Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET II

The problems in this Example Sheet are harder than in ES-I. They follow the topics I have covered in my lectures (energy principle, waves, magnetic reconnection, tearing mode) and on each of these topics you are offered an opportunity to work through a a fairly serious calculation of the type you may encounter in your research work. I urge you to invest some of your time in these problems as they represent bits of theory that, in a longer course, could have made into lectures. These problems will be discussed in the 2nd Examples Class (2.03.05, 14:30 in MR5).

1. Convective instabilities in a gravitational field. In this problem, you will study the stability of a straight-field equilibrium in a gravitational field: $\mathbf{B}_0 = B_0(z)\hat{\mathbf{x}}$, $p_0 = p_0(z)$, $\rho_0 = \rho_0(z)$, $\mathbf{g} = -g\hat{\mathbf{z}}$, where

$$\frac{d}{dz}\left(p_0 + \frac{B_0^2}{8\pi}\right) = -\rho_0 g. \tag{1}$$

Consider displacements in the form $\xi = \hat{\xi}(z)\exp(ik_x x + ik_y y)$. The general expression for the perturbation of the potential energy is then

$$\delta W = \frac{1}{2}\int dz \left[\gamma p_0 |\nabla\cdot\xi|^2 + \frac{|\mathbf{Q}|^2}{4\pi} + (\nabla\cdot\xi^*)\,\xi\cdot\nabla p_0 + \frac{\mathbf{j}_0\cdot(\xi^*\times\mathbf{Q})}{c} + (\xi^*\cdot\mathbf{g})\,\nabla\cdot(\rho_0\xi)\right], \tag{2}$$

where $\mathbf{j}_0 = (c/4\pi)\nabla\times\mathbf{B}_0$ and $\mathbf{Q} = \delta\mathbf{B} = \nabla\times(\xi\times\mathbf{B}_0) = -\xi\cdot\nabla\mathbf{B}_0 + \mathbf{B}_0\cdot\nabla\xi - \mathbf{B}_0\nabla\cdot\xi$.

1. First consider the case with no magnetic field: $B_0 = 0$. Write δW for this equilibrium. Observe that it depends only on two scalar quantities: $\nabla\cdot\xi$ and ξ_z (and their conjugates). By minimising δW with respect to $\nabla\cdot\xi$ (and $\nabla\cdot\xi^*$), derive *the Schwarzschild stability criterion* for convection:

$$\text{Stability} \Leftrightarrow \frac{d}{dz}\ln\left(\frac{p_0}{\rho_0^\gamma}\right) > 0 \tag{3}$$

   If this is broken, you get the so-called *interchange instability.* Can you think of a physical picture of this instability?

2. Now restore magnetic field. Consider a class of displacements with $k_x = 0$ (show that these displacements do not bend the field lines). Calculate δW and again observe that it depends only on $\nabla\cdot\xi$ and ξ_z and their conjugates. Minimise with respect to $\nabla\cdot\xi$ and $\nabla\cdot\xi^*$ and show that

$$\frac{d}{dz}\ln\left(\frac{p_0}{\rho_0^\gamma}\right) + \frac{B_0^2}{4\pi p_0}\frac{d}{dz}\ln\left(\frac{B_0}{\rho_0}\right) < 0, \tag{4}$$

   is a sufficient condition for instability (*magnetised interchange instability*). Would you be justified in claiming stability if this condition is not satisfied?

