THE STABILITY OF AN INCOMPRESSIBLE ELECTRICALLY CONDUCTING FLUID ROTATING ABOUT AN AXIS WHEN CURRENT FLOWS PARALLEL TO THE AXIS

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1. The problem of the stability of a fluid rotating about an axis to an axisymmetric disturbance has been examined in the inviscid case by Rayleigh [**1**], who derived a simple criterion based on an analogy with the stability of plane stratified fluid of variable density. Later a complete discussion of the stability of viscous motion between rotating cylinders for small axisymmetric disturbances was given by G. I. Taylor [**2**]. More recently, the problem of magneto-hydrodynamic stability has claimed the attention of several workers, and, amongst other problems, the stability of a rotating fluid, when a constant magnetic field is applied in the direction of the axis of rotation, has been examined by Chandrasekhar [**3**].

In this note we are interested in the case in which a current is applied in the direction of the axis, so that the magnetic lines form circles about the axis, with the field strength an arbitrary function of the distance from the axis. The main discussion is restricted to the case where the fluid is inviscid and perfectly conducting, the intention being to look for a simple modification to the Rayleigh criterion that applies in the absence of the magnetic field. $\underline{B} = B_\phi \hat{e}_\phi$

2. The addition of electromagnetic stress in the fluid will not change the monotonic nature of instability when it occurs, and it is quite easy, from the following considerations, to find the distribution of current and fieldstrength which give neutral stability when the fluid is at rest and the disturbance is axisymmetric.

Assuming that the fluid obeys Ohm's Law, the equation for the magnetic field (Batchelor [**4**]) when the magnetic diffusivity λ is zero, is

$$\frac{d}{dt}\mathbf{H} = (\mathbf{H} \, . \, \nabla)\, \mathbf{v}, \tag{1}$$

where $\mathbf{H}$ and $\mathbf{v}$ are the magnetic field and velocity vectors respectively and $\frac{d}{dt}$ represents the time rate following the particles of the fluid.

In this case let the x axis be the axis of symmetry, y the distance from the axis and the applied field $H_0(y)$ in the direction $\boldsymbol{\phi} = \mathbf{i} \times \mathbf{k}$, where $\mathbf{i}$ and $\mathbf{k}$ are unit vectors in the x and y directions respectively. Write

$$\mathbf{v} = u\mathbf{i} + v\mathbf{k} + w\boldsymbol{\phi}.$$

so $y = r$, $\phi = \theta$, $x = z$

When u, v, w are independent of ϕ equation (1) becomes

$$\frac{dH}{dt} = \frac{H}{y} v, \text{ where } \mathbf{H} = H\boldsymbol{\phi},$$

and since $v = \dfrac{dy}{dt}$ we have $\dfrac{d}{dt}(H/y) = 0$. i.e. $\frac{H}{y} = $ cnst

Hence H/y remains constant for each particle of fluid in any axisymmetric motion, *i.e.* the strength of the line is proportional to its length.

A case of special interest arises in which the applied current density $j\mathbf{i}$ is uniform, so that $H = Cy$, C being a constant proportional to j. In this case the value of H/y for all particles is the same, so that, whatever axisymmetric motion be imposed on the fluid, the magnetic field strength remains the same at every point. The effect of the electro-magnetic stress in this case is simply to modify the hydrostatic pressure; and any axisymmetric flow, viscous or inviscid, which is dynamically possible in the absence of the magnetic field, is unaffected by this distribution of H. Furthermore this magnetic field would give no opportunity for magnetic diffusion and these conditions will hold whatever the diffusivity coefficient λ.

From the point of view of stability, we can say that, if the fluid has any axisymmetric motion, the stability of such motion is unaltered by the addition of a uniform current parallel to the axis. In particular if the fluid is in hydrostatic equilibrium with this current, the system is neutrally stable to any axisymmetric disturbance.

These considerations, though of use in pointing out the state of neutral stability as far as the magnetic field is concerned, do not provide us with a definite criterion of stability. This criterion is derived in the following work by considering small disturbances to the steady velocity and magnetic fields.

3. Restricting ourselves to the inviscid perfectly conducting case, the equations for $\mathbf{v}$ and $\mathbf{H}$ in addition to equation (1) are

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho} \operatorname{grad} p + \frac{\mu}{4\pi} [\operatorname{Curl} \mathbf{H} \times \mathbf{H}], \tag{2}$$

$$\operatorname{Div} \mathbf{v} = 0, \tag{3}$$

$$\operatorname{Div} \mathbf{H} = 0. \tag{4}$$

The undisturbed vectors are

$$\mathbf{v} = U_0(y)\boldsymbol{\phi},$$

$$\mathbf{H} = H_0(y)\boldsymbol{\phi}.$$

Suppose that in a small disturbance these become

$$\mathbf{v} = u\overset{z}{\mathbf{i}} + v\overset{r}{\mathbf{k}} + (U_0 + w)\overset{\phi}{\boldsymbol{\phi}},$$

$$\mathbf{H} = (H_0 + h)\boldsymbol{\phi},$$

where u, v, w, h are independent of ϕ.

