Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET I: Problems 8-9

These problems will be discussed in the 1st Examples Class (9.02.05, 14:30 in MR5).

8. Conservation Laws for RMHD. In Problem 3, you derived the RMHD equations. Show that these equations conserve the following three integral invariants.

$$\text{Energy} \quad E = \int d^2x \left(\frac{\rho u^2}{2} + \frac{B^2}{8\pi}\right) = \int d^2x \left(\frac{\rho|\nabla\phi|^2}{2} + \frac{|\nabla\psi|^2}{8\pi}\right), \tag{1}$$

$$\text{Cross-helicity} \quad C = \int d^2x\, \mathbf{u}\cdot\mathbf{B} = \int d^2x\, (\nabla\phi)\cdot(\nabla\psi), \tag{2}$$

$$\text{2D magnetic invariant} \quad I = \int d^2x\, \psi^2. \tag{3}$$

Write the evolution equations for all three of these quantities, including viscous and resistive terms. Show that energy and the "ψ^2-stuff" always decay with time (in the absence of sources).

9. The Grad-Shafranov Equation. Consider MHD equilibrium in cylindrical coordinates (r, θ, z). Assume axial symmetry: $\partial/\partial\theta = 0$.

1. Use solenoidality of the magnetic field to show that
$$B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \quad B_z = \frac{1}{r}\frac{\partial\psi}{\partial z}. \tag{4}$$
ψ is called the poloidal flux function.

2. Use Ampère's law to express the components of the current $\mathbf{j} = (j_r, j_\theta, j_z)$ in terms of ψ and of $F = rB_\theta$. The latter is called the poloidal current function.

3. Write the θ component of the force balance $(1/c)\mathbf{j}\times\mathbf{B} = \nabla p$. Show that it is equivalent to $\nabla F \times \nabla\psi = 0$. Argue that this implies $F = F(\psi)$ (F is a function of ψ only).

4. Note that you can now express any derivatives of F in terms of its derivative with respect to ψ, e.g.,
$$\frac{\partial F}{\partial r} = \frac{dF}{d\psi}\frac{\partial\psi}{\partial r}. \tag{5}$$
Now write the r and z components of the force balance. From the two resulting expressions, obtain $\nabla p \times \nabla\psi = 0$, whence $p = p(\psi)$.

5. In either r or z component of the force balance, express the pressure gradient in terms of $dp/d\psi$ and obtain the following *Grad-Shafranov equation*
$$-\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) = 4\pi r^2\frac{dp}{d\psi} + F\frac{dF}{d\psi} \tag{6}$$

6. Now show that if cylindrical symmetry is assumed ($\partial/\partial\theta = 0$, $\partial/\partial z = 0$), this equation reduces to the equation of cylindrical equilibrium derived in class:
$$\frac{\partial}{\partial r}\left(p + \frac{B^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}. \tag{7}$$