Magnetohydrodynamics and Turbulence  
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EXAMPLE SHEET III

These problems will be discussed in the 3d Examples Class (9.03.05, 14:30 in MR5).

1. Anisotropic $k$-Space Correlation Functions. Consider the correlation function if the velocity field in $k$ space:

$$\langle u_i(k)u_j(k') \rangle = (2\pi)^3 \delta(k + k') C_{ij}(k)$$  \hspace{1cm} (1)

Suppose there is one special direction in space, defined by the unit vector $\hat{b}$ (this can be the direction of an imposed magnetic field or the axis of rotation or the direction of gravity). Then the general form of the tensor $C_{ij}$ is

$$C_{ij}(k) = C_1 \delta_{ij} + C_2 \hat{k}_i \hat{k}_j + C_3 \hat{b}_i \hat{b}_j + C_4 \hat{b}_i \hat{k}_j + C_5 \hat{k}_i \hat{b}_j ,$$  \hspace{1cm} (2)

where $\hat{k}_i = k_i / k$ and $C_1, \ldots, C_5$ are functions of $k$ and of $\xi = \hat{b} \cdot \hat{k} = \cos \theta$ ($\theta$ is the angle between $k$ and $\hat{b}$, so $k_i = \xi k$).

1. Assuming mirror symmetry, $C_{ij}(k) = C_{ij}(-k)$, and incompressibility of the velocity field, show that $C_{ij}$ can be written in the form

$$C_{ij}(k) = C_{iso}(k, \xi) \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right) + C_{aniso}(k, \xi) \left[ \hat{b}_i \hat{b}_j + \xi^2 \hat{k}_i \hat{k}_j - \xi \left( \hat{b}_i \hat{k}_j + \hat{k}_i \hat{b}_j \right) \right] .$$  \hspace{1cm} (3)

Express $C_{iso}$ and $C_{aniso}$ in terms of $C_1, \ldots, C_5$. Thus, second-order velocity correlator depends on two scalar functions only. We can get back the isotropic result by setting $C_{aniso} = 0$.

2. An alternative pair of scalar functions is often useful: the correlation function $C_{\parallel}(k, \xi)$ of the velocities along $\hat{b}$ and the correlation function $C_{\perp}(k, \xi)$ of the velocities in the plane perpendicular to $\hat{b}$. Give definitions for these functions that you think are appropriate and express them in terms of $C_{iso}$ and $C_{aniso}$.

3. Suppose all variation of the velocity along $\hat{b}$ is suppressed. What happens to the tensor $C_{ij}$?

2. Scalar Turbulence. Part I: Yaglom’s $\frac{4}{3}$ Law. Consider the equation for the evolution of passive scalar $\theta(t, x)$ (this can be temperature, or concentration of an admixture like a dye or salt, or, in 2D hydrodynamics, the vorticity field, or, in RMHD, the magnetic flux function, etc.):

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta + f ,$$  \hspace{1cm} (4)

where $\mathbf{u}$ is the (turbulent) velocity field, $\kappa$ is the scalar diffusivity, and $f$ is the source function (scalar “forcing”). We will assume that $f$ varies at some (large) scale $L_\theta < L$ ($L$ is the outer scale of the turbulence).

1. Define the scalar variance $\mathcal{E}_\theta = \langle \theta^2 \rangle / 2$ (“energy” of the scalar field), the scalar correlation function $C(y) = \langle \theta(x_1) \theta(x_2) \rangle$, and the scalar structure function $S(y) = \langle \delta \theta^2 \rangle$, where $\delta \theta = \theta(x_2) - \theta(x_1)$ and $y = x_2 - x_1$. Express $S(y)$ in terms of $C(y)$ and $\mathcal{E}$.  


2. Define a mixed 3d-order correlation function \( F_i(y) = \langle u_i(x_1)\theta(x_1)\theta(x_2) \rangle = F(y)\hat{y}_i \) and the corresponding structure function \( G_i(y) = \langle \delta u_i\delta \theta^2 \rangle = G(y)\hat{y}_i \), where \( \delta u_i = u_i(x_2) - u_i(x_1) \) and \( \hat{y} = y/y \). Show that \( G(y) = 4F(y) \).

Hint. Any one-point average that is a first-rank tensor (vector) is zero by isotropy (why?). Also, \( \langle u(x_1)\alpha(x_2) \rangle = 0 \) for any scalar field \( \alpha \) (at which point in my lecture on the \( \frac{4}{3} \) Law did I prove this?).

3. Now, proceeding analogously to the derivation of the \( \frac{4}{3} \) Law in my lectures, derive the analog of the von Kármán–Howarth equation for the passive scalar:

\[
\frac{\partial S}{\partial t} = 4\frac{d\varepsilon}{dt} - 4\epsilon_{\theta}(y) - \frac{1}{y^{d-1}} \frac{\partial}{\partial y} y^{d-1} G(y) + 2\kappa \frac{1}{y^{d-1}} \frac{\partial}{\partial y} y^{d-1} \frac{\partial S}{\partial y},
\]

where \( \epsilon_{\theta}(y) = \langle \theta(x_1) f(x_2) \rangle \).

