Dimensional Analysis. A. Schekochihin

UNIQ-2011 Lectures 11,13.07.11 (2 hours)

(1) Measurement, units, dimensions

A fundamental feature of any advanced (or even not so advanced) technological civilisation is measuring things.

- Distance from Oxford to London ~ 80 km
- Speed of a car ~ 60 miles/hour
- Speed of light 300 000 km/sec
- Volume of a bottle ~ 0.75 litre
- Mass of water in a glass ~ 200 gram
- Radius of the Earth ~ 6400 km etc. etc.

Acceleration of gravity $g = 9.8 \frac{m}{s^2}$

What do all these statements mean?

Usually, they mean that we have chosen some units in which to measure a quantity and then can compare it with some standard (e.g. standard of a meter kept in the Bureau of Weights and Measures in Paris).

Or we can define a way to measure some quantity in terms of several units: e.g. speed: in units of length per unit time.

Thus, there are independent and dependent (derived) units.

For any given class of physical phenomena (e.g. mechanics), we can choose a fundamental set of units: length, time, mass — and find that we can express everything else in terms of these.

E.g. velocity $= \frac{\text{length}}{\text{time}}$, acceleration $= \frac{\text{length}}{\text{time}^2}$ etc.

SI system: meter, kg, sec $\Rightarrow$ velocity $\to \frac{m}{sec}$
length mass time etc.

What constitutes an adequate system of units depends on the range of physical phenomena we are interested in.

E.g. Geometry (size of objects) : just length

Kinematics (moving objects) : length & time

Dynamics (objects moving and subject to forces) :
length, time, mass

E&M : have to add a unit of charge
(in SI Coulomb)

NB: The choice of a system of units is not unique.

E.g. we could use velocity & time instead of length and time. Then length becomes a dependent unit, expressed as speed · time (e.g. of: distances measured in ~~[illegible]~~ light years)   1 knot $\approx 2 \frac{km}{hr}$

What if I change the units themselves ~~[illegible]~~ (rather than what they measure)?

E.g. use km, tonne, hour (truck driver's units)
length mass time

Then all ~~many~~ quantities previously expressed in m, kg, sec must be multiplied or divided by some conversion factors:

indep. $\begin{cases} \text{length} \to \text{length}/L & L = 10^3 \text{ (m in 1 km)} \\ \text{time} \to \text{time}/T & T = 3600 \text{ (s in 1 hr)} \\ \text{mass} \to \text{mass}/M & M = 10^3 \text{ (kg in 1 tonne)} \end{cases}$

dep. $\begin{cases} \text{velocity} \to \text{velocity}/(L/T) \\ \text{acceleration} \to \text{acceleration}/(L/T^2) \end{cases}$ $\quad \text{density} \to \dfrac{\text{density}}{M/L^3}$

This allows us to introduce the concept of <u>dimension</u> of a physical quantity: it is the function that determines the conversion factor by which a physical quantity changes if we change units of ~~mea~~ measurement:

$[l] = L \quad [t] = T \quad [m] = M$

$[v] = LT^{-1} \quad [a] = LT^{-2} \quad [\rho] = ML^{-3} \quad \dfrac{kg}{m^3}$

What if we used a different system, say TVM instead of LMT?

$[t] = T \quad [v] = V \quad [m] = M$

$[l] = VT \quad [a] = VT^{-1} \quad [\rho] = MV^{-3}T^{-3}$

$\dfrac{kg}{\text{knots}^3 \, \sec^3}$

So: units are independent if we cannot derive their dimensions from each other.

~~T~~wo important ~~[illegible]~~ exercises.

- What are the dimensions of force?

$\boxed{f = ma}$, so $[f] = MLT^{-2}$ 2

↑ Newton's 2nd law.

amusing example: driving time in min = distance in km + # of lights (breaks down in other units)

<u>Physical laws are independent of ~~[illegible]~~ the units</u>

and so both sides of equations that express them must have the same dimensions.

This is the key principle, which will allow us to discover some amazing things shortly.

