Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET I: Problems 4-7

These problems will be discussed in the 1st Examples Class (9.02.05, 14:30 in MR5).

4. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with density $n = 1\ \mathrm{cm}^{-3}$ undergoing a gravitational collapse to a spherical star with density $n = 10^{26}\ \mathrm{cm}^{-3}$. The magnetic field in the galactic plasma is $\sim 3 \times 10^{-6}$ G. Assuming that flux is frozen, estimate the magnetic field in a star.

5. Consider a uniform magnetic field in the z direction and a line current $I(t)$ also in the z direction producing an azimuthal field:

$$\mathbf{B} = B_z \hat{\mathbf{z}} + \frac{2I(t)}{cr}\,\hat{\theta}, \tag{1}$$

where $\hat{\mathbf{z}}$ and $\hat{\theta}$ are unit vectors along the z and θ directions in the cylindrical coordinate system (r, θ, z). Now assume that this set-up is embedded into an ideal MHD fluid ($\eta = 0$). Can you find a radial velocity field, $\mathbf{u} = u_r(t, r)\hat{\mathbf{r}}$, that preserves both radial and axial flux? Can you find one that preserves the field lines?

6. Flux Concentration. Consider a simple 2D model of incompressible convective motion:

$$\mathbf{u} = U\left(-\sin\frac{\pi x}{L}\,\cos\frac{\pi z}{L},\ 0,\ \cos\frac{\pi x}{L}\,\sin\frac{\pi x}{L}\right). \tag{2}$$

1. In the neighbourhood of the stagnation point $(0, 0, 0)$, linearise the flow, assume vertical magnetic field, $\mathbf{B} = (0, 0, B(t, x))$ and derive an equation for $B(t, x)$. Suppose $B(0, x) = B_0 = \mathrm{const}$ It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\mathrm{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.

2. Find a steady solution of your equation. Use flux conservation and $B(x) = B(-x)$ to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?

3. Can you think of a heuristic argument based on the induction equation that would tell you that these answers were to be expected?

7. Zeldovich Antidynamo Theorem. Consider the case of an arbitrary 2D velocity field: $\mathbf{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (specifically, you may work in a periodic box), this velocity field is not a dynamo, i.e., any initial magnetic field will always eventually decay. This is one of the classical antidynamo results: the Zeldovich Theorem.
Hint. Consider separately the equations for B_z and for the (x, y)-plane magnetic field. You will find that the latter satisfies an equation similar to the one in RMHD (see Problem 3).