Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET II

These problems will be discussed in the 2nd Examples Class (14.11.05, 14:00, room TBD).

1. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with density $n = 1\ \mathrm{cm}^{-3}$ undergoing a gravitational collapse to a spherical star with density $n = 10^{26}\ \mathrm{cm}^{-3}$. The magnetic field in the galactic plasma is $\sim 3\times 10^{-6}$ G. Assuming that flux is frozen, estimate the magnetic field in a star.

2. Flux Concentration. Consider a simple 2D model of incompressible convective motion:

$$\mathbf{u} = U\left(-\sin\frac{\pi x}{L}\cos\frac{\pi z}{L},\ 0,\ \cos\frac{\pi x}{L}\sin\frac{\pi z}{L}\right). \tag{1}$$

1. In the neighbourhood of the stagnation point $(0,0,0)$, linearise the flow, assume vertical magnetic field, $\mathbf{B} = (0,0,B(t,x))$ and derive an equation for $B(t,x)$. Suppose $B(0,x) = B_0 = \mathrm{const}$ It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\mathrm{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.
2. Find a steady solution of your equation. Use flux conservation and $B(x) = B(-x)$ to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?
3. Can you think of a heuristic argument based on the induction equation that would tell you that these answers were to be expected?

3. Zeldovich Antidynamo Theorem. Consider the case of an arbitrary 2D velocity field: $\mathbf{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (specifically, you may work in a periodic box), this velocity field is not a dynamo, i.e., any initial magnetic field will always eventually decay. This is one of the classical antidynamo results: the Zeldovich Theorem.
Hint. Consider separately the equations for B_z and for the (x,y)-plane magnetic field. Show that B_z decays (consider time evolution of the volume integral of B_z^2), then write B_x, B_y in terms of one scalar function (this is possible because $\partial B_x/\partial x + \partial B_y/\partial y = 0$) and show that it deacays as well.

4. X-Point Collapse. Let us set up the following initial magnetic-field configuration:

$$\mathbf{B}_0(\mathbf{r}_0) = B_0\hat{\mathbf{z}} + \hat{\mathbf{z}}\times\nabla_0\psi(x_0,y_0), \tag{2}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$, $B_0 = \mathrm{const}$, and

$$\psi(x_0,y_0) = \frac{1}{2}\left(x_0^2 - y_0^2\right). \tag{3}$$

1. Draw the field lines in the (x_0, y_0) plane to see that this is an X-point.
2. Use Lagrangian MHD

$$\rho_0\frac{\partial^2\mathbf{r}}{\partial t^2} = -J\left(\nabla_0\mathbf{r}\right)^{-1}\cdot\nabla_0\left(\frac{p_0}{J^\gamma} + \frac{|\mathbf{B}_0\cdot\nabla_0\mathbf{r}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\mathbf{B}_0\cdot\nabla_0\left(\frac{\mathbf{B}_0}{J}\cdot\nabla_0\mathbf{r}\right), \tag{4}$$

where $\mathbf{r}(t, \mathbf{r}_0) = (x, y, z)$ and $J = |\det \nabla_0 \mathbf{r}|$, and seek solutions in the form

$$x = \xi(t) x_0, \quad y = \eta(t) y_0, \quad z = z_0. \tag{5}$$

Show that $\xi(t)$ and $\eta(t)$ satisfy the following equations

$$\frac{d^2\xi}{dt^2} = \eta \left(\frac{1}{\eta^2} - \frac{1}{\xi^2} \right), \tag{6}$$

$$\frac{d^2\eta}{dt^2} = \xi \left(\frac{1}{\xi^2} - \frac{1}{\eta^2} \right). \tag{7}$$

3. Consider the possibility that, as time goes on, $\eta(t) \to 0$ (becomes small) and $\xi(t) = \xi_c + \dots$, where ξ_c is some constant. Find solutions that satisfy this assumption. The answer is

$$\xi(t) \approx \xi_c + \frac{9}{4} \left(\frac{2}{9\xi_c} \right)^{1/3} (t_c - t)^{4/3}, \tag{8}$$

$$\eta(t) \approx \left(\frac{9\xi_c}{2} \right)^{1/3} (t_c - t)^{2/3} \tag{9}$$

as $t \to t_c$, where t_c is some finite time constant. This is called *the Syrovatskii solution* for the X-point collapse.

4. Calculate the magnetic field as a function of time and convince yourself that the initial X-point configuration collapses explosively (in a finite time) to a sheet along the x axis. What do you think happens after t reaches t_c?