- How many dim-lly independent quantities are there in this set: p, ρ, v?

pressure, density, velocity?

force/area

$[v] = LT^{-1}$ $[\rho] = ML^{-3}$ $[p] = \left[\frac{f}{L^2}\right] = ML^{-1}T^{-2}$

So $\left[\frac{p}{\rho}\right] = \frac{L^2}{T^2} = [v^2]$ $\Rightarrow$ $\sqrt{\frac{p}{\rho}}$ has units of velocity

What is this velocity?

$c_s = \sqrt{\gamma \frac{p}{\rho}}$ is speed of sound

1.4 in air at room temp.

↑ constant cannot be determined from dim. analysis.

So, just by considering the dimensions, we have been able to discover that a fluid or a gas has a special speed associated with it!

This was the first example of dim. analysis. Ex. 1

I could have asked the question so:

~~[illegible]~~ what is the speed of sound in any given medium? Clearly it must depend on ρ and p. What can a speed be equal to if it depends on ρ and p? ~~[illegible]~~ It must be proportional to $\sqrt{p/\rho}$, so

$$\boxed{c_s = \sqrt{\frac{p}{\rho}} \cdot \text{constant}}$$

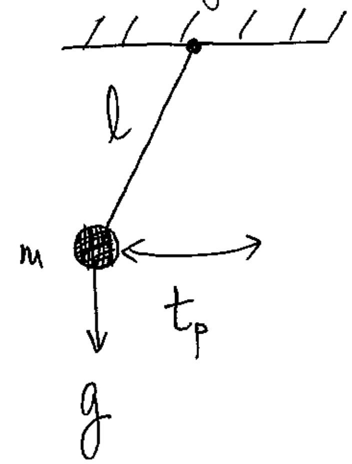
to get this we only need one good measurement!

(you can be confident of this because the relationship between c_s and p and ρ is a physical law and it cannot change if we change units – so, scaling units on the rhs must produce the same scaling factor ~~[illegible]~~ on the lhs etc.)

Note that we did not have to solve any equations (~~[illegible]~~ of motion, wave propagation etc.) to get this result. And since it has no choice but to hold, we can even ~~[illegible]~~ test if we solve them correctly by ~~[illegible]~~ checking the dimensions of the results of our calculations.

~~[illegible]~~

A systematic example: **The Pendulum.** Ex. 2



What is the period t_p of (small) oscillations of a pendulum? Let us find it w/o recourse to solving any equations.
What can t_p depend on?

l, m, g

$[l] = L \quad [m] = M \quad [g] = LT^{-2} \quad [t_p] = T$

$\Pi = \frac{t_p}{\sqrt{l/g}}$ is dim-less (i.e., if I change units, Π will not change)

In principle, it may be that $\Pi = \Pi(l, m, g)$ - but is it?

Change units of mass: $m \to m/M$, but Π is unchanged. So indep. of mass.

Similarly with l and g.

So $\frac{t_p}{\sqrt{l/g}} = \Pi = \text{const} \Rightarrow \boxed{t_p = \underbrace{\text{const}}_{= 2\pi}\sqrt{\frac{l}{g}}}$

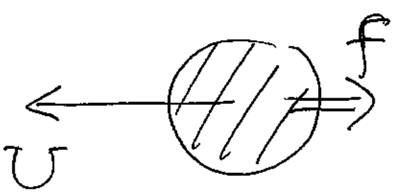
$= 2\pi$ (only need one measurement!)

We solved an interesting physics problem from nothing! (just by analysing dimensions)

$\boxed{\text{Ex. 3}}$

But things are not always quite so simple...

Drag force on a moving body



Consider a body (say, a sphere) moving through gas at high speed (constant). What is the drag it feels? In other words, how much power do we need to move it?

Let's not worry about friction (we will worry about that later on in these lectures) — so the force will be all due to inertia of the gas as it is being pushed apart by the body.

