted lines are  $\tau_{dyn} = 1 - M_0$  (equation 44) and  $\tau_0 = M_0 \ln(1/M_0)$  (equation 45). Right-hand panel: PDF of the time  $\tau$  it took those realizations to grow to M = 1. Three cases,  $M_0 = 0.1$ , 0.01, 0.001 are shown here, with  $\tau$  rescaled by  $\tau_0$ . The slope corresponding to  $\tau^{-3/2}$  is shown for reference. Inset: same PDFs, but without rescaling by  $\tau_0$  and shown on a log-linear plot to highlight the exponential cutoff at  $\tau \gtrsim 1 \sim \tau_{dyn}$ .

field to reach M = 1 much more quickly than predicted by the estimate (44).

#### 4.3.2 Time to saturation

Consider a realization starting with  $M = M_0 \ll 1$  and quickly finding itself at the mirror threshold. Then the initial growth rate is  $\tilde{\sigma} \sim 1/M_0$  (equation 41) and so  $M \sim M_0 \exp(\tau/M_0)$ . Allowing for a (relatively rare) instance in which this value of  $\tilde{\sigma}$  persisted longer than the typical correlation time (of order  $M_0$  initially, but getting longer as M grows), we find that M = 1 is achieved after

$$\tau_0 \sim M_0 \ln\left(\frac{1}{M_0}\right), \quad \text{or} \quad t_0 \sim \frac{p}{\epsilon\beta_0} \ln\left(\frac{\sigma_0\beta_0}{\nu_{ii}}\right) \ll t_{\text{dyn}}.$$
(45)

This means that some number of realizations will get there much earlier than suggested by the estimate (44). These are the fast-growth episodes seen in Fig. 4.

Fig. 5 quantifies their contribution to the overall field growth using the numerical solution of equations (38)–(40) (typically for  $N = 10^5$  realizations). The left-hand panel shows that the time  $\tau_f$ needed for a given fraction *f* of these realizations to achieve M = 1actually decreases at small  $M_0$ , in contrast to the more conservative prediction (44). The PDF of the time for the realizations starting at  $M = M_0$  to reach M = 1,  $P_1(\tau | M_0)$  (right-hand panel), has a peak around  $\tau = \tau_0$ , followed by a power-law tail with a scaling that appears to be  $\sim \tau^{-3/2}$  (cf. the MHD case, equation A17), and then an exponential cutoff at  $\tau \gtrsim 1$  (i.e.  $t \gtrsim t_{dyn}$ , as per equation (44); see the inset in the left-hand panel of Fig. 5). Eventually all realizations reach M = 1 (the dynamo is 100 per cent efficient), with a (relatively small) fraction arriving much earlier than others (of order 10 per cent; these are the realizations with  $\tau$  up to the left of the peak of the PDF).

#### 4.3.3 Stochastic non-linear plasma dynamo

It is interesting, for completeness, to examine what happens at times  $\tau_0 \ll \tau \ll \tau_{dyn} \sim 1$ , i.e. during the period when the system settles into self-similar evolution, as suggested by the power-law behaviour of  $P_1(\tau|M_0)$ . Since at this point we are getting deeper into the study of the precise properties of the particular zero-dimensional model we have chosen (equations 38–40) – properties that may or may not

carry over quantitatively to the more realistic situations, – we have exiled the more detailed treatment to Appendix C. Let us discuss its results in qualitative terms.

As we explained in Section 4.2, the dynamo is non-linear at all times and so, in some typical sense,  $\partial_{\tau} M$  becomes a constant independent of M (equation 43). In the stochastic system given by equations (38)–(40), this behaviour can be teased out by formally introducing a new stochastic variable,  $\lambda = \tilde{\sigma} M$ , so equation (38) becomes

$$\partial_{\tau} M = \lambda$$
 (46)

and equation (40) can be transformed accordingly into an equation for  $\lambda$ . It then turns out that  $\lambda$  has a stationary distribution with most probability around  $\lambda \sim 1$  (as should be expected from the estimate  $\tilde{\sigma} \sim 1/M$ ). This distribution is strongly intermittent: the PDF of  $\lambda$ has a power-law tail,  $P(\lambda) \sim \lambda^{-2}$  at  $\lambda \gg 1$ , and so a logarithmically divergent mean. Since from equation (46) it follows that  $\langle M \rangle = \langle \lambda \rangle \tau$ , the mean magnetic energy also diverges; indeed, its PDF turns out to have a power-law tail,  $P(M) \sim M^{-1}$  at  $\tau \ll M \ll 1$ . The PDFs of  $\lambda$  and M obtained by numerical solution of equations (38)–(40) are shown in Fig. 6; note that the PDF of M evolves self-similarly in time.

Obviously, the divergence of  $\langle M \rangle$  does not mean that the magnetic energy is infinite or, worse still, that its PDF is non-normalizable as an  $M^{-1}$  tail would imply. In fact, the distribution of M is regularized at  $M \sim 1$  because for  $M \gtrsim 1$ , the rate of strain  $\tilde{\sigma}$  no longer spends most of the time outside the stable interval [-2M, M] and the dynamics start to look more akin to the conventional dynamo of Section 2. Furthermore — and more to the point, physically, the field growth must saturate at  $M \gtrsim 1$  via a non-linear mechanism not included in our model (in our numerical simulations, we simply remove the realizations that reach M = 1). Physically, the divergence of  $\langle M \rangle$  on time-scales  $\tau \sim 1$  is a statistical expression of the fact that individual realizations can have periods of intense growth that take them to M = 1 in very short times  $\tau \sim \tau_0 \ll 1$ , as discussed in Section 4.3.2.

