MHD Turbulence in Galaxies and Clusters

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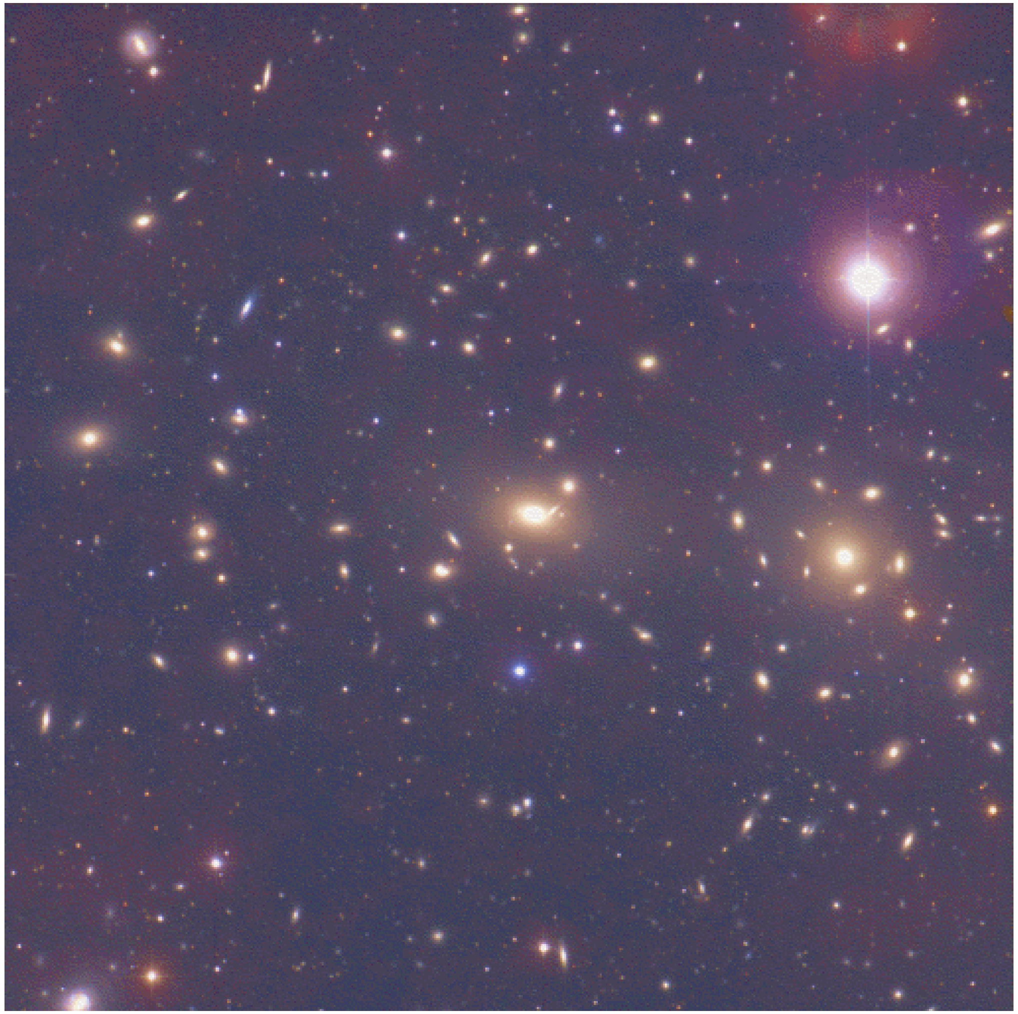
This is a cluster of galaxies…

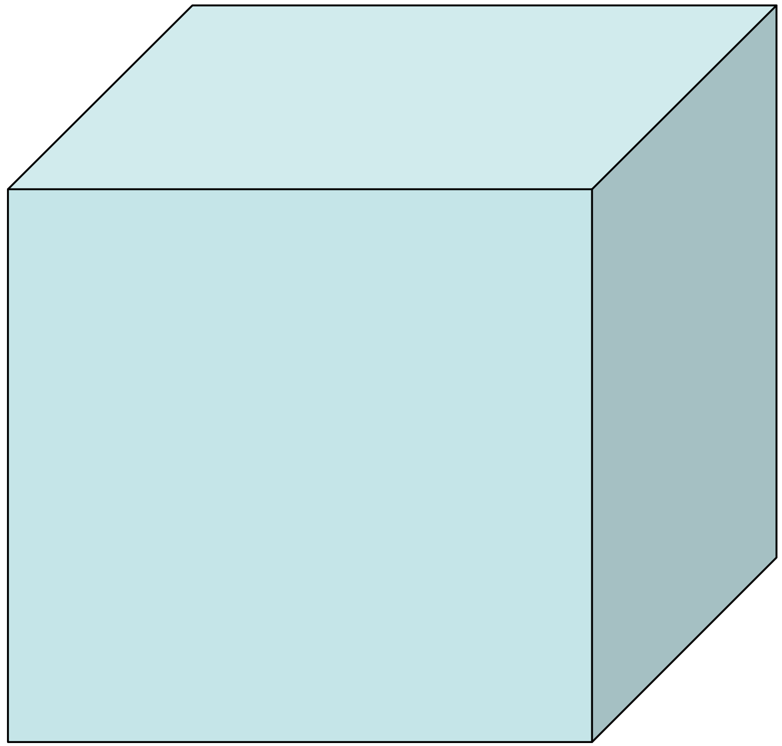


The Coma Cluster

This is a galaxy…

M51
The Whirlpool Galaxy

…and this is the Matrix



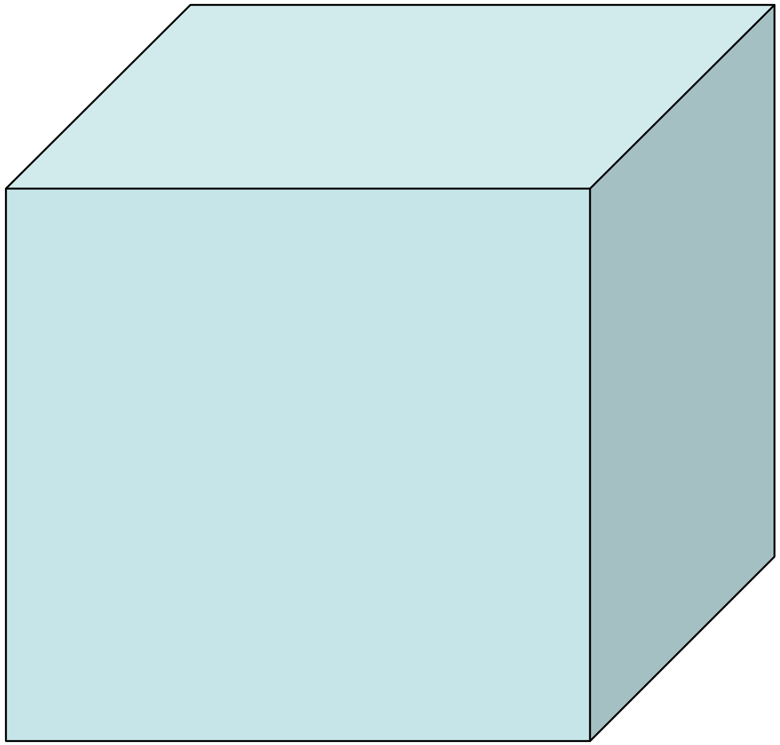
N^3

The Periodic Box

MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0$$

$$\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B}$$



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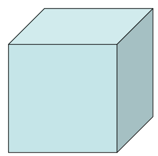

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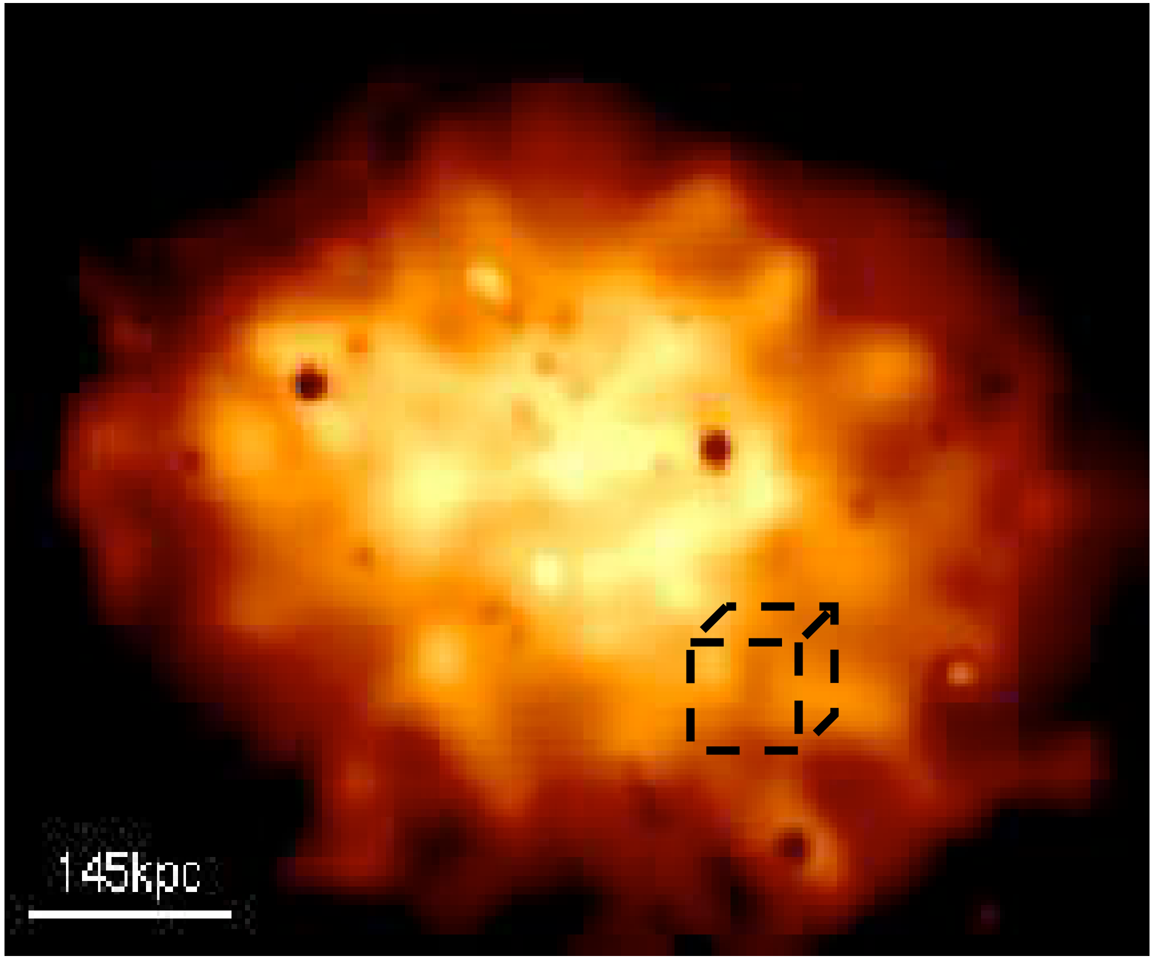


Coma Cluster
from
Schuecker *et al.*
astro-ph/0404132

MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$
$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$

Clusters are wonderful:

- **amorphous plasma**
- **no rotation (nonhelical!)**
- **subsonic turbulence (from mergers/structure formation)**
- **tangled magnetic fields**

… a case of "pure" turbulence

(more about them later…)

MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$
$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$

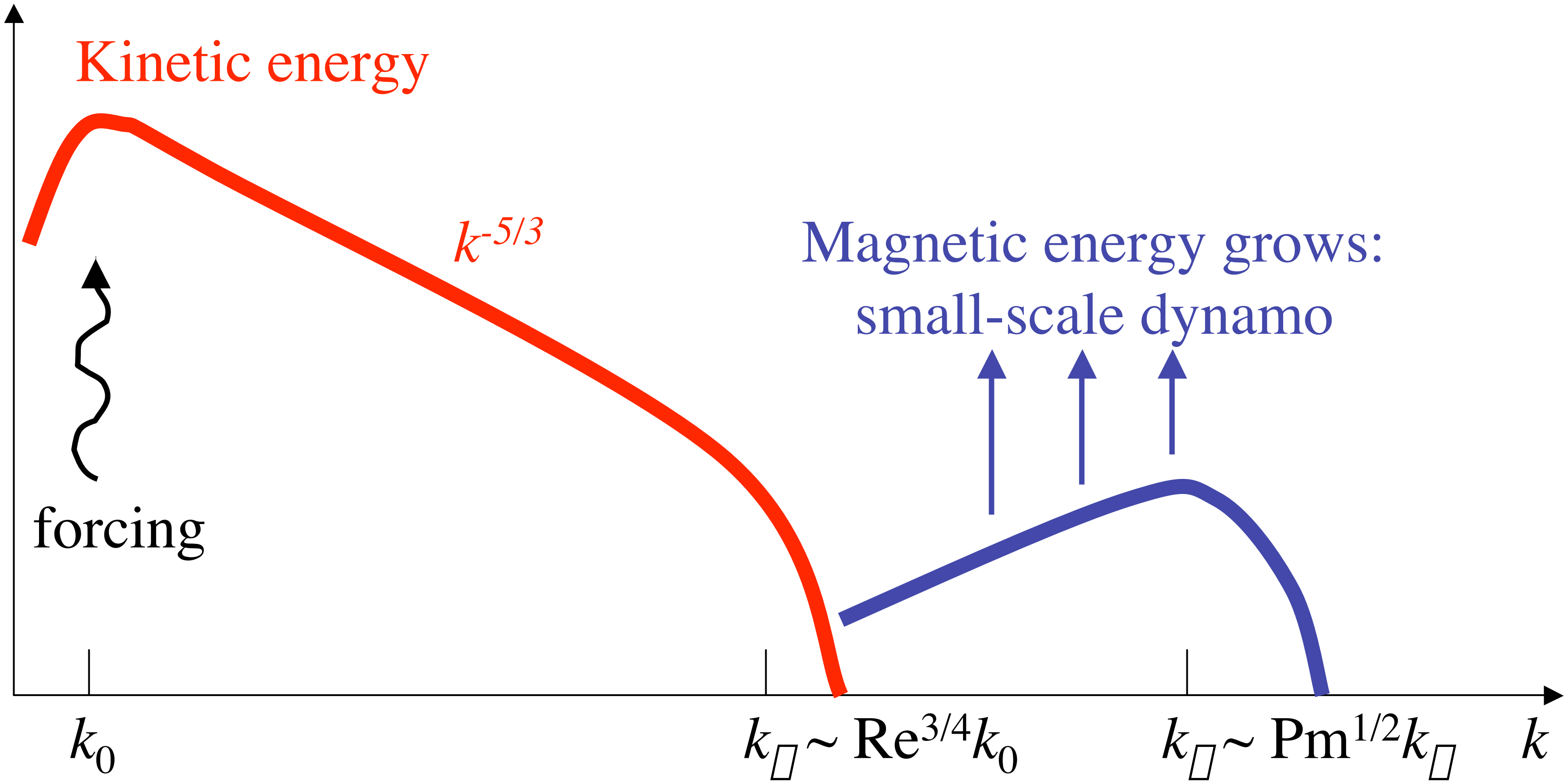
Set $\langle \mathbf{B} \rangle = 0$, start with small seed field. There are two main issues:

1. Is there small-scale dynamo, i.e., does magnetic energy $\langle B^2 \rangle$ grow?
2. If it does, **how does it saturate**: what is the long-term state of the fully-developed isotropic MHD turbulence?

Can we simulate this problem in our box?

I.e., what are the scales ranges and what happens at smallest scales?

Two Scale Ranges in the Problem

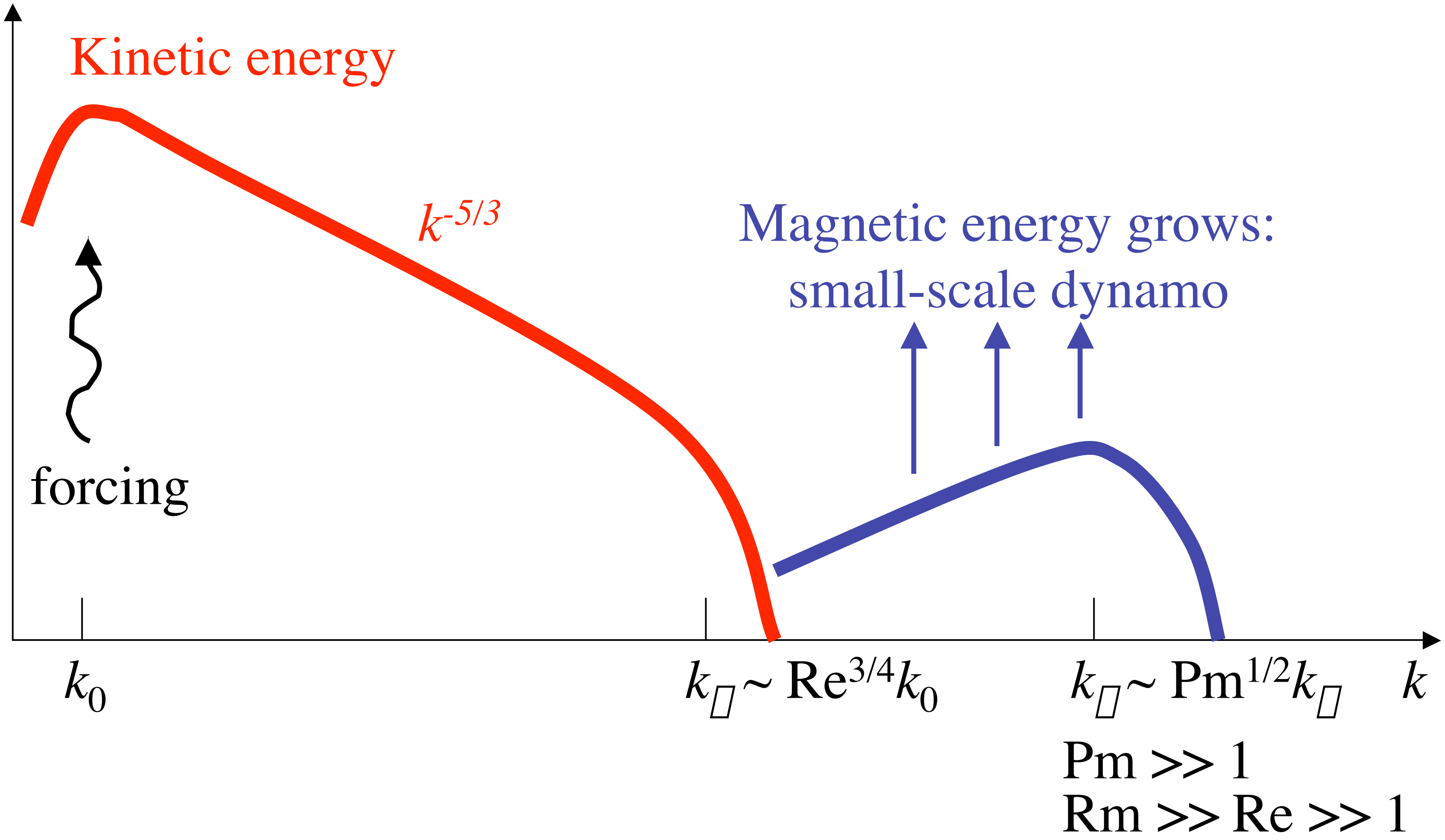


<u>A key parameter: ***magnetic Prandtl number***</u>

Pm = $\nu/\eta \sim 2.6\times10^{-5}\, T^4/n$ (ionised) or $1.7\times10^{7}\, T^2/n$ (with neutrals)

- **Pm >>1**: galaxies (10^{14}), clusters (10^{29})
- **Pm <<1**: planets (10^{-5}), stars ($10^{-7}\ldots10^{-4}$), protostellar discs (10^{-8}), liquid-metal laboratory dynamos

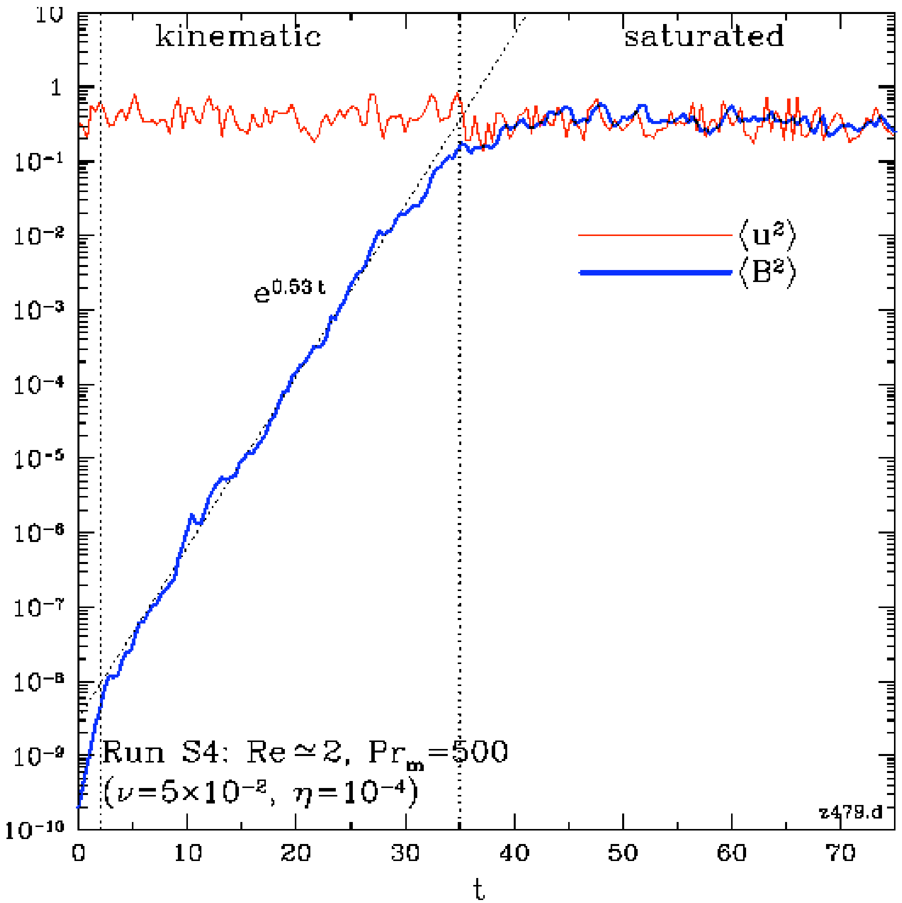
Small-Scale Dynamo at Pm ≥ 1

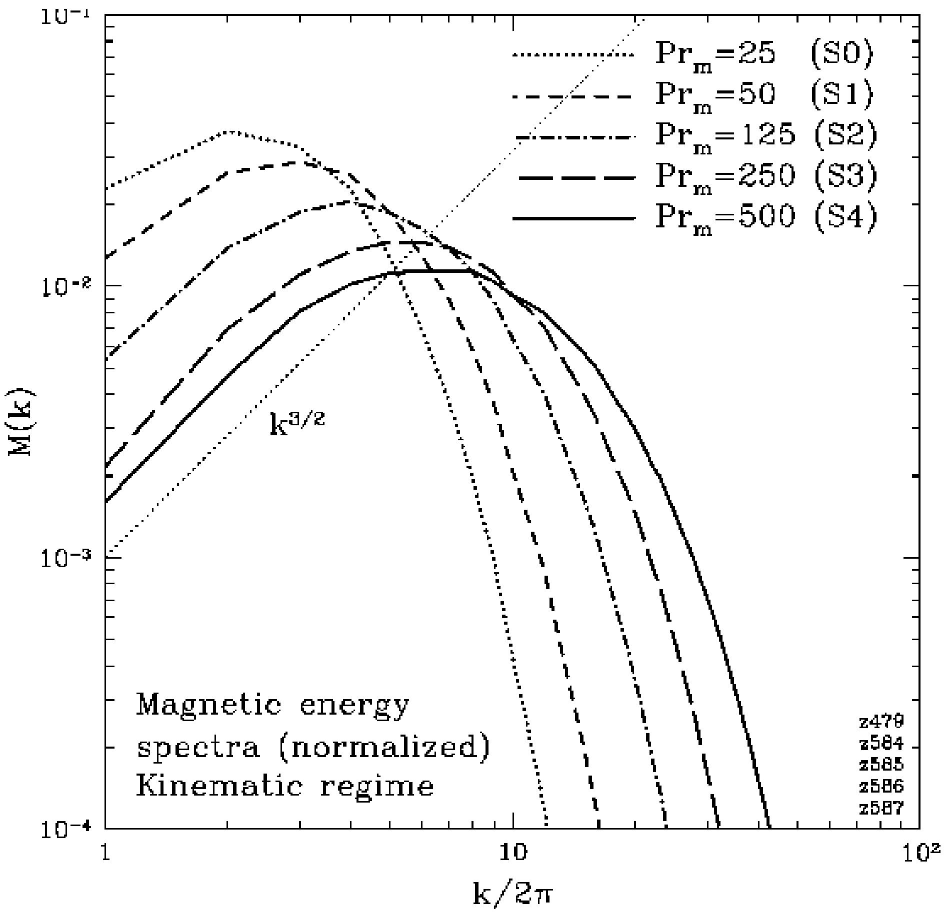


Pm ≥ 1: it is well established numerically that an initial weak seed field will grow at the smallest scales provided $Rm > Rm_c \sim 10^2$

[Meneguzzi, Frisch & Pouquet 1981, *PRL* **47**, 1060]

Small-Scale Dynamo: DNS



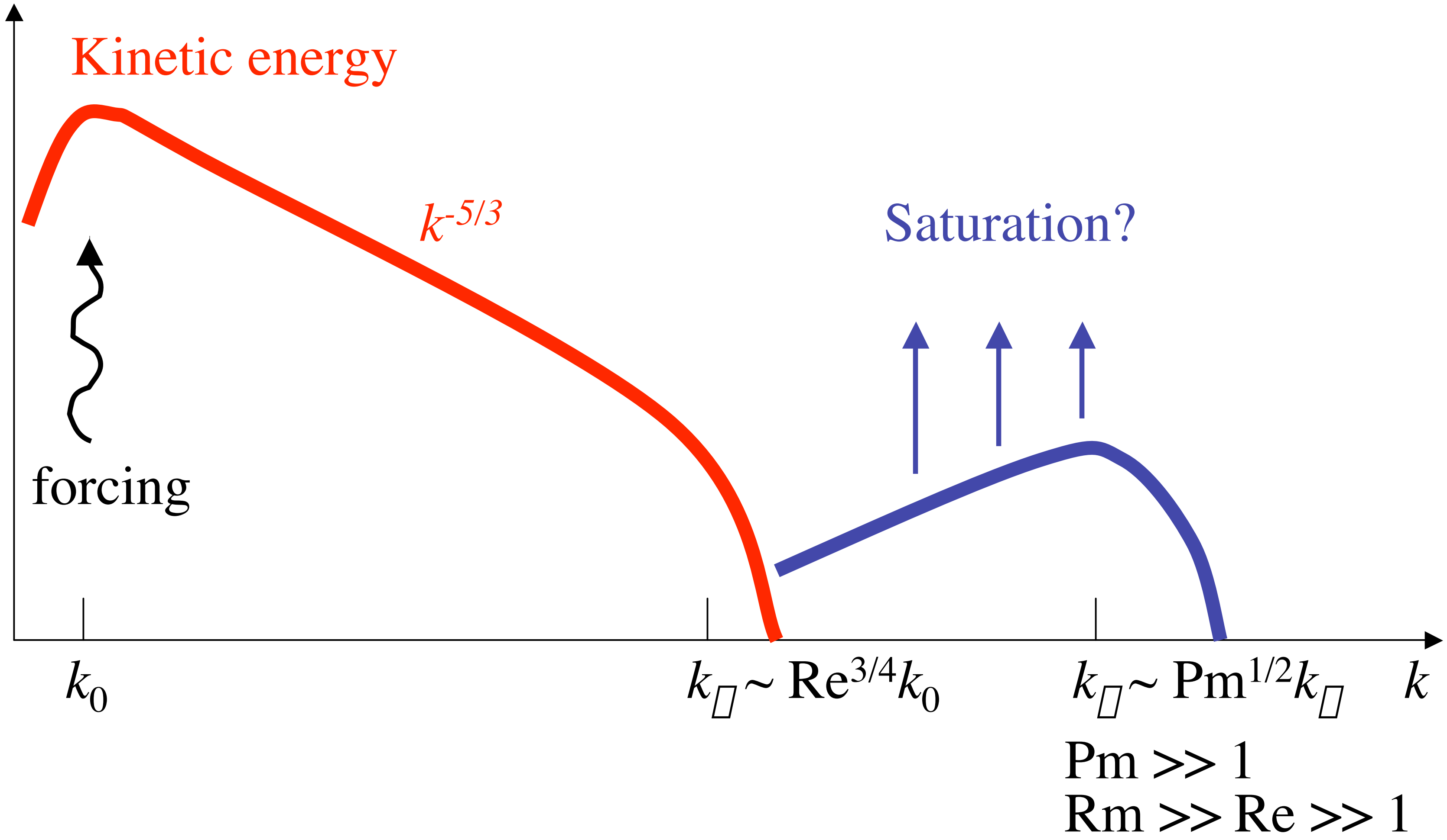


- **$\langle B^2\rangle$ grows exponentially, then saturates**
- **Field at the resistive scale ($k_\eta \sim$ Pm$^{1/2} k_\nu$)**

Fairly sophisticated analytical treatment of this regime is possible

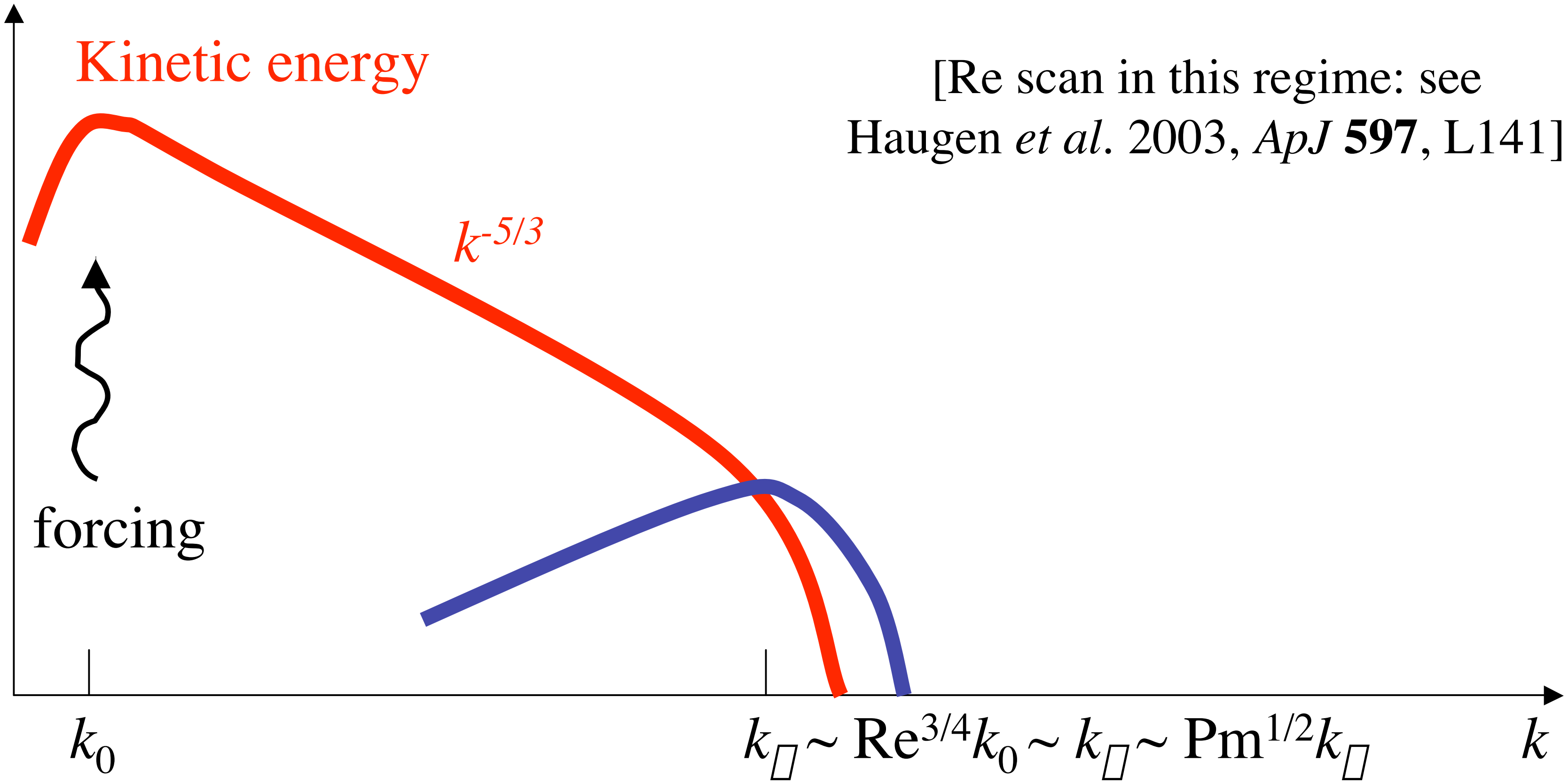
[AAS *et al*. 2004, *ApJ* **612**, 276 and references therein]

Numerical Approaches



Do we really need to resolve two scale ranges **<u>in saturation</u>**?