Parameters that matter:

$$[\rho] = \frac{M}{L^3} \qquad [p] = \frac{M}{LT^2} \qquad [U] = \frac{L}{T} \qquad [d] = L$$

density of gas — pressure — velocity of body — diameter of body

$$[f] = \frac{ML}{T^2} \quad \text{drag force.}$$

Form a dim-less combination involving force:

$$\frac{f}{\rho U^2 d^2} = \Pi\underbrace{(\rho, p, U, d)} = \Pi(\underbrace{\rho, U, d}, Ma)$$

but these are not independent! — these are indep. $(\rho, U, d \Rightarrow LMT)$

$\frac{p}{\rho} \sim c_s^2$ speed of sound, so we have another dim-less combination: $Ma = \frac{U}{c_s}$ Mach Number.

By the same argument as before, Π cannot depend on ρ, U, d — but it can (and does!) depend on Ma

So we have learned that

$$\boxed{f = \rho U^2 d^2 \, \Pi(\mathrm{Ma})}$$

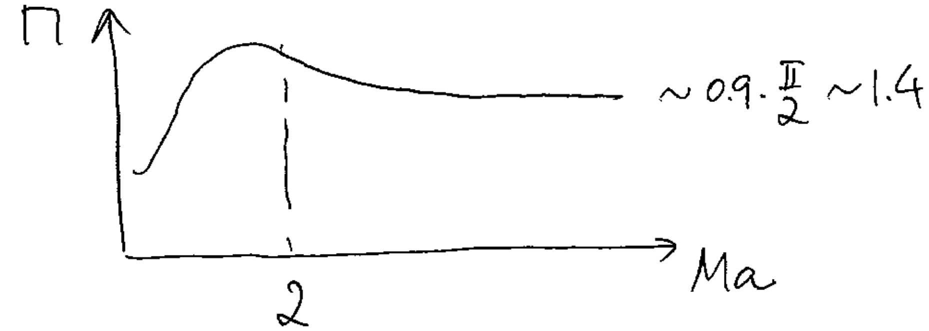
unknown function, which we cannot find from dim. analysis.

This is less conclusive than with pendulum, but still very useful: a priori, we have $f = f(\rho, p, U, d)$ function of 4 parameters. We have now reduced the problem to finding just one function of one dim-less parameter: $\Pi(\mathrm{Ma})$ [we have also figured out what matters physically].

It is often possible to solve such problems completely in some limit. E.g. consider supersonic motion $\mathrm{Ma} \gg 1$. If $\Pi(\mathrm{Ma}) \to$ finite limit as $\mathrm{Ma} \to \infty$, we get

$$f = \mathrm{const} \cdot \rho U^2 d^2 \quad \text{as} \quad U \gg c_s$$

In this case, it works: experimentally,



Note.: Power needed to move body:

$$P \sim f \cdot U \sim \rho U^3 d^2$$

quite a strong scaling

So, general recipe:

1) Find parameters on which quantity of interest depends
[here one needs to have some physical insight into what is relevant and what is not]
↳ e.g. we neglected friction

2) Find parameters with independent dimensions

3) Find dimensionless combinations. Then

Dim-less combination involving quantity of interest = function of all other dimless. combinations.

lecture 1 ended here

(3) The Π Theorem.

What has above been shown to work by example can be formally generalized.

Here are the steps (w/o proof)

(1) The dimension function is always a power-law monomial, i.e., the dimension of any physical quantity a is

$[a] = L^{\alpha} M^{\beta} T^{\gamma} \ldots$ (and other units if appropriate, e.g. charge Q)

(2) Recall that quantities $a_1, a_2, \ldots, a_k$ have independent dimensions if the dim. of none of them can be expressed as product of dimensions of others.

If we have a system of k indep. (fundamental) units (e.g. $k=3$ for LMT) and k quantities $a_1 \ldots a_k$

 (expressible in terms of those units)

with independent dimensions, it is always possible to change to a system of units that have the same dim's as $a_1 \dots a_k$ — and so, we can then always change units so that any one of the a_i's changes by some specified factor, while all other a_i's remain unchanged.