The disturbance term in $\mathbf{H}$ has only a ϕ component because lines of $\mathbf{H}$ move with the fluid, and since initially the lines are circles about the axis, they will remain so for an axisymmetric motion. I don't get this argument but it satisfies the induction eqn.

Inserting these values in equations (1), (2) and (3), and neglecting the squares and products of the disturbances, we derive the following equations.

$\partial_t \delta B_\phi + \delta v_r \partial_r B_\phi = \frac{B_\phi}{r} \delta v_r$

$$\frac{\partial h}{\partial t} + v\frac{dH_0}{dy} = \frac{H_0}{y} v, \tag{5}$$

z mom:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\mu}{4\pi\rho} H_0 \frac{\partial h}{\partial x}, \tag{6}$$

$\frac{\delta B_\phi}{r} \partial_r (r B_\phi)$ $\frac{B_\phi}{r} \partial_r (r \delta B_\phi)$

r mom:

$$\frac{\partial v}{\partial t} - 2\frac{U_0}{y} w = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\mu}{4\pi\rho}\frac{h}{y}\frac{d}{dy}(yH_0) - \frac{\mu H_0}{4\pi\rho y}\frac{\partial}{\partial y}(yh), \tag{7}$$

$\partial_t \delta v_\phi + \delta v_r (\partial_r U_\phi + U_\phi / r)$

ϕ mom:

$$\frac{\partial w}{\partial t} + v\left(\frac{dU_0}{dy} + \frac{U_0}{y}\right) = 0, \tag{8}$$

$$\nabla\cdot \underline{u} = \frac{1}{r}\partial_r (r\delta v_r) + \partial_z \delta v_z \equiv \frac{1}{y}\frac{\partial}{\partial y}(yv) + \frac{\partial u}{\partial x} = 0. \tag{9}$$

Let the disturbance be resolved into Fourier components in the x direction and consider the component of wavelength $\frac{2\pi}{\beta}$ which is proportional to $e^{i\beta x}$.

Equations (6) and (9) then become

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} i\beta p - \frac{\mu}{4\pi\rho} H_0 i\beta h, \tag{10}$$

$$\frac{1}{y}\frac{\partial}{\partial y}(yv) + i\beta u = 0. \tag{11}$$

From equations (10), (11) and (5) we derive

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = -\frac{1}{\beta^2 y}\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial y}(yv)\right] - \frac{\mu H_0}{4\pi\rho}\left(\frac{H_0}{y} - \frac{dH_0}{dy}\right) v. \tag{12}$$

Then eliminating $\overset{\delta v_\phi}{w}$ and p from equation (7) by using (8) and (12), we derive an equation for v,

$$\frac{\partial^2}{\partial t^2}\left\{\frac{1}{\beta^2}\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial}{\partial y}(yv)\right) - v\right\} = \left\{\frac{1}{y^3}\frac{\partial}{\partial y}(U_0 y)^2 - \frac{\mu y}{4\pi\rho}\frac{\partial}{\partial y}\left(\frac{H_0}{y}\right)^2\right\} v. \tag{13}$$

$\delta v_r \propto e^{i\omega t}$

Hence, if v varies as e^{ict},

ω

$$\frac{c^2}{\beta^2}\frac{d}{dy}\left(\frac{1}{y}\frac{d}{dy}(yv)\right)-c^2 v=-\left\{\frac{1}{y^3}\frac{d}{dy}(U_0 y)^2-\frac{\mu y}{4\pi\rho}\frac{d}{dy}\left(\frac{H_0}{y}\right)^2\right\}v,$$

k_z

or

$$\frac{1}{\beta^2}\frac{d}{dy}\left[\frac{1}{y}\frac{d}{dy}(yv)\right]-v=-\lambda g(y)\, v, \tag{14}$$

where

$$g(y)=\frac{1}{y^3}\frac{d}{dy}(U_0 y)^2-\frac{\mu y}{4\pi\rho}\frac{d}{dy}\left(\frac{H_0}{y}\right)^2,$$

Sturm-Louiville prob

and

$$\lambda=\frac{1}{c^2}.$$

eigenvalue $\frac{1}{\omega^2}$

$\delta v_r = 0$ on the boundaries

If the flow takes place between two cylinders at radii a and b, the boundary conditions are that $v=0$ at $y=a$, $y=b$. The problem is then to find the characteristic values of λ. If λ is real and positive the system is stable. Otherwise, if λ is real and negative, or complex, then it is unstable.