# 4.4 Constraints on the seed field

We now itemize a set of constraints on the magnetic field in the same way as we did in Section 3.5 for Model I, with typical ICM time-scales given by (24)–(26).



**Figure 6.** Left-hand panel: PDF of  $\lambda = \tilde{\sigma} M$  for  $\lambda > 0$  in Model II, equations (38)–(40) with initial magnetic energy M = 0.01. The PDF is shown at  $\tau = 0.25$ , 0.5, 1 and 2 and is approximately stationary. Slopes corresponding to  $\lambda^{-3/4}$  and  $\lambda^{-2}$  are shown for reference (they are derived in Appendix C). Right-hand panel: PDFs of M for the same simulation, shown for the same times, with M rescaled by the time  $\tau$  (the PDF evolves self-similarly; see Appendix C). The slope corresponding to  $M^{-1}$  is shown for reference. Note that realizations that reach M = 1 are removed from our simulation, so this PDF is just for the remaining ones at any given time.

(i) Plasma must remain magnetized with respect to the increased effective collisionality:  $v_{\text{eff}} \sim \sigma \beta \sim (\epsilon/p)\beta^2 \ll \Omega_i$  (see equations 36 and 41). This gives

$$\frac{1}{\beta} \gg \left(\frac{\varepsilon}{p\Omega_1}\right)^{2/5} \sim 10^{-7}, \text{ or } M \gg \frac{\nu_{ii}^{3/5}}{\sigma_0^{1/5}\Omega_1^{2/5}} \sim 10^{-5},$$
(47)

a much more stringent constraint than equation (27), which used the 'bare' Coulomb collision rate.

(ii) The same lower bound on the magnetic field is obtained (coincidentally) by demanding that the rate of strain remain slower than the peak growth rates (28) and (29) of the firehose and mirror instabilities. The bound is the same as for Model I, equation (31), because the maximum rate of strain affecting the magnetic field is the same in Model II and in the version of Model I proposed in Section 3.4 (see equations 41 and 23) – indeed, as discussed in Section 4.2, setting the effective collisionality to pin the pressure anisotropy at the marginal level is equivalent to requiring the magnetic field to be just dynamically significant at the corresponding 'viscous' scale, the principle used to set that scale in Section 3.4.

(iii) The above constraints are in fact overriden by an even stronger requirement that the effective collision rate (not just the rate of strain) be smaller than the mirror's growth rate: imposing  $v_{\rm eff} \sim (\varepsilon/p)\beta^2 \ll \gamma_{\rm m} \sim \Omega_1/\beta^{3/2}$  gives

$$\frac{1}{\beta} \gg \left(\frac{\varepsilon}{p\Omega_1}\right)^{2/7} \sim 10^{-5}, \text{ or } M \gg \frac{\nu_{ii}^{5/7}}{\sigma_0^{3/7}\Omega_1^{2/7}} \sim 10^{-3}.$$
 (48)

Thus, as with Model I, we again conclude that a relatively simple closure scheme represented by Model II is not really adequate for describing the evolution of truly weak magnetic fields. We will discuss this issue further in Section 5.1.

## 4.5 Summary for Model II

In this model, the pressure anisotropy is kept marginal by the enhancement of the effective collisionality of the plasma, owing to alleged scattering of particles off the microscale magnetic fluctuations caused by the pressure-anisotropy-driven instabilities. This results in a dramatic decrease of the local viscosity of the plasma wherever and whenever the field is weak and changing. The turbulent cascade in such places can extend to very small scales, so the rate of strain becomes very large and very fast field growth results. Dynamical saturation can be achieved, at least by some fraction of realizations, on time-scales that are smaller for weaker seeds than for strong ones (equation 45) – and by the majority of realizations after one large-scale turnover time (equation 44). Both the field and its rate of growth are extremely intermittent, with shallow power-law distributions (Section 4.3.3), as one might intuitively expect in a physical system where the local value of viscosity depended on the local instantaneous magnetic field and its stretching rate. The dynamo is always non-linear in the sense that the magnetic-energy density is comparable to the kinetic-energy density of the motions that are dominantly amplifying it at any given moment in time (Section 4.2).

Thus, the dynamo in Model II is fast (much faster than the regular MHD dynamo)<sup>8</sup> and very efficient. However, in order for the model to be valid, scale separation is required between the smallest spatial and time-scales of the fluid motions benefitting from diminished viscosity and the spatial and time-scales of the mirror and firehose fluctuations that are making this diminished viscosity possible. This requirement constrains the field (under the most pessimistic estimate, equation 48) to values just over an order of magnitude below the observed strength, rather like in Model I. Thus, here again, the model of field evolution provided by the enhanced-collisionality closure is more suitable for studying the magnetized dynamics of the current state of the ICM than the genesis of the observed field from a tiny primordial seed (see further discussion in Section 5.1).

# **5 DISCUSSION**

It is perhaps fair to say that the foregoing represents more an account of the state of our ignorance about the nature of plasma dynamo (or, more generally, the dynamics of magnetic field in weakly collisional environments) than a definitive solution even to a subset of the problem. Nevertheless, we believe it was useful to compose this catalogue of seemingly sensible modelling choices, a priori limitations of their validity (see Sections 3.5 and 4.4), and their possible consequences – not least because moving these from the category

<sup>&</sup>lt;sup>8</sup> See discussion in Section 5.4 on why this is not seen in current numerical studies employing the enhanced-collisionality closure.

of unknown unknowns to that of the known ones allows for a more informed discussion of the problems at hand.