E.g. $\underset{k=3}{LMT}$, in the drag force problem
$a_1 = \rho \quad a_2 = U \quad a_3 = d$ were indep.

So we could measure everyth. in units of density, velocity and length — and could scale these units independently.

(3) Now consider any given physical problem. It always reduces to finding some relationship(s) of the form

$$\underset{\substack{\text{desired}\\\text{quantity}\\\text{(e.g. drag}\\\text{force)}}}{a} = f(\overbrace{\underbrace{a_1 \dots a_k}_{\substack{\text{indep.}\\\text{e.g.}\\\rho, U, d}}, \underbrace{b_1 \dots b_m}_{\substack{\text{dependent}\\\text{e.g. } p}}}^{\text{governing parameters (e.g. } \rho, U, d, p)})$$

$\Downarrow$

Can always express

$$\begin{cases} [b_1] = [a_1]^{\alpha_1} [a_2]^{\beta_1} \dots \\ \vdots \\ [b_m] = [a_1]^{\alpha_m} [a_2]^{\beta_m} \dots \\ \text{and } [a] = [a_1]^{\alpha} [a_2]^{\beta} \dots \end{cases}$$

How to find the exponents? Just by solving a system of simultaneous linear equations:

$$[f] = M^1 L^1 T^{-2} = [\rho]^\alpha [U]^\beta [d]^\gamma =$$
$$= (M L^{-3})^\alpha (L T^{-1})^\beta L^\gamma =$$
$$= M^\alpha L^{-3\alpha+\beta+\gamma} T^{-\beta}$$

So $\alpha = 1$

$-3\alpha+\beta+\gamma = 1 \Rightarrow \gamma = 1+3-2 = 2$

$-\beta = -2 \Rightarrow \beta = 2$

$\Rightarrow [f] = [\rho U^2 d^2]$

$$[p] = M^1 L^{-1} T^{-2} = [\rho]^{\alpha_1} [U]^{\beta_1} [d]^{\gamma_1} =$$
$$= (M L^{-3})^{\alpha_1} (L T^{-1})^{\beta_1} L^{\gamma_1}$$
$$= M^{\alpha_1} L^{-3\alpha_1+\beta_1+\gamma_1} T^{-\beta_1}$$

$\alpha_1 = 1$

$-3\alpha_1+\beta_1+\gamma_1 = -1 \Rightarrow \gamma_1 = -1+3-2 = 0$

$-\beta_1 = -2 \Rightarrow \beta_1 = 2$

$\Rightarrow [p] = [\rho U^2]$

So, this means we can introduce m+1 dim-less combinations:

$$\Pi = \frac{a}{a_1^{\alpha} a_2^{\beta} \cdots} \qquad \Pi_1 = \frac{b_1}{a_1^{\alpha_1} a_2^{\beta_1} \cdots} \quad \vdots \quad \Pi_m = \frac{b_m}{a_1^{\alpha_m} a_2^{\beta_m} \cdots}$$

e.g. $\Pi = \frac{f}{\rho U^2 d^2}$, $\Pi_1 = \frac{p}{\rho U^2} = \frac{1}{Ma^2}$

and recast our physical relationship as

$$\Pi = \frac{f(a_1 \ldots a_k, b_1, \ldots b_m)}{a_1^{\alpha} a_2^{\beta} \cdots} = \frac{f(a_1 \ldots a_k, \Pi_1 a_1^{\alpha_1} a_2^{\beta_1} \ldots, \ldots, \Pi_m a_1^{\alpha_m} a_2^{\beta_m} \ldots)}{a_1^{\alpha} a_2^{\beta} \cdots}$$
$$= \mathcal{F}(a_1 \ldots a_k, \Pi_1, \ldots \Pi_m)$$