If we write $\chi=y^{\frac{1}{2}}v$, (14) becomes

$$\frac{d^2\chi}{dy^2}=\left\{\frac{3}{4y^2}+\beta^2-\lambda\beta^2 g(y)\right\}\chi, \tag{15}$$

and $\chi=0$ at $y=a$ and b.

If $\bar{\chi}$ denotes the conjugate of χ, multiply (15) by $\bar{\chi}$ and integrate throughout the interval (a, b). We then have

$$\lambda\beta^2\int_a^b g(y)\,\chi\bar{\chi}\,dy=\int_a^b\left[\left(\frac{3}{4y^2}+\beta^2\right)\chi\bar{\chi}+\chi'\bar{\chi}'\right]dy, \tag{16}$$

where χ', $\bar{\chi}'$ denote $\frac{d\chi}{dy}$, $\frac{d\bar{\chi}}{dy}$ respectively.

I think this means ω is either purely real or imaginary

The integrals in (16) are real so that λ is real. Hence the instability where it exists will be monotonic. Further, since the right-hand side of (16) is positive the sign of λ is that of $\int_a^b g(y)\,\chi\bar{\chi}\,dy$.

If $g(y)$ has the same sign throughout (a, b), then λ will have the sign of $g(y)$ and in this case the system will be stable or unstable according as $g(y)>0$ or $g(y)<0$.

! if $g(y)$ is -ve anywhere then there is instability.

If $g(y)$ changes in sign, it has been established in Sturm-Liouville theory, which applies here, that both positive and negative values of λ occur. (See Ince [5]). Hence in this case the system is unstable.

Thus in general, if $g(y)$ is anywhere negative, the system is unstable.

When $H_0=0$ this immediately gives us the Rayleigh criterion for stability, viz. $\frac{d}{dy}(U_0 y)^2 \geqslant 0$ at all points.

When $U_0 = 0$ the condition for stability is that

$$\frac{d}{dy}\left(\frac{H_0}{y}\right)^2 < 0,$$

← for $\Omega = 0$ $((B_\phi)^2)' < 0$ → stable i.e. $B_\phi = r^\alpha$ & $\alpha < 1$ → stability

i.e. H_0/y should continuously decrease in magnitude from the axis. The neutral case, we have already noted, is when H_0/y is constant. If $H_0 = ky^n$ the system is stable if $n < 1$, which means that the current density decreases in magnitude from the centre.

If both U_0 and H_0 fields are independently stable, then the combined fields will be stable. On the other hand, if the U_0 and H_0 fields are independently unstable, the combined field will be unstable. But unstable characteristics in one field may be overcome by the stability of the other. ← Good!

Suppose for example that the field H_0 is due to a line current down the axis so that $H_0 = C_1/y$, C_1 being constant. This field is in itself stable. The condition for stability with U_0 is that

$$\frac{1}{y^3}\frac{d}{dy}(U_0 y)^2 > \frac{\mu y}{4\pi\rho}\frac{d}{dy}\left(\frac{C_1}{y^2}\right)^2,$$

i.e.
$$\frac{d}{dy}(U_0 y)^2 > -\frac{C_1^2}{\pi\rho y}.$$

Thus a velocity U_0 for which

$$0 > \frac{d}{dy}(U_0 y)^2 > -\frac{\mu C_1^2}{\pi\rho y}.$$

is in itself unstable but is rendered stable by the line current. On the other hand if

$$\frac{d}{dy}(U_0 y)^2 < -\frac{C_1^2}{\pi\rho y}$$

the line of current does not suffice to remove the hydrodynamic instability.

4. When the system is unstable the question arises as to what form the instability will take.

Suppose now that $g(y)$ is not everywhere positive, so that negative characteristic values of λ exist. Since the time variation is with $e^{\pm(i/\sqrt{\lambda})t}$, the mode which will give the most rapid amplification in time is the one with the numerically least negative value of λ. This will ultimately be the dominant mode, and since when $\lambda = 0$ the solutions are non-oscillatory, the mode of maximum amplification will be one having no zeros between a and b.

Thus ultimately the disturbance will appear like $v = e^{i\beta x} f(y)$ where $f(y)$ is of the same sign in the range (a, b). This means that, for any value of x, the fluid is all moving either radially outwards or radially inwards and it is clear that this motion is of the same type as the cellular system of vortices observed and described by Taylor [2] in the case where the cylinders rotate in the same direction. We may infer from this that similar vortical systems may be set up by unstable magneto-hydrostatic fields, and by unstable combinations of rotating fluid with magnetic field.

References.

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