What are these problems that motivate the need to model the ICM at all? Here we focus on three of them; for the specific conclusions from our investigation of Models I and II, we refer the reader to the summaries provided in Sections 3.6 and 4.5.

#### 5.1 Cosmic magnetogenesis and collisionless plasma dynamo

It is a long-standing question how the magnetic field in the Universe has managed to grow from a small primordial seed to the observed value (in the  $\mu$ G range in galaxy clusters). This requires an amplification by at least three and possibly many more orders of magnitude over a period of a few billions of years (the estimates for the primordial field vary from  $10^{-21}$  to, very optimistically,  $10^{-9}$  G; see Durrer & Neronov 2013). The plasma is magnetized and weakly collisional (dilute) already for  $B \gtrsim 10^{-17}$  G, so conventional MHD description is unsuitable and a model of field evolution incorporating pressure-anisotropy-driven microinstabilities is required.

Normally one assumes that there would be a healthy scale separation between the chaotic motions of the ICM that amplify the field and the firehose and mirror instabilities that limit the pressure anisotropy arising from any attempt to change B. The two models of how they do that considered above were concerned with the relationship between the pressure anisotropy and the rate of change of the field while taking the action of the instabilities to be instantaneous. Two very different field-evolution scenarios have emerged (slow for Model I, fast for Model II) - but what the models have in common is the dramatic reduction in the field-parallel viscous stress (which is equal to the pressure anisotropy). As a consequence, the turbulent rates of strain responsible for amplifying the field are expected to be much larger and to occur at much smaller scales than is usually assumed for the ICM. Keeping these scales separated from the scales (in time and space) at which the firehose and mirror fluctuations occur is only possible for magnetic fields merely one or two orders of magnitude below the observed level. Thus, in order to understand how the field can grow from a primordial seed to values of order  $10^{-8} - 10^{-7}$  G, we need a theory of a fully collisionless dynamo operating with no scale separation between plasma flows and pressure-anisotropy-driven instabilities. Is there such a dynamo? How fast is it? These questions are open - and they cannot be answered by solving fluid equations with any microphysical closure that assumes an instantaneous adjustment of the pressure anisotropy to marginal level, but rather require a fully kinetic treatment. This is a hard problem, but the good news is that at least its numerical solution appears less challenging if no scale separation between the motions and the instabilities is required.

#### 5.2 Understanding observed magnetic field and ICM motions

As the field grows closer to the observed level, the instabilities do become instantaneous and so the dynamics of the field and the plasma can perhaps be described by MHD equations with a microphysical closure represented by one of the two models we have studied. Understanding the structure of the saturated field – and of the turbulence into which it is embedded – is a fascinating problem, also quite open. In particular, it is entirely unclear what sets the spatial scale of the observed magnetic field. Arguably this is actually a more interesting problem than the magnetogenesis as the field appears to be of dynamically important strength everywhere it has been measured, so theories of how it got there are not observationally falsifiable (except perhaps in the laboratory; see, e.g. Spence, Reuter & Forest 2009). In contrast, turbulence measurements in the ICM over a range of scales are a growing industry (e.g. Schuecker et al. 2004; Vogt & Enßlin 2005; Churazov et al. 2012; Sanders & Fabian 2013), so a good understanding of its magnetofluid dynamics is quite indispensable – and not possible without a model of how magnetic field can change in a moving weakly collisional plasma. This point applies with even greater force to the multitudinous observations and attendant modelling of various large- and medium-scale ordered motions in the ICM, which invariably require dragging magnetic field around (e.g. rising bubbles; see Churazov, Ruszkowski & Schekochihin 2013, and references therein).

#### 5.3 Heating of the ICM

An interesting and important non-trivial consequence of the closure one assumes for the pressure anisotropy and for the evolution of the magnetic field is the thermal stability of the ICM and hence the existence or absence of the so-called cooling catastrophe (see Kunz et al. 2011, and references therein). The viscous heating rate per unit volume in a magnetized, subsonically moving plasma is

$$Q_{\rm visc} = (p_{\perp} - p_{\parallel}) \gamma \sim v_{ii} p \Delta^2 \sim \frac{v_{ii} p}{\beta^2}, \tag{49}$$

where the last expression follows by assuming that the pressure anisotropy is marginal with respect to firehose or mirror instability conditions (cf. Lyutikov 2007). Thus, even though the viscous heating comes from the dissipation of plasma motions, the final expression for it does not appear to be related to them except via the local value of  $\beta$  (which, one assumes, is set by the saturated level of the magnetic field and, therefore, related to the kinetic-energy density of the turbulence).

Under the assumptions of Model I, the collision rate in equation (49) is the Coulomb collision rate and so we have a specific local relationship between the heating and the local values of magnetic field *B*, density *n* and temperature *T* of the ICM. Balancing  $Q_{\text{visc}}$  with the radiative cooling rate of the ICM, which depends on *n* and *T*, produces a definite relationship between these parameters and the magnetic field and leads to a thermally *stable* equilibrium with reasonable values of  $Q_{\text{visc}}$  and *B* for typical cluster-core conditions (Kunz et al. 2011).