Numerical Approaches: Re >> 1, Pm ~ 1



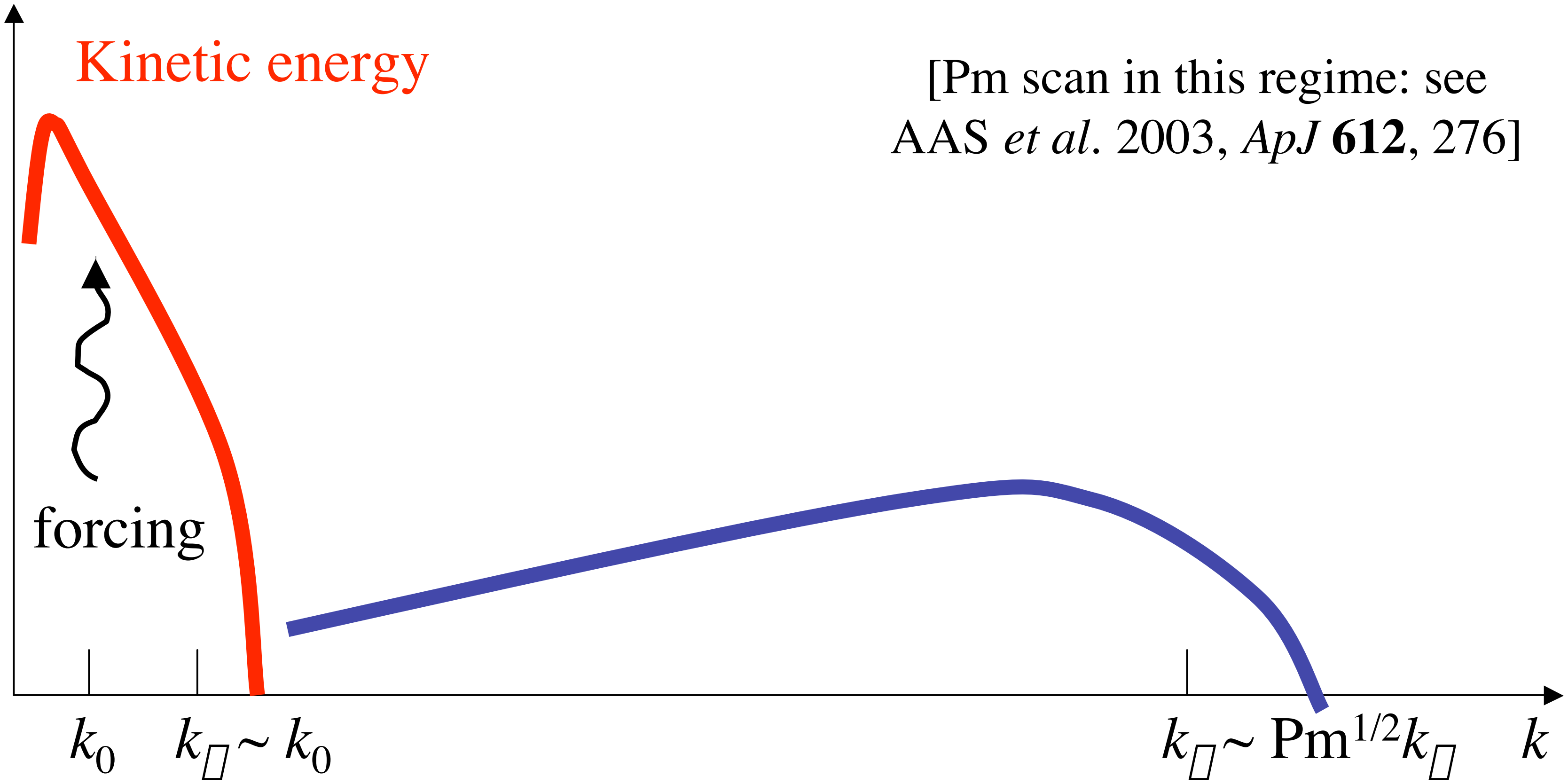
[Re scan in this regime: see Haugen *et al*. 2003, *ApJ* **597**, L141]

Do we really need to resolve two scale ranges in saturation?

***Approach I*: Re >> 1, Pm ~ 1 (sacrifice subviscous range)**

Numerical Approaches: Re ~ 1, Pm >> 1

[Pm scan in this regime: see AAS *et al*. 2003, *ApJ* **612**, 276]



Do we really need to resolve two scale ranges in saturation?

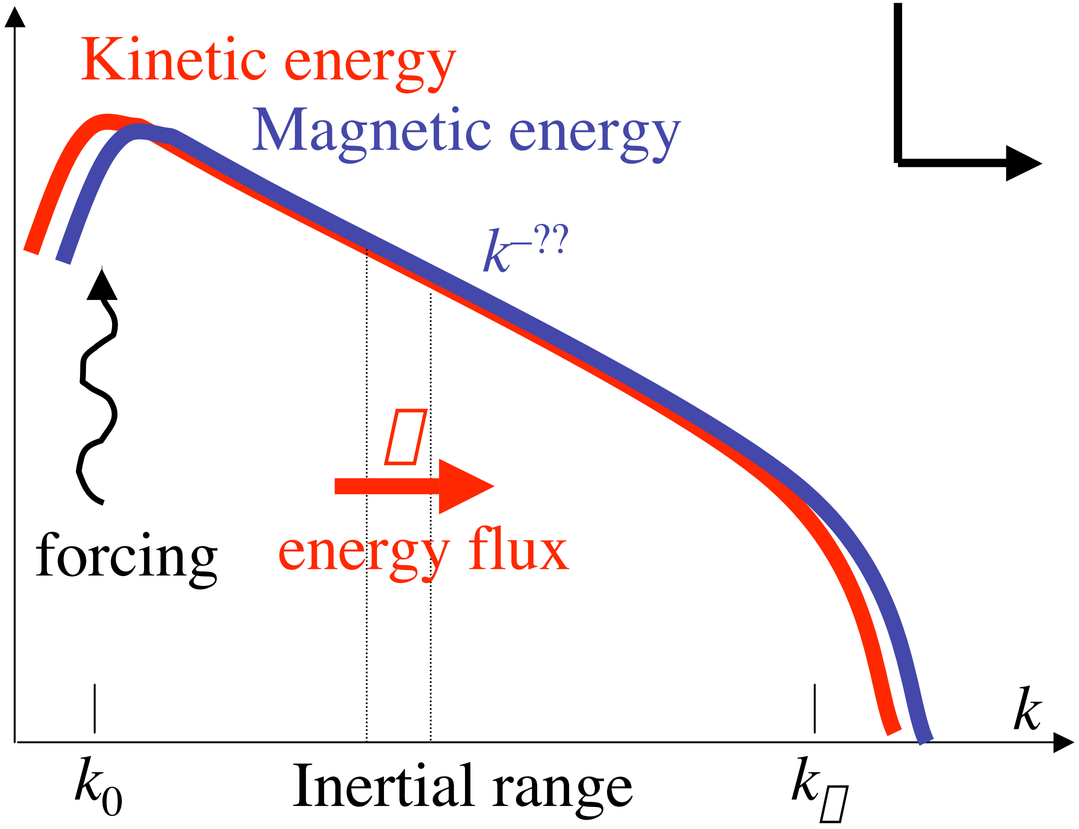
Approach I: Re >> 1, Pm = 1 (sacrifice subviscous range)

***Approach II*: Re ~ 1, Pm >> 1 (sacrifice inertial range)**

MHD Turbulence: Standard Picture

Standard approach: forget about dissipation scales and

assume
- **magnetic energy is large-scale dominated**
- **elementary motions are Alfvénic,**
  so $u_k \sim B_k$
- **interactions are local in *k* space**



Scaling arguments
à la Kolmogorov

MHD Turbulence: Standard Picture

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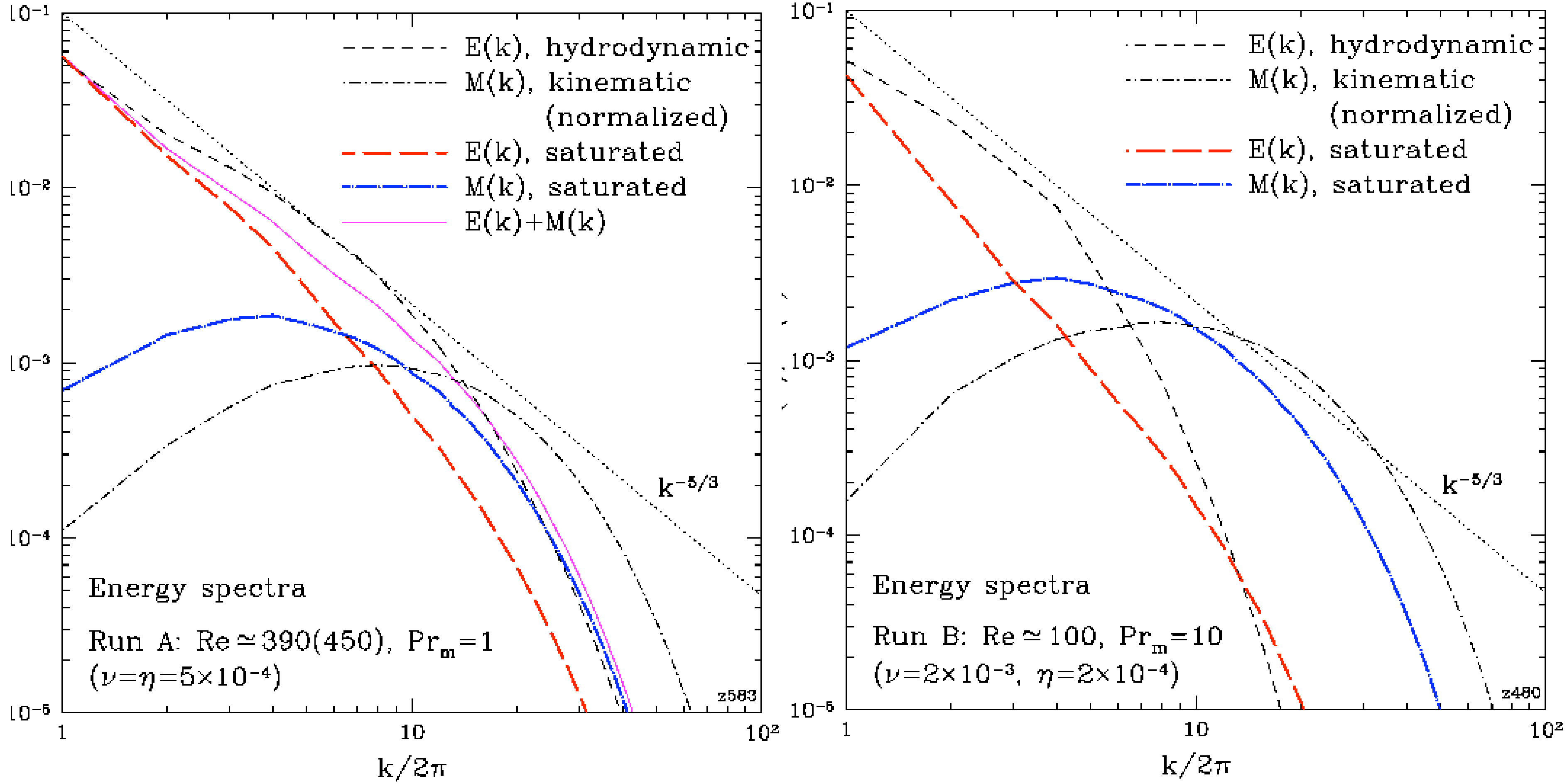
- **interactions are local in k space**

With further assumptions, obtain various scaling laws…

- **IK63/65:** weak interactions, isotropy → $E(k) \sim (\varepsilon v_A)^{1/2}\, k^{-3/2}$
- **Weak turbulence:** weak interactions, no cascade in $k_{||}$ (extreme anisotropy) → $E(k) \sim (\varepsilon k_{||} v_A)^{1/2}\, k_{\perp}^{-2}$
- **GS95:** strong interactions/critical balance (Alfvén time ~ turnover time) → $E(k) \sim \varepsilon^{2/3}\, k_{\perp}^{-5/3}$, $k_{||} \sim \varepsilon^{1/3} v_A^{-1}\, k_{\perp}^{2/3}$

Isotropic MHD Turbulence: DNS

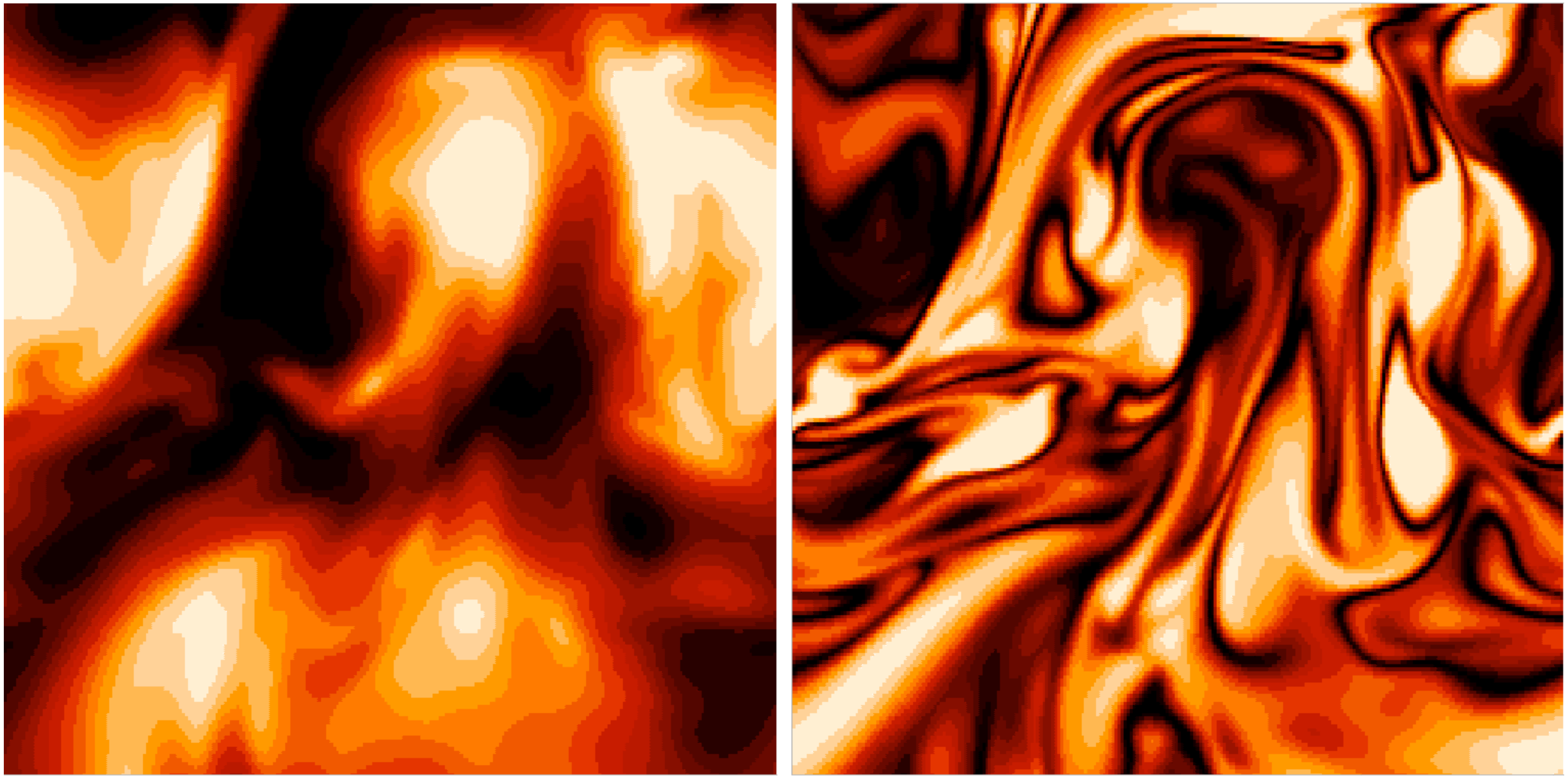


There is no evidence of scale-by-scale equipartition!

Excess of magnetic energy at small scales

[Maron *et al*. 2004, *ApJ* **603**, 569; AAS *et al*. 2004, *ApJ* **612**, 276;
Highest resolution to date (1024^3) : Haugen *et al*. 2003, *ApJ* **597**, L141]

Isotropic MHD Turbulence: DNS



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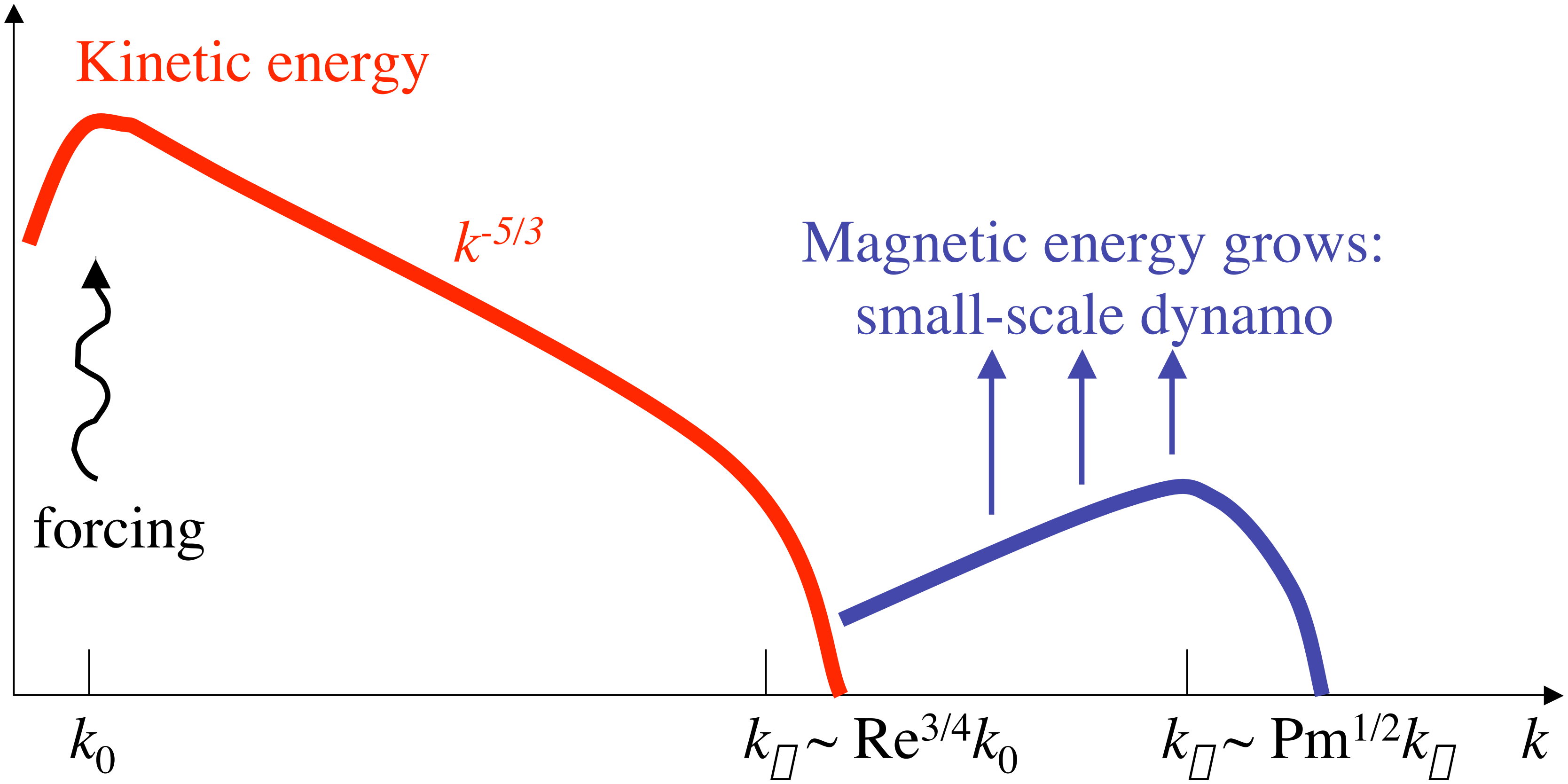
Excess of magnetic energy at small scales

[AAS *et al*. 2004, *ApJ* **612**, 276]

Small-Scale Dynamo at Pm ≥ 1



Pm >> 1
Rm >> Re >> 1

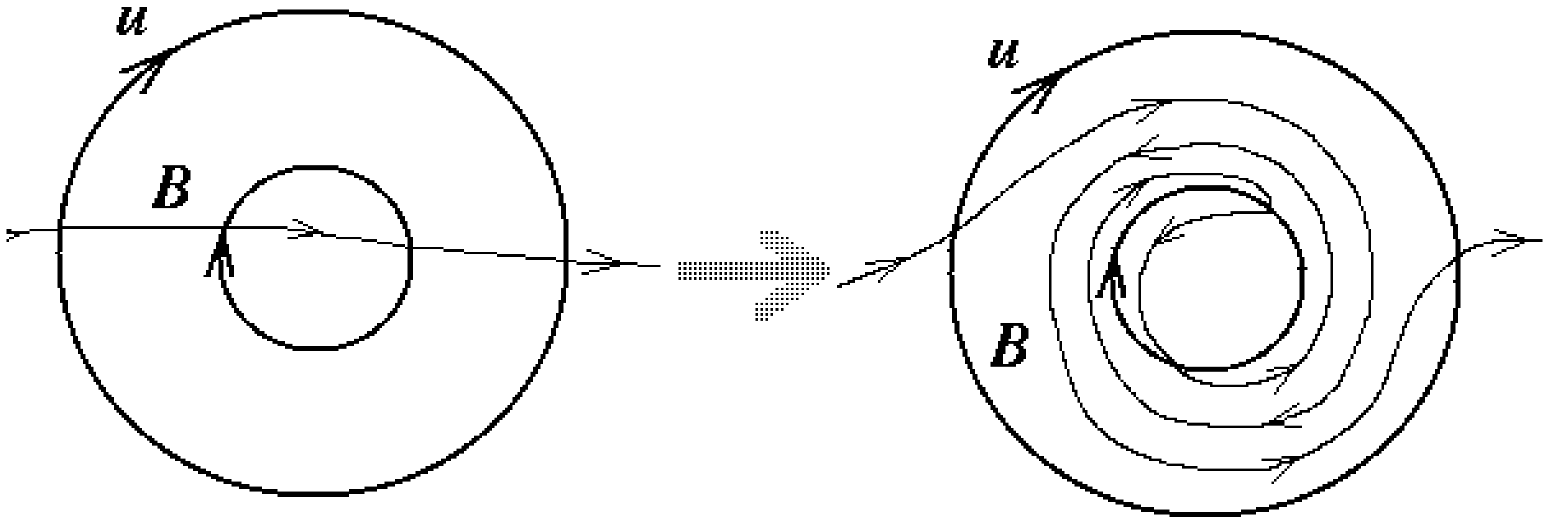
Go back to small-scale dynamo and ask some basic questions…

Folded Structure: Common Sense

What sort of fields does the small-scale dynamo make?

When Rm >> 1, field "frozen" into the flow (cf. material lines).