4. Consider the statistically steady state and show that for \( y \ll L_\theta \),

\[
G(y) = -\frac{4}{3} \epsilon_{\theta} y + 2\kappa S(y),
\]

where \( \epsilon_{\theta} = \epsilon_{\theta}(0) = \langle \theta f \rangle \) the input variance per unit time. Show from Eq. (4) that \( \epsilon_{\theta} = \kappa \langle |\nabla\theta|^2 \rangle \) (scalar dissipation per unit time). Equation (6) for \( d = 3 \) is the Yaglom’s \( \frac{1}{2} \) Law.

5. Show that if \( f = 0 \) and we consider a self-similar decay of the scalar \( (\partial S/\partial t = 0) \), Eq. (6) is still satisfied. What is \( \epsilon_{\theta} \) in this case?

3. Scalar Turbulence. Part II: The Oboukhov-Corrsin Spectrum. Now you are going to develop a dimensional theory of scalar turbulence à la K41 theory I described in my lectures.

1. Let us figure out when the diffusive term in Eq. (6) is negligible. Assume that \( S(y) \sim \delta \theta^2 \) and (dimensionally) \( \bar{\epsilon}_{\theta} \sim \delta \theta^2/\tau_1 \) (flux of scalar variance), where \( \delta \theta \) is the scalar variation across scale \( l = y \) and \( \tau_1 \) is some cascade time. Show that the diffusive term is negligible if

\[
\frac{\kappa \tau_1}{l^2} \ll 1.
\]

2. Assume that \( \tau_1 \sim l/\delta u_l \) (why?) and show that, for \( \delta u_l \) satisfying the K41 scaling, Eq. (7) reduces to \( l \gg l_\kappa = \text{Sc}^{-3/4}/\nu \), where \( l_\nu = (\nu^3/\epsilon)^{1/4} \) is the viscous scale, \( \epsilon \) is the Kolmogorov flux, and \( \text{Sc} = \nu/\kappa \) is called the Schmidt number.

Note that, since you have used K41 inertial-range scaling for the cascade time, your estimates are only correct for \( \text{Sc} \ll 1 \) (why?).

3. Show that an equivalent expression for the diffusive scale is \( l_\kappa \sim \text{Pe}^{-3/4} L_\theta \) (provided the characteristic scale of the scalar source is \( L_\theta < L \)), where \( \text{Pe} = \delta u_{\text{Lu}} L_\theta/\kappa \) is called the Péclet number (analog of the Reynolds number for scalars).

The scale range of \( l \) such that \( L > L_\theta \gg l \gg l_\nu \gg l_\nu \) is called the inertial-convective range. It is non-empty if \( \text{Re} \gg \text{Pe} \gg 1 \).

4. Using the Yaglom’s law, show that, for \( l \) in the inertial-convective range,

\[
\delta \theta \sim \bar{\epsilon}_{\theta}^{1/2} \epsilon^{-1/6} l^{1/3},
\]

or, for the spectrum of scalar variance,

\[
E_{\theta}(k) \sim \bar{\epsilon}_{\theta} \epsilon^{-1/3} k^{-5/3}
\]

(the Oboukhov-Corrsin spectrum). Sketch the spectra of the kinetic energy and of the scalar variance, indicating all relevant wavenumbers \( k \sim 1/l \) and slopes.
5. Show that Eq. (8) can be derived purely dimensionally (without recourse to Yaglom’s law) by assuming that the flux of scalar variance $\bar{\epsilon}_\theta$ is independent of $l$ in the inertial-convective range.

4. Scalar Turbulence. Part III: The Batchelor Spectrum. What if $Sc \gg 1$? Then $l_\kappa$ we calculated in Problem 3 is smaller than $l_\nu$. Our dimensional theory only applies to $l \gg l_\nu$. Let us figure out what the scalar does at $l \ll l_\nu$.

1. Use the scaling of $\tau_l \sim l/\delta u_l$ in the viscous range ($l < l_\nu$) derived in my lectures to show that Eq. (7) reduces to $l \gg l_\kappa = Sc^{-1/2}l_\nu$ — the new expression for the diffusive scale in the limit $Sc \gg 1$.

The scale range $l_\nu \gg l \gg l_\kappa$ is called the viscous-convective range (or subviscous range).

2. In a manner analogous to what you did in Problem 3, use Yaglom’s law or the assumption that $\bar{\epsilon}_\theta$ is independent of $l$ to show that, for $l$ in the viscous-convective range,

$$\delta \theta \sim \bar{\epsilon}_\theta^{1/2} \epsilon_\theta^{-1/4} \nu^{1/4},$$

(independent of scale!) or, for the spectrum of scalar variance,

$$E_\theta(k) \sim \bar{\epsilon}_\theta \epsilon_\theta^{-1/2} \nu^{1/2} k^{-1}$$

(the Batchelor spectrum). This spectrum is the result of these two properties of the viscous-convective range: (i) flux of scalar variance is independent of $l$, (ii) cascade time is independent of $l$ (and equal to the turnover time of the viscous-scale eddies — confirm this is so!).

3. Thus, in the inertial-convective range, we have the Oboukhov-Corrsin spectrum, in the viscous-convective range, we have the Batchelor spectrum. Sketch the spectra of the kinetic energy and of the scalar variance in the case $Sc \gg 1$, indicating all relevant wavenumbers $k \sim 1/l$ and slopes.