Turning to Model II, we must replace in equation (49)  $\nu_{ii} \rightarrow \nu_{\text{eff}} \sim \sigma \beta$ . Thus, the collisionality is no longer set by the equilibrium state of the plasma but adjusts to the local rate of strain. Using equation (41) and  $\sigma_0 \sim (\varepsilon \nu_{ii}/p)^{1/2}$ , we find simply that  $\nu_{\text{eff}} \sim \varepsilon \beta^2/p$  and so<sup>9</sup>

$$Q_{\rm visc} \sim \varepsilon.$$
 (50)

Thus, whatever energy is injected into turbulence will be dissipated by heating, with no more microphysical constraints on the heating rate. This is a situation more familiar to those used to dealing with standard fluid turbulence problems, although here it arises not because the scale of the motions adjusts to accommodate a given dissipation rate but because the collisionality of the plasma does. This means that whatever determines the thermal stability of the ICM in this scenario has to do with how large-scale energy is deposited into the ICM turbulence – an outcome that may be conceptually satisfying to anyone who feels that microphysics should never matter (although it does matter for deciding whether this scenario is in fact correct).

<sup>&</sup>lt;sup>9</sup> In Model I, this equation also holds but is non-trivial and allows one to determine the injection scale of the turbulence (Kunz et al. 2011), whereas in Model II, it is automatic and provides no further information.

# 5.4 A comment on existing numerical simulations with microphysical closures

We are not aware of any studies so far to simulate the ICM under the assumptions of Model I. The effective enhancement of the collision rate that underpins Model II is more straightforward to implement in MHD-CGL equations and this has been done by several groups (Sharma et al. 2006; Kunz et al. 2012; Meng et al. 2012; Santos-Lima et al. 2014). Relatively little qualitative difference with the standard collisional MHD case was found, suggesting that the effect of plasma microphysics is simply to render collisionless plasma effectively collisional (Santos-Lima et al. 2014). This is a tempting conclusion, which, if true, would relieve the ICM modelling community of a serious headache. Putting aside the question of how likely the firehose and especially mirror instabilities in fact are to cause enhanced particle scattering (see Section 1.1), if they do, this still appears to produce a highly complex situation with spatially and temporally intermittent local viscosity and hence very different field evolution than in collisional MHD (Section 4). However, one can only capture this complexity in a numerical simulation if it is the intermittent local viscosity and not numerical grid dissipation or some other form of fixed-scale regularization that determines the cutoff scale for the turbulence. If, on the other hand, a fixed-scale regularization (effectively, a small isotropic viscosity similar to the one present in the collisional MHD) is present and, given limited resolution, overrides the plasma viscosity (much diminished owing to enhanced collisionality), the effect of the microphysical closure is simply to disconnect the pressure anisotropy from the stretching rate and, therefore, from having any influence on the evolution of the magnetic field - either directly or via its effect on the local viscosity of the plasma. Note that the pressure anisotropy will still have a role in modifying the tension force - an order-unity enhancement at the mirror threshold (a prefactor of 3/2 when  $\Delta = 1/\beta$ ; see equation 21) and a full suppression at the firehose threshold ( $\Delta = -2/\beta$ ). The latter effect appears potentially to be the more important, but in turbulent and dynamo situations, the mirror-unstable regions tend to dominate (Sharma et al. 2006; Santos-Lima et al. 2014).<sup>10</sup>

To conclude, the results presented above highlight the extent to which weakly collisional magnetized plasmas have the potential to surprise us and the importance of getting to grips with the rich microphysical world that, while unobservable directly (except perhaps in the solar wind), underlies all observable large-scale dynamics and thermodynamics of these plasmas.

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<sup>10</sup> Note, however, the crucial role that the firehose threshold appears to play in magnetic reconnection (Schoeffler, Drake & Swisdak 2011; Matteini et al. 2013). If reconnection of the magnetic fields must be understood quantitatively in order to predict the structure of the saturated dynamo states, the firehose regions may turn out to be more important than their spatial sparseness might suggest.

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#### APPENDIX A: ZERO-DIMENSIONAL DYNAMO

Here we provide the full solution of the 'zero-dimensional dynamo' introduced in Section 2. The calculation is standard but, as far as we know, is not readily available in textbooks in this form.

We wish to consider the following equations:

$$\partial_t B = \sigma B, \tag{A1}$$

$$\partial_t \sigma = -\tau_c^{-1} \sigma + \sqrt{2D} \chi(t), \tag{A2}$$

where in the Langevin equation (A2),  $\tau_c$  is the correlation time and *D* the diffusion coefficient [equation (6) has  $\tau_c = 1/2\sigma_0$  and  $D = 2\sigma_0^3$ ].

# A1 Fokker–Planck equation

The joint time-dependent PDF of  $\sigma$  and B is  $P(t, \sigma, B) = \langle \tilde{P} \rangle$ , where  $\tilde{P} = \delta(\sigma - \sigma(t))\delta(B - B(t))$ , where  $\sigma$  and B are random variables, whereas B(t) and  $\sigma(t)$  are solutions of equations (A1) and (A2). Then

$$\partial_t \tilde{P} = -\partial_\sigma \tilde{P} \partial_t \sigma(t) - \partial_B \tilde{P} \partial_t B(t)$$
  
=  $-\partial_\sigma \left( -\tau_c^{-1} \sigma + \sqrt{2D} \chi \right) \tilde{P} - \partial_B \sigma B \tilde{P}.$  (A3)

Averaging this, we get

$$\partial_t P = -\sqrt{2D} \,\partial_\sigma \langle \chi \,\tilde{P} \rangle + \tau_{\rm c}^{-1} \partial_\sigma \sigma \,P - \sigma \,\partial_B B P. \tag{A4}$$