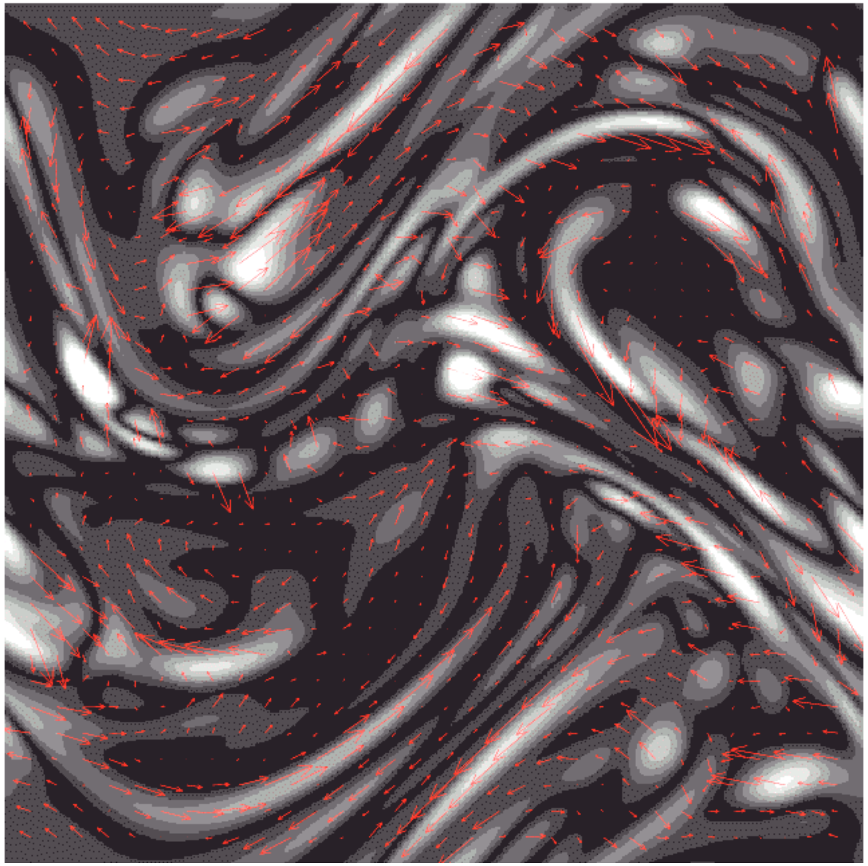
The flow winds up the field into <u>folds</u>:



Straight field   Folded field

- **Direction reversals at the resistive scale,** $k_\perp \sim k_\eta$
- **Field varies slowly along itself:** $k_\parallel \sim k_{\text{flow}}$

Folded Structure: DNS (Pm >> 1)



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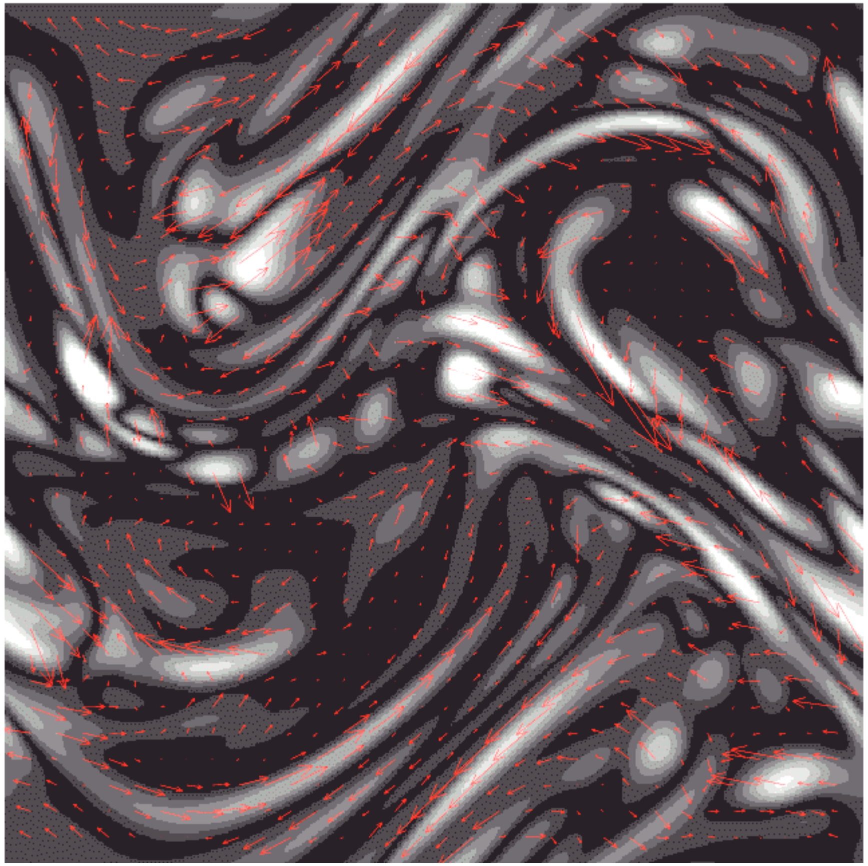

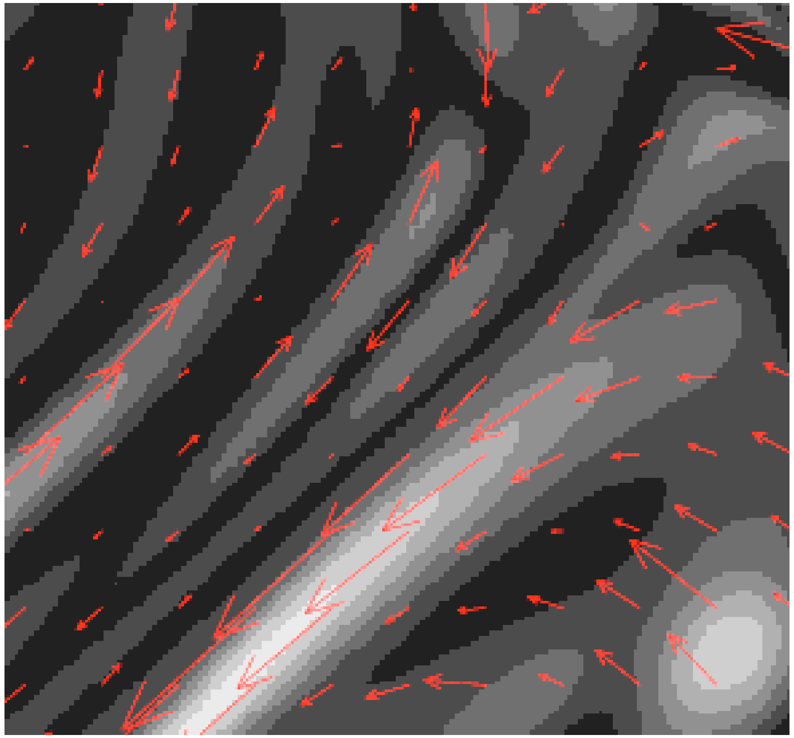

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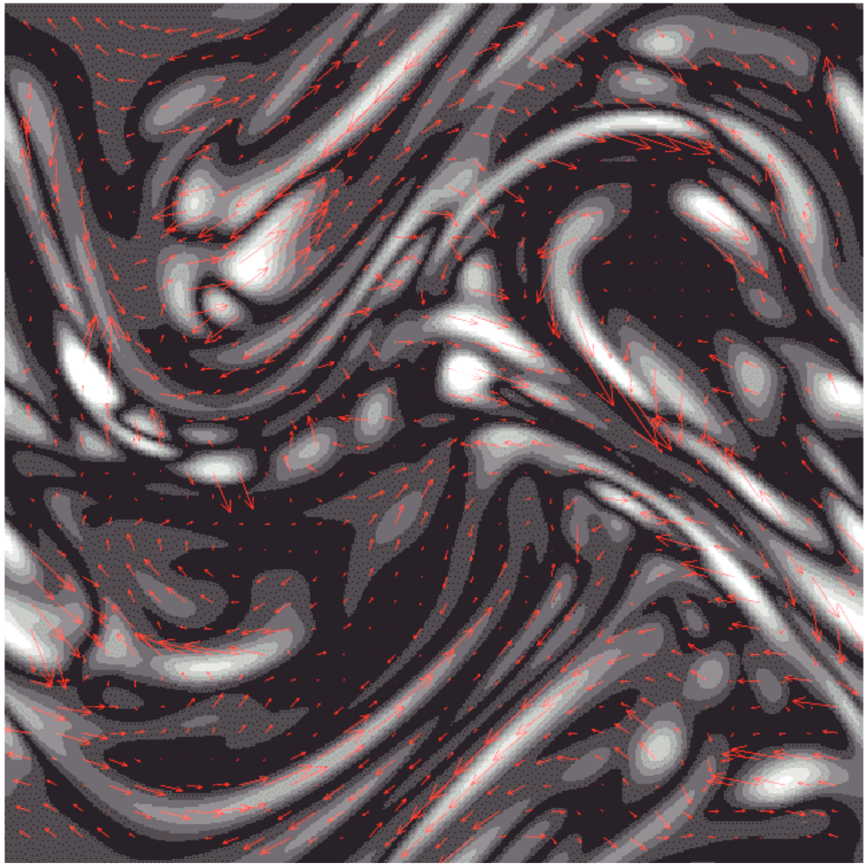
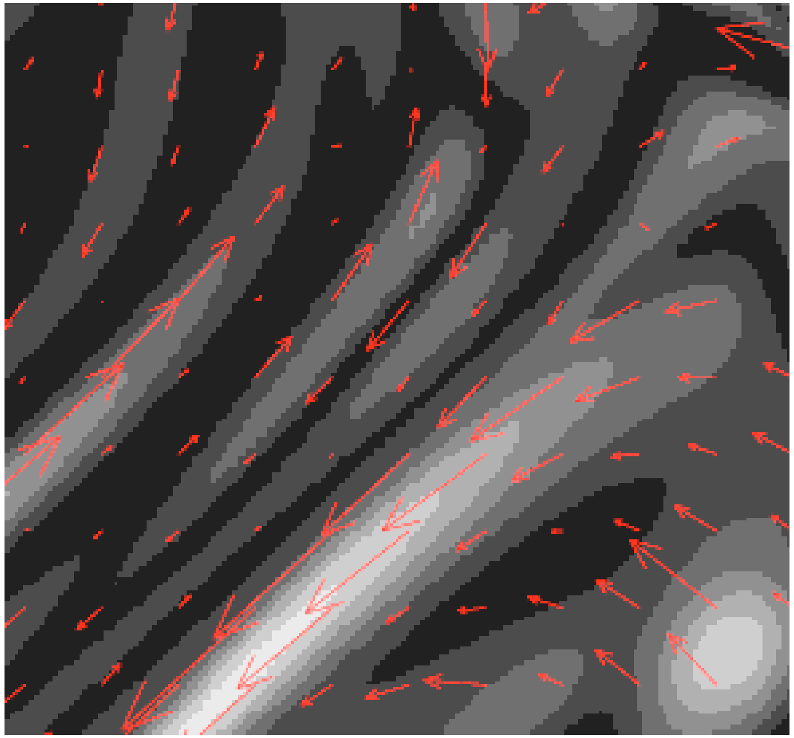


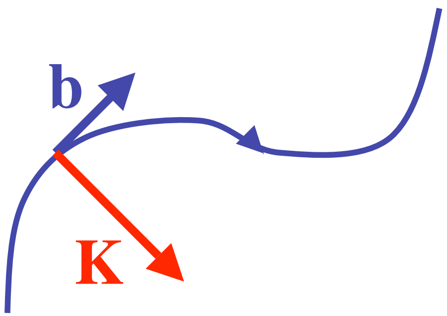


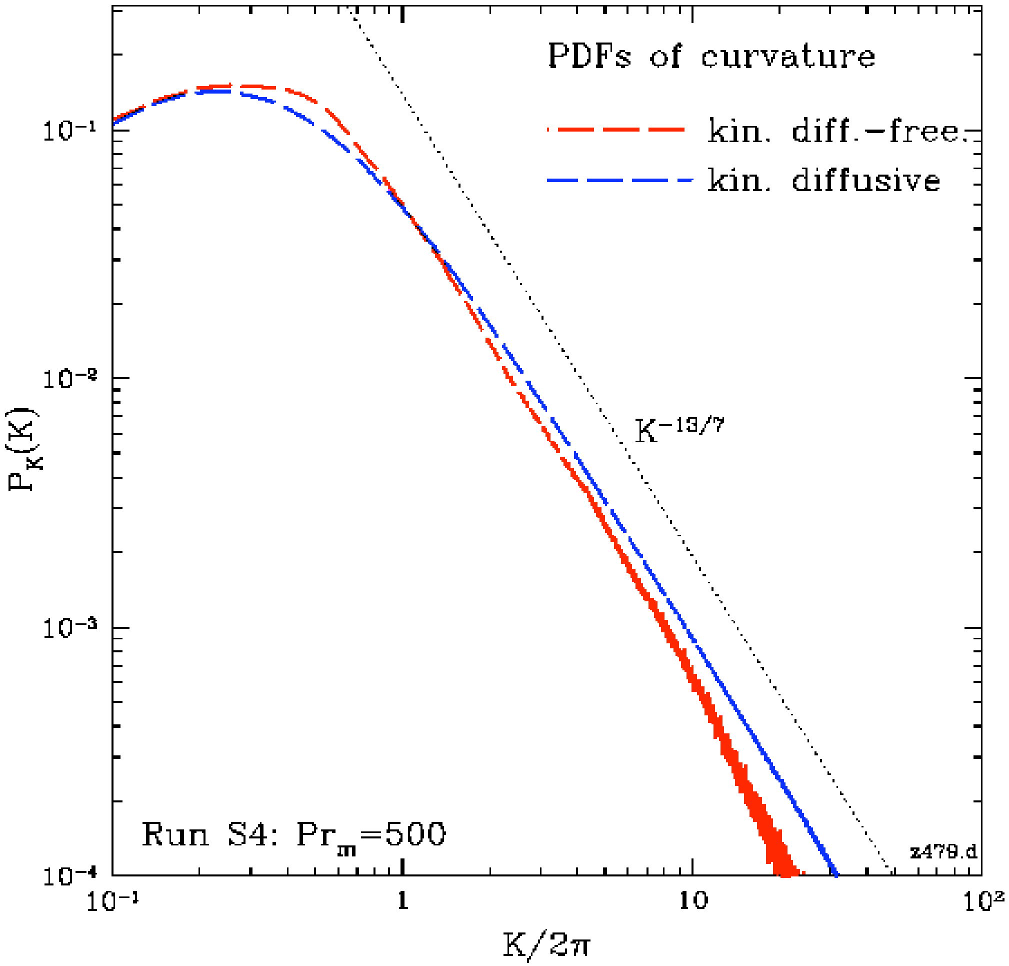
Many ways of diagnosing this structure:

- ***Extreme flux cancellation*** [Ott & coworkers 1988-98, Cattaneo 1994]
- ***Anisotropic two-point correlators*** [Chertkov *et al*. 1999, *PRL* **83**, 4065]
- ***Statistics of field-line curvature***
  [AAS *et al*. 2002, *PRE* **65**, 016305; 2004, *ApJ*, **612**, 276]

Field Line Curvature Statistics

<u>More detailed information</u>:
geometry of field lines described by the PDF of their curvature
K=b·□b





$$P_K(K) = \frac{6}{7}\frac{K}{(1+K^2)^{10/7}} \quad \sim \text{power tail } K^{-13/7}$$

[AAS *et al*. 2002, *PRE* **65**, 016305; 2004, *ApJ*, **612**, 276
cf. work on **material lines**: e.g., Drummond & Münch 1991, *JFM* **225**, 529]

Curvature and Field Strength

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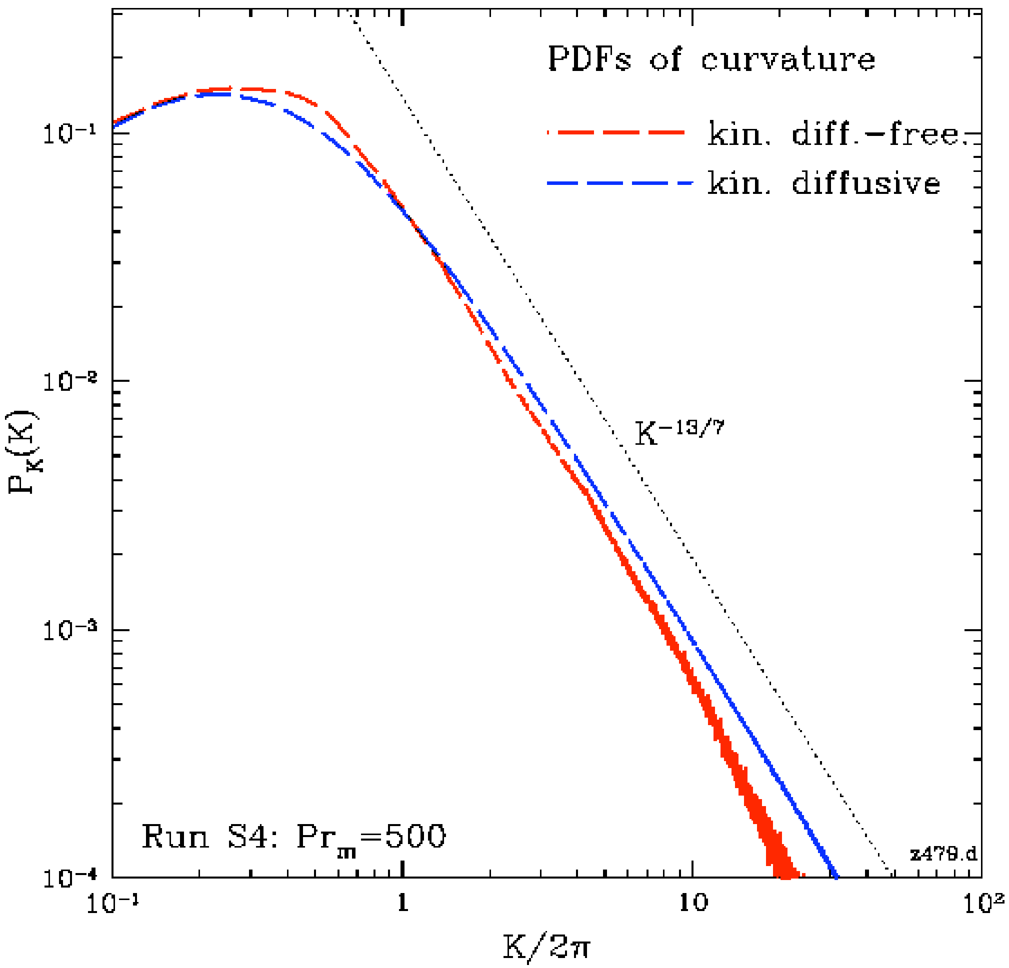
$\mathbf{K}=\mathbf{b}\cdot\nabla\mathbf{b}$

Curvature and field strength are anticorrelated



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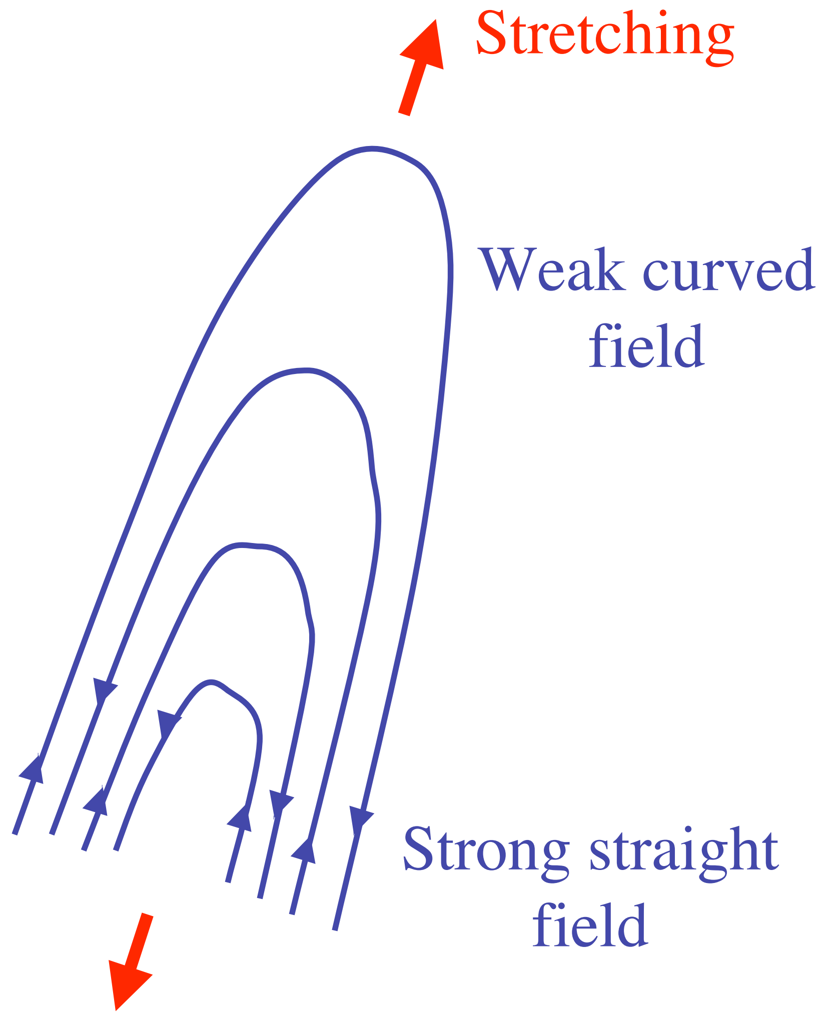
…which is clear from simple geometry of field stretching (and can be shown both **analytically** and **numerically**)



[AAS *et al*. 2002, *PRE* **65**, 016305; 2004, *ApJ*, **612**, 276
cf. work on **material lines**: e.g., Drummond & Münch 1991, *JFM* **225**, 529]

Characteristic Wavenumbers

The crudest diagnostics:

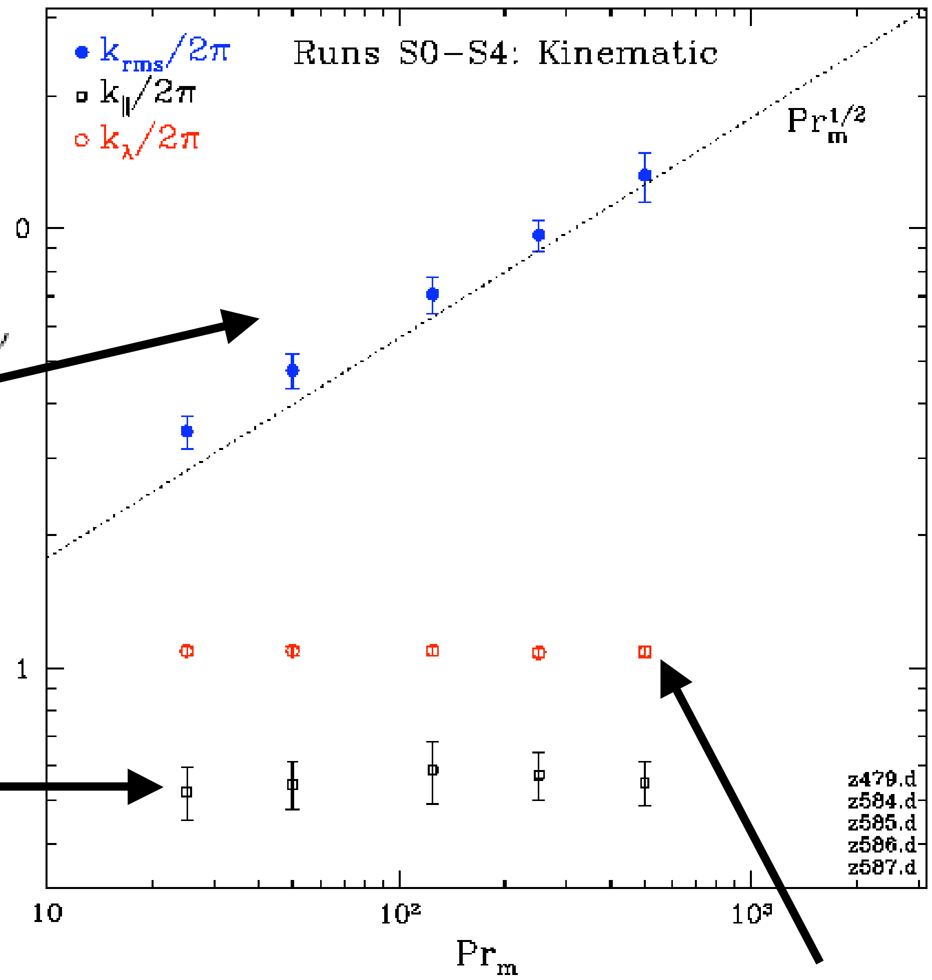
$$k_{\rm rms} = \left[\frac{2}{\langle B^2\rangle}\int_0^\infty {\rm d}k\, k^2 M(k)\right]^{1/2}$$

$$= \left[\frac{\langle|\nabla \mathbf{B}|^2\rangle}{\langle B^2\rangle}\right]^{1/2} \sim k_\eta \sim {\rm Pr}^{1/2} k_\nu$$

(**direction reversals**, get this already from spectral theory)

$$k_\parallel = \left[\frac{\langle|\mathbf{B}\cdot\nabla\mathbf{B}|^2\rangle}{\langle B^4\rangle}\right]^{1/2} \sim k_\nu$$

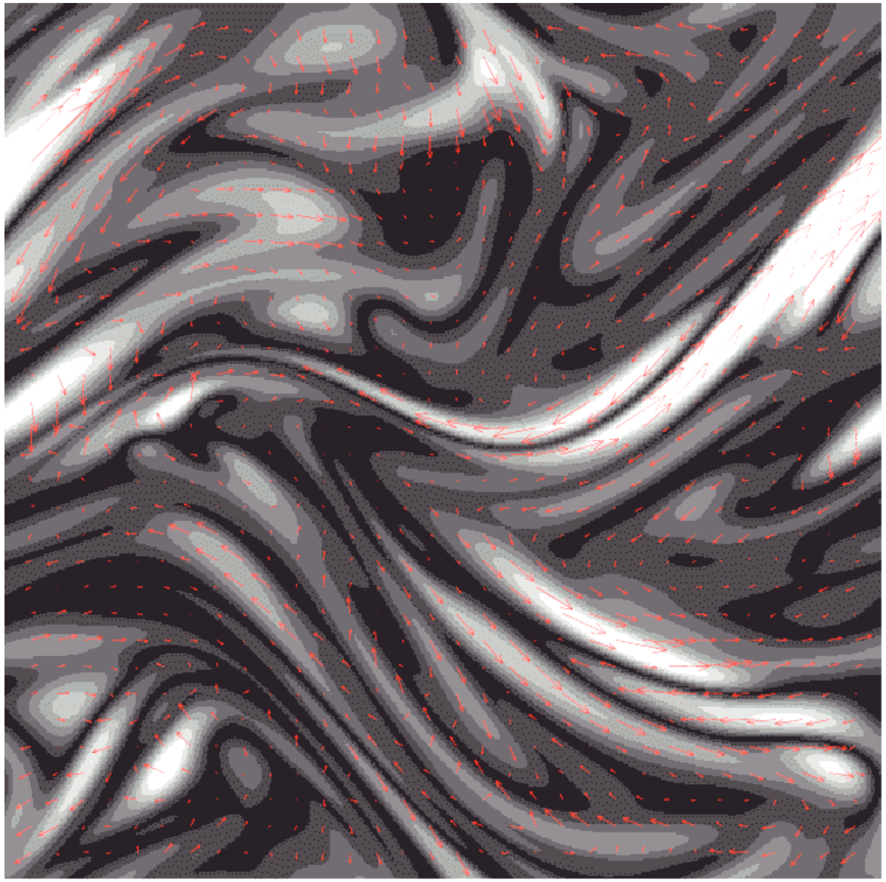
(inverse **"fold length"**, cannot get it from the spectrum!)

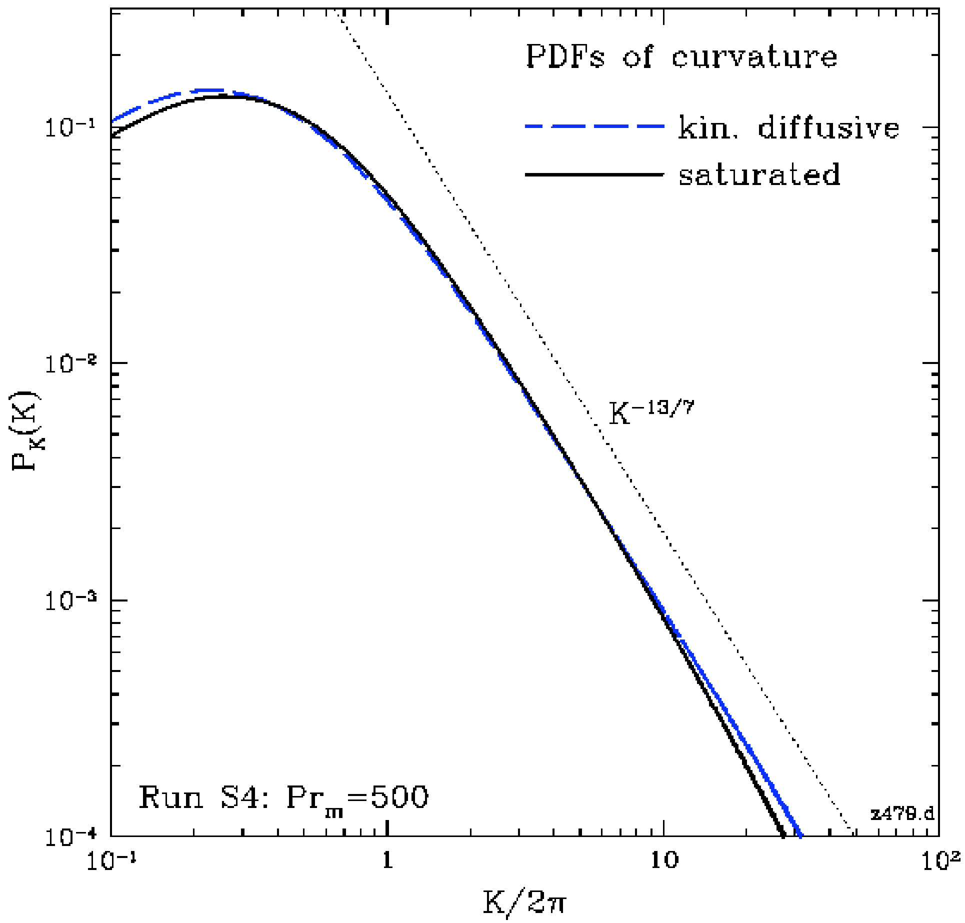


Compare them with the inverse Taylor microscale:

$$k_\lambda = \left[\frac{\langle|\nabla \mathbf{u}|^2\rangle}{\langle u^2\rangle}\right]^{1/2} = \frac{\sqrt{5}}{\lambda}$$

Folded Structure Preserved in Saturation

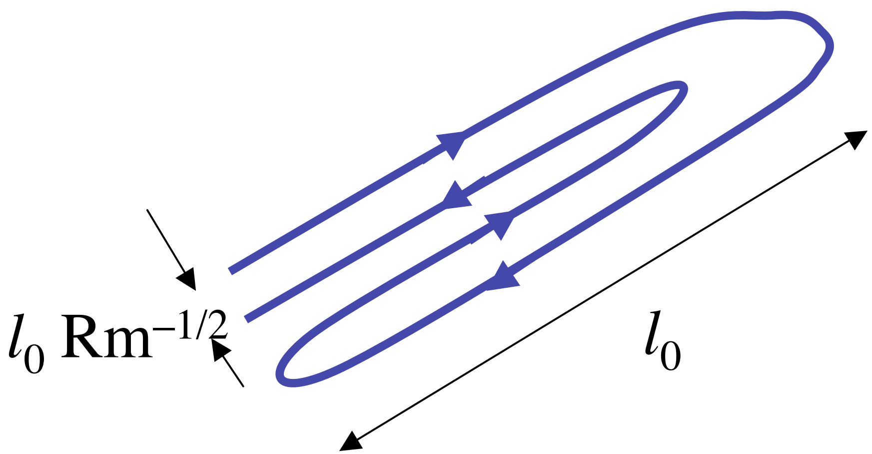




All the same features of **field-line geometry** and **field-strength anticorrelation with curvature** as in kinematic dynamo

[AAS *et al*. 2004, *ApJ* **612**, 276]

Saturation via Anisotropy



$$\mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}B^2$$

Folds provide a direction in space that is locally coherent at the scale of the flow

[AAS *et al*. 2004, *PRL* **92**, 084504]

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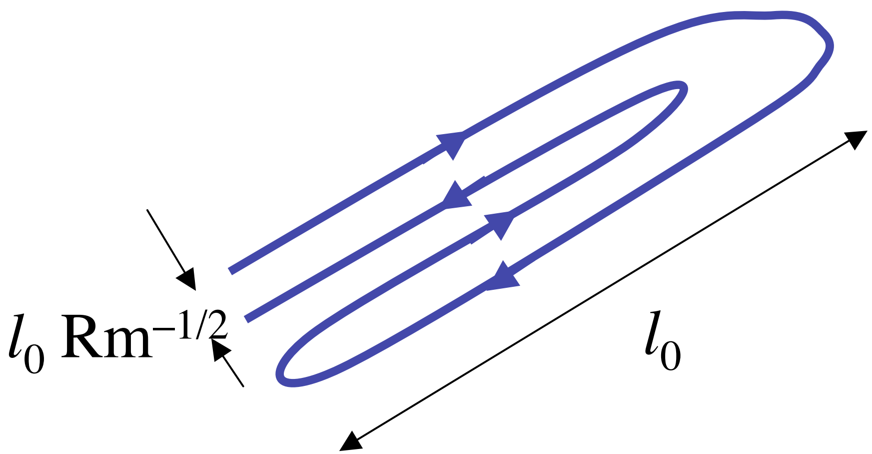


$$\mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}B^2$$

Folds provide a direction in space that is locally coherent at the scale of the flow

It is possible to construct a Fokker-Planck-type model of saturated spectra based on the idea that **saturation occurs via partial two-dimensionalisation of the velocity gradients with respect to the local direction of the folds**

This weakens stretching and enhances mixing, so dynamo saturates at marginally stable balance of the two

[AAS *et al*. 2004, *PRL* **92**, 084504]

A Fokker-Planck Model of Saturation

Build on the Kazantsev formalism and model saturation by

making velocity statistics depend on the local field direction $b^i b^j$:

$$\kappa^{ij}(\mathbf{k}) = \kappa^{(\mathrm{i})}(k,|\mu|)\left(\delta^{ij} - \hat{k}_i\hat{k}_j\right) + \kappa^{(\mathrm{a})}(k,|\mu|)\left(\hat{b}^i\hat{b}^j + \mu^2\hat{k}_i\hat{k}_j - \mu\hat{b}^i\hat{k}_j - \mu\hat{k}_i\hat{b}^j\right)$$