5. Now do a similar calculation for incompressible Lagrangian MHD ($J = 1$). Remember that total pressure is now determined by the condition $J = 1$. Show that the solution in this case is

$$\xi(t) = e^{S(t)}, \quad \eta(t) = e^{-S(t)}, \tag{10}$$

where $S(t)$ is an arbitrary function of time. If, e.g., $S(t) = \Lambda t$, show that this means the X-point collapses exponentially. This is called *the Chapman-Kendall solution.*

5. Conservation Laws for Incompressible MHD. Consider equations of incompressible MHD:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \tag{11}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}, \tag{12}$$

where $\mathbf{f}$ is some forcing function and $\rho = \text{const}$. Derive the evolution equations for the kinetic energy $\int d^3x\, \rho u^2/2$, magnetic energy $\int d^3x\, B^2/8\pi$ and cross-helicity $\int d^3x\, \mathbf{u} \cdot \mathbf{B}$ directly from these equations and make sure your result is consistent with the incompressible limit of the more general equations I derived in class (you should find the incompressible derivation much simpler). Assume that all surface integrals vanish. Can you interpret all terms in your energy equations? Under what assumption are the total energy (kinetic + magnetic) and cross-helicity conserved? When energy is "lost" in this system, where does it go?

6. Ambipolar and Viscous Damping. Consider the incompressible MHD for a plasma with a neutral component:

$$\begin{aligned}
\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i &= -\frac{\nabla p}{\rho_i} + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi\rho_i} - \mu_{in}\left(\mathbf{u}_i - \mathbf{u}_n\right), \quad \nabla\cdot\mathbf{u}_i = 0, && (13)\\
\frac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{u}_n &= -\frac{\nabla p_n}{\rho_n} + \nu_n \nabla^2 \mathbf{u}_n - \mu_{ni}\left(\mathbf{u}_n - \mathbf{u}_i\right), \quad \nabla\cdot\mathbf{u}_n = 0, && (14)\\
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{B} &= \mathbf{B}\cdot\nabla\mathbf{u}_i, && (15)
\end{aligned}$$

where ion viscosity and magnetic diffusivity have been ignored, subscripts i and n refer to ion and neutral quantities, respectively, $\nu_n \sim v_{\rm th}\lambda_{\rm mfp}$ is the neutral viscosity, $\mu_{in} \sim v_{\rm th}/\lambda_{\rm mfp}$ is the ion-neutral collision rate, $\mu_{ni} = (\rho_i/\rho_n)\mu_{in} = \mu_{in}\chi/(1-\chi)$ is the neutral-ion collision rate, $\chi = \rho_i/(\rho_n+\rho_i)$ is the degree of ionisation of the plasma.

Assume a straight-field static equilibrium, $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. Write the ion and neutral velocities in terms of ion and neutral displacements and work out the dispersion relation:

$$i\omega^3 - \omega^2\left(\nu_n k^2 + \frac{\mu_{in}}{1-\chi}\right) - i\omega\left(k_\parallel^2 v_A^2 + \mu_{in}\nu_n k^2\right) + k_\parallel^2 v_A^2\left(\nu_n k^2 + \frac{\chi}{1-\chi}\,\mu_{in}\right) = 0. \qquad (16)$$

Consider various asymptotic solutions of this dispersion relation. You will find that the following are the interesting limits:

1. $\beta \ll 1$ (show that this implies $\nu_n k^2 \ll k_\parallel^2 v_A^2/\mu_{in}$)
   (a) $k\lambda_{\rm mfp} \ll \beta^{1/2}$ (show that this implies $k_\parallel v_A \ll \mu_{in}$)
   — Alfvén waves plus ambipolar damping.
   (b) $\beta^{1/2} \ll k\lambda_{\rm mfp} \ll 1$ (show that this implies $\nu_n k^2 \ll \mu_{in} \ll k_\parallel v_A$)
   — Alfvén waves plus collisional damping.
   (c) $k\lambda_{\rm mfp} \gg 1$ (i.e., $\nu_n k^2 \gg \mu_{in}$)
   — Undamped Alfvén waves.

2. $\beta \gg 1$ (i.e., $\nu_n k^2 \gg k_\parallel^2 v_A^2/\mu_{in}$)
   (a) $k\lambda_{\rm mfp} \ll 1/\beta^{1/2}$ (show that this implies $\nu_n k^2 \ll k_\parallel v_A$)
   — Alfvén waves plus viscous damping.
   (b) $1/\beta^{1/2} \ll k\lambda_{\rm mfp} \ll 1$ (show that this implies $k_\parallel v_A \ll \nu_n k^2 \ll \mu_{in}$)
   — Viscous relaxation (non-oscillatory).
   (c) $1 \ll k\lambda_{\rm mfp} \ll \beta^{1/2}$ (i.e., $\nu_n k^2 \gg \mu_{in} \gg k_\parallel v_A$)
   — Ambipolar relaxation (non-oscillatory).

For each of these cases, find all three solutions of the dispersion relation. Which solution has the weakest damping? (meaning that it is the long-term solution) Do the verbal descriptions of these solutions given above make sense to you? (if not, you might have made a mistake!)

Physically speaking, do you trust all of these results equally well?