Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET III

These problems will be discussed in the 3rd Examples Class (to be scheduled).

1. Kinetic Alfvén Waves. There is an approximation, often used at very small scales, in which one assumes that magnetic field lines are frozen not into the mass flow $\mathbf{u}_i$ (ion velocity) but into the electron flow velocity $\mathbf{u}_e$, which can be expressed in terms the ion velocity $\mathbf{u}_i$ and the current density $\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$, where e is electron charge and $n = n_i = n_e$ is the ion/electron density. This is called the Electron (or Hall) MHD.

1. Under the above assumption, write a closed system of equations for $\mathbf{B}$ and $\mathbf{u}_i$, assuming incompressibility and neglecting viscosity and Ohmic diffusion.
2. Consider the static equilibrium with a straight uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}} = \text{const}$. Derive the dispersion relation for waves in this system. You will find the following definitions useful: $v_A = B_0/(4\pi n m_i)^{1/2}$ is the Alfvén speed (m_i is the ion mass), $\omega_{pi} = (4\pi e^2 n/m_i)^{1/2}$ is the plasma frequency.
3. Obtain an explicit formula for the frequency $\omega = \omega(\mathbf{k})$ from your dispersion relation. Under what conditions do you recover the Alfvén waves?
4. The quantity $d_i = c/\omega_{pi}$ is called the plasma skin depth. Assume $kd_i \gg 1$ (k is the absolute value of the wave vector) and find the corresponding limiting form of the dispersion relation. The waves you have obtained are called the kinetic Alfvén waves (KAW).

2. Reduced Electron MHD. In Problem 1, if we had assumed from the outset that the characteristic scale of all fields $l \ll d_i$ and the charateristic time is $\tau \ll l/v_A$, we would have found that the equation for the magnetic field in this limit is

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n}\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]\,. \tag{1}$$

Again let us consider perturbations about a straight-field equilibrium, $\mathbf{B} = B_0\hat{\mathbf{z}} + \delta\mathbf{B}$. For infinitesimal perturbations, the linearised Eq. (1) again gives KAW. In this Problem, you will consider perturbations that are small, but not infinitesimally so. In a way similar to my derivation of the Reduced MHD equations, let us assume that the perturbations are highly anisotropic, $k_\parallel \ll k_\perp$ (this is confirmed by numerical simulations of EMHD turbulence). Let us further assume that the wave frequency and the nonlinear interaction time are same order.

1. Show that this implies

$$\frac{\delta B}{B_0} \sim \frac{k_\parallel}{k_\perp}. \tag{2}$$

   This is the ordering you will now use to derive a reduced version of EMHD.

2. Show that the magnetic field can be represented as follows:

$$\frac{\delta \mathbf{B}}{B_0} = \frac{1}{v_A}\,\hat{\mathbf{z}} \times \nabla_\perp \Psi + \hat{\mathbf{z}}\,\frac{\delta B_\parallel}{B_0}. \tag{3}$$

3. Show that the evolution equations for Ψ and $\delta B_\parallel$ are

$$\frac{\partial \Psi}{\partial t} = v_A^2 d_i \frac{\mathbf{B}}{B_0} \cdot \nabla \frac{\delta B_\parallel}{B_0}, \tag{4}$$
$$\frac{\partial}{\partial t}\frac{\delta B_\parallel}{B_0} = -d_i \frac{\mathbf{B}}{B_0} \cdot \nabla \nabla_\perp^2 \Psi, \tag{5}$$

where

$$\frac{\mathbf{B}}{B_0} \cdot \nabla = \frac{\partial}{\partial z} + \frac{\delta \mathbf{B}_\perp}{B_0} \cdot \nabla_\perp = \frac{\partial}{\partial z} + \frac{1}{v_A}\{\Psi, \cdots\}. \tag{6}$$

The algebra here is quite unpleasant. You will probably need the NRL Plasma Formulary to deal with some of the multiple vector products.

4. Check that these equations give the right dispersion relation for KAW.

3. Conservation Laws in Reduced MHD. In class, I derived evolution equations for 5 scalar quantities in anisotropic MHD: Ψ, Φ, $\delta B_\parallel$, $u_\parallel$ and $\delta\rho$. These equations have 5 quadratic conserved quantities (5 cascades that do not exchage energy). Work them out.