Can then derive an equation for magnetic-energy spectrum in (almost) the usual way:

$$\boxed{\partial_t M = \frac{1}{8}\gamma_\perp\frac{\partial}{\partial k}\left[(1+2\sigma_\|)k^2\frac{\partial}{\partial k} - (1+4\sigma_\perp+10\sigma_\|)\,k\right]M + 2(\sigma_\perp+\sigma_\|)\gamma_\perp M - 2\eta k^2 M}$$

where $\gamma_\perp = \int d^3k\, k_\perp^2 \kappa_\perp \sim \left[\langle|\nabla_\perp \mathbf{u}_\perp|^2\rangle\right]^{1/2}$ "mixing rate"

$$\sigma_\perp = \frac{1}{\gamma_\perp}\int d^3k\, k_\|^2\kappa_\perp \sim \frac{\langle|\nabla_\| \mathbf{u}_\perp|^2\rangle}{\langle|\nabla_\perp \mathbf{u}_\perp|^2\rangle}, \quad \sigma_\| = \frac{1}{\gamma_\perp}\int d^3k\, k_\|^2\kappa_\| \sim \frac{\langle|\nabla_\| \mathbf{u}_\||^2\rangle}{\langle|\nabla_\perp \mathbf{u}_\perp|^2\rangle}$$

Solution in the limit $\eta \to +0$ is

$$\boxed{M(k) \simeq k^s e^{\gamma t} K_0(k/k_\eta),}$$

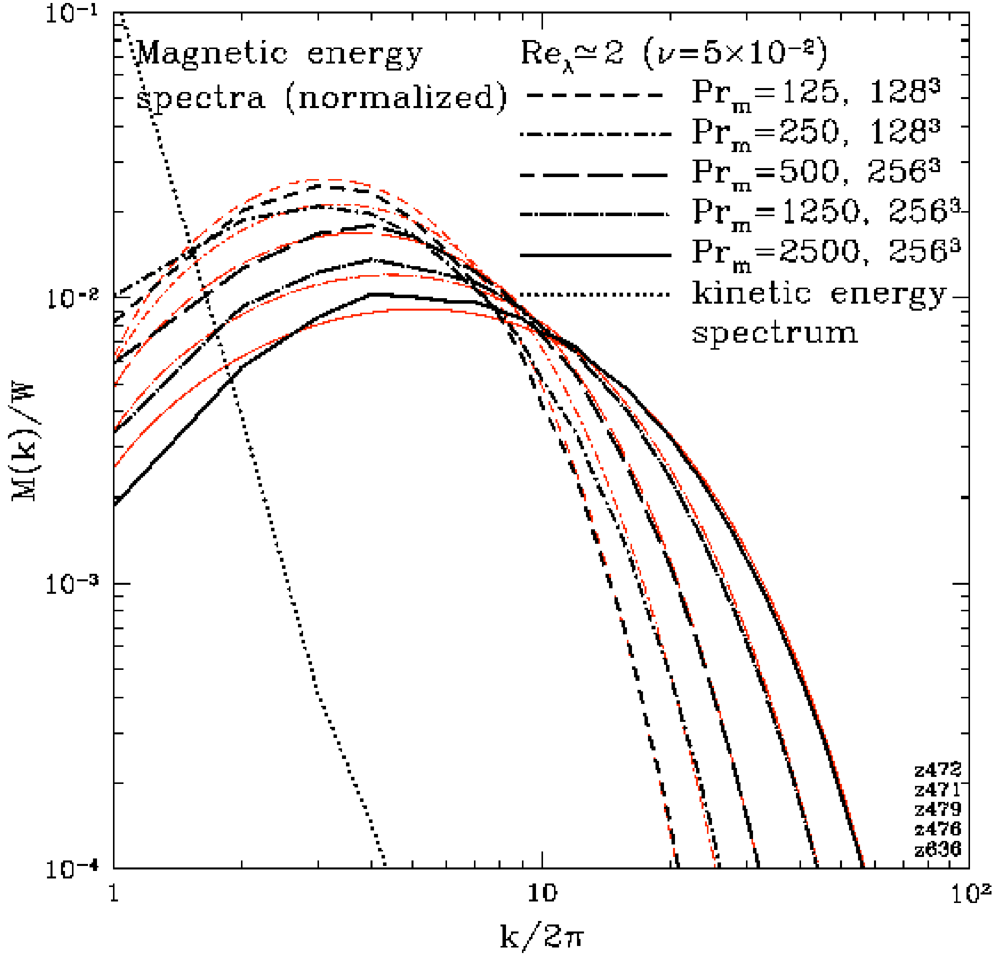
$$s = 2\frac{\sigma_\perp + 2\sigma_\|}{1+2\sigma_\|}, \quad \gamma = \gamma_\perp\left[2(\sigma_\perp+\sigma_\|) - \frac{(1+2\sigma_\perp+6\sigma_\|)^2}{8(1+2\sigma_\|)}\right], \quad k_\eta = \frac{1}{4}\left[\frac{(1+2\sigma_\|)\gamma_\perp}{\eta}\right]^{1/2}$$

γ = 0 at some sufficiently small $\sigma_\perp$, $\sigma_\|$ $\Rightarrow$ saturation purely by means of anisotropy!

[AAS *et al*. 2004, *PRL* **92**, 084504]

Saturated Spectra: Theory vs. DNS

We can solve the model with simulation parameters: these nonasymptotic solutions fit an entire sequence of spectra in runs with Re ~ 1, Pm >>1



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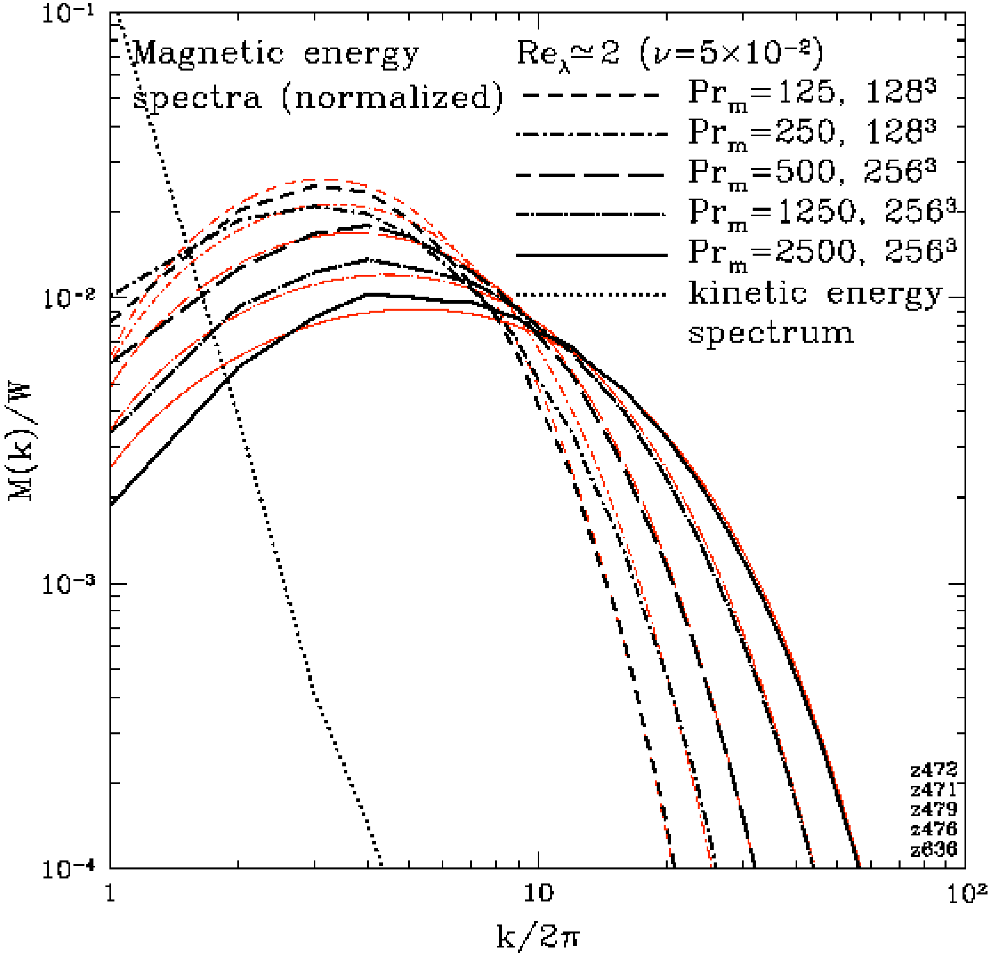
Saturated Spectra: Theory vs. DNS

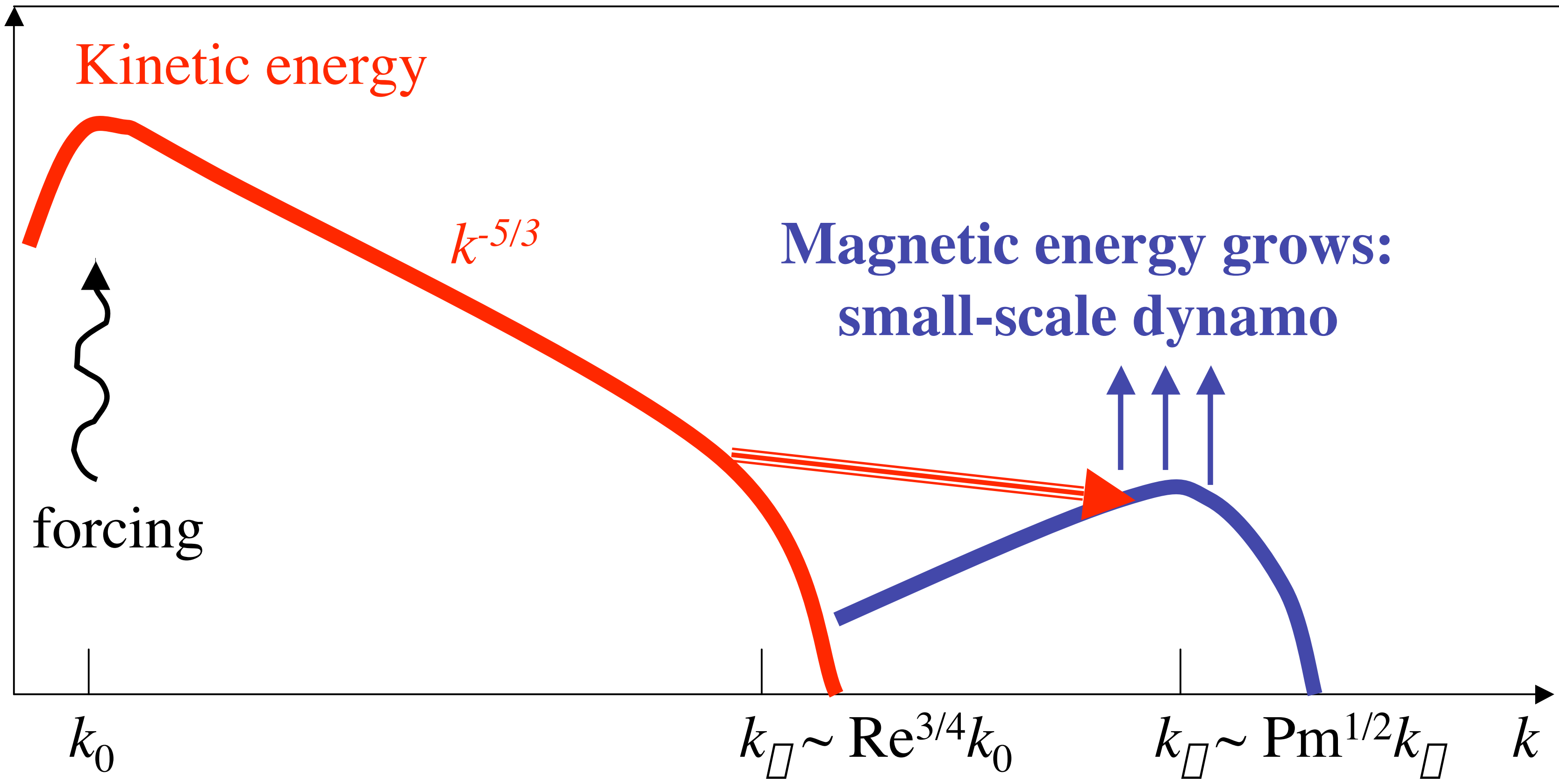
We can solve the model with simulation parameters:
these nonasymptotic solutions fit an entire sequence of spectra
in runs with Re ~ 1, Pm >>1

This is a pleasant surprise: apparently, the saturation mechanism is simple and robust enough to be captured by such an elementary model!



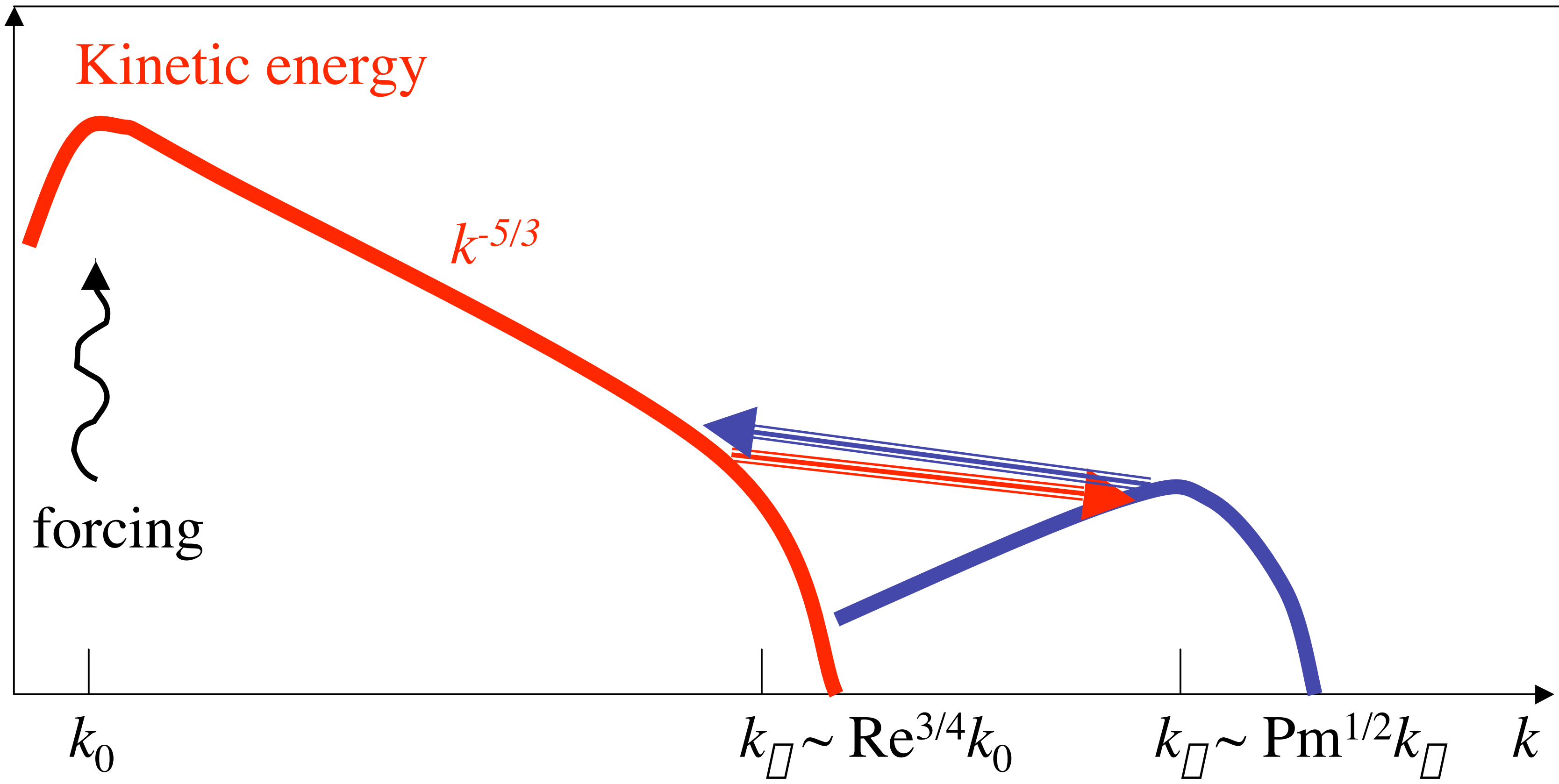
[AAS *et al*. 2004, *PRL* **92**, 084504]

MHD Turbulence: Multiscale Flow



We have thus far considered dynamo in a *single-scale* random flow
True turbulence has a range of scales

Onset of Back Reaction



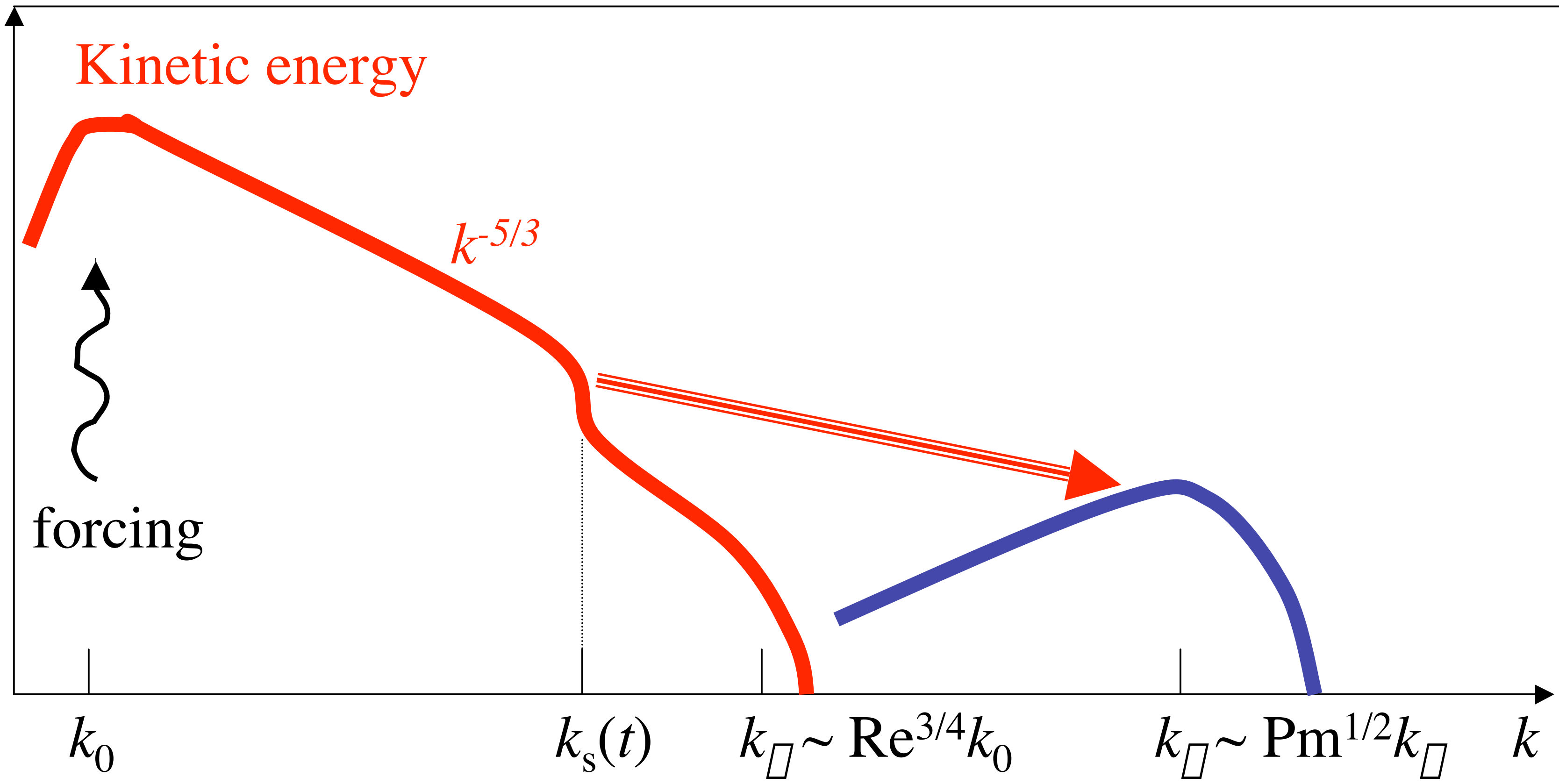
Kinematic growth
continues until $\mathbf{B}\cdot\nabla\mathbf{B} \sim \mathbf{u}\cdot\nabla\mathbf{u}$

$k_{\|} B^2 \sim k_{\nu} u^2$ i.e., $B^2 \sim u^2$

Mag. energy ~ visc. eddies energy

[AAS *et al*. 2002, *PRE* **65**, 016305]

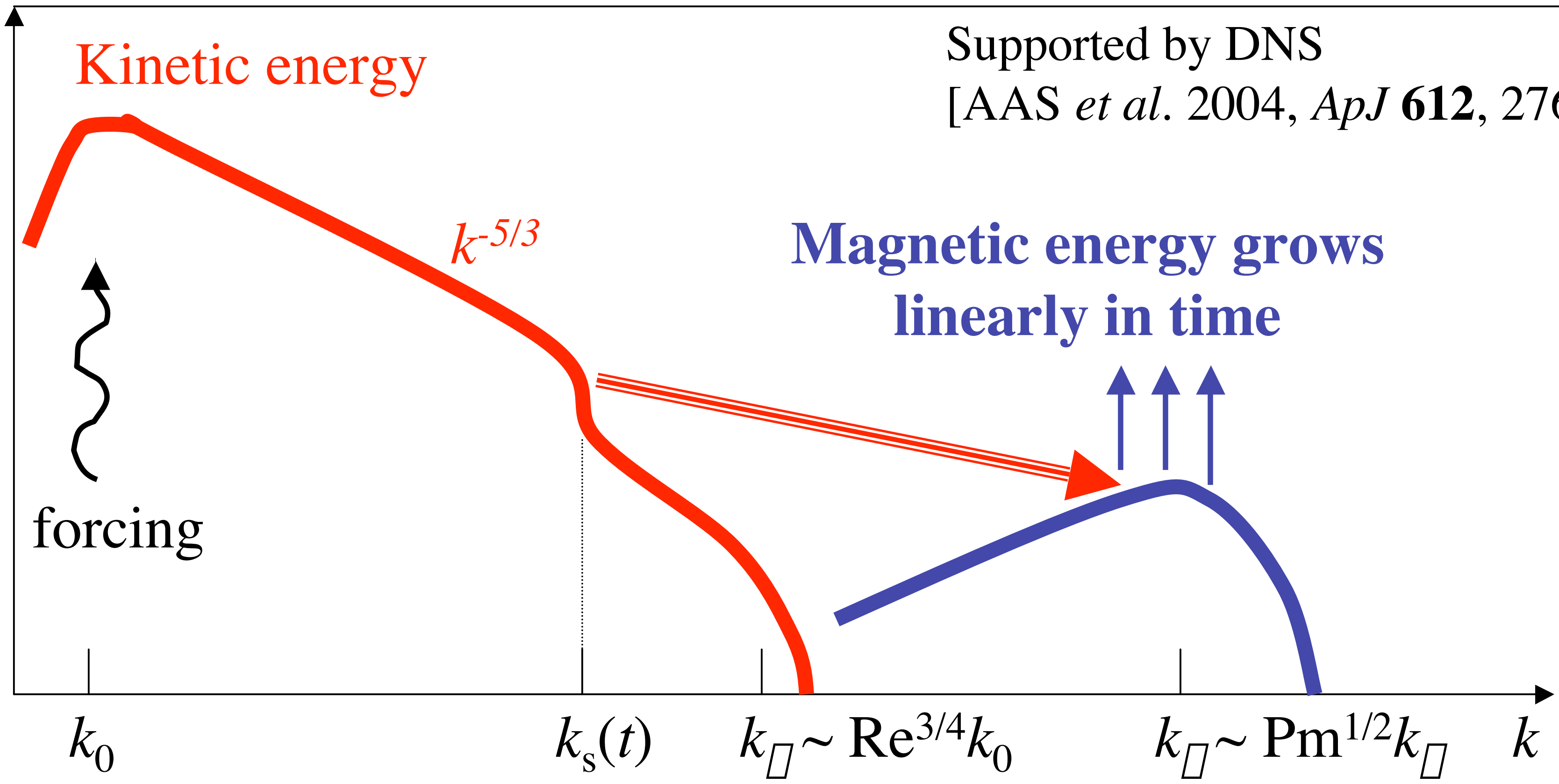
Intermediate Nonlinear Growth



Define *stretching scale* $l_s(t)$: $u_{\ell_s}^2 \sim \langle B^2 \rangle$

[AAS *et al*. 2002, *NJP* **4**, 84; Maron *et al*. 2004, *ApJ* **603**, 569]

Intermediate Nonlinear Growth



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$$\frac{d}{dt}\langle B^2\rangle \sim \frac{u_{\ell_s}}{\ell_s}\langle B^2\rangle \sim \frac{u_{\ell_s}^3}{\ell_s} \sim \epsilon = \mathrm{const} \longrightarrow \boxed{\langle B^2\rangle(t) \sim \epsilon t}$$

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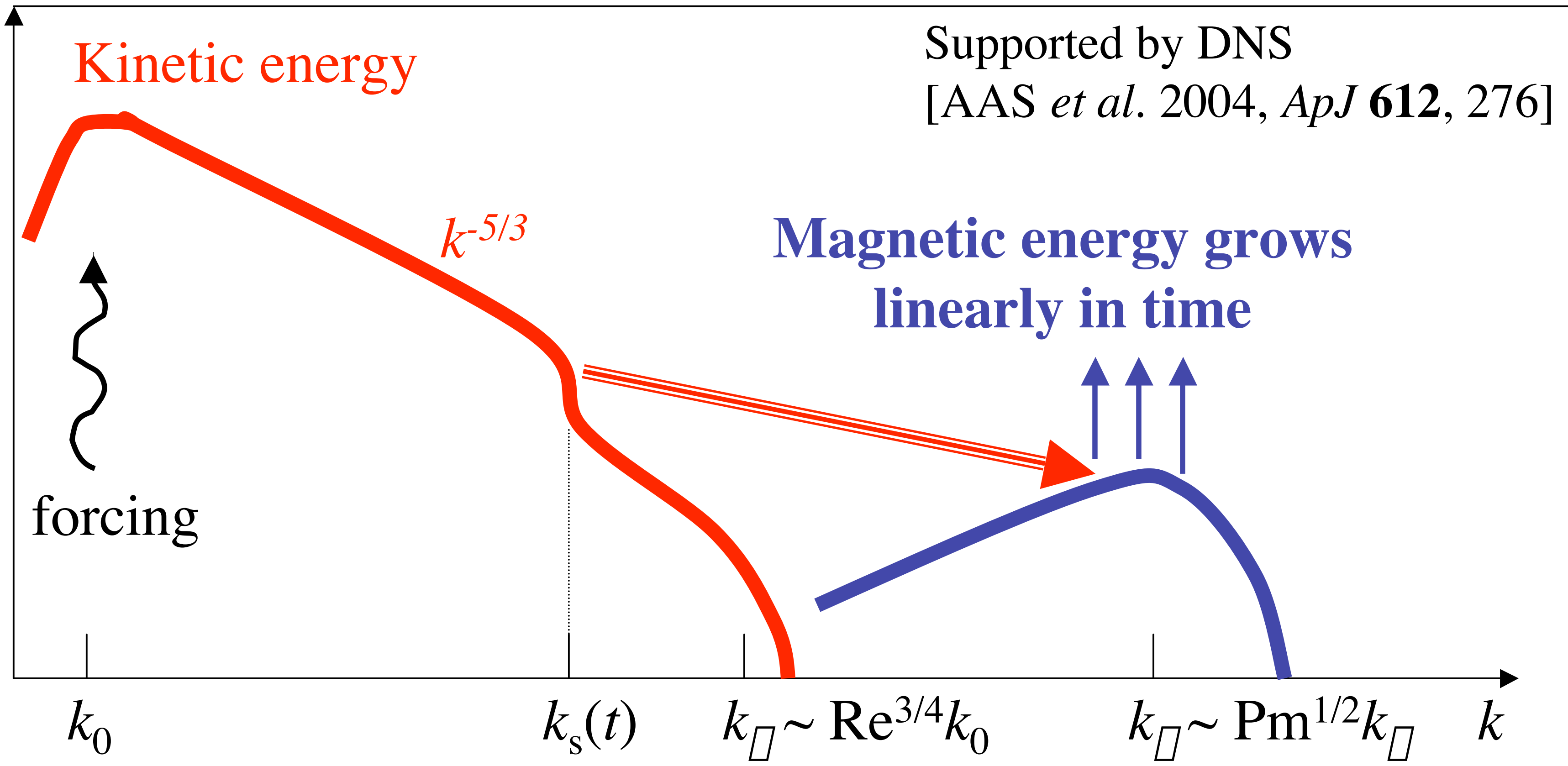