The average can be calculated by formally integrating equation (A3),

$$\tilde{P}(t) = \int^{t} dt' \left[ \partial_{\sigma} \left( \tau_{c}^{-1} \sigma - \sqrt{2D} \chi(t') \right) - \partial_{B} \sigma B \right] \tilde{P}(t'), \quad (A5)$$

and using the fact that  $\chi(t)$  and  $\tilde{P}(t')$  are uncorrelated for  $t \ge t'$ ( $\tilde{P}$  only depends on the past values of  $\chi$ , but not the present or the future),  $\langle \chi \rangle = 0$  and  $\langle \chi(t)\chi(t') \rangle = \delta(t - t')$ :

$$\langle \chi(t)\tilde{P}(t)\rangle = -\frac{1}{2}\sqrt{2D}\,\partial_{\sigma}P. \tag{A6}$$

Combined with equation (A4), this gives the Fokker–Planck equation for the joint PDF:

$$\partial_t P = D \,\partial_\sigma^2 P + \tau_c^{-1} \partial_\sigma \sigma P - \sigma \partial_B B P. \tag{A7}$$

#### A2 Moments of B

Now let  $P_n(\sigma) = \int_0^\infty dB B^n P(\sigma, B)$ . Then  $P_0(\sigma)$  is the PDF of  $\sigma$ and  $\int d\sigma P_n(\sigma) = \langle B^n \rangle$  are the moments of *B*.  $P_n$  satisfies

$$\partial_t P_n = D \,\partial_\sigma^2 P_n + \tau_c^{-1} \partial_\sigma \sigma P_n + n\sigma P_n. \tag{A8}$$

If we look for solutions in the form

$$P_n = \psi_n(x) \exp\left(\gamma_n t - \frac{\sigma^2}{4D\tau_c}\right), \quad x = \frac{\sigma - 2D\tau_c^2 n}{\sqrt{2D\tau_c}}, \tag{A9}$$

then  $\psi(x)$  satisfies

$$\psi_n'' - x^2 \psi_n = -(1 - 2\gamma_n \tau_c + 2D\tau_c^3 n^2) \psi_n.$$
 (A10)

This is a Schrödinger equation for a harmonic oscillator. The nonoscillating solution is the ground state, corresponding to the expression in the parentheses on the right-hand side (the energy) being equal to 1. Then  $\gamma_n = D\tau_c^2 n^2$  and  $\psi_n = C_n e^{-x^2/2}$ , where  $C_n$  are constants. Assembling this with equation (A9), we get

$$P_n = \frac{\hat{C}_n}{\sqrt{2\pi D\tau_c}} \exp\left[D\tau_c^2 n^2 t - \frac{(\sigma - D\tau_c^2 n)^2}{2D\tau_c}\right],\tag{A11}$$

where some  $\sigma$ - and *t*-independent factors have been absorbed into the new constant  $\hat{C}_n$ . Note that for n = 0 and  $\hat{C}_0 = 1$ , this gives a Gaussian distribution for  $\sigma$ , with  $\langle \sigma^2 \rangle = D\tau_c$ . For  $\tau_c = 1/2\sigma_0$  and  $D = 2\sigma_0^3$ , we get  $\langle \sigma^2 \rangle = \sigma_0^2$ , as stated in Section 2.

Integrating equation (A11) over  $\sigma$ , we obtain the time evolution of the moments of the magnetic field:

$$\langle B^n \rangle = \hat{C}_n e^{D\tau_c^2 n^2 t}.$$
 (A12)

For  $\tau_c = 1/2\sigma_0$  and  $D = 2\sigma_0^3$ , this becomes the expression quoted in Section 2. We see that the constants  $\hat{C}_n = \langle B_0^n \rangle$  are the moments of the initial distribution of *B*. For simplicity, we may assume that the field starts with the same value  $B_0$  in all realizations and that *B* is normalized to that value; then all  $\hat{C}_n = 1$ .

#### A3 PDF of B

The quickest way to calculate the PDF of *B* is to notice that there was nothing in the calculation above that required *n* to be discrete or even real. Therefore, letting  $n = i\lambda$  in equation (A12), we find

$$\langle \mathbf{e}^{i\lambda\ln B} \rangle = \mathbf{e}^{-D\tau_{\rm c}^2\lambda^2 t}.\tag{A13}$$

The left-hand side is the characteristic function (the Fourier transform) of the PDF of  $\ln B$ . Inverse-Fourier transforming in  $\lambda$  and expressing the result as the PDF of *B*, we get

$$P(B) = \frac{1}{B\sqrt{4\pi D\tau_{\rm c}^2 t}} \exp\left(-\frac{\ln^2 B}{4D\tau_{\rm c}^2 t}\right),\tag{A14}$$

the lognormal distribution quoted in Section 2 (with  $4D\tau_c^2 = 2\sigma_0$ ). Note that a typical realization grows subexponentially:  $\ln B \sim 2\tau_c \sqrt{Dt}$  even though all the moments  $\langle B^n \rangle$  grow exponentially, a property that implies that a very large number of realizations must be used in a numerical solution in order to capture the correct intermittent dynamo behaviour over any given time t (cf. Artyushkova & Sokoloff 2006). This is, however, a feature to some extent particular to the zero-dimensional model: in 3D, a typical growing realization does grow exponentially (see discussion and references at the beginning of Section 3.3), although its growth rate is still different from the growth rate of the magnetic energy and the point about the necessity of good statistics stands.