But now, since both sides are dimless, scaling any of the parameters a_1 by an arbitrary factor is equiv. to simply changing units, so should not change values of $\Pi, \Pi_1 \ldots \Pi_m$ because they are dim-less or values of the rest of a_i's because they are independent. Therefore, $\mathcal{F}$ is indep of $a_1 \ldots a_k$ and we obtain the Π theorem:

$$\Pi = \mathcal{F}(\Pi_1, \ldots, \Pi_m)$$

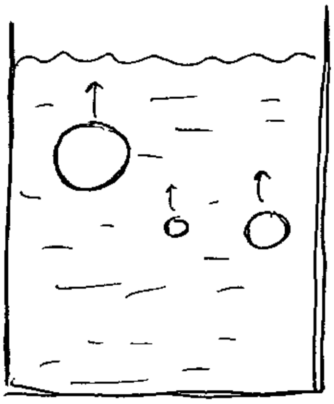
or $$\boxed{a = a_1^{\alpha} a_2^{\beta} \ldots \ \mathcal{F}(\Pi_1, \ldots, \Pi_m)}$$

Edgar Buckingham 1914 Phys. Rev. 4, 345

e.g. $f = \rho U^2 d^2 \mathcal{F}(Ma)$.

So, if we have k ~~independent~~ units in our fundamental system of units ($k=3$ for LMT) and n governing parameters in the ~~[illegible]~~ problem under scrutiny, we expect to be able to ~~express~~ reduce the answer to ~~an~~ an undetermined function of $m = n-k$ dim-less combinations.

$\boxed{\text{Ex. 5}}$ Rising Bubbles (and related stories)



How fast do bubbles rise depending on their size?

Find U (velocity) as a function of d (bubble diameter)

Let us first try a "quick and dirty" solution.

$[U] = \frac{L}{T}$ velocity depends on L, T

If we had two governing parameters with independent dims involving only L and T, we'd know what to do. OK, these are

$[d] = L$ (bubble size) and $[g] = \frac{L}{T^2}$ acceleration of gravity

So, immediately, $\boxed{U = \text{const}\sqrt{gd}}$!

(bubble 4 times the size rises twice as fast)

Does this make sense? Well, it's just force balance:

$$\underset{\text{(buoyancy)}}{\text{Archimedian force}} \sim \rho d^3 g = \text{drag force} \sim \rho U^2 d^2 \text{ from Ex. 3}$$

So, indeed,

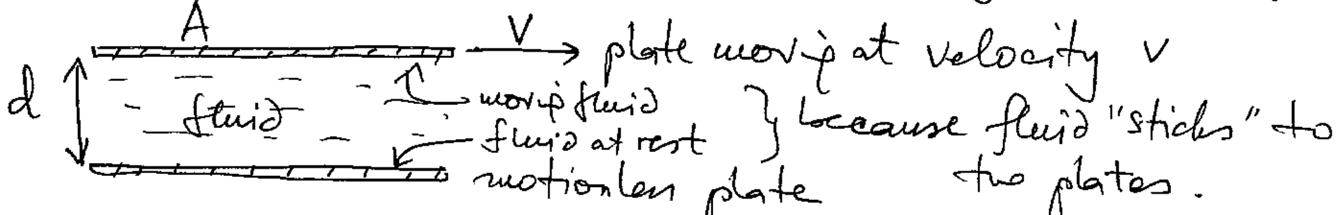
$$U^2 \sim gd.$$

But recall that the drag force result involved assuming "high speed" — we did not quantify this assumption, which was necessary to neglect viscosity of the fluid. But surely this was a dodgy assumption? – especially for bubbles, which in our experience rise rather slowly and at quite different speeds in fluids of varying viscosity.

So, we need to include <u>the effect of viscosity</u>.

– which means that we need to introduce some quantity that characterises the viscosity of a fluid, a quantity that could be measured for any given fluid.