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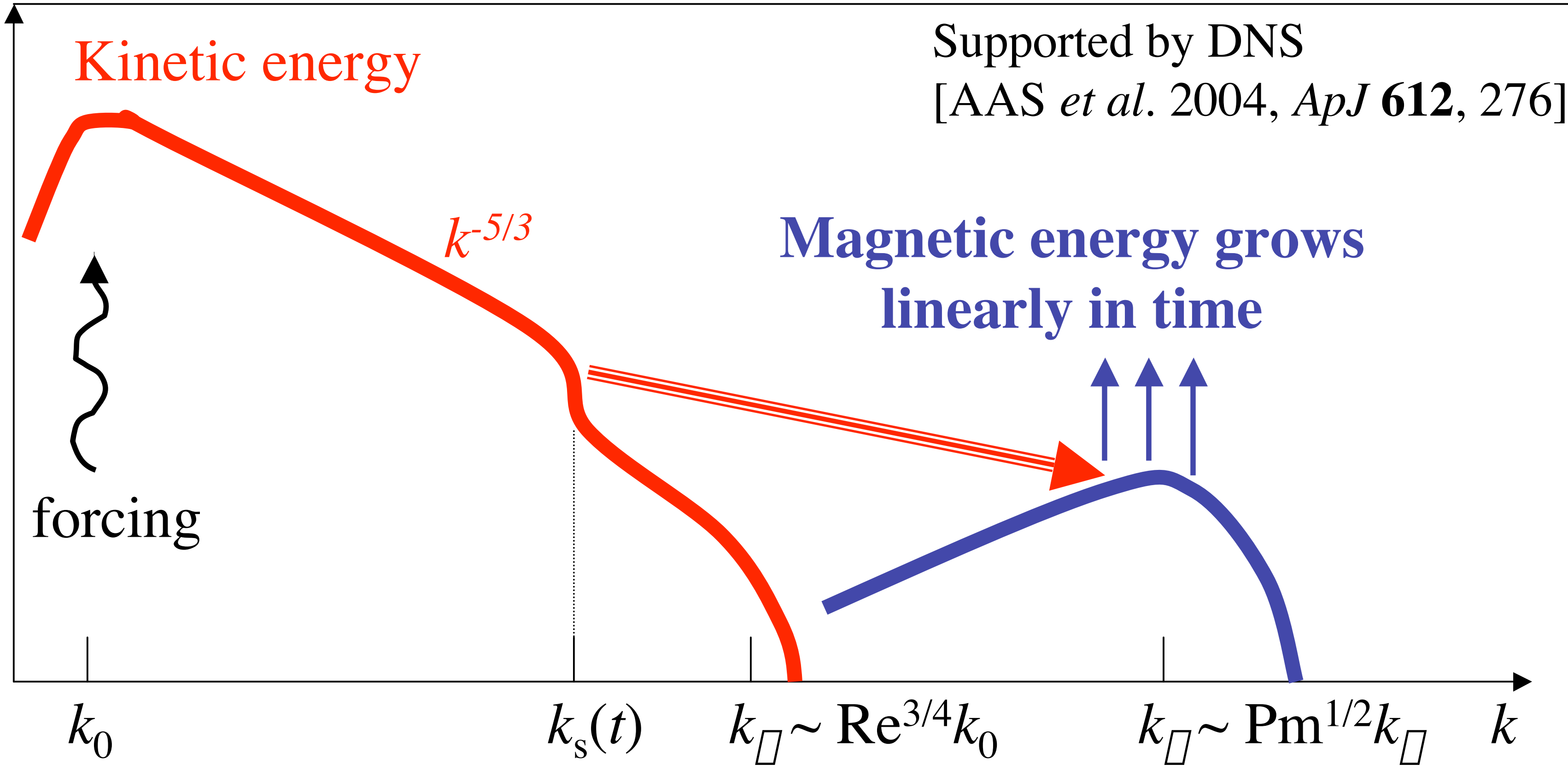
$$\mathbf{u}\cdot\nabla\mathbf{u} \sim \frac{u_{\ell_s}^2}{\ell_s} \sim \mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}\langle B^2\rangle \longrightarrow \boxed{k_{||} \sim \ell_s^{-1}}$$

Intermediate Nonlinear Growth



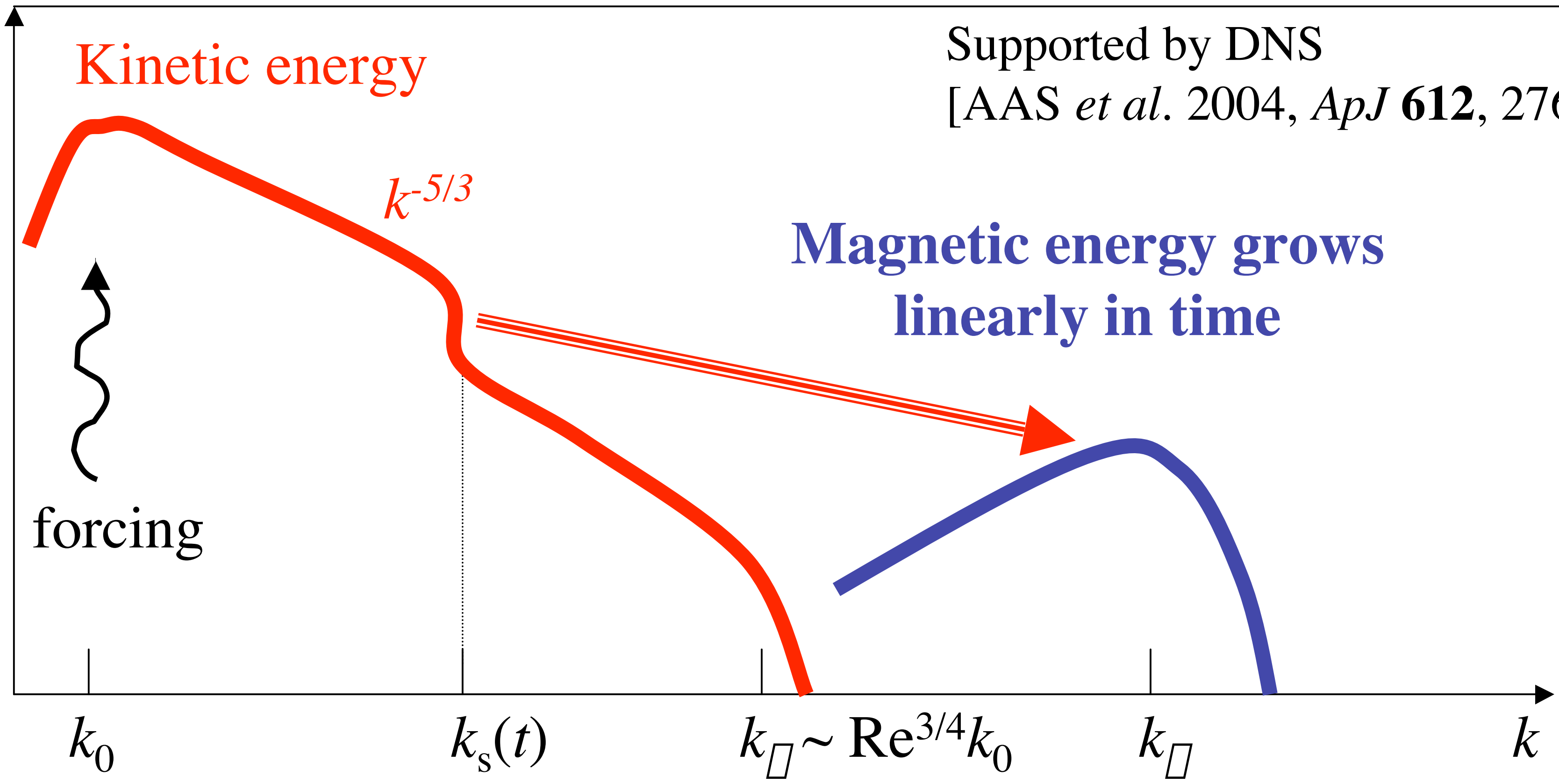
$$\frac{u_{\ell_s}}{\ell_s} \sim \frac{\eta}{\ell_\eta^2} \longrightarrow \ell_\eta \sim \left(\frac{\eta \ell_s}{u_{\ell_s}}\right)^{1/2} \sim (\epsilon t)^{1/2}$$ **selective decay**

It is possible to construct a Fokker-Planck model of this self-similar intermediate growth stage

[AAS *et al*. 2002, *NJP* **4**, 84]

Intermediate Nonlinear Growth

Supported by DNS
[AAS *et al*. 2004, *ApJ* **612**, 276]



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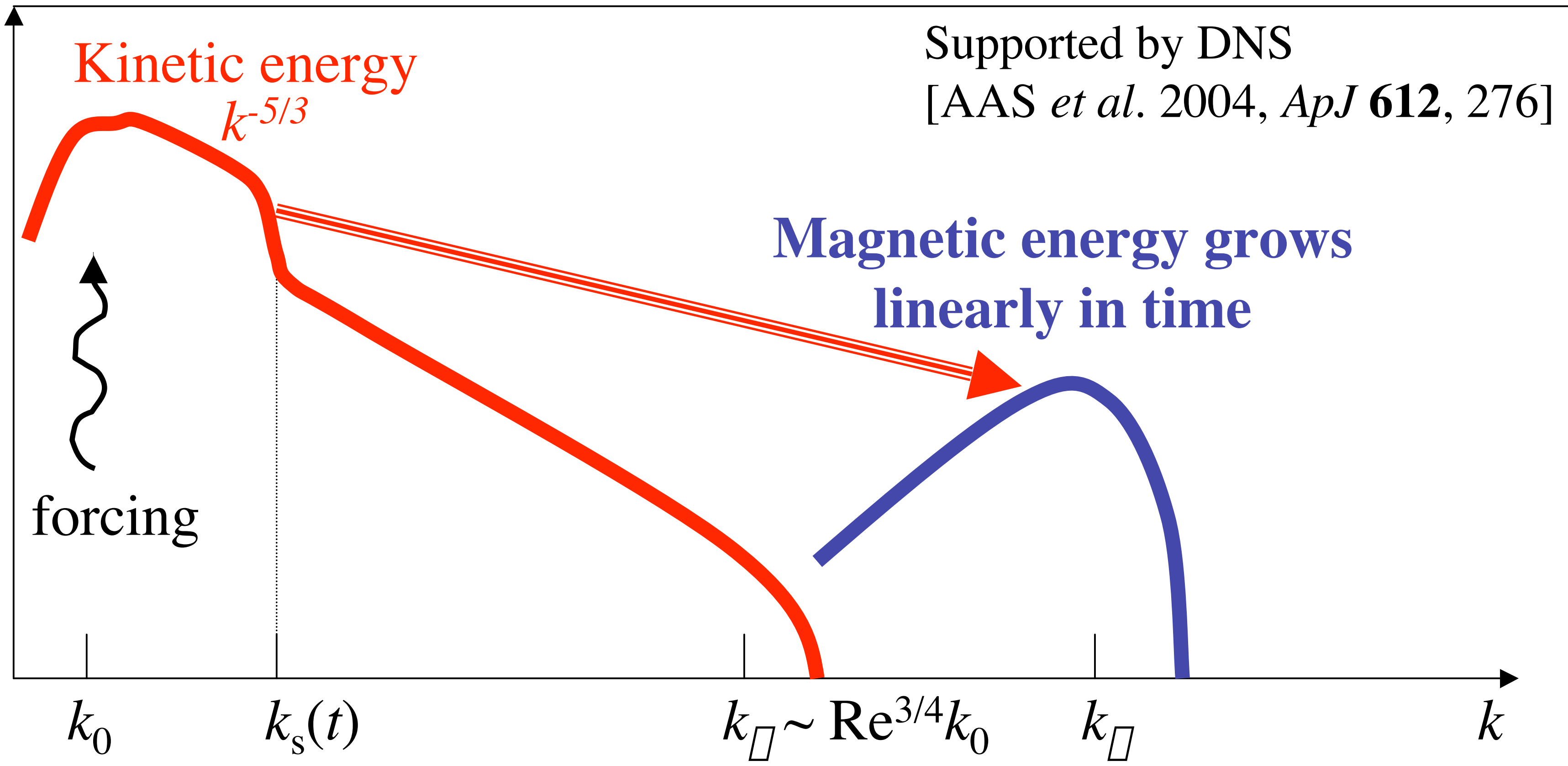

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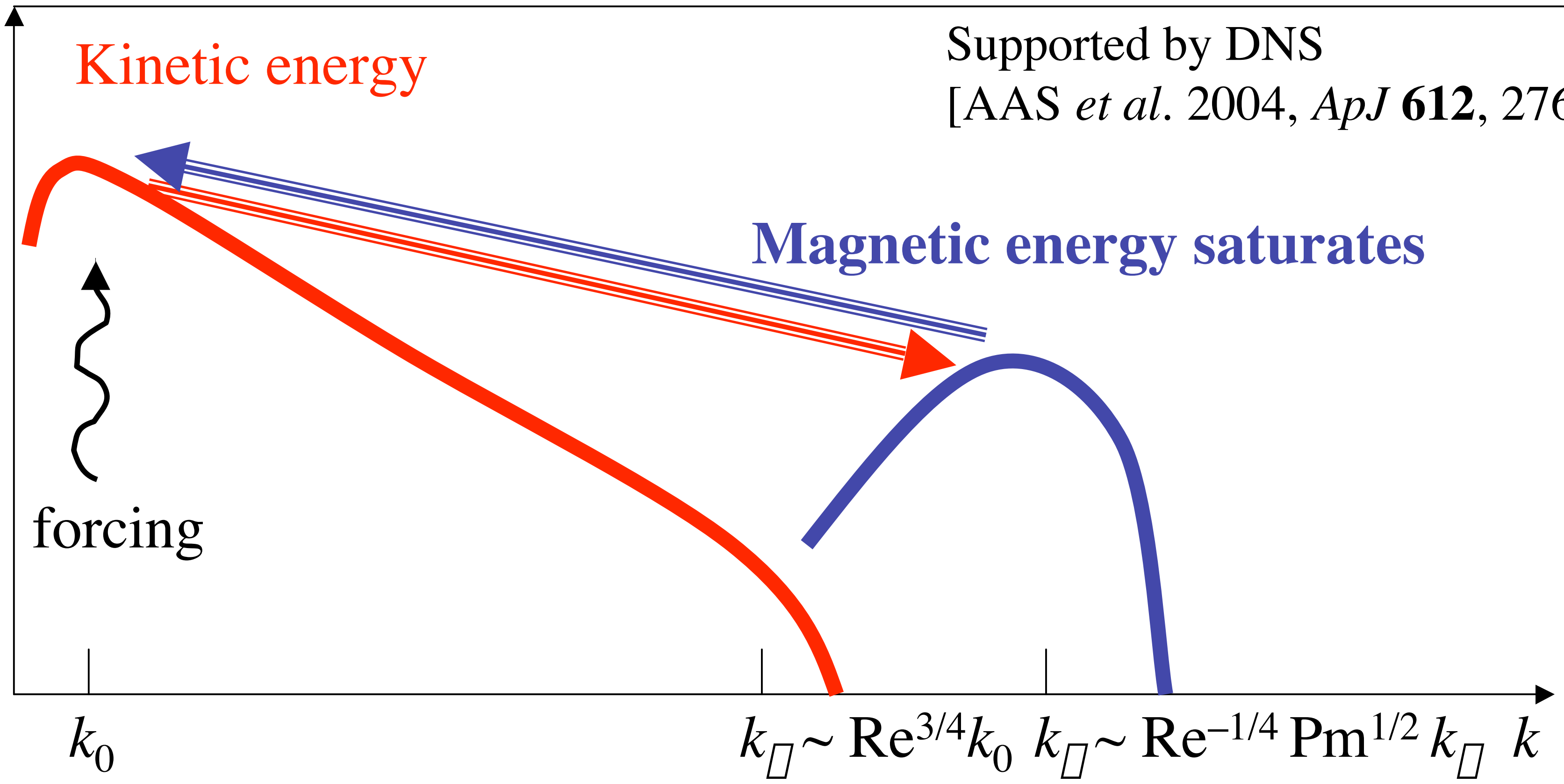
Saturation



Nonlinear growth/selective decay/fold elongation continue until

$l_s \sim l_0$ ⟶ $\langle B^2 \rangle \sim \langle u^2 \rangle$

and $l_\eta \sim \mathrm{Rm}^{-1/2}\, l_0 \sim \mathrm{Re}^{1/4}\,\mathrm{Pm}^{-1/2}\, l_\nu$

[AAS *et al*. 2002, *NJP* **4**, 84]

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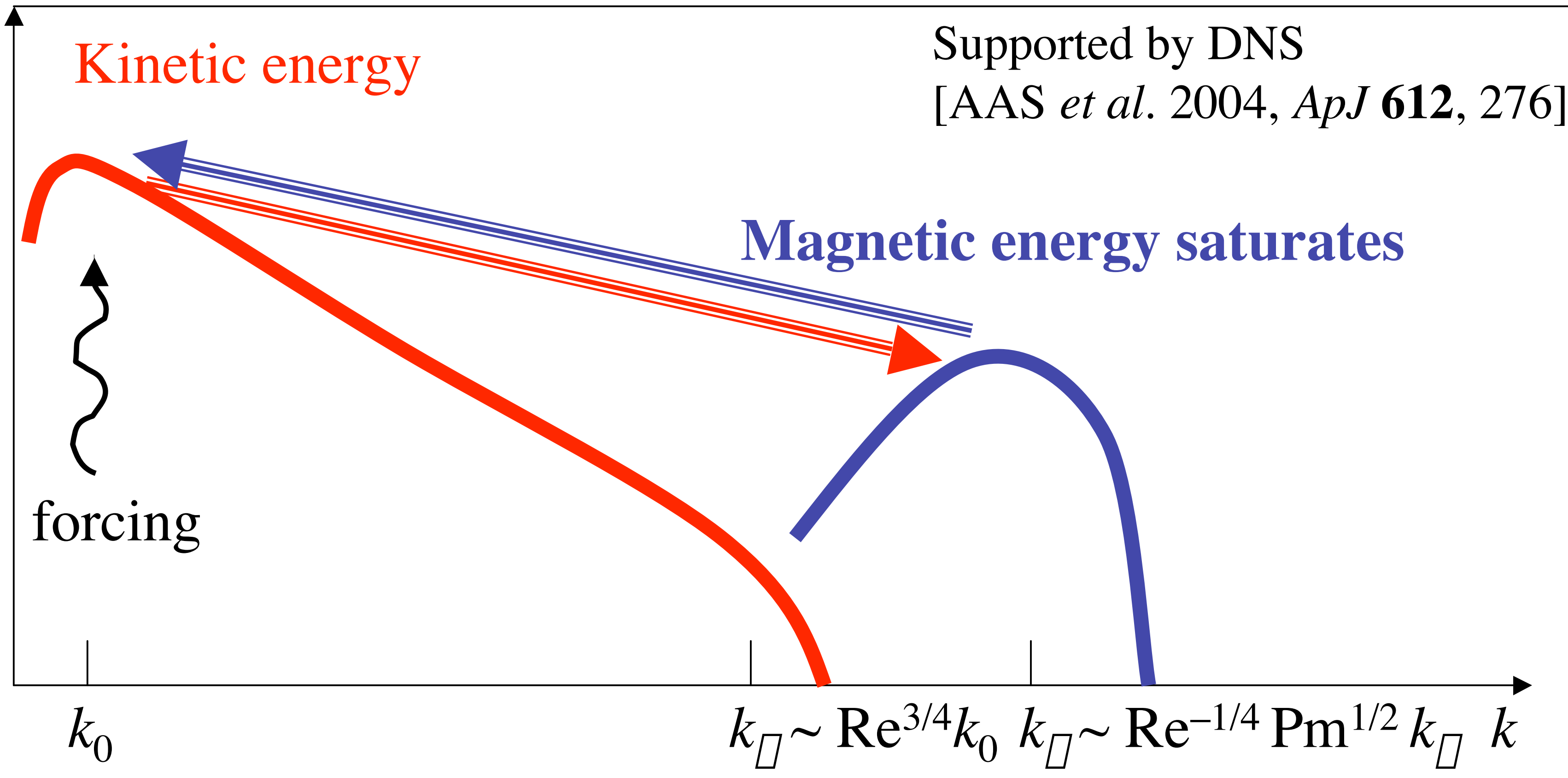

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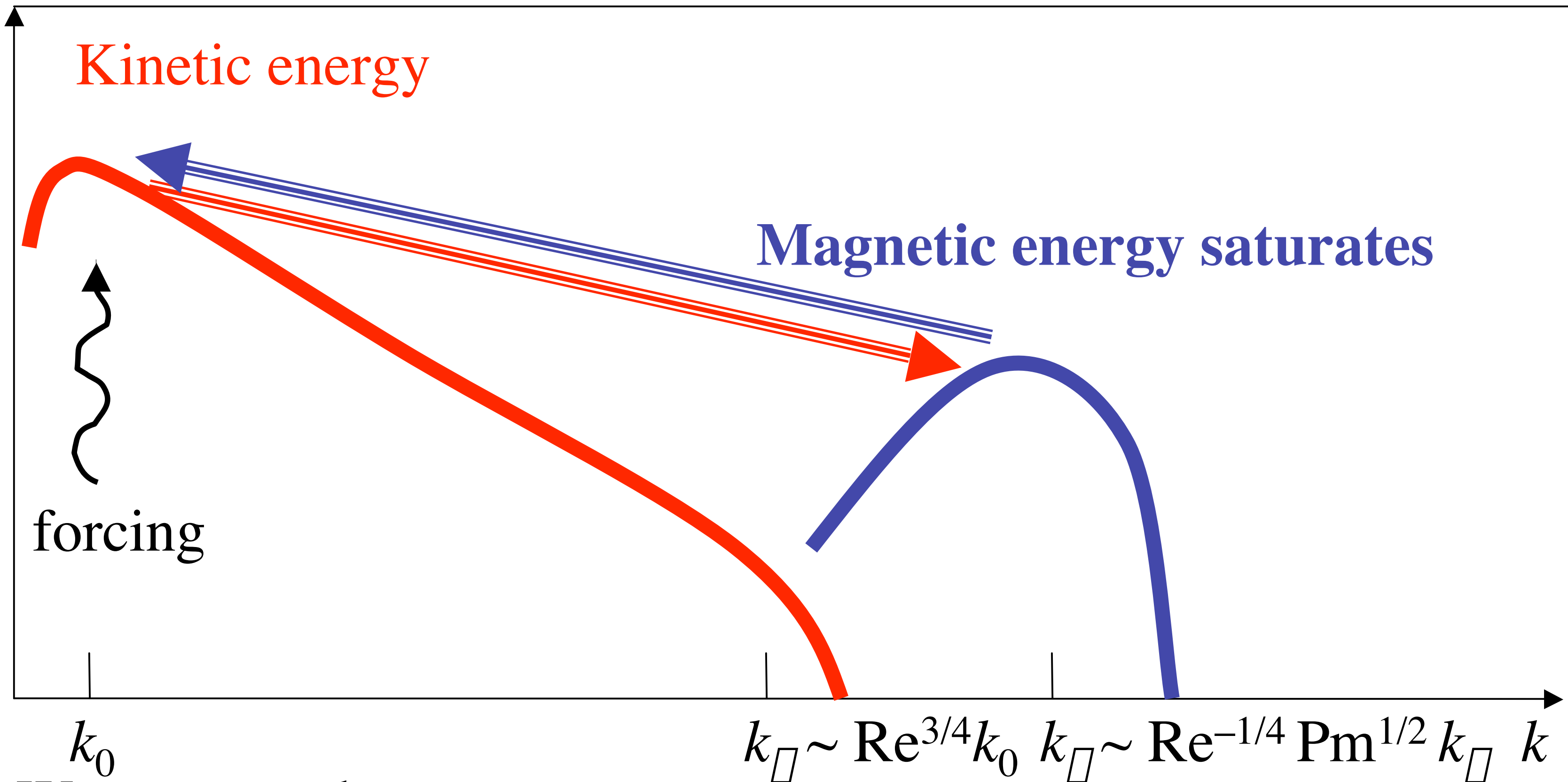
and $l_\eta \sim \mathrm{Rm}^{-1/2}\, l_0 \sim \mathrm{Re}^{1/4}\,\mathrm{Pm}^{-1/2}\, l_\nu$

NB: l_ν and l_η distinguishable only if **Pm >> Re$^{1/2}$ >> 1**!!!

Saturation

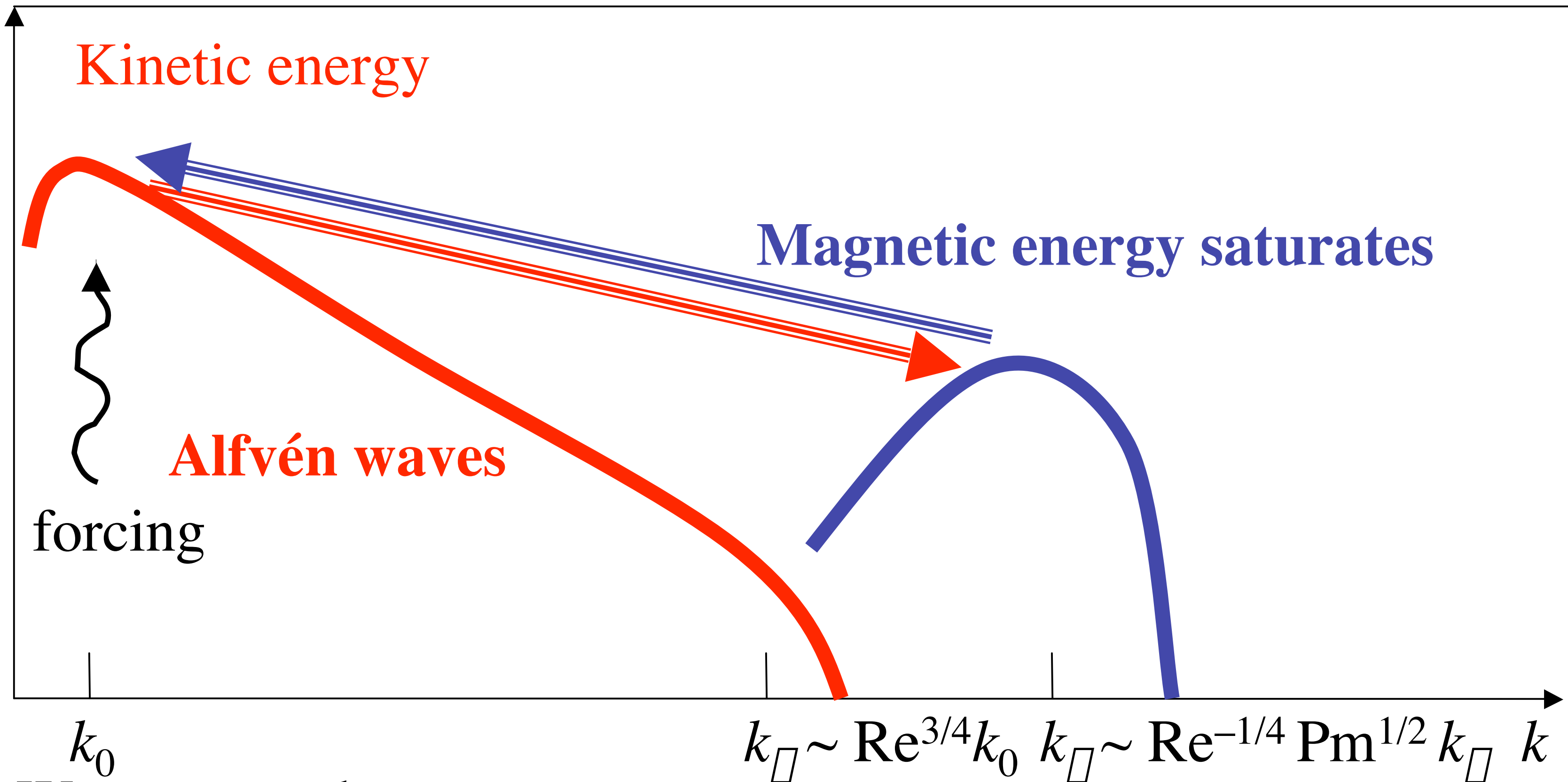


We propose that

• ***saturation** is a balance between stretching and mixing by the outer-scale motions and Ohmic diffusion of the folded field*

[Maron *et al*. 2004, *ApJ* **603**, 569; AAS *et al*. 2004, *ApJ* **612**, 276]

Alfvén Waves and Folded Fields

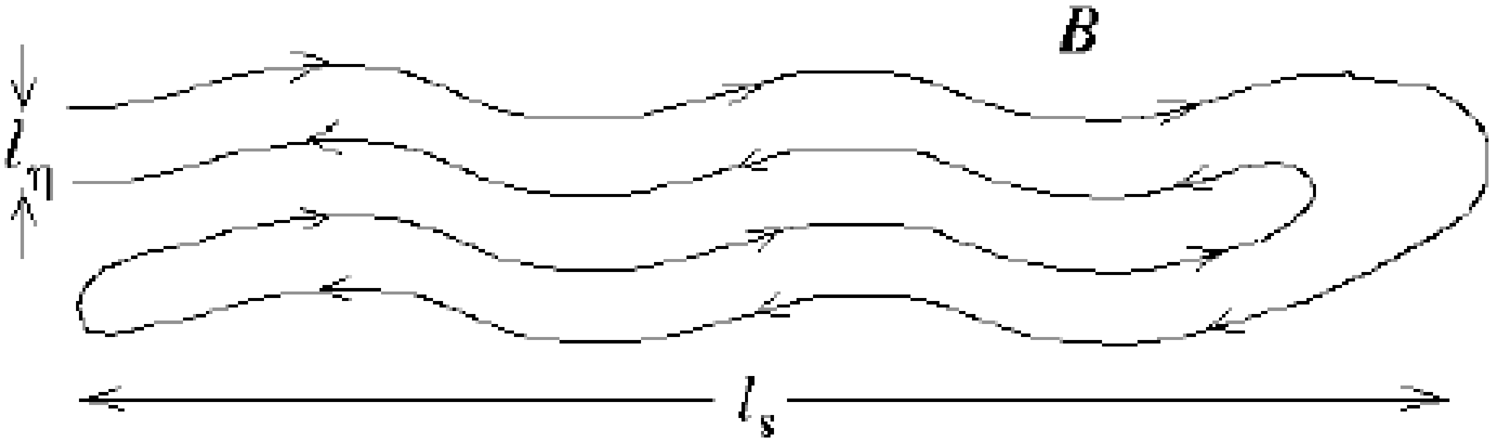


We propose that

- ***saturation** is a balance between stretching and mixing by the outer-scale motions and Ohmic diffusion of the folded field*
- *the fully developed isotropic MHD turbulence **in the inertial range** is a superposition of folded magnetic fields and Alfvén waves*

[AAS *et al*. 2004, *ApJ* **612**, 276]