# A4 Time to saturation

In terms of the dimensionless quantities used in most of this paper,  $M = v_{ii}/\sigma_0\beta_0$  and  $\tau = 2\sigma_0 t$ , the PDF of the magnetic energy is

$$P(\tau, M|M_0) = \frac{1}{M\sqrt{4\pi\tau}} \exp\left[-\frac{\ln^2(M/M_0)}{4\tau}\right],$$
 (A15)

which is also the probability of reaching magnetic energy M at time  $\tau$  starting with  $M_0$  at  $\tau = 0$ . Given  $P(\tau, M|M_0)$ , one can calculate the probability  $P_1(\tau|M_0)$  of reaching the dynamical level M = 1 (equation 7) for the first time at time  $\tau$  (the 'first-passage time') via the standard relation

$$P(\tau, 1|M_0) = \int_0^{\tau} \mathrm{d}\tau' P_1(\tau'|M_0) P(\tau - \tau', 1|1).$$
 (A16)

Inverting the integral operator on the right-hand side via the Laplace transform, we get

$$P_1(\tau | M_0) = \frac{|\ln M_0|}{2\sqrt{\pi}\tau^{3/2}} \exp\left(-\frac{\ln^2 M_0}{4\tau}\right).$$
 (A17)

This immediately allows us to calculate the fraction of the realizations that eventually reach M = 1:

$$f_{\max} = \int_0^\infty d\tau \, P_1(\tau | M_0) = 1 \tag{A18}$$

(i.e. they all do in this model). Note that the mean time for this to happen,

$$\langle \tau \rangle = \frac{1}{f_{\text{max}}} \int_0^\infty \mathrm{d}\tau \,\tau \, P_1(\tau | M_0), \tag{A19}$$

diverges because  $P_1 \sim \tau^{-3/2}$  at  $\tau \gg 1$  (so the 100 per cent 'dynamo efficiency' implied by equation (A18) takes a long time to consummate). The characteristic time for a typical realization to reach dynamically relevant fields can be read off from the exponential factor in equation (A17):  $\tau \sim \ln^2 M_0$ . However, the exponential growth of the mean magnetic energy (equation A12) implies that the realizations that dominantly contribute to  $\langle M \rangle$  (and all other moments) only require  $\tau \sim \ln M_0$  (hence equation 8).<sup>11</sup>

<sup>11</sup> This highlights the point that what is meant by a 'dynamo' and how fast that dynamo is considered to be is to an extent a matter of definition: do we wish the mean magnetic energy  $\langle B^2 \rangle$  to grow exponentially? Do we wish a typical realization to do so? In the rather simplistic zero-dimensional model adopted here, all realizations do eventually reach the non-linear threshold (7), albeit at subexponential rates (see Appendix A4), but in a standard model of a more realistic 3D MHD dynamo taking into account also the effect of resistivity, most realizations in fact decay superexponentially while both the typical growing realizations and the magnetic energy grow exponentially, albeit at different rates (Zeldovich et al. 1984; Chertkov et al. 1999; Schekochihin et al. 2004; Schekochihin & Cowley 2007). In our treatment of Model I, we encounter a case of very difficult field growth (Section 3.2).

# APPENDIX B: MAGNETIC FIELD STATISTICS AND GROWTH TIMES IN MODEL I WITH ONE-SCALE FLOW

Here we describe a very simple way to understand the behaviour of the one-scale version of Model I (Sections 3.2 and 3.3). Let  $M_0 \leq M \ll 1$ . Since  $\tilde{\sigma}$  is order unity (equation 16), the rate of strain will spend most of the time outside the stable interval [-2M, M] and so, as  $\tilde{\sigma}$  fluctuates between positive and negative values, the effective stretching rate will be alternately pinned at the mirror or firehose threshold. Thus, we may approximately replace equation (15) with

$$\partial_{\tau} M = \begin{cases} M^2, & \tilde{\sigma} > 0, \\ -2M^2, & \tilde{\sigma} < 0. \end{cases}$$
(B1)

Furthermore, let us take the long-time limit,  $\tau \gg 1$ , treat  $\tau$  as a discrete counter with step size  $\Delta \tau \sim 1$  and  $\tilde{\sigma}(\tau)$  as a sequence of discrete independent trials with either positive or negative outcome, each with probability p = 1/2 (this is reasonable because the correlation time of  $\tilde{\sigma}$  is order unity, but the crude nature of the model will leave us with the need to fit the numerically obtained distribution to the analytical result by choosing a suitable value of  $\Delta \tau$ , which will indeed turn out to be of order unity; see Appendix B1). Integrating equation (B1), we get

$$\frac{1}{M_0} - \frac{1}{M(\tau)} = \Delta \tau \sum_{i=1}^{\tau/\Delta \tau} x_i, \quad x_i = \begin{cases} 1, & p = 1/2, \\ -2, & p = 1/2. \end{cases}$$
(B2)

# B1 PDF of M

The mean  $\langle x_i \rangle = -1/2$  and the variance  $(\langle x_i^2 \rangle - \langle x_i \rangle^2)^{1/2} = 3/2$ , so, by Central Limit Theorem, the quantity

$$X = \frac{2}{3}\sqrt{\frac{\tau}{\Delta\tau}} \left[ \frac{1}{\tau} \left( \frac{1}{M_0} - \frac{1}{M} \right) + \frac{1}{2} \right]$$
(B3)

has the unit normal distribution

$$P(X) = \frac{1}{\sqrt{2\pi}} e^{-X^2/2}.$$
 (B4)

The PDF of X found in the numerical solution of equations (15) and (16) is shown in Fig. B1; we find that  $\Delta \tau = 1.4$  gives the best fit. Thus, the PDF of the magnetic energy at time  $\tau$ , having started with



**Figure B1.** PDF of the quantity X given by equation (B3). Several times are shown for the case with  $M_0 = 0.05$ , together with the unit normal PDF (B4). The best fit is obtained for  $\Delta \tau = 1.4$ .

