Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET I: Problems 1-3

These problems will be discussed in the 1st Examples Class (9.02.05, 14:30 in MR5).

NB: *Results of Problems 2 and 3 will be used in future Examples Sheets and Lectures, so do work them out!*

1. Go through your notes on the kinetic derivation of MHD equations and formulate an intelligent question.

2. Electron MHD. Consider Ohm's law with the Hall term:

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{\mathbf{j}\times\mathbf{B}}{cen} + \frac{1}{\sigma}\mathbf{j}. \tag{1}$$

1. Estimate the relative size of the two terms on the right-hand side. Use $\sigma = e^2 n/m_e\nu_e$, where ν_e is the electron collision frequency. Express the ratio of the two terms as a ratio of certain time scales and as a ratio of certain length scales. These quatities may prove useful:

$$\Omega_s = \frac{q_s B}{cm_s},\quad \rho_s = \frac{v_{\mathrm{th},s}}{\Omega_s},\quad v_{\mathrm{th},s} = \left(\frac{T_s}{m_s}\right)^{1/2},\quad \lambda_{\mathrm{mfp},s} = \frac{v_{\mathrm{th},s}}{\nu_s}, \tag{2}$$

   where $s = i, e$ is the species index.

2. If you have figured out how to do these estimates, take the more general form of Ohm's law derived in my lectures,

$$\mathbf{E} + \frac{1}{c}\mathbf{u}\times\mathbf{B} = \frac{\mathbf{j}\times\mathbf{B}}{cen} + \frac{1}{\sigma}\mathbf{j} - \frac{\nabla p_e}{en} + \frac{\nabla\cdot\hat{\Pi}_e}{en} - \frac{m_e}{e}\left(\frac{\partial\mathbf{u}_e}{\partial t} + \mathbf{u}_e\cdot\nabla\mathbf{u}_e\right), \tag{3}$$

   and estimate the size of all terms on the rhs compared to the $\mathbf{u}\times\mathbf{B}/c$ term. You may use $p_e \sim nT_e$, $\hat{\Pi}_e \sim \mu_e\mathbf{u}_e/l$, where the electron viscosity is $\mu_e \sim m_e n v_{\mathrm{th},e}\lambda_{\mathrm{mfp},e}$. Try to work out the answers in terms of dimensionless ratios of physical quantities such as, for example, ρ_i/l, where ρ_i is the ion Larmor radius, l is the characteristic length scales of the fluid quantities like $\mathbf{u}$ and $\mathbf{B}$.

3. Derive an equation for the evolution of the magnetic field using Ohm's law (1), Faraday's law and Ampère's law. This is the induction equation in Hall MHD (or Electron MHD, EMHD).

4. Suppose the resistive term is negligible (what does this mean, physically?). Does your induction equation with the Hall term and $\eta = 0$ allow magnetic field lines to be broken?
   Hint: If you answer this question correctly, you will understand why EMHD is called EMHD.

3. Reduced MHD. Let us consider a plasma threaded by a very strong uniform magnetic field pointing in the z (vertical) direction: $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. Such a field will have the effect of suppressing all gradients along itself, so let us neglect them, $\nabla = (\partial_x, \partial_y, 0)$ (no z variation in any of the fields). Let us write the total magnetic field in the form $\mathbf{B} = B_0\hat{\mathbf{z}} + \delta\mathbf{B}$ ($|\delta\mathbf{B}| \ll B_0$) and assume that both

$\delta\mathbf{B}$ and the velocity field are two-dimensional: $\delta\mathbf{B} = (B_x, B_y, 0)$, $\mathbf{u} = (u_x, u_y, 0)$. Since the magnetic field is solenoidal, we can write $\delta\mathbf{B}$ in terms of one scalar function ψ (called the flux function):

$$B_x = -\frac{\partial\psi}{\partial y}, \quad B_y = \frac{\partial\psi}{\partial x}, \quad \text{or} \quad \delta\mathbf{B} = \hat{\mathbf{z}} \times \nabla\psi \tag{4}$$

(confirm that the last formula is equivalent to the first two; show that contours of constant ψ are the field lines of the field $\delta\mathbf{B}$; note that $\psi = -A_z$, the z component of the vector potential). Now you will derive the so-called equations of Reduced MHD (RMHD):

1. Show that $\mathbf{j} = (c/4\pi)\hat{\mathbf{z}}\nabla^2\psi$.

2. Write Ohm's law neglecting $\delta\mathbf{B}$ compared to $\mathbf{B}_0$,

$$\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}_0}{c} = \frac{1}{\sigma}\mathbf{j}, \tag{5}$$

   and show that $\mathbf{u} = c(\mathbf{E} \times \mathbf{B}_0)/B_0^2$. Express the perpendicular electric field in terms of the electrostatic potential, $\mathbf{E}_\perp = -\nabla\varphi$, and show that the velocity field can be written as

$$u_x = -\frac{\partial\phi}{\partial y}, \quad u_y = \frac{\partial\phi}{\partial x}, \tag{6}$$

   where $\phi = -(c/B_0)\varphi$ is called the stream function (note that this velocity field is incompressible).

3. Reduce the induction equation to the following equation for the flux function:

$$\begin{aligned} \frac{\partial\psi}{\partial t} + [\phi, \psi] &= \eta\nabla^2\psi, \qquad\qquad (7) \\ \text{where} \quad [\phi, \psi] &= \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} \quad \text{is called the Poisson bracket.} \end{aligned}$$

4. Show that the vorticity is $\omega = \nabla \times \mathbf{u} = \hat{\mathbf{z}}\nabla^2\phi$. Write the momentum equation and show that it leads to the following equation for the stream function:

$$\frac{\partial}{\partial t}\nabla^2\phi + [\phi, \nabla^2\phi] = \frac{1}{4\pi\rho}[\psi, \nabla^2\psi] + \nu\nabla^4\phi. \tag{8}$$

Equations (8) and (7) are the equations of RMHD in two dimensions.