Alfvén Waves and Folded Fields



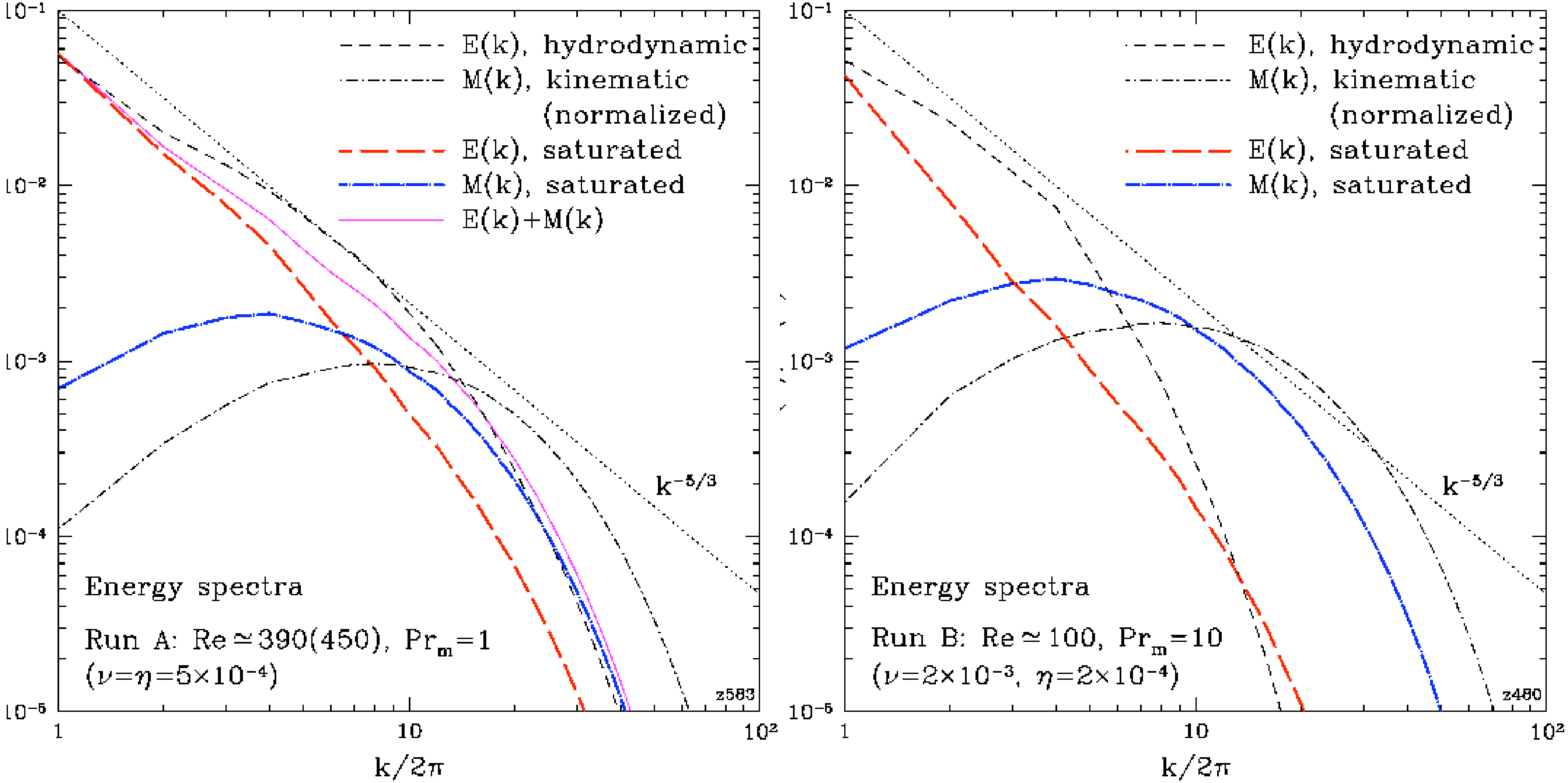
Dispersion relation: $\boxed{\omega^2 = \mathbf{kk} : \hat{\mathbf{b}}\hat{\mathbf{b}} \langle B^2 \rangle}$ for $k_\nu \ll k \ll k_{||} \sim \ell_s^{-1}$

We propose that

- ***saturation*** *is a balance between stretching and mixing by the outer-scale motions and Ohmic diffusion of the folded field*
- *the fully developed isotropic MHD turbulence* ***in the inertial range*** *is a superposition of folded magnetic fields and Alfvén waves*

[AAS *et al*. 2004, *ApJ* **612**, 276]

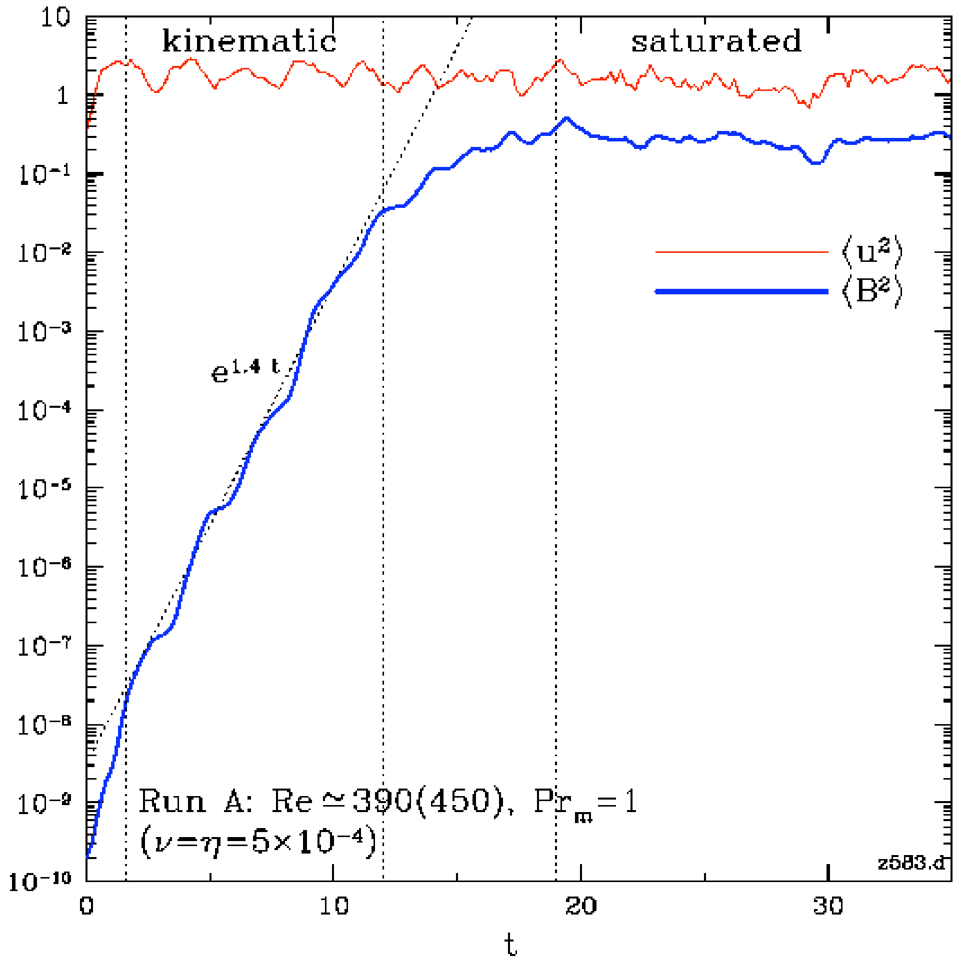
DNS: Saturated Spectra



- **Folds** account for the predominance of large-*k* modes in magnetic-energy spectra
- **Alfvén waves** should show up *in the velocity spectra*

[AAS *et al*. 2004, *ApJ* **612**, 276]

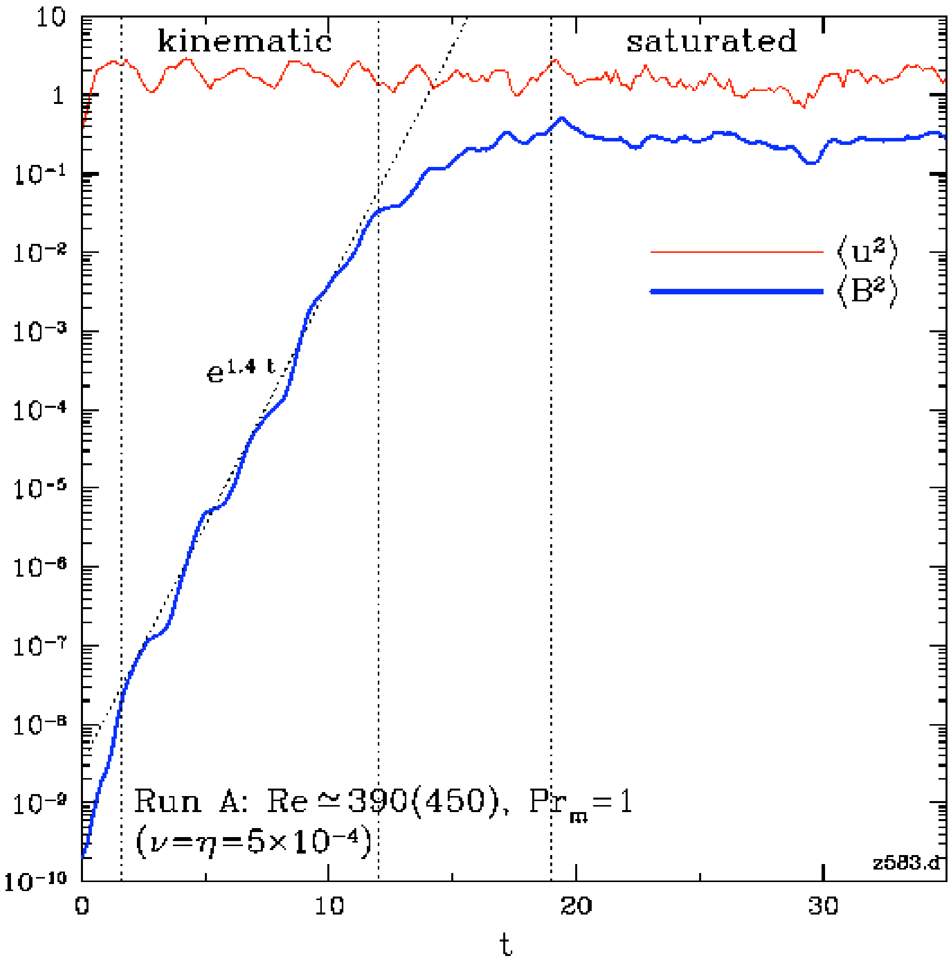
DNS: Intermediate Growth



Slower than exponential growth

[AAS *et al*. 2004, *ApJ* **612**, 276]

DNS: Intermediate Growth



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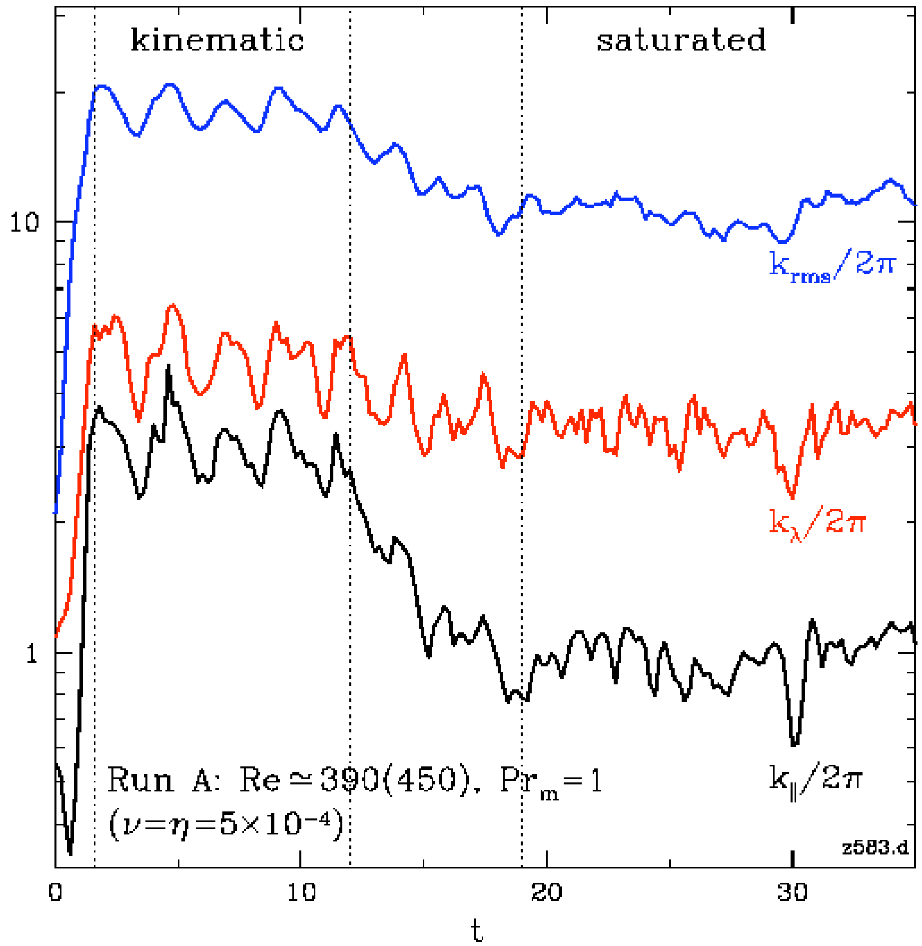

Selective decay and fold elongation

[AAS *et al*. 2004, *ApJ* **612**, 276]

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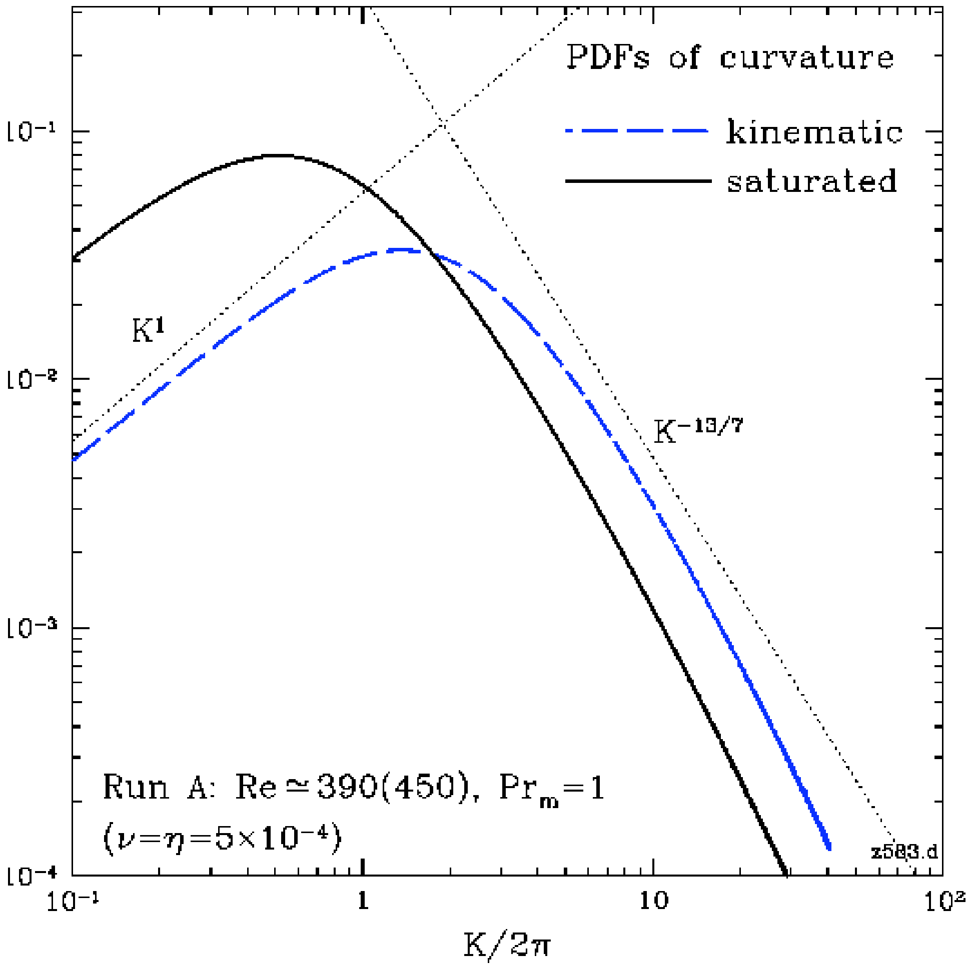

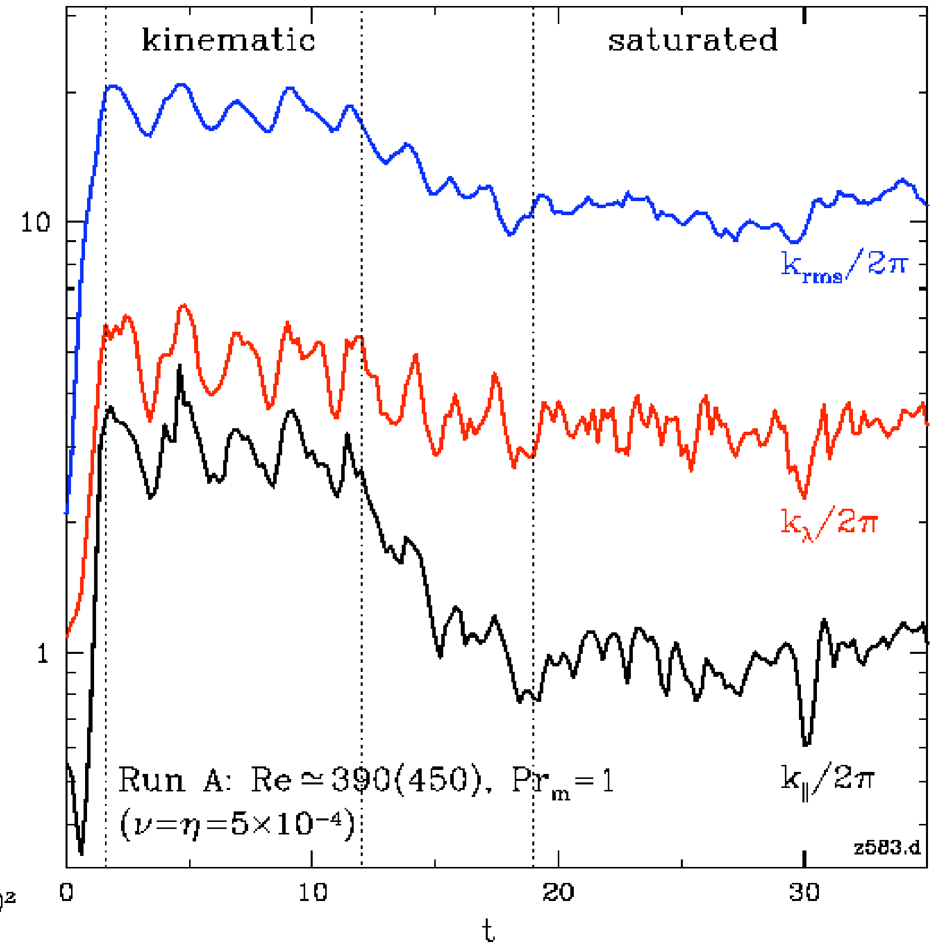

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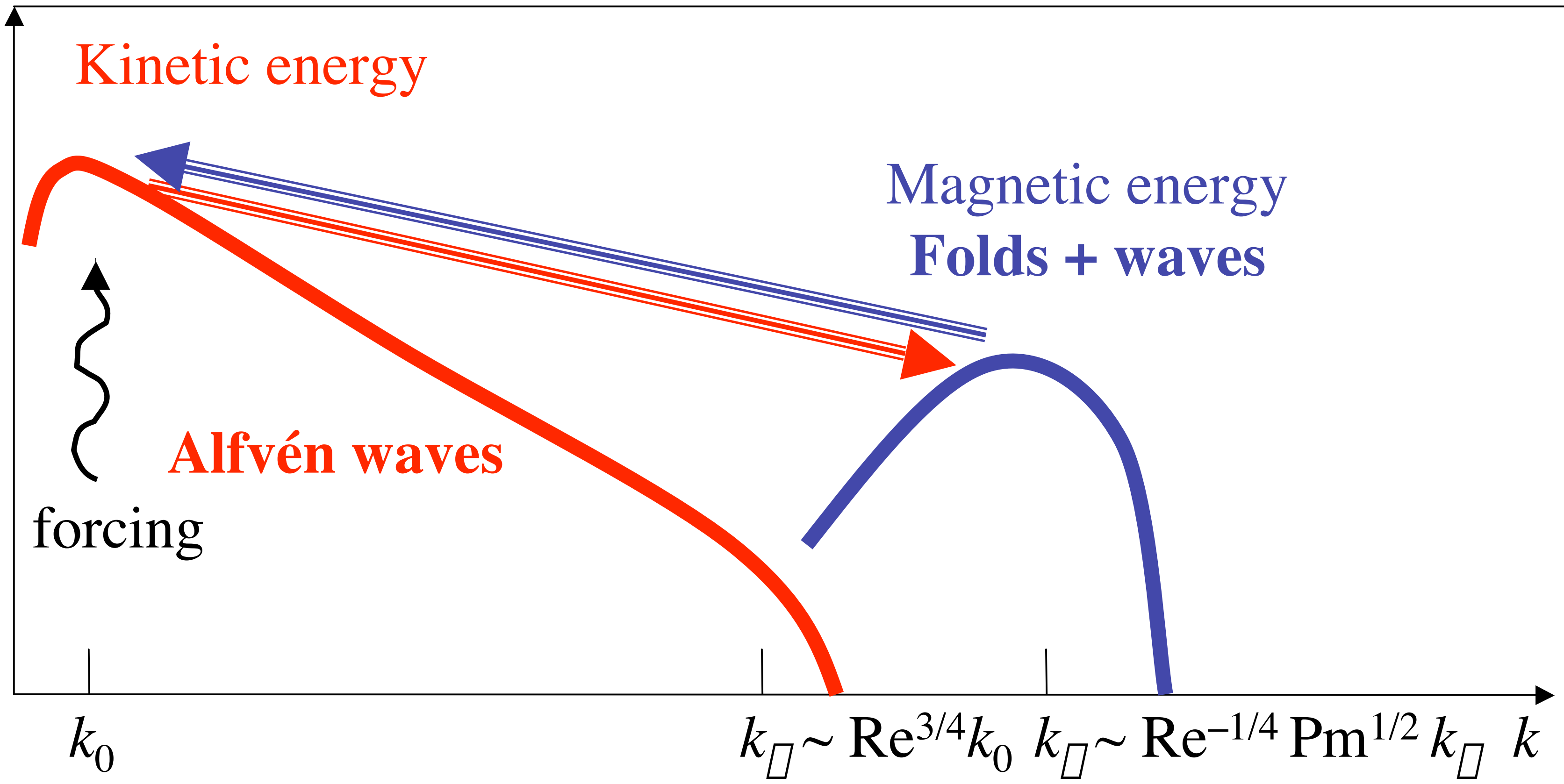
Alfvén Waves and Folded Fields

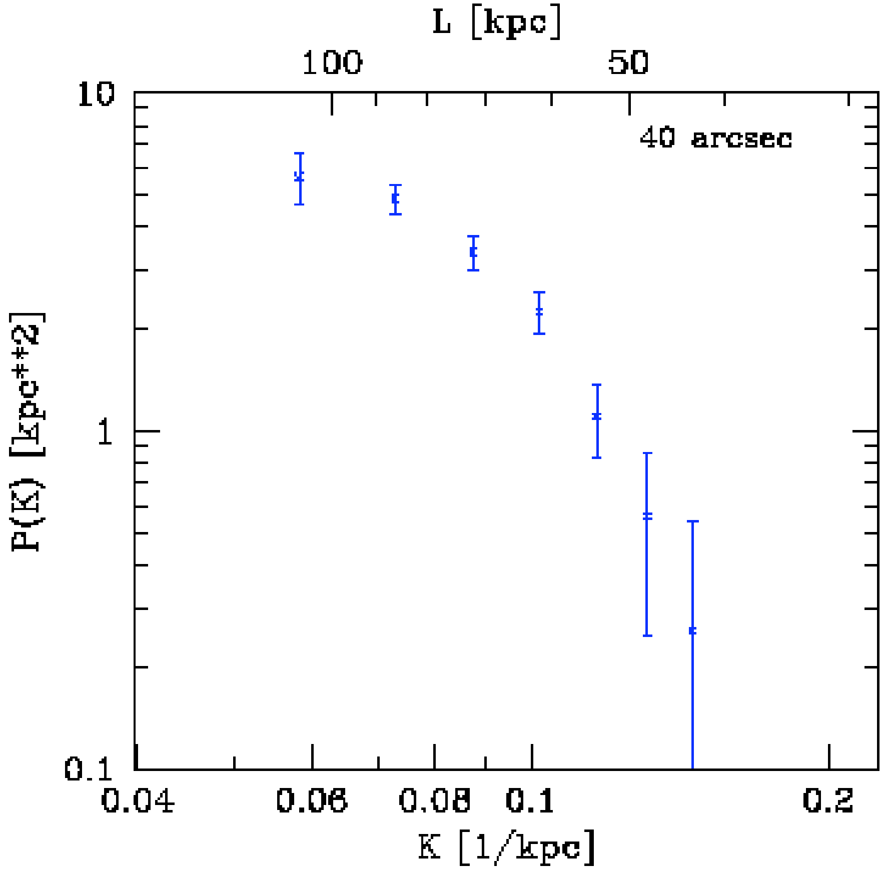


NB: The assumption of locality in k space has been abandoned!

[AAS *et al*. 2004, *ApJ* **612**, 276]

Cluster Turbulence

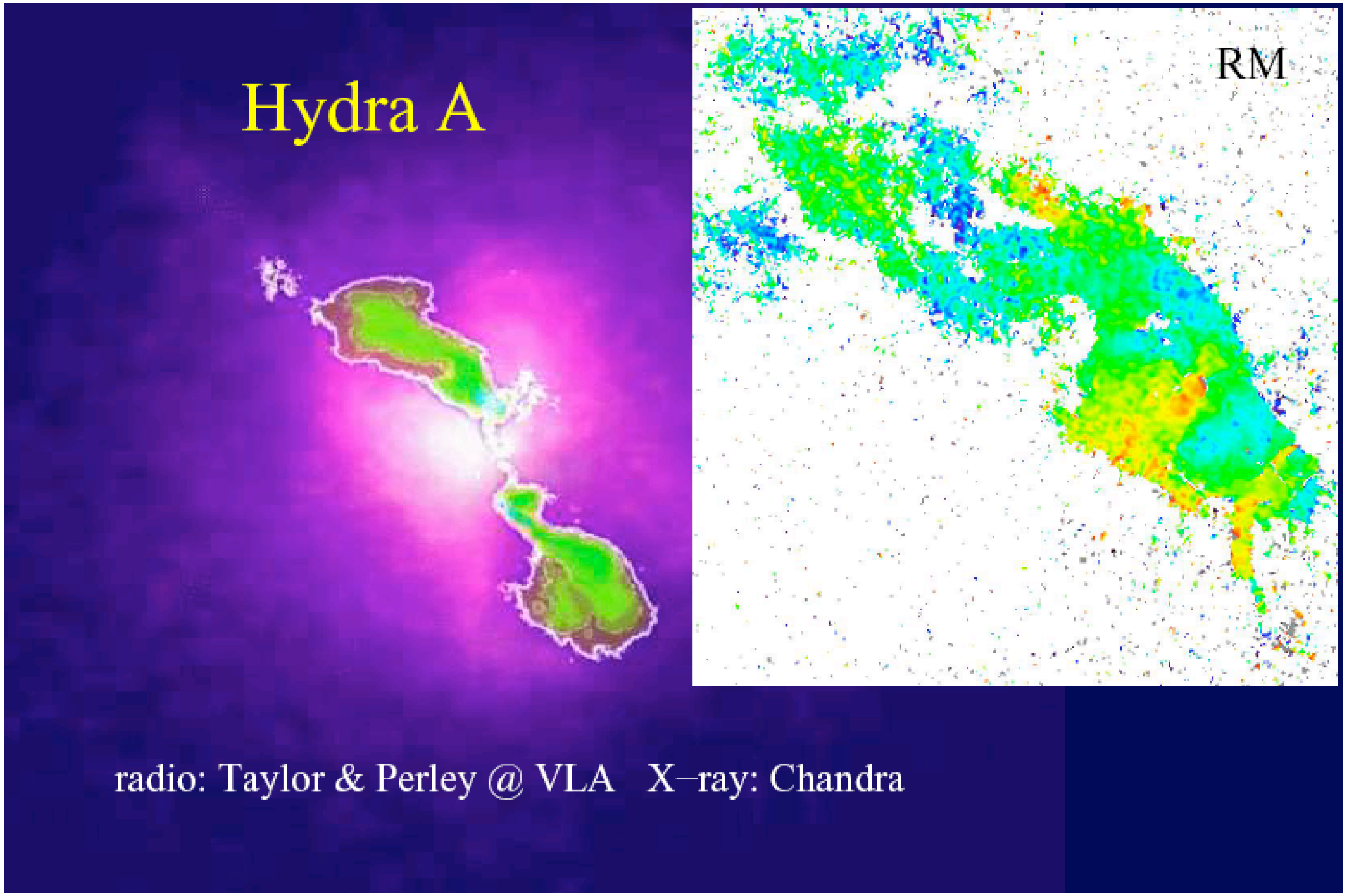
TURBULENCE
Coma cluster
[Schuecker *et al.*, astro-ph/0404132]



- Driven by mergers
- Subsonic below outer scale
- Outer scale ~ $10^2 \ldots 10^3$ kpc
- Viscous scale ~ 10…30 kpc
- Re ~ $10^2 \ldots 10^3$

Cluster Magnetic Fields

Faraday Rotation data from extended sources allows one to measure spatial structure (spectra) of magnetic fields in clusters

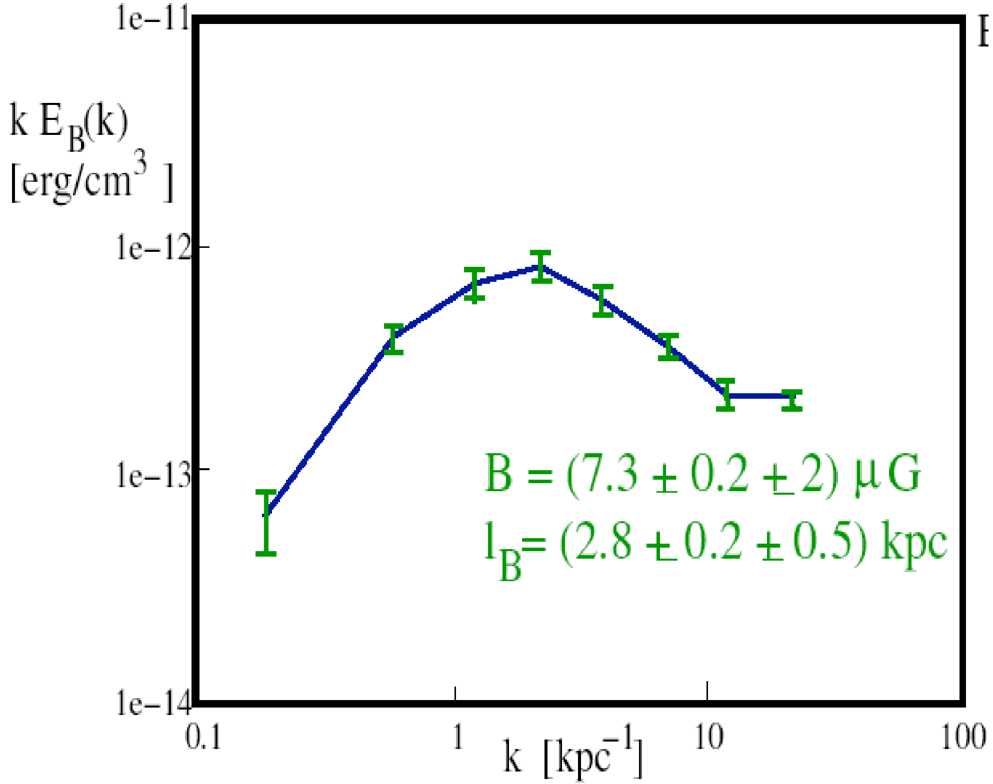


[picture courtesy of T. Enßlin]