2. Ambipolar and Viscous Damping. Consider the incompressible MHD for a plasma with a neutral component:

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{\nabla p_i}{\rho_i} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho_i} - \mu_{in}\left(\mathbf{u}_i - \mathbf{u}_n\right), \quad \nabla \cdot \mathbf{u}_i = 0, \tag{5}$$

$$\frac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{u}_n = -\frac{\nabla p_n}{\rho_n} + \nu_n \nabla^2 \mathbf{u}_n - \mu_{ni}\left(\mathbf{u}_n - \mathbf{u}_i\right), \quad \nabla \cdot \mathbf{u}_n = 0, \tag{6}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u}_i, \tag{7}$$

where ion viscosity and magnetic diffusivity have been ignored, subscripts i and n refer to ion and neutral quantities, respectively, $\nu_n \sim v_{\rm th}\lambda_{\rm mfp}$ is the neutral viscosity, $\mu_{in} \sim v_{\rm th}/\lambda_{\rm mfp}$ is the ion-neutral collision rate, $\mu_{ni} = (\rho_i/\rho_n)\mu_{in} = \mu_{in}\chi/(1-\chi)$ is the neutral-ion collision rate, $\chi = \rho_i/(\rho_n + \rho_i)$ is the degree of ionisation of the plasma.

Assume a straight-field static equilibrium, $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. Write the ion and neutral velocities in terms of ion and neutral displacements and work out the dispersion relation:

$$i\omega^3 - \omega^2\left(\nu_n k^2 + \frac{\mu_{in}}{1-\chi}\right) - i\omega\left(k_\parallel^2 v_A^2 + \mu_{in}\nu_n k^2\right) + k_\parallel^2 v_A^2\left(\nu_n k^2 + \frac{\chi}{1-\chi}\,\mu_{in}\right) = 0. \tag{8}$$

Consider various asymptotic solutions of this dispersion relation. You will find that the following are the interesting limits:

1. $\beta \ll 1$ (show that this implies $\nu_n k^2 \ll k_\parallel^2 v_A^2/\mu_{in}$)
   - (a) $k\lambda_{\rm mfp} \ll \beta^{1/2}$ (show that this implies $k_\parallel v_A \ll \mu_{in}$)
     — Alfvén waves plus ambipolar damping.
   - (b) $\beta^{1/2} \ll k\lambda_{\rm mfp} \ll 1$ (show that this implies $\nu_n k^2 \ll \mu_{in} \ll k_\parallel v_A$)
     — Alfvén waves plus collisional damping.
   - (c) $k\lambda_{\rm mfp} \gg 1$ (i.e., $\nu_n k^2 \gg \mu_{in}$)
     — Undamped Alfvén waves.
2. $\beta \gg 1$ (i.e., $\nu_n k^2 \gg k_\parallel^2 v_A^2/\mu_{in}$)
   - (a) $k\lambda_{\rm mfp} \ll 1/\beta^{1/2}$ (show that this implies $\nu_n k^2 \ll k_\parallel v_A$)
     — Alfvén waves plus viscous damping.
   - (b) $1/\beta^{1/2} \ll k\lambda_{\rm mfp} \ll 1$ (show that this implies $k_\parallel v_A \ll \nu_n k^2 \ll \mu_{in}$)
     — Viscous relaxation (non-oscillatory).
   - (c) $1 \ll k\lambda_{\rm mfp} \ll \beta^{1/2}$ (i.e., $\nu_n k^2 \gg \mu_{in} \gg k_\parallel v_A$)
     — Ambipolar relaxation (non-oscillatory).

For each of these cases, find all three solutions of the dispersion relation. Which solution has the weakest damping? (meaning that it is the long-term solution) Do the verbal descriptions of these solutions given above make sense to you? (if not, you might have made a mistake!)

Physically speaking, do you trust all of these results equally well?

3. ***X*-Point Collapse.** Let us set up the following initial magnetic-field configuration:

$$\mathbf{B}_0(\mathbf{r}_0) = B_0\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \nabla_0\psi(x_0, y_0), \tag{9}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$, $B_0 = \text{const}$, and