4. Advection of Magnetic Field in Ideal MHD. If Ohmic diffusion is ignored, the randomly advected magnetic field $\tilde{\mathbf{B}}(t)$ satifies the following equation (in the Lagrangian frame):

$$\frac{\partial \tilde{B}^i}{\partial t} = \sigma^i_m \tilde{B}^m, \tag{7}$$

where $\sigma^i_m(t) = \partial u^i/\partial x^m$ is the gradient of the velocity field. Note that there is no explicit dependence on the space variable anywhere. Take the velocity field to be a Gaussian random field white in time, three-dimensional and isotropic, and

$$\langle \sigma^i_m(t)\sigma^j_n(t')\rangle = \delta(t-t')\kappa_2 T^{ij}_{mn}, \quad T^{ij}_{mn} = \left[\delta^{ij}\delta_{mn} - \frac{1}{4}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right)\right]. \tag{8}$$

1. Let $\tilde{P}(\mathbf{B}, t) = \delta(\mathbf{B} - \tilde{\mathbf{B}}(t))$. Define the PDF of the magnetic field $P(B,t) = \langle \tilde{P}(\mathbf{B}, t)\rangle$. Derive a closed equation for this PDF using the Furutsu-Novikov formula

$$\langle \sigma^i_m(t)\tilde{P}(t)\rangle = \int dt' \langle \sigma^i_m(t)\sigma^j_n(t')\rangle \left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t')}\right\rangle. \tag{9}$$

2. Because of isotropy, P only depends on the absolute value $B = |\mathbf{B}|$, so the normalisation rule is

$$1 = \int d^3\mathbf{B} P(B) = 4\pi \int dB\, B^2 P(B). \tag{10}$$

Define $F(B) = 4\pi B^2 P(B)$ and derive an equation for $F(B)$. The result should be

$$\frac{\partial F}{\partial t} = \frac{\gamma}{5}\frac{\partial}{\partial B}\left(B^2 \frac{\partial F}{\partial B} - 2BF\right), \tag{11}$$

where $\gamma = (5/4)\kappa_2$.

3. What is the expression for magnetic energy $\langle B^2\rangle(t)/8\pi$ in terms of the function $F(B,t)$? Derive an equation for $\langle B^2\rangle(t)$ and show that $\langle B^2\rangle(t)$ grows exponentially at the rate 2γ.

4. Now derive the growth rates of higher moments of the magnetic field: let $\langle B^n \rangle \propto e^{\gamma_n t}$. Calculate γ_n in terms of γ and n. From this, find how the quantity $\langle B^4 \rangle / \langle B^2 \rangle^2$ changes with time. How would you interpret this quantity? What does the time evolution you have determined tell you about the volume-filling properties of the field?

5. By a simple change of variables, convert Eq. (11) into an equation with constant coefficients. Assuming that at $t = 0$, $F(B, 0) = \delta(B - B_0)$, show that the solution of the equation is

$$F(B,t) = \frac{e^{-(9/20)\gamma t}}{B_0\sqrt{(4/5)\pi\gamma t}} \left(\frac{B}{B_0}\right)^{1/2} \exp\left\{-\frac{[\ln(B/B_0)]^2}{(4/5)\gamma t}\right\}, \tag{12}$$

so the PDF is a lognormal spreading with time. In your solution, you can use the fact that the diffusion equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{13}$$

has the following Green's function solution

$$f(t,x) = \int dx' \frac{f(t',x')}{\sqrt{4\pi D(t-t')}} \exp\left[-\frac{(x-x')^2}{4D(t-t')}\right], \quad t' < t. \tag{14}$$