Cluster Magnetic Fields

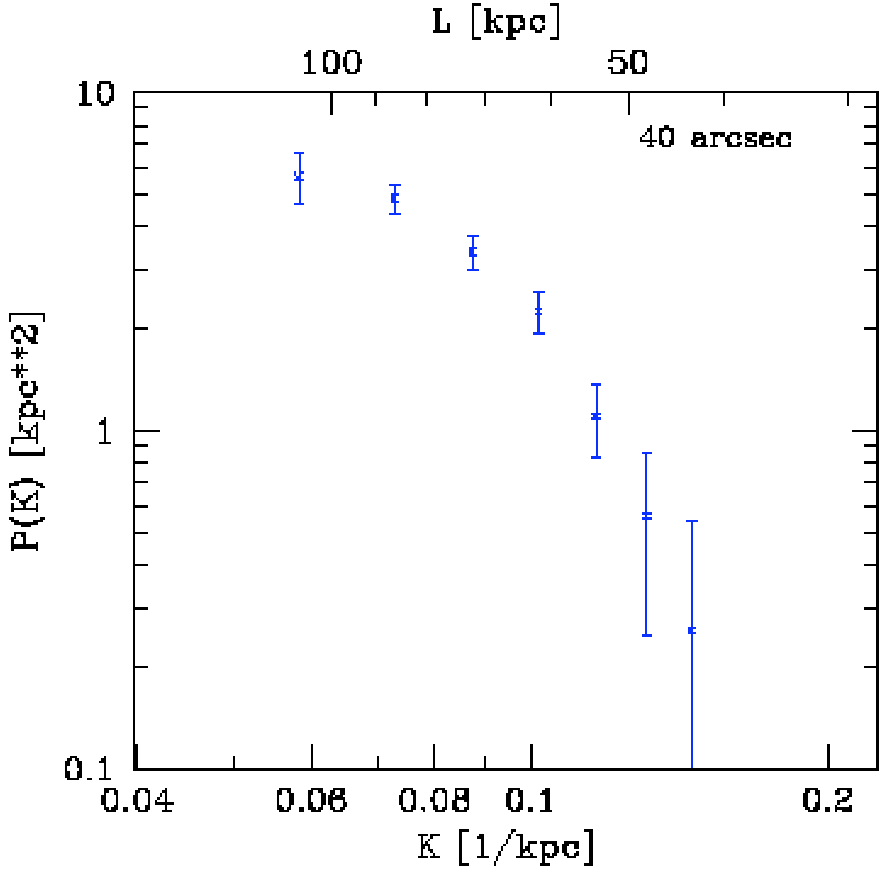
MAGNETIC FIELDS
Hydra A Cluster
[Vogt & Enßlin 2004,
picture courtesy of T. Enßlin]

- *B* ~ 1…10 G
  (equipartition
  strength ~ 100 G)
- Mostly disordered
- Scale ~ 1 kpc

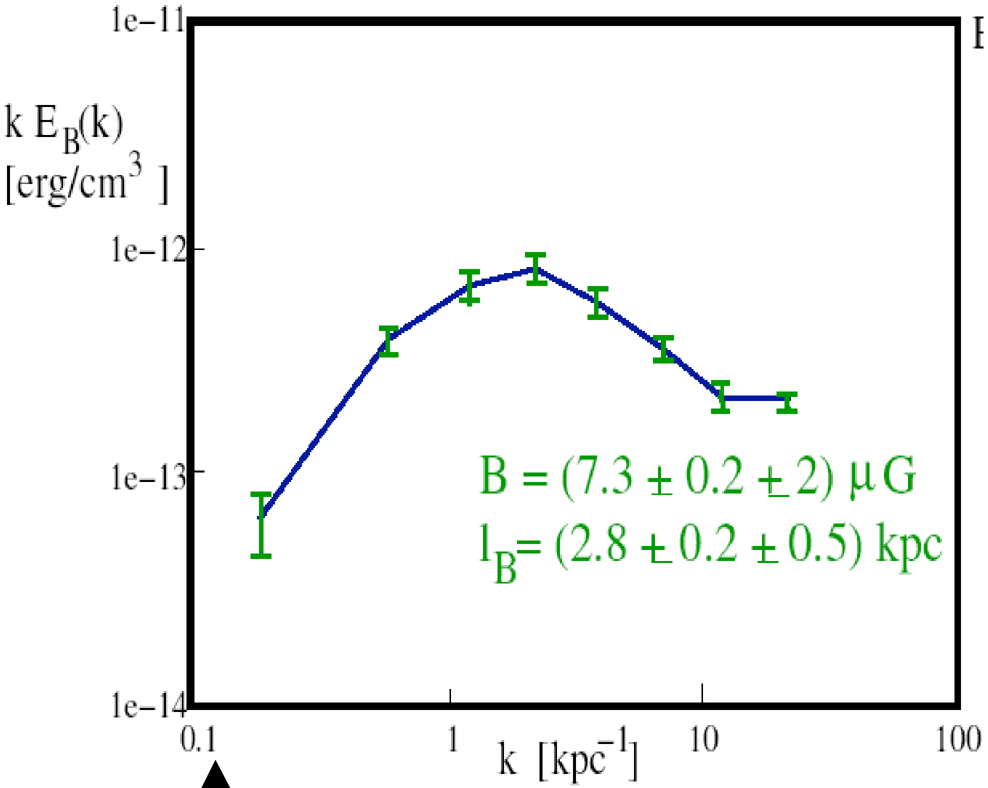


Cluster MHD

TURBULENCE
Coma cluster
[Schuecker *et al*., astro-ph/0404132]

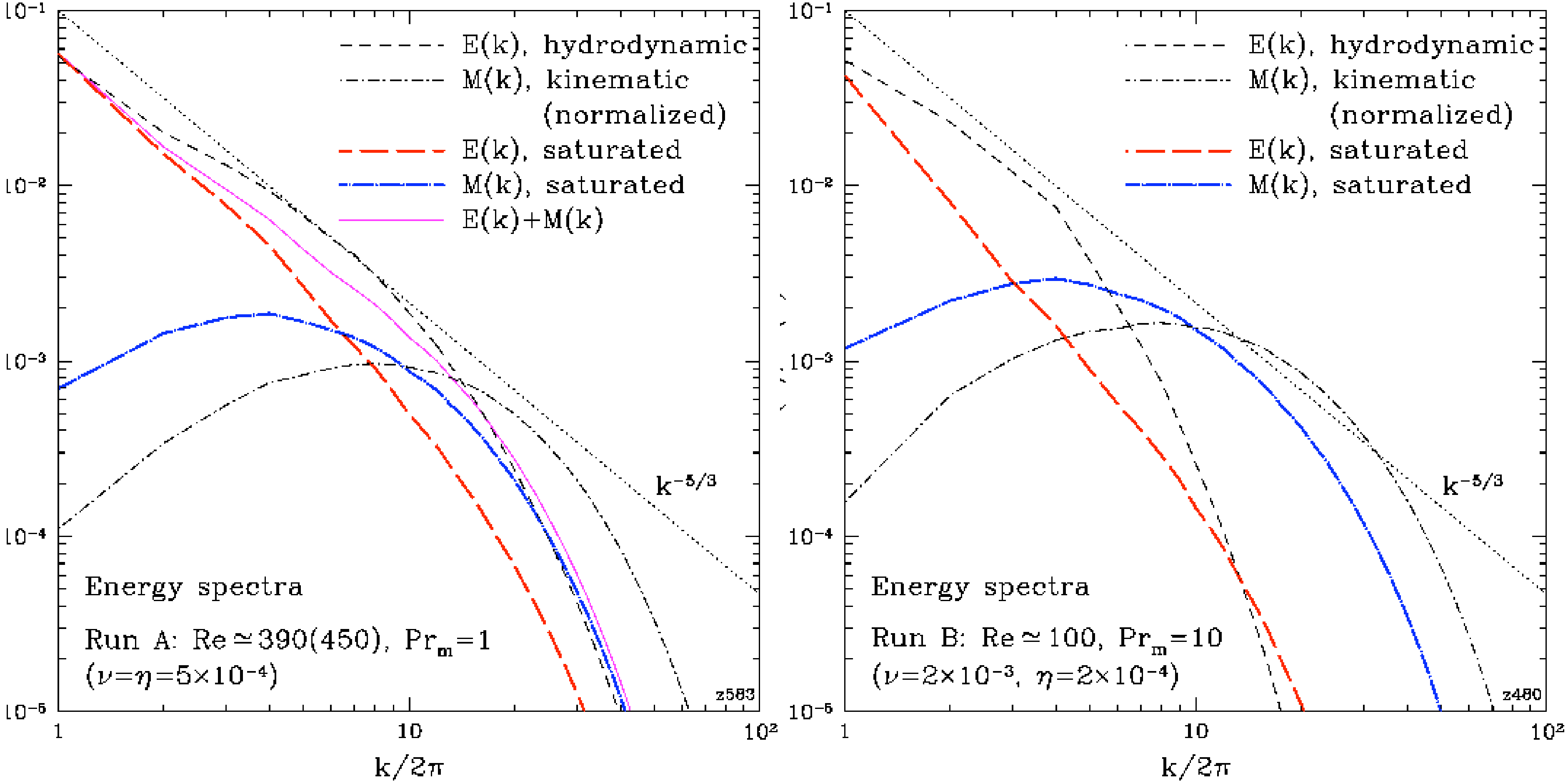


MAGNETIC FIELDS
Hydra A Cluster
[Vogt & Enßlin 2004,
picture courtesy of T. Enßlin]



Viscous scale is around here (~10 kpc)

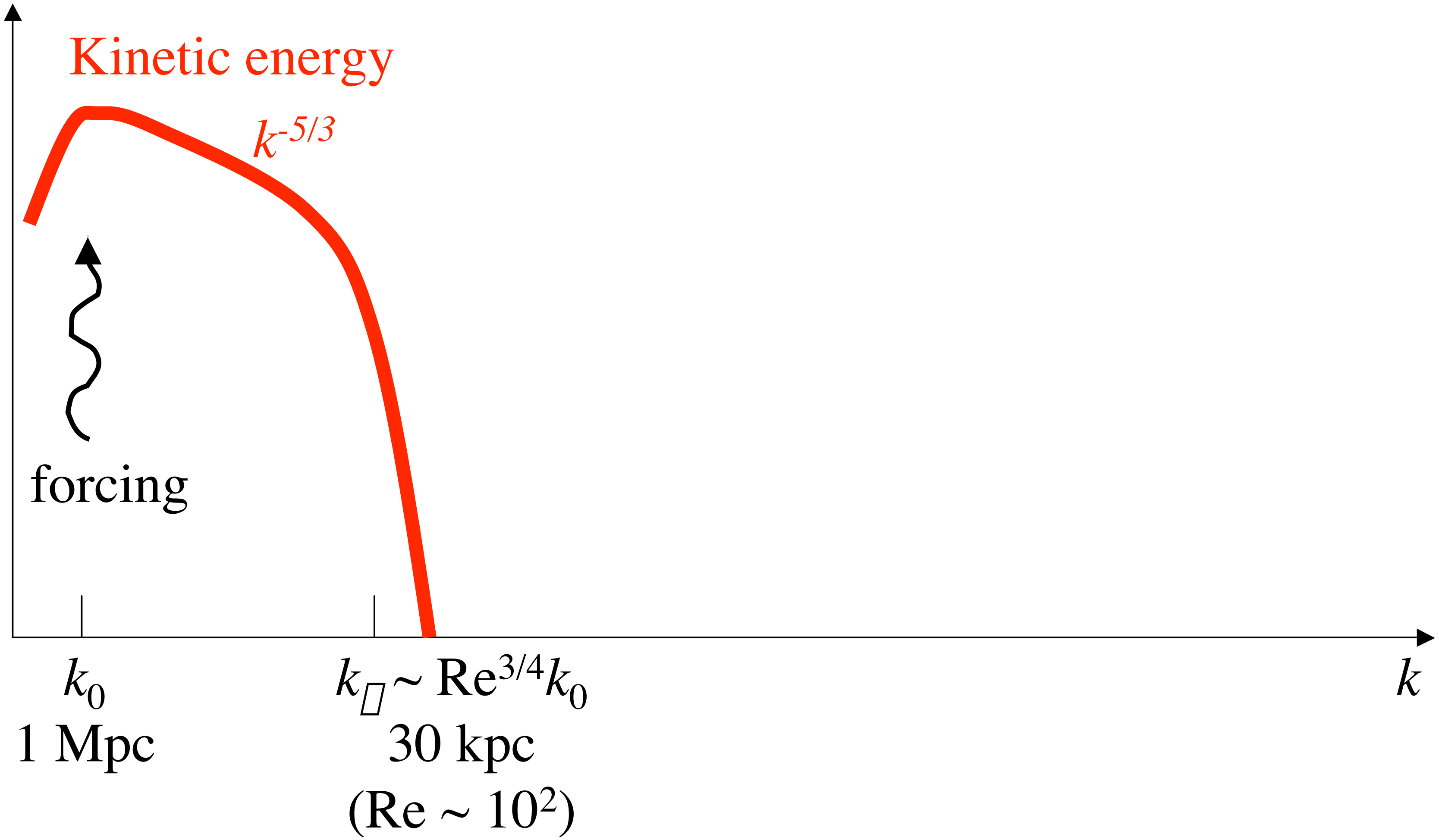
Cluster MHD



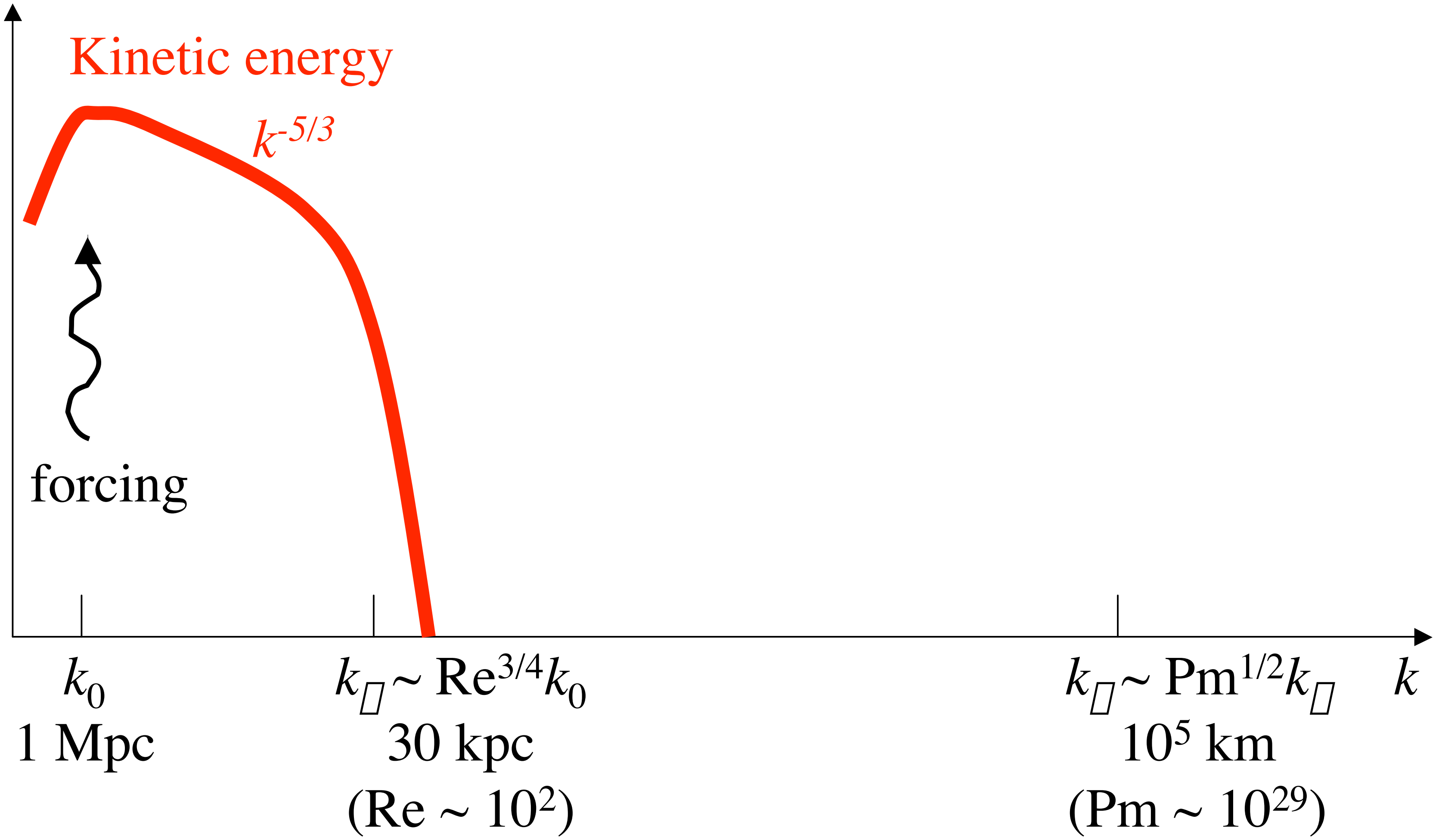
Kinetic- and magnetic-energy spectra look (at least qualitatively) quite similar to ourspectra of simulated MHD turbulence

Does this mean we've got it right?

Cluster MHD

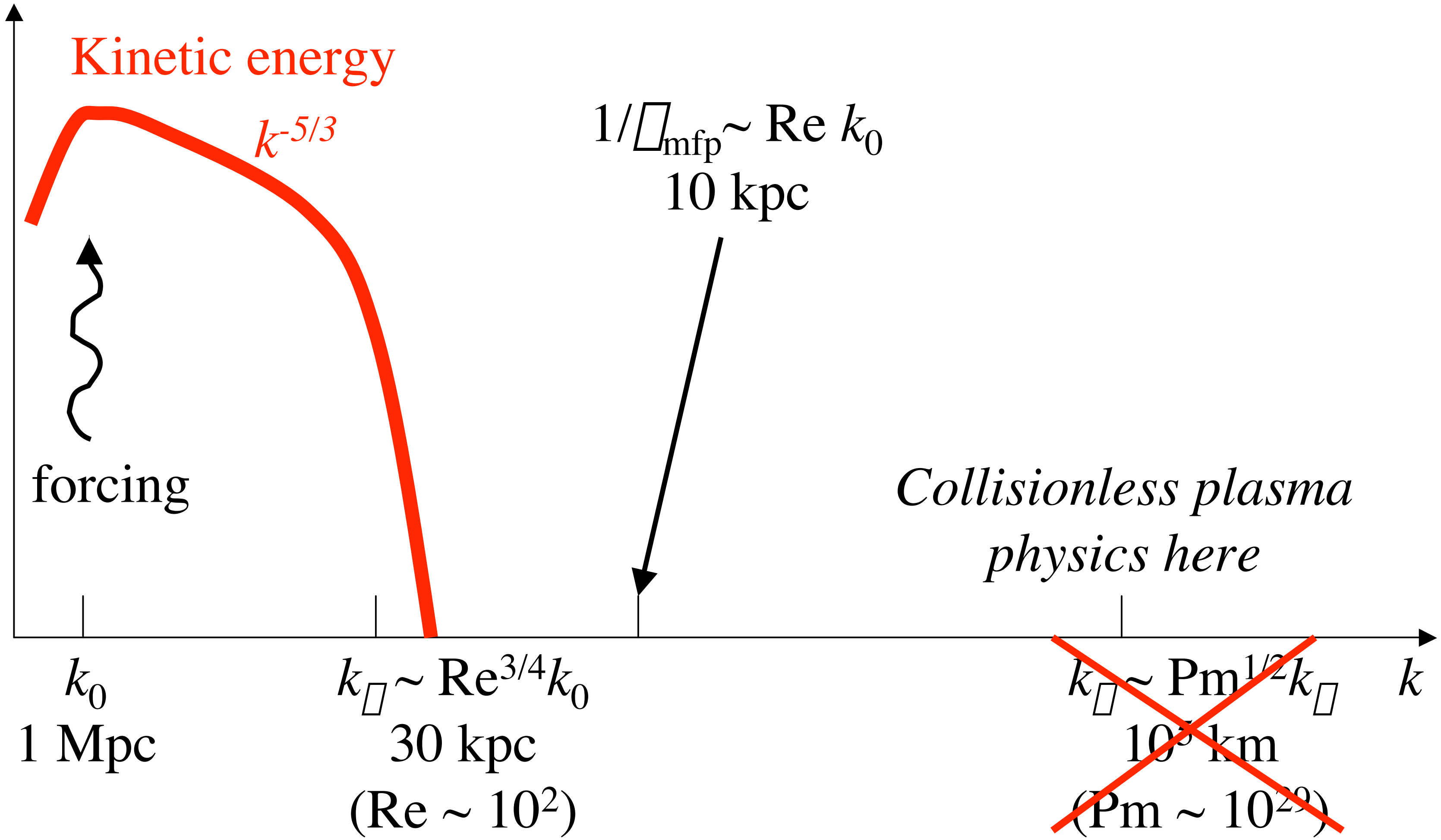


Cluster MHD



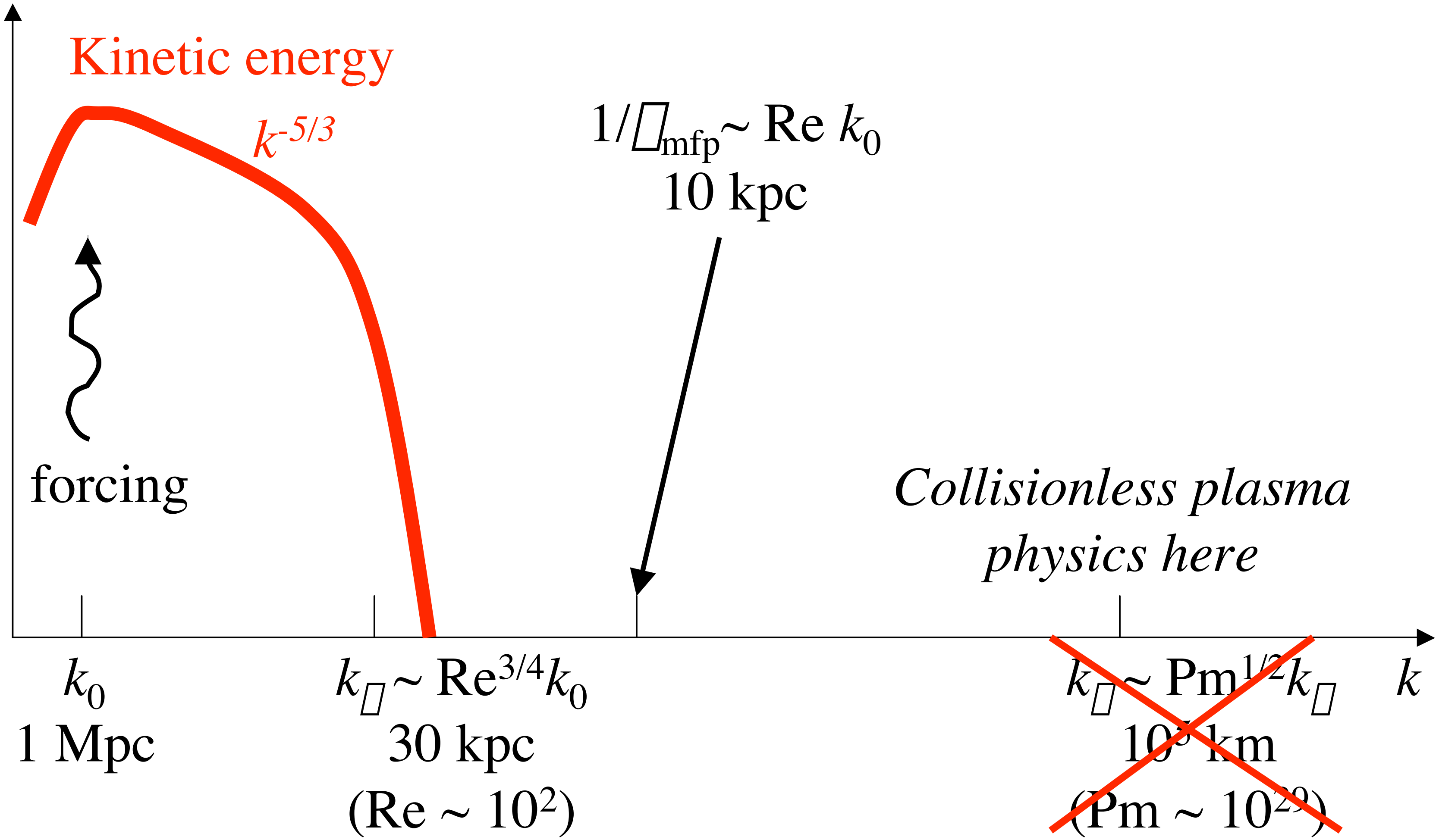
This is a ludicrous distance!

Cluster ~~MHD~~



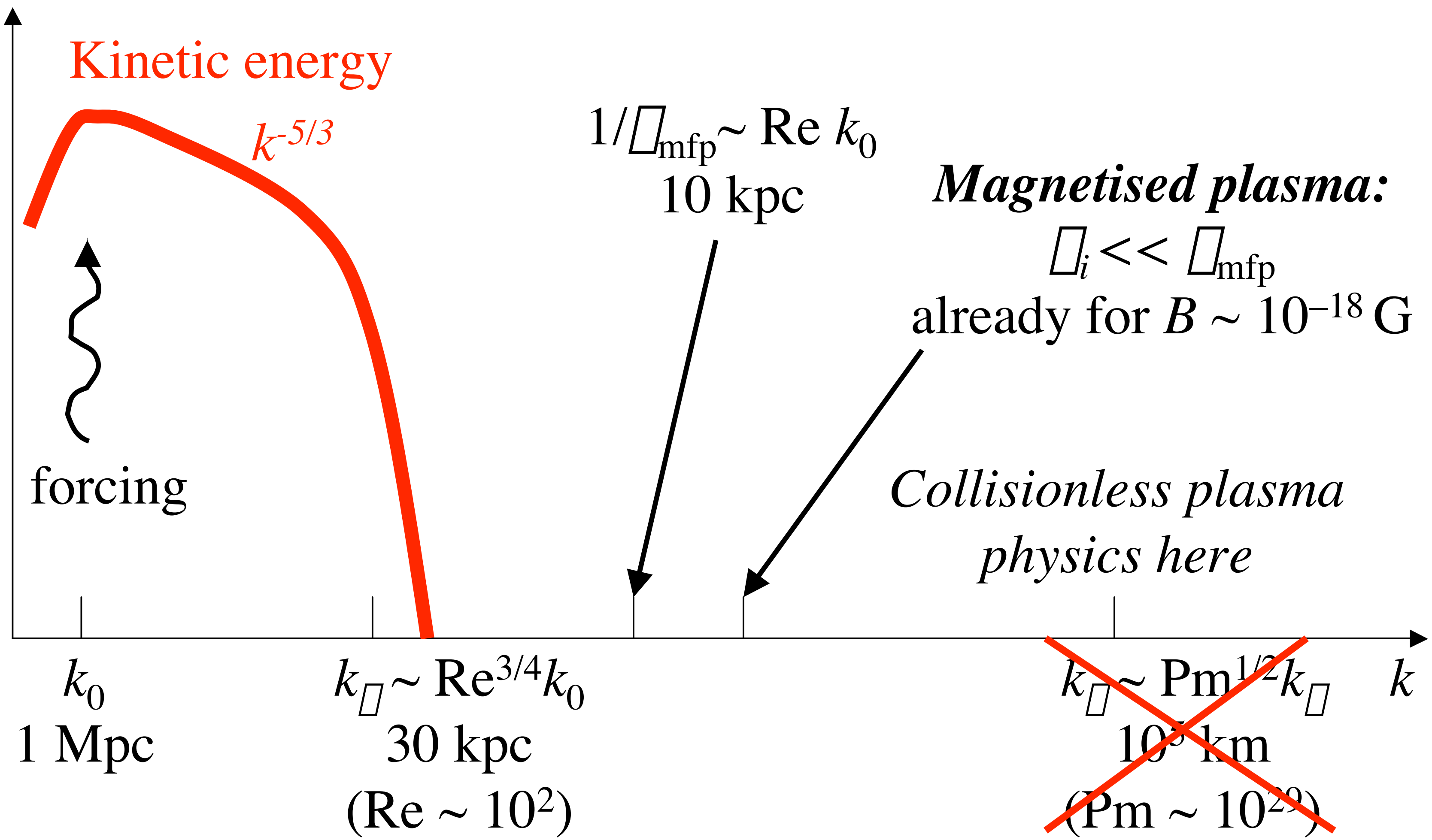
<u>What is the effective magnetic cutoff?</u>

Cluster Plasma Physics



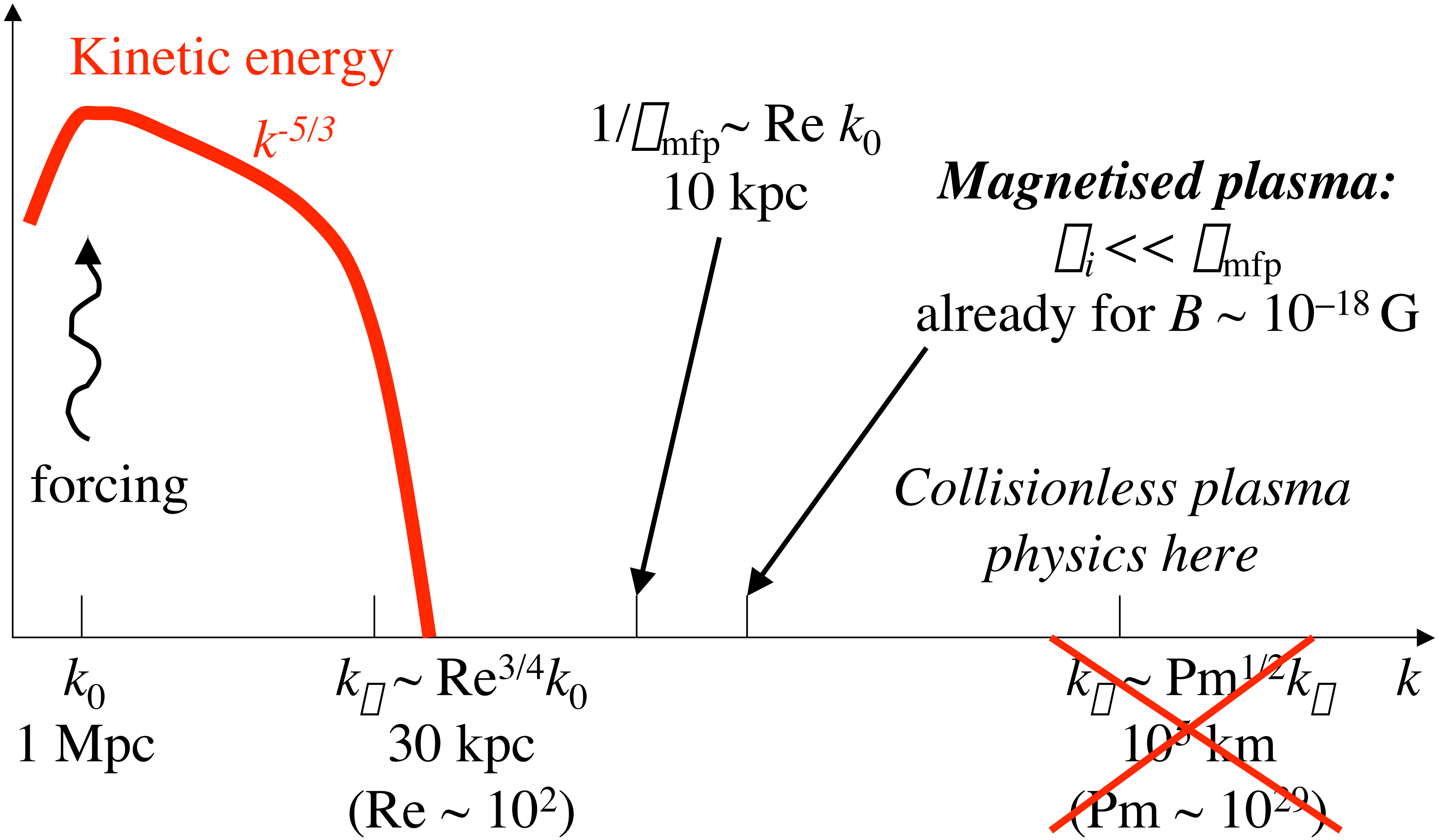
$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \cdot \hat{\mathbf{P}} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \quad \text{with} \quad \hat{\mathbf{P}} = \sum m\mathbf{v}\mathbf{v}$$

Cluster Plasma Physics



$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \cdot \hat{\mathbf{P}} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \quad \text{with} \quad \hat{\mathbf{P}} = (\hat{\mathbf{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})p_\perp + \hat{\mathbf{b}}\hat{\mathbf{b}}p_\parallel$$

Cluster Plasma Physics



$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \tilde{p} + \nabla \cdot \left[\hat{\mathbf{b}}\hat{\mathbf{b}}(p_\perp - p_{||})\right] + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi} \qquad \nabla \cdot \mathbf{u} = 0$$

where $p_\perp - p_{||} = 3\rho\nu_B \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{u}$

MHD with Braginskii Viscosity

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \tilde{p} + \nabla \cdot \left[\hat{\mathbf{b}}\hat{\mathbf{b}}(p_\perp - p_{||})\right] + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi} \qquad \nabla \cdot \mathbf{u} = \dot{0}$$

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$$\frac{d\mathbf{B}}{dt} = \mathbf{B}\cdot\nabla\mathbf{u}$$

[Braginskii 1965, *Rev. Plasma Phys.* **1**, 205]

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<u>Physics</u>: *conservation of the first adiabatic invariant* $\mu = mv_\perp^2/2B$

Changes in field strength □ pressure anisotropy

[Braginskii 1965, *Rev. Plasma Phys.* **1**, 205]

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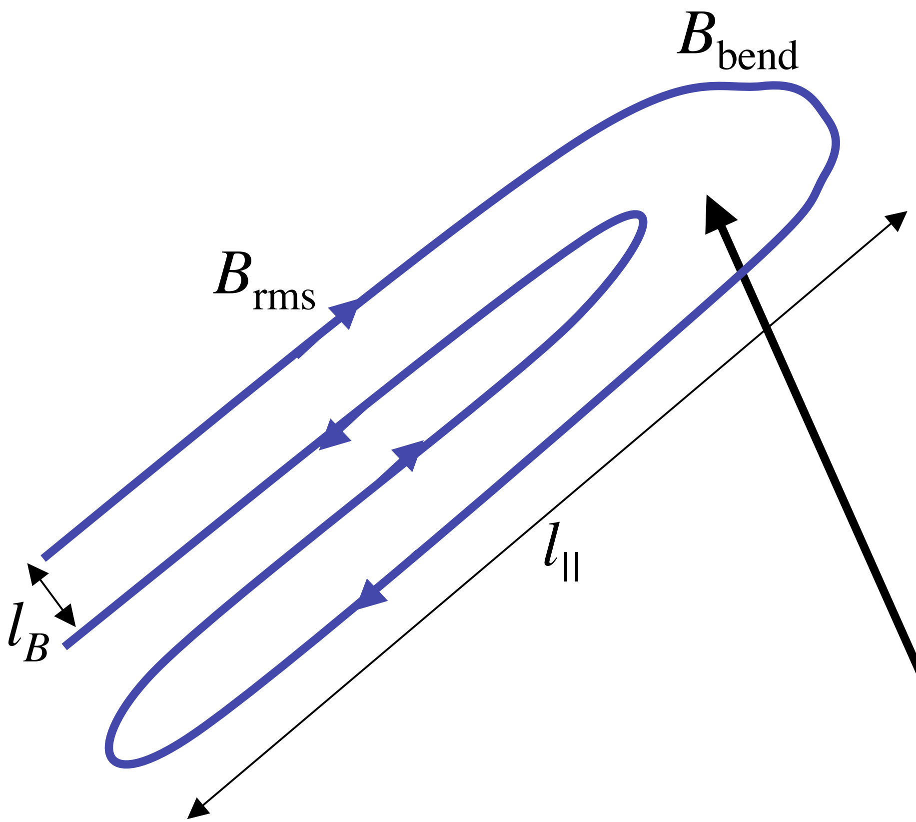
<u>Physics</u>: *conservation of the first adiabatic invariant* $\mu = m v_{\perp}^2 / 2B$

Changes in field strength pressure anisotropy

<u>Can we solve/simulate this modified system of equations?</u>

[Braginskii 1965, *Rev. Plasma Phys.* **1**, 205]

Firehose Instability



There is a very fast-growing instability (faster than ∇u!)

$$\gamma = k_{||}(-3\nu_B \mathbf{bb}:\nabla\mathbf{u} - v_A^2)^{1/2}$$

valid both for $k_\perp << k_{||} < 1/\lambda_{\mathrm{mfp}}$ and $k_{||} > 1/\lambda_{\mathrm{mfp}}$

amplifies shear-Alfvén-polarised perturbations

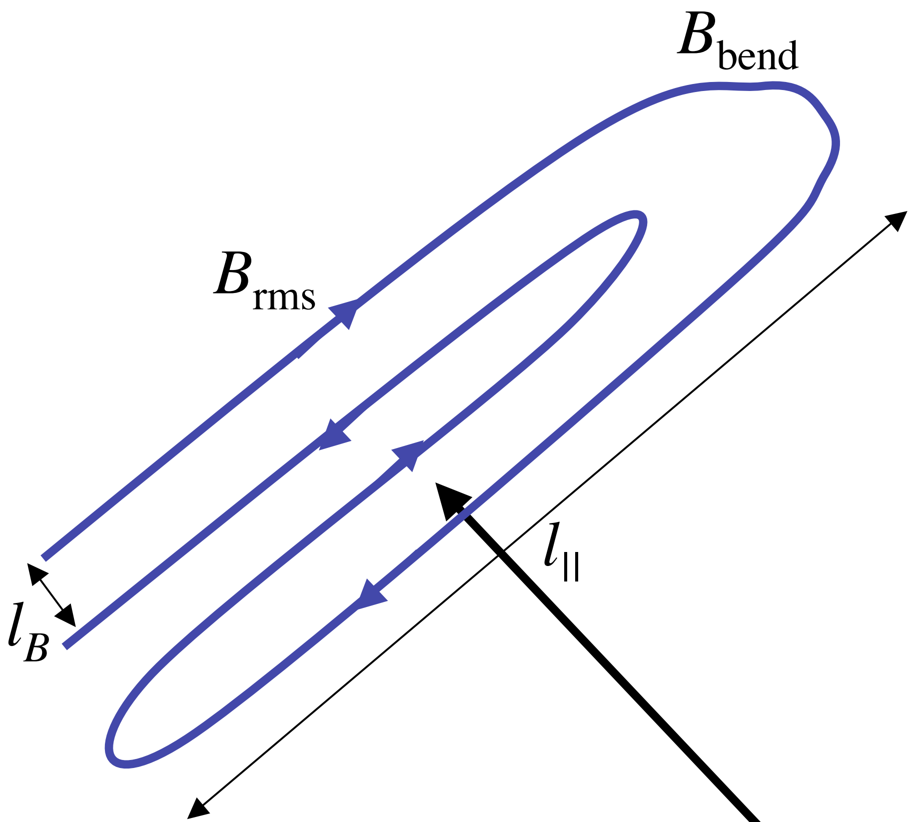
occurs in the regions of decreasing field:

$$\frac{1}{B}\frac{dB}{dt} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} < 0$$

bends of the folds

[Rosenbluth 1956, *LANL Report 2030*;
Vedenov & Sagdeev 1958, *Doklady* **3**, 278;
Parker 1958, *Phys. Rev.* **109**, 1874;
Chandrasekhar *et al*. 1958, *Proc. Roy. Soc.* **245**, 435]

Mirror Instability



There is another instability (slower than firehose, still faster than ∇u)

$$\gamma = (2/\pi)^{1/2} k_{||} \square \left[(3\square_B \mathbf{bb}{:}\nabla\mathbf{u}/v_{th})(1-k_{||}^2/2k_{\perp}^2) - v_A^2(1+k_{||}^2/k_{\perp}^2) \right]$$

valid for $k_{\perp} > k_{||}/\sqrt{2} >> 1/\lambda_{mfp}$
(collisionless regime)

occurs in the regions of increasing field:

$$\frac{1}{B}\frac{dB}{dt} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} > 0$$

straight segments of the folds

[Rosenbluth 1956, *LANL Report 2030*;
Vedenov & Sagdeev 1958, *Doklady* **3**, 278;
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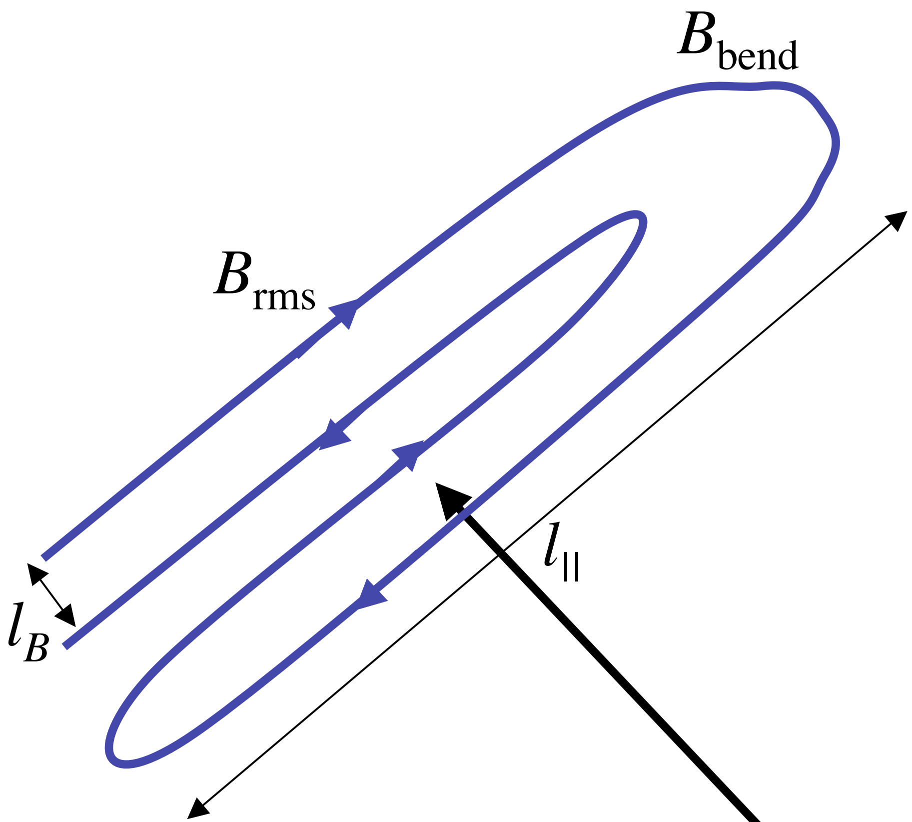

STABILISED
when $v_A^2 \sim u^2$
(at the viscous scale)

There is another instability (slower than firehose, still <u>faster than □u</u>)

$$\square = (2/\square)^{1/2} k_{||} \square$$
$$[(3\square_B \mathbf{bb}{:}\square\mathbf{u}/v_{\text{th}})(1 - k_{||}^2/2k_{\square}^2)$$
$$- v_A^2(1 + k_{||}^2/k_{\square}^2)]$$

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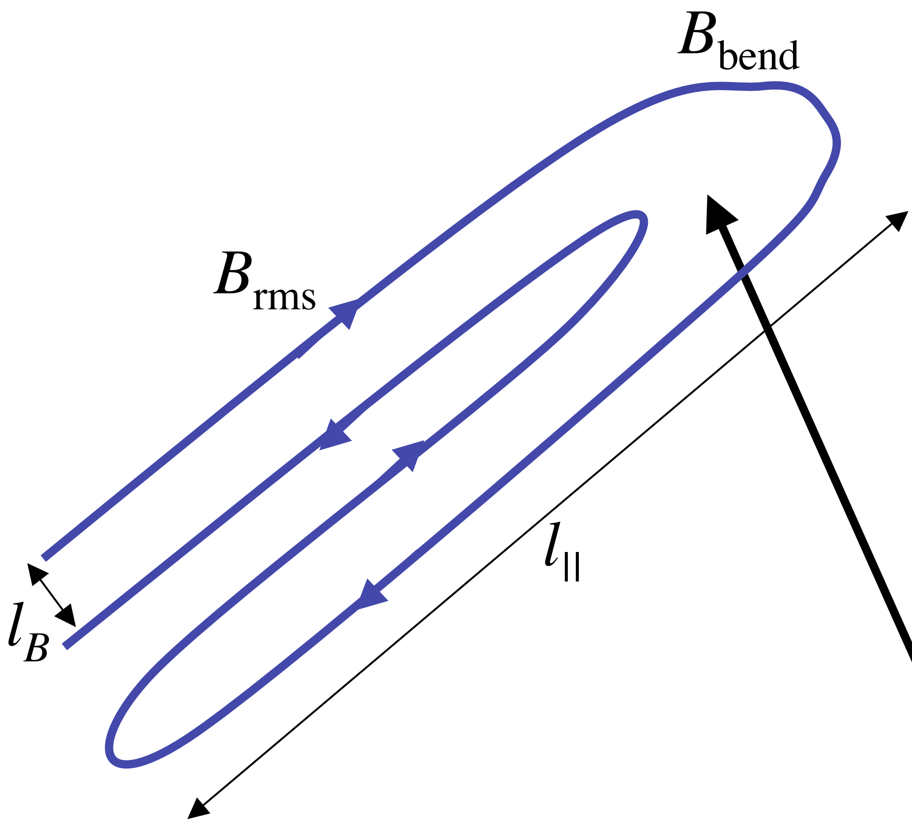
straight segments of the folds

Firehose Instability



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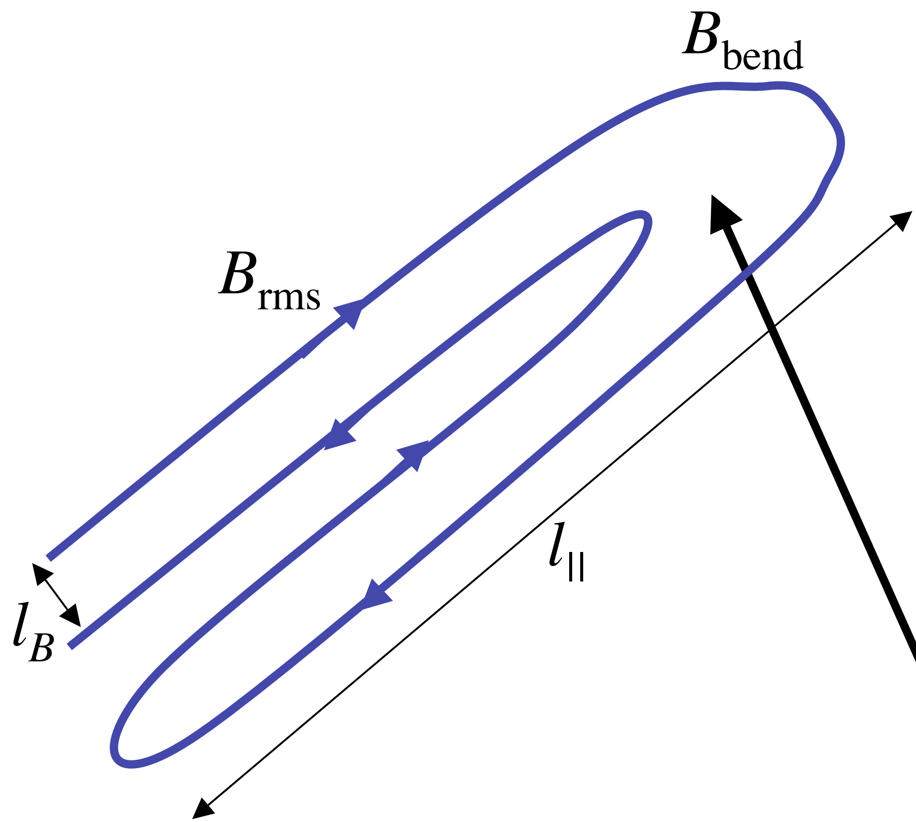
bends of the folds

NOT STABILISED:
in the bends $v_A^2 << u^2$
even in saturation

Firehose Instability



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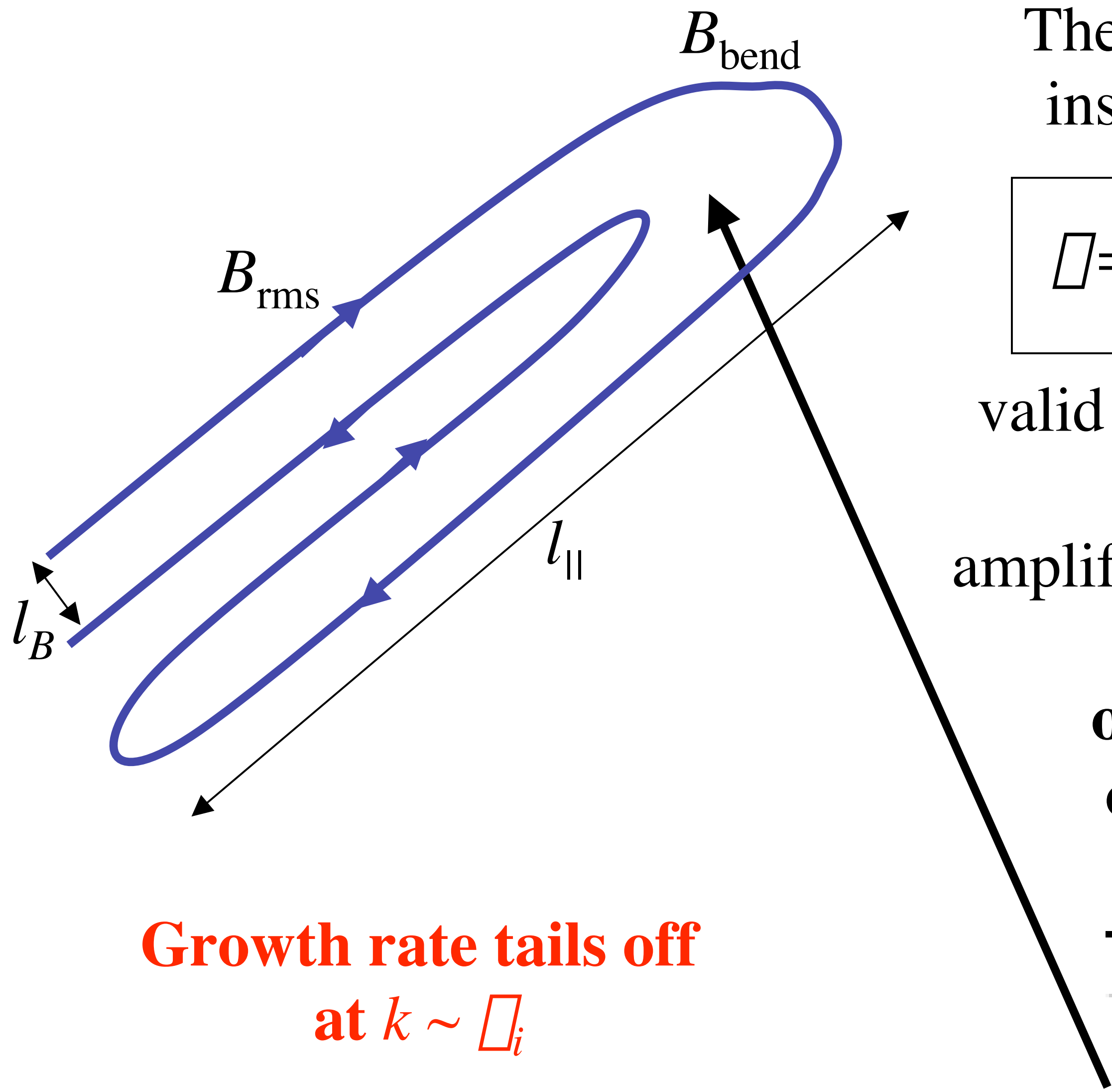
bends of the folds

So a simulation of MHD with Braginskii viscosity will blow up at the grid scale!

[Maron 2002, unpublished]

Firehose Instability



There is a very fast-growing instability (faster than ∇u!)

$$\gamma = k_{||}(-3\nu_B \mathbf{bb}:\nabla\mathbf{u} - v_A^2)^{1/2}$$

valid both for $k_{\perp} << k_{||} < 1/\lambda_{\text{mfp}}$
and $k_{||} > 1/\lambda_{\text{mfp}}$
amplifies shear-Alfvén-polarised perturbations

occurs in the regions of decreasing field:

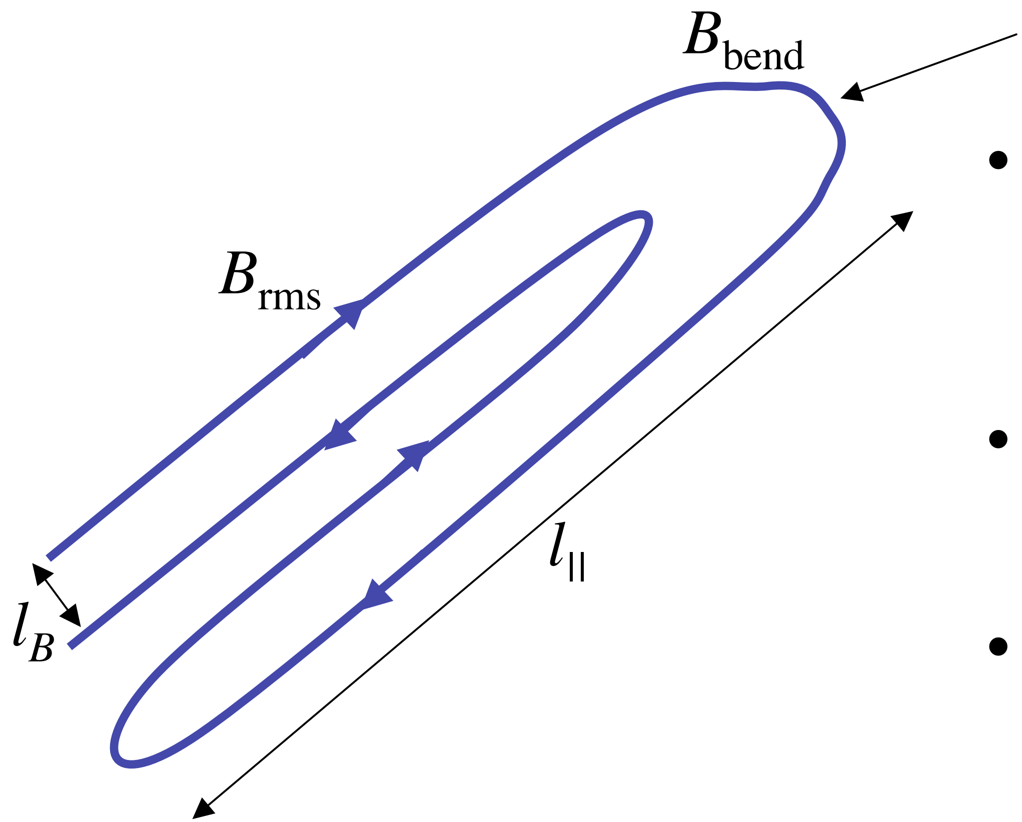
$$\frac{1}{B}\frac{dB}{dt} = \hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} < 0$$

bends of the folds

Growth rate tails off at $k \sim \rho_i$

[Davidson & Völk 1968, *Phys. Fluids* **11**, 2259]

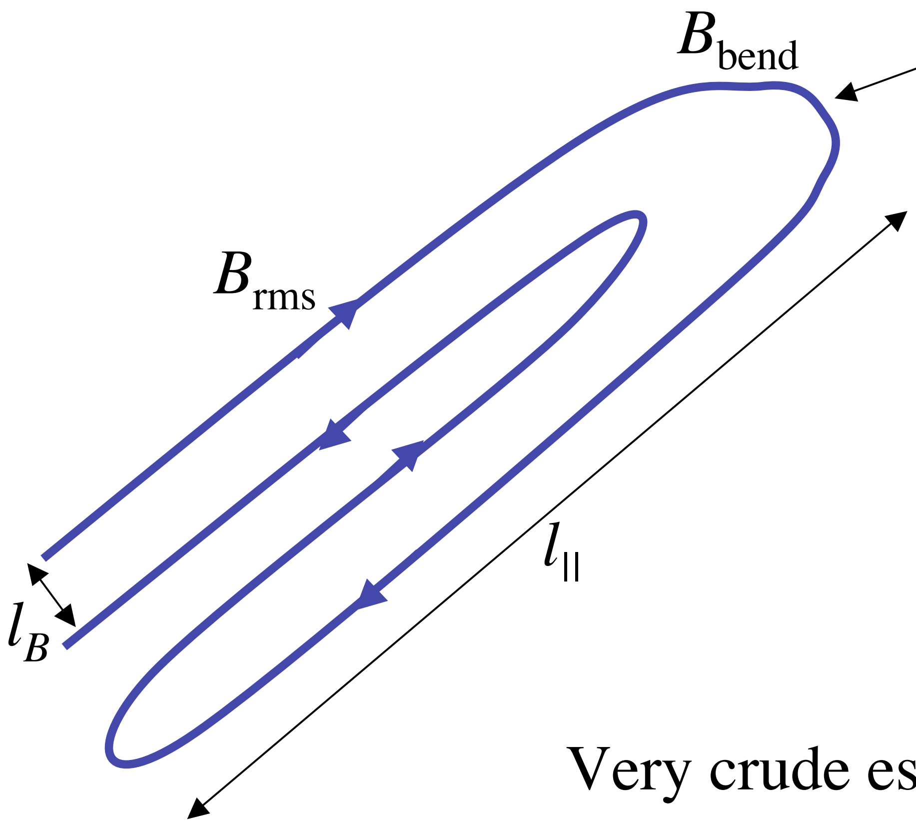
Effective Magnetic Cutoff



Curvature $K_{\rm bend} \sim 1/\rho_{i,\,\rm bend}$

- Can show that $BK^{1/2} \sim$ const throughout the fold, so
  $B_{\rm bend}/B_{\rm rms} \sim (\rho_{i,\,\rm bend}/l_{||})^{1/2}$
- But $\rho_i \propto 1/B$, so
  $\rho_{i,\,\rm bend}/\rho_{i,\,\rm rms} \sim B_{\rm rms}/B_{\rm bend}$
- Also $l_B/l_{||} \sim B_{\rm bend}/B_{\rm rms}$

Effective Magnetic Cutoff



Curvature $K_{\text{bend}} \sim 1/\square_{i,\,\text{bend}}$

- Can show that $BK^{1/2} \sim$ const throughout the fold, so
  $B_{\text{bend}}/B_{\text{rms}} \sim (\square_{i,\,\text{bend}}/l_{||})^{1/2}$
- But $\square_i$ μ $1/B$, so
  $\square_{i,\,\text{bend}}/\square_{i,\,\text{rms}} \sim B_{\text{rms}}/B_{\text{bend}}$
- Also $l_B/l_{||} \sim B_{\text{bend}}/B_{\text{rms}}$

Very crude estimates give

$$\frac{l_B}{1\ \text{kpc}} \sim 10^{-2}\left(\frac{T}{10^8\ \text{K}}\right)^{1/6}\left(\frac{B_{\text{rms}}}{1\ \mu\text{G}}\right)^{-1/3}\left(\frac{l_{||}}{1\ \text{Mpc}}\right)^{2/3} \sim 10\ \text{pc}$$

A bit too small, but not unreasonable…

NB: l_B tends to decrease as instrument resolution increases

[cf. Kim *et al*. 1990, *ApJ* **355**, 29; Feretti *et al*. 1995, *A&A* **302**, 680]

Poetry in Lieu of Summary

James Clerk Maxwell on the inevitability of kinetic description of cluster plasmas

At quite uncertain times and places,
The atoms left their heavenly path,
And by fortuitous embraces,
Engendered all that being hath.
And though they seem to cling together,
And form "associations" here,
Yet, soon or late, they burst their tether,
And through the depths of space career.

1874