Alfvén Wave Cascades in MHD Turbulence

An overview of theoretical uncertainties

Alexander Schekochihin *(DAMTP/Cambridge)*

1st CMPD Winter School on Advanced Plasma Theory
UCLA 3-7 January 2005

The Unknown

As we know,
There are known knowns.
There are things we know we know.
We also know
There are known unknowns.
That is to say
We know there are some things
We do not know.
But there are also unknown unknowns,
The ones we don't know
We don't know.

D. H. Rumsfeld
12.02.02, DoD news briefing
as quoted by www.slate.com

Outline

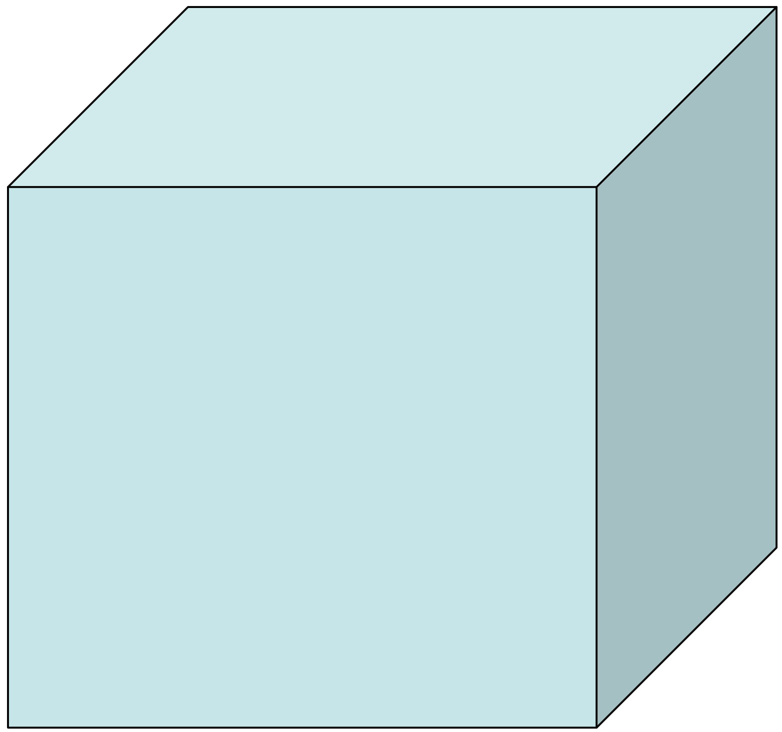
We do not know very much about MHD turbulence.

I will ask a very basic question:
WHAT ARE THE KINETIC AND MAGNETIC ENERGY SPECTRA?
and review very simple arguments that lead to various answers (none of which has been compellingly proven to be true) pointing out the difficulties that arise.

I will also show some of the numerical and observational evidence available to us today.

MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$
$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$

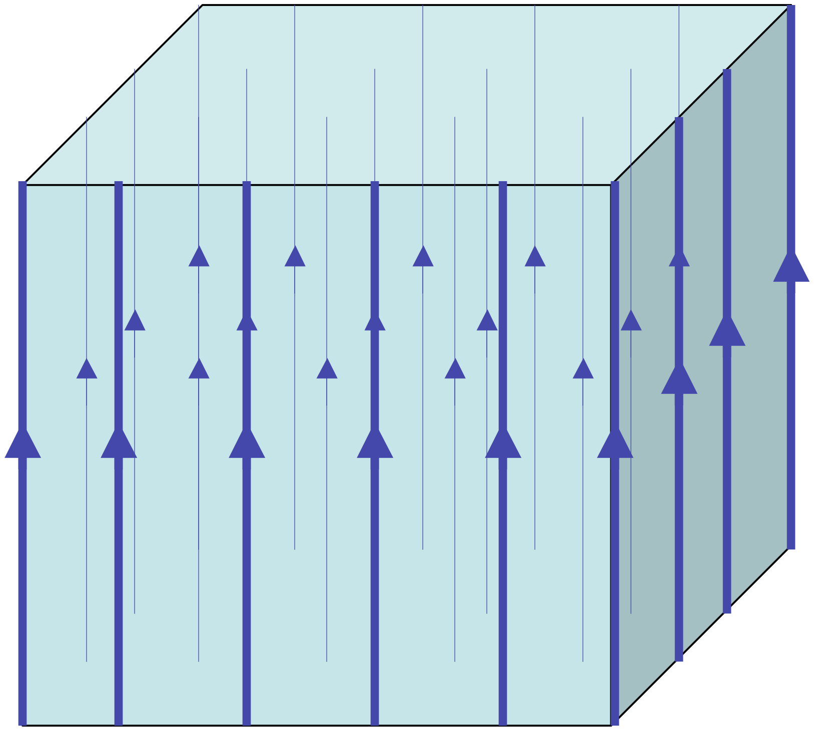


MHD Turbulence: The Fundamental Problem