 $M_0$  at time 0, is

$$P(\tau, M|M_0) = \frac{M^{-2}}{\sqrt{9\pi\tau\Delta\tau/2}} \exp\left[-\frac{2}{9\tau\Delta\tau} \left(\frac{1}{M_0} - \frac{1}{M} + \frac{\tau}{2}\right)^2\right].$$
(B5)

To understand the time evolution of this distribution, note that, from equation (B3),

$$M = \frac{1}{1/M_0 + \tau/2 - 3X\sqrt{\tau\Delta\tau}/2}.$$
 (B6)

We see that not much happens for  $\tau \ll 1/M_0$ ; for  $\tau \gg 1/M_0$ , the typical realizations decay secularly with time. The growing ones are due to the events with  $X \gtrsim \sqrt{\tau}$ , which are increasingly rare as time goes on (the mean tendency for *M* to decay due to the asymmetry between the firehose and mirror thresholds wins in the long run). The explosively growing realizations are typically those with  $X \sim \sqrt{\tau_{dyn}} \sim 1/\sqrt{M_0}$  and their fraction is  $f \sim \int_{\sqrt{\tau_{dyn}}}^{\infty} dX P(X) \sim \exp(-1/M_0)$ , in line with equation (14).

Finally, letting  $\tau \gg 1$  and averaging powers of *M* using equation (B6) and the distribution (B4), we find that all moments of the magnetic energy decay with time

$$\langle M^n \rangle \approx \frac{1}{\left(1/M_0 + \tau/2\right)^n} \to 0.$$
 (B7)

#### **B2** Time to saturation

Since we know the  $M_0 \rightarrow M$  transition probability (B5), we can calculate the probability to reach M = 1 in time  $\tau$  in the way described in Appendix A4 (via equation A16). The result is

$$P_1(\tau | M_0) = \frac{\left| \tau_{\rm dyn} \right|}{\sqrt{9\pi\Delta\tau\tau^3/2}} \exp\left[ -\frac{2}{9\Delta\tau\tau} \left( \tau_{\rm dyn} + \frac{\tau}{2} \right)^2 \right], \qquad (B8)$$

where  $\tau_{dyn} = 1/M_0 - 1$ . Therefore, the fraction of realizations that ever make it to M = 1 is (assuming  $M_0 < 1$ )

$$f_{\max} = \exp\left[-\frac{4}{9\Delta\tau} \left(\frac{1}{M_0} - 1\right)\right] \tag{B9}$$

and the mean time for them to do it is

$$\langle \tau \rangle = 2 \left| \frac{1}{M_0} - 1 \right| = 2 |\tau_{\rm dyn}|.$$
 (B10)

These are the quantitative versions of the estimates (14) and (13), respectively. Note that equation (B8) implies that at long times,  $\tau \gg \tau_{\rm dyn}$ , the PDF of the time to M = 1 is  $\propto \tau^{-3/2} \exp(-\tau/18\Delta\tau)$  (see the right-hand panel of Fig. 1).

# B3 Case of non-zero mean stretching rate

The above calculations are easily generalized to the case of non-zero mean stretching rate (Section 3.3), which amounts to letting  $x_i = 1$  with probability p > 1/2 and  $x_i = -2$  with probability 1 - p in equation (B2). Then  $\langle x_i \rangle = 3p - 2$ ,  $(\langle x_i^2 \rangle - \langle x_i \rangle^2)^{1/2} = 9p(1-p)$  and so (B3) becomes

$$X = \sqrt{\frac{\tau}{9p(1-p)\Delta\tau}} \left[\frac{1}{\tau} \left(\frac{1}{M_0} - \frac{1}{M}\right) - (3p-2)\right], \qquad (B11)$$

which is distributed normally (equation B4). Therefore,

$$M = \frac{1}{1/M_0 - (3p-2)\tau - 3X\sqrt{p(1-p)\tau\Delta\tau}/2}.$$
 (B12)

If p > 2/3, this explodes at  $\tau = 1/(3p - 2)M_0$ . Accordingly, assuming  $\tau \gg 1$ , we find that all moments of *M* explode

$$\langle M^n \rangle \approx \frac{1}{[1/M_0 - (3p-2)\tau]^n}.$$
 (B13)

The generalized version of equation (B8) for the probability of reaching M = 1 at time  $\tau$  is

$$P_{1}(\tau|M_{0}) = \frac{|\tau_{\rm dyn}|}{\sqrt{18\pi p(1-p)\Delta\tau\tau^{3}}} \exp\left\{-\frac{[\tau_{\rm dyn}-(3p-2)\tau]^{2}}{18p(1-p)\Delta\tau\tau}\right\}.$$
(B14)

The fraction of the realizations that make it is the integral of the above:

$$f_{\max} = \exp\left[-\frac{2(2-3p)}{9p(1-p)\Delta\tau}\left(\frac{1}{M_0}-1\right)\right], \quad p < \frac{2}{3},$$
 (B15)

$$f_{\max} = 1, \quad p \ge \frac{2}{3} \tag{B16}$$

and  $\langle \tau \rangle = |\tau_{dyn}|/|3p-2|$ . Thus, for p > 2/3, all realizations reach M = 1 on the characteristic time-scale  $\sim \tau_{dyn}$ .