Viscosity is basically a measure of how difficult it is to move fluid differentially wrt itself



It is known empirically that force on moving plate

$$f \text{ is } \propto \frac{v}{d} A \quad \leftarrow \text{area of plate}$$

and the (<u>dimensional</u>!) const of proportionality is, within some class of fluids and ambient conditions, approx. independent of v or d or A. So let

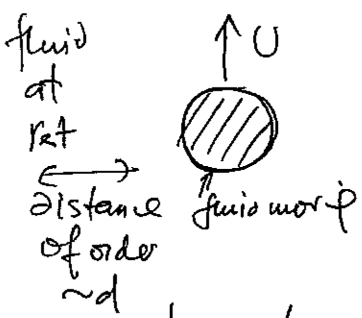
$$\boxed{f = \mu \frac{v}{d} A} \qquad [\mu] = \left[\frac{fd}{vA}\right] = \frac{ML}{T^2}\frac{L\,T}{L\,L^2}$$

$\uparrow$ viscosity

$$= \frac{M}{LT}$$

Note: This is one of the ways physics moves forward: empirical laws are parametrised, then understood on a deeper level (e.g. viscosity from kin. theory)

Physically, viscosity is relevant to the determination of the drag force on a moving object because there are two sets of forces opposing motion:



1) inertial forces – object pushes the medium apart as it moves

2) viscous forces – fluid in the immediate vicinity of the object has to move at the speed of the object, while fluid far away must be at rest, so the object sets up a differential flow.

So, let's repeat our drag force calculation

$$[f] = \frac{ML}{T^2} \quad [\rho] = \frac{M}{L^3} \quad [p] = \frac{M}{LT^2} \quad [U] = \frac{L}{T} \quad [d] = L$$

desired quantity

$$[\mu] = \frac{M}{LT}$$

NB: neglect 1) weight of air in the bubble
2) pressure changes with height
3) ~~surface tension effect on shape of bubble~~ also expansion of bubble

$n = 5$ gov. parameters.
$k = 3$ indep., say ρ, U, d, as before
So $m = 2$ – we'll have 2 dim-less combinations!

Find them:

$[p] = [\rho U^2]$ we already know (p. 11), so $\boxed{Ma = \dfrac{U}{\sqrt{p/\rho}}}$ Mach #

$$[\mu] = [\rho]^\alpha [U]^\beta [d]^\gamma = M^\alpha L^{-3\alpha+\beta+\gamma} T^{-\beta} = \frac{M}{LT}$$

So $\alpha = 1$ $\beta = 1$ $\gamma = 1$ $\boxed{Re = \dfrac{\rho U d}{\mu}}$ Reynolds number

From the Π-theorem, some function of 2 dim-less numbers.

$$f = \rho U^2 d^2 \mathcal{F}(Ma, Re)$$

So we now see what it meant to move "fast".
We needed $Re \gg 1$ and implicitly assumed that $\mathcal{F}(Ma, \infty)$ was finite (NB: this sort of thing is not always vindicated)

Clearly, for bubbles $Ma \ll 1$ ($U \ll c_s$).
So let's assume that $\mathcal{F}(0, Re)$ is finite and we only need to figure out the Re dependence.

1) Familiar limit is $Re \gg 1$ $\Rightarrow$ $f = \underbrace{\text{const}}_{\mathcal{F}(0,\infty)} \rho U^2 d^2$

It turns out that this is a situation in which the fluid behind the bubble becomes turbulent, so viscous forces no longer matter ("turbulent drag")

2) Opposite limit: $Re \ll 1$

It's clear $\mathcal{F}(0,0)$ cannot be finite because then we'd get the same answer indep. of viscosity.
So we need a little physical insight to guess what $\mathcal{F}(Re)$ looks like at small Re.
[this is the only way to deal with things that do not follow from dim. analysis formally]

If viscosity is large, let's argue that drag force should not depend on density — because inertia of the fluid is no longer important.