$$\psi(x_0, y_0) = \frac{1}{2}\left(x_0^2 - y_0^2\right). \tag{10}$$

1. Draw the field lines in the (x_0, y_0) plane to see that this is an X-point.
2. Use Lagrangian MHD

$$\rho_0\frac{\partial^2\mathbf{r}}{\partial t^2} = -J\left(\nabla_0\mathbf{r}\right)^{-1}\cdot\nabla_0\left(\frac{p_0}{J^\gamma} + \frac{|\mathbf{B}_0\cdot\nabla_0\mathbf{r}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\mathbf{B}_0\cdot\nabla_0\left(\frac{\mathbf{B}_0}{J}\cdot\nabla_0\mathbf{r}\right), \tag{11}$$

where $\mathbf{r}(t, \mathbf{r}_0) = (x, y, z)$ and $J = |\det\nabla_0\mathbf{r}|$, and seek solutions in the form

$$x = \xi(t)x_0, \quad y = \eta(t)y_0, \quad z = z_0. \tag{12}$$

Show that $\xi(t)$ and $\eta(t)$ satisfy the following equations

$$\frac{d^2\xi}{dt^2} = \eta\left(\frac{1}{\eta^2} - \frac{1}{\xi^2}\right), \tag{13}$$

$$\frac{d^2\eta}{dt^2} = \xi\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right). \tag{14}$$

3. Consider the possibility that, as time goes on, $\eta(t) \to 0$ (becomes small) and $\xi(t) = \xi_c + \dots$, where ξ_c is some constant. Find solutions that satisfy this assumption. The answer is

$$\xi(t) \approx \xi_c + \frac{9}{4}\left(\frac{2}{9\xi_c}\right)^{1/3}(t_c - t)^{4/3}, \tag{15}$$

$$\eta(t) \approx \left(\frac{9\xi_c}{2}\right)^{1/3}(t_c - t)^{2/3} \tag{16}$$

as $t \to t_c$, where t_c is some finite time constant. This is called *the Syrovatskii solution* for the X-point collapse.

4. Calculate the magnetic field as a function of time and convince yourself that the initial X-point configuration collapses explosively (in a finite time) to a sheet along the x axis. What do you think happens after t reaches t_c?
5. Now do a similar calculation for incompressible Lagrangian MHD ($J = 1$). Remember that total pressure is now determined by the condition $J = 1$. Show that the solution in this case is

$$\xi(t) = e^{S(t)}, \quad \eta(t) = e^{-S(t)}, \tag{17}$$

where $S(t)$ is an arbitrary function of time. If. e.g., $S(t) = \Lambda t$, show that this mean the X-point collapses exponentially. This is called *the Chapman-Kendall solution.*

4. Tearing Mode with Viscosity. Work through my notes on the tearing mode. Now restore the viscous term ($\nu \nabla_\perp^4 \phi$) in the ϕ equation. This introduces an extra time scale, $\tau_{\rm visc} = (\nu k^2)^{-1}$, and an extra spacial scale, $\delta_{\rm visc} = (\nu/\gamma)^{1/2}$. Assume that

$$\gamma \tau_{\rm visc} \gg 1 \quad \text{and} \quad \delta_{\rm visc} \ll 1/k \sim \text{system scale.} \tag{18}$$

Suppose that the viscous scale is larger than the resistive scale:

$$\delta_{\rm visc} \gg \delta = (\eta/\gamma)^{1/2}. \tag{19}$$

1. Work out the equation for the inner solution and find how γ, δ and $\delta_{\rm visc}$ scale with η and ν. The answer is

$$\gamma \sim \eta^{5/6} \nu^{-1/6}, \quad \delta \sim (\eta\nu)^{1/6}, \quad \delta_{\rm visc} \sim \nu^{7/12} \eta^{-5/12} \tag{20}$$

(note that you do not need to calculate the exact solution in the inner region to get these scalings!).

2. Show that, assuming a large Lundquist number $S = v_A L/\eta \gg 1$ (here $L \sim 1/k$ is the system scale), assumptions (18) and (19), and, therefore, your scalings, hold provided

$$S^{-2/5} \ll \text{Pm} \ll S^{2/7}, \tag{21}$$

where $\text{Pm} = \nu/\eta$ is called the magnetic Prandtl number.