# APPENDIX C: STOCHASTIC NON-LINEAR PLASMA DYNAMO IN MODEL II

Here we derive some analytical results for the dynamo model given by equations (38)–(40), to support the qualitative summary in Section 4.3.3.

Consider equations (38) and (40). The expectation from the qualitative discussion in Section 4.2 is that the magnetic energy will, in some typical sense, grow linearly in time at a rate that is of order unity in rescaled variables,  $M \sim \tau$  (equation 43). In a stochastic system, this will be a random process, so, anticipating the form it will take, we introduce a new stochastic variable  $\lambda = \tilde{\sigma} M$ . Then equations (38) and (40) become, for  $\xi |\lambda| > M^2$ ,

$$\partial_{\tau}M = \lambda,$$
 (C1)

$$\partial_{\tau}\lambda = \frac{\lambda^2 - \sqrt{\xi|\lambda|}\,\lambda}{M} + \frac{\sqrt{2}\,(\xi|\lambda|)^{3/4}}{\sqrt{M}}\,\chi(\tau). \tag{C2}$$

The Fokker–Planck equation for the joint PDF of M and  $\lambda$  is obtained in the same fashion as it was done in Appendix A for the PDF of Band  $\sigma$ . The result is

$$M\left(\partial_{\tau}P + \lambda\partial_{M}P\right) = \xi^{3/2}\partial_{\lambda}|\lambda|^{3/4}\partial_{\lambda}|\lambda|^{3/4}P + \partial_{\lambda}\left(\sqrt{\xi|\lambda|}\lambda - \lambda^{2}\right)P \equiv \hat{L}P.$$
(C3)

This equation has a self-similar solution

$$P(\tau, M, \lambda) = \frac{1}{\tau} f(m, \lambda), \quad m = \frac{M}{\tau},$$
 (C4)

where  $f(m, \lambda)$  satisfies

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$$-m\partial_m (m-\lambda) f = \hat{L}f.$$
(C5)

Note that the PDF of  $\lambda$  is, therefore, stationary, while the PDF of *M* is self-similar,  $P(\tau, M) = (1/\tau) \int d\lambda f(M/\tau, \lambda)$ .

The shape of these PDFs is not hard to work out. Equation (C5) has the following simple solutions for  $m \gg \lambda$  and  $m \ll \lambda$ :

$$n \gg \lambda$$
:  $f = \frac{1}{m} f_0(\lambda),$  (C6)

$$m \ll \lambda$$
:  $f = f_0(\lambda),$  (C7)

where  $f_0$  satisfies

$$\hat{L}f_0 = 0. \tag{C8}$$

This solution suggests that if we integrate out the  $\lambda$  dependence, we should get a PDF of *M* that is constant at  $M \ll \tau$  and has an  $M^{-1}$  tail at  $M \gg \tau$  – as is indeed confirmed by the numerical simulations (see the right-hand panel of Fig. 6). Note that, technically speaking, this PDF is not normalizable, but, as we explained in Section 4.3.3, implicitly we assume a cutoff at  $M \sim 1$ .

The PDF of  $\lambda$ , which is the solution of equation (C8), is obtained via direct integration, with a stipulation that  $f_0 \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Introducing a new variable  $x = |\lambda|/\xi$ , equation (C8) becomes

$$\partial_x x^{3/4} \left( \partial_x + 1 \mp \sqrt{x} \right) x^{3/4} f_0^{\pm} = 0,$$
 (C9)

where the upper (lower) sign is for  $\lambda > 0$  ( $\lambda < 0$ ). At the mirror threshold (the upper sign),

$$f_0^+ = \frac{\text{const}}{x^{3/4}} \,\mathrm{e}^{\frac{2}{3}x^{3/2} - x} \int_x^\infty \frac{\mathrm{d}y}{y^{3/4}} \,\mathrm{e}^{-\frac{2}{3}y^{3/2} + y}; \tag{C10}$$

at the firehose threshold (the lower sign),

,

$$f_0^- = \frac{\text{const}}{x^{3/4}} \,\mathrm{e}^{-\frac{2}{3}x^{3/2} - x} \left[ \int_0^x \frac{\mathrm{d}y}{y^{3/4}} \,\mathrm{e}^{\frac{2}{3}y^{3/2} + y} + c_0 \right]. \tag{C11}$$

The integration constants (the prefactors and  $c_0$ ) are fixed by normalization and matching  $f_0^{\pm}$  to the behaviour at  $x \leq M^2/\xi^2$ , where the rate of strain is within the stability interval and so equation (39) must be used – we will not go into these complications here. The distributions (C10) and (C11) are power laws both at small and large *x*:

$$x \ll 1: f_0^+ \sim \frac{1}{x^{3/4}}, \quad f_0^- \sim \frac{c_0}{x^{3/4}} + \frac{4}{\sqrt{x}},$$
 (C12)

$$x \gg 1: f_0^+ \sim f_0^- \sim \frac{1}{x^2}.$$
 (C13)

Fig. 6 (left-hand panel) shows  $f_0^+$  found in our numerical simulations, with both power laws in reasonable agreement with theory.

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