5. Scalar Turbulence. Part IV: Spectrum of Scalar Variance in the Viscous-Convective Range. Consider the equation for the evolution of passive scalar $\theta(t, \mathbf{x})$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = \eta\nabla^2\theta. \tag{15}$$

where η is the scalar diffusivity (sorry about change of notation! — I need κ for velocity correlators). Consider scalar decay in a linear velocity field:

$$\mathbf{u}^i = \sigma^i_m(t)x^m. \tag{16}$$

(When is is this a reasonable model?) and take the velocity field to be a Gaussian white noise:

$$\langle \sigma^i_m(t)\sigma^j_n(t') \rangle = \delta(t-t')\kappa_2 \left[\delta^{ij}\delta_{mn} - \frac{1}{d+1}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right)\right]. \tag{17}$$

1. Construct a calculation leading to the equation for the passive scalar spectrum in the way exactly analogous to my calculation for the dynamo (see my lecture notes):

   (a) Write the solution of Eq. (15) as a superposition of plane waves. Find evolution equations for the amplitudes and wavevectors of these waves.

   (b) Define the joint PDF of the amplitudes and wavevectors. Derive a closed equation for this PDF using Furutsu-Novikov formula. Note that, because of isotropy, the PDF only depends on the absolute value of the wavevector — this will simplify your equation.

   (c) Show that the spectrum of the scalar variance is a superposition of spectra of the plane waves. Derive the equation for the spectrum $T(t, k)$:

$$\frac{\partial T}{\partial t} + 2\eta k^2 T = -\frac{\partial}{\partial k}\mathcal{F}(k) = D\frac{\partial}{\partial k}\left[k^2\frac{\partial T}{\partial k} - (d-1)kT\right], \tag{18}$$

   where $D = \kappa_2(d-1)/2(d+1)$ and $\mathcal{F}(k)$ is the flux of scalar variance. This equation was first derived by Kraichnan in 1968 (in a different way).

(d) Seek eigenfunction solutions of this equation, $T(t, k) = e^{-\lambda D t}\Phi(k/k_\eta)$, where $k_\eta = (D/2\eta)^{1/2}$. Solve for Φ.

If we introduce some cut-off wavenumber $k_* \ll k_\eta$, λ is determined by the boundary condition on the flux $\mathcal{F}(k_*)$.

2. Let us consider a forced scalar problem for a moment (as in Problem 4 of Example Sheet I). The forcing is pumping a constant flux $\bar{\epsilon}_\theta$ at some large scale. All this flux must be dissipated, so we must have $\mathcal{F}(k_*) = \bar{\epsilon}_\theta$. Find the solution that satisfies this boundary condition and show that it has the Batchelor k^{-1} scaling at $k_* < k \ll k_\eta$.

3. Now consider the decaying case. As in the dynamo case, we might think a zero-flux boundary condition should be imposed: $\mathcal{F}(k_*) = 0$. Calculate λ in this case. What is the slope of the spectrum at $k_* < k \ll k_\eta$? Argue that your prediction for the decay rate means it is of the order of the turnover time of the viscous eddies.

4. These results had been thought to describe the scalar decay correctly until numerical experimental evidence showed the decay to be much slower: this effect is called *the strange mode.* In fact, the decay rate of the scalar is set by the decay rate of the slowest-decaying mode, which is a box-scale mode not described by the viscous-convective-range theory. It decays at the rate of turbulent diffusion associated with the box size $L_{\rm box}$, so we have, in fact,

$$\lambda \sim \frac{\text{decay rate of the box mode}}{\text{viscous eddy turnover rate}} \sim \frac{\delta u_L L/L_{\rm box}^2}{\delta u_{l_\nu}/l_\nu} \sim \left(\frac{L}{L_{\rm box}}\right)^2 \mathrm{Re}^{-1/2} \ll 1, \tag{19}$$

where $L \le L_{\rm box}$ is the outer scale of the turbulence. Do you understand this estimate? Derive the last expression.

Assuming that λ is set by Eq. (19) and is equal to some small number, show from your solution that the scalar variance spectrum at $k_* < k \ll k_\eta$ scales as $k^{-1+\lambda/d}$ — only slightly shallower than Batchelor's spectrum.

These results are due to Fereday & Haynes, *Phys. Fluids* **16**, 4359 (2004) and Schekochihin, Haynes & Cowley, *Phys. Rev. E* **70**, 046304 (2004), but do try to derive them youself before you look!