Since $f = \rho U^2 d^2 \mathcal{F}(Re)$ and $Re = \frac{\rho U d}{\mu}$,

the only way to arrange for this is

$$\mathcal{F}(Re) \simeq \frac{\text{const}}{Re}, \text{ so}$$

$$\boxed{f = \text{const} \cdot \mu U d}$$

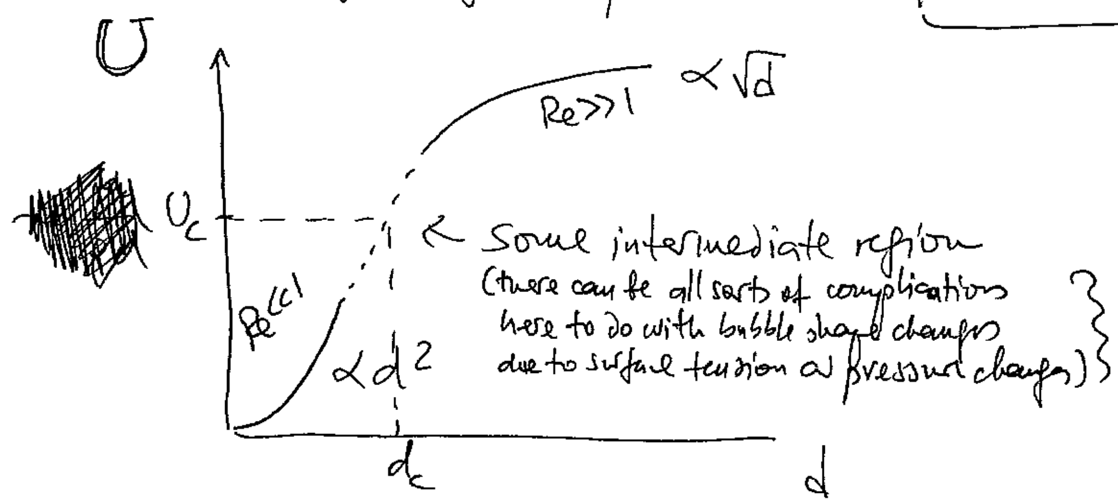
Stokes formula (in fact, already Aristotle thought $f \propto U$ - before Newton knew better)

for $Re \ll 1$

NB the many ways of achieving this limit (so for slow velocities, large viscosities, low densities and/or small bubbles)

Balance with Archimedes force (buoyancy):

$$\rho d^3 g \sim \mu U d \quad \Rightarrow \quad \boxed{U = \text{const}\, \frac{\rho g}{\mu} d^2}$$



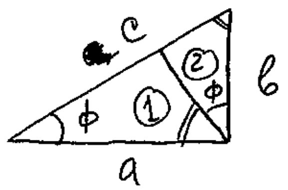
So speed of rising bubbles increases quite fast with their size if they are small.

NB: $Re \sim \frac{\rho^2 g d^3}{\mu^2} \sim 1$ when $d_c \sim \frac{\mu^{2/3}}{\rho^{2/3} g^{1/3}}$,

(transition around these values) $U_c \sim (\mu g/\rho)^{1/3}$

end of lecture 2

Ex.6 Pythagoras Theorem (this is quite amusing)



Area of a right triangle is completely determined by its hypotenuse c and one (let's say the smaller) of its acute angles ϕ

Dim-lly, $A = c^2 f(\phi)$

some fn of ϕ

Dividing it into 2 triangles ① and ②,

$A_1 = a^2 f(\phi)$, $A_2 = b^2 f(\phi)$

But $A = A_1 + A_2$, so

$$c^2 \cancel{f(\phi)} = a^2 \cancel{f(\phi)} + b^2 \cancel{f(\phi)}$$

$$\boxed{c^2 = a^2 + b^2} \quad \text{q.e.d.}$$

(Angles are dim-less because they are fraction of a circle (i.e. there is a special $\phi_{max} = 2\pi = 360^\circ$. So think of angles as ϕ/ϕ_{max})

Note that this is based on operating in flat space. If we were in curved space - e.g. a triangle on the surface of a sphere, there would be another parameter - r, radius of the sphere, so we would have $A = c^2 f(\phi, \frac{c}{r})$

dim-less parameter

So we'd have

$$c^2 f(\phi, \frac{c}{r}) = a^2 f(\phi, \frac{a}{r}) + b^2 f(\phi, \frac{b}{r})$$

Can't cancel f! - unless $\frac{c}{r}, \frac{a}{r}, \frac{b}{r} \ll 1$, so we take the limit $f(\phi, 0)$ and recover previous result.