Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET IV

This problem and any other questions you have about the course will be discussed in the 4th Examples Class.

1. Scalar Turbulence. Part IV: Spectrum of Decaying Scalar Variance in the Viscous-Convective Range. Consider the equation for the evolution of passive scalar $\theta(t, \mathbf{x})$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \eta \nabla^2 \theta. \tag{1}$$

where η is the scalar diffusivity (sorry about change of notation! — I need κ for velocity correlators). Consider scalar decay in a linear velocity field:

$$\mathbf{u}^i = \sigma^i_m(t) x^m. \tag{2}$$

(When is is this a reasonable model?) and take the velocity field to be a Gaussian white noise:

$$\langle \sigma^i_m(t) \sigma^j_n(t') \rangle = \delta(t - t') \kappa_2 \left[\delta^{ij} \delta_{mn} - \frac{1}{d+1} \left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m \right) \right]. \tag{3}$$

1. Construct a calculation leading to the equation for the passive scalar spectrum in the way exactly analogous to my calculation for the dynamo (see my lecture notes):

   (a) Write the solution of Eq. (1) as a superposition of plane waves. Find evolution equations for the amplitudes and wavevectors of these waves.

   (b) Define the joint PDF of the amplitudes and wavevectors. Derive a closed equation for this PDF using Furutsu-Novikov formula. Note that, because of isotropy, the PDF only depends on the absolute value of the wavevector — this will simplify your equation.

   (c) Show that the spectrum of the scalar variance is a superposition of spectra of the plane waves. Derive the equation for the spectrum $T(t, k)$:

   $$\frac{\partial T}{\partial t} + 2\eta k^2 T = -\frac{\partial}{\partial k} \mathcal{F}(k) = D \frac{\partial}{\partial k} \left[k^2 \frac{\partial T}{\partial k} - (d-1) k T \right], \tag{4}$$

   where $D = \kappa_2 (d-1)/2(d+1)$ and $\mathcal{F}(k)$ is the flux of scalar variance. This equation was first derived by Kraichnan in 1968 (in a different way).

   (d) Seek eigenfunction solutions of this equation, $T(t, k) = e^{-\lambda D t} \Phi(k/k_\eta)$, where $k_\eta = (D/2\eta)^{1/2}$. Solve for Φ.

   If we introduce some cut-off wavenumber $k_* \ll k_\eta$, λ is determined by the boundary condition on the flux $\mathcal{F}(k_*)$.

2. Let us consider a forced scalar problem for a moment (as in Problem 4 of Example Sheet III). The forcing is pumping a constant flux $\bar{\epsilon}_\theta$ at some large scale. All this flux must be dissipated, so we must have $\mathcal{F}(k_*) = \bar{\epsilon}_\theta$. Find the solution that satisfies this boundary condition and show that it has the Batchelor k^{-1} scaling at $k_* < k \ll k_\eta$.

3. Now consider the decaying case. As in the dynamo case, we might think a zero-flux boundary condition should be imposed: $\mathcal{F}(k_*) = 0$. Calculate λ in this case. What is the slope of the spectrum at $k_* < k \ll k_\eta$? Argue that your prediction for the decay rate means it is of the order of the turnover time of the viscous eddies.

4. These results had been thought to describe the scalar decay correctly until numerical experimental evidence showed the decay to be much slower: this effect is called *the strange mode.* In fact, the decay rate of the scalar is set by the decay rate of the slowest-decaying mode, which is a box-scale mode not described by the viscous-convective-range theory. It decays at the rate of turbulent diffusion associated with the box size $L_{\rm box}$, so we have, in fact,

$$\lambda \sim \frac{\text{decay rate of the box mode}}{\text{viscous eddy turnover rate}} \sim \frac{\delta u_L L/L_{\rm box}^2}{\delta u_{l_\nu}/l_\nu} \sim \left(\frac{L}{L_{\rm box}}\right)^2 \mathrm{Re}^{-1/2} \ll 1, \tag{5}$$

where $L \leq L_{\rm box}$ is the outer scale of the turbulence. Do you understand this estimate? Derive the last expression.

Assuming that λ is set by Eq. (5) and is equal to some small number, show from your solution that the scalar variance spectrum at $k_* < k \ll k_\eta$ scales as $k^{-1+\lambda/d}$ — only slightly shallower than Batchelor's spectrum.

These results are due to Fereday & Haynes, *Phys. Fluids* **16**, 4359 (2004) and Schekochihin, Haynes & Cowley, *Phys. Rev. E* **70**, 046304 (2004), but do try to derive them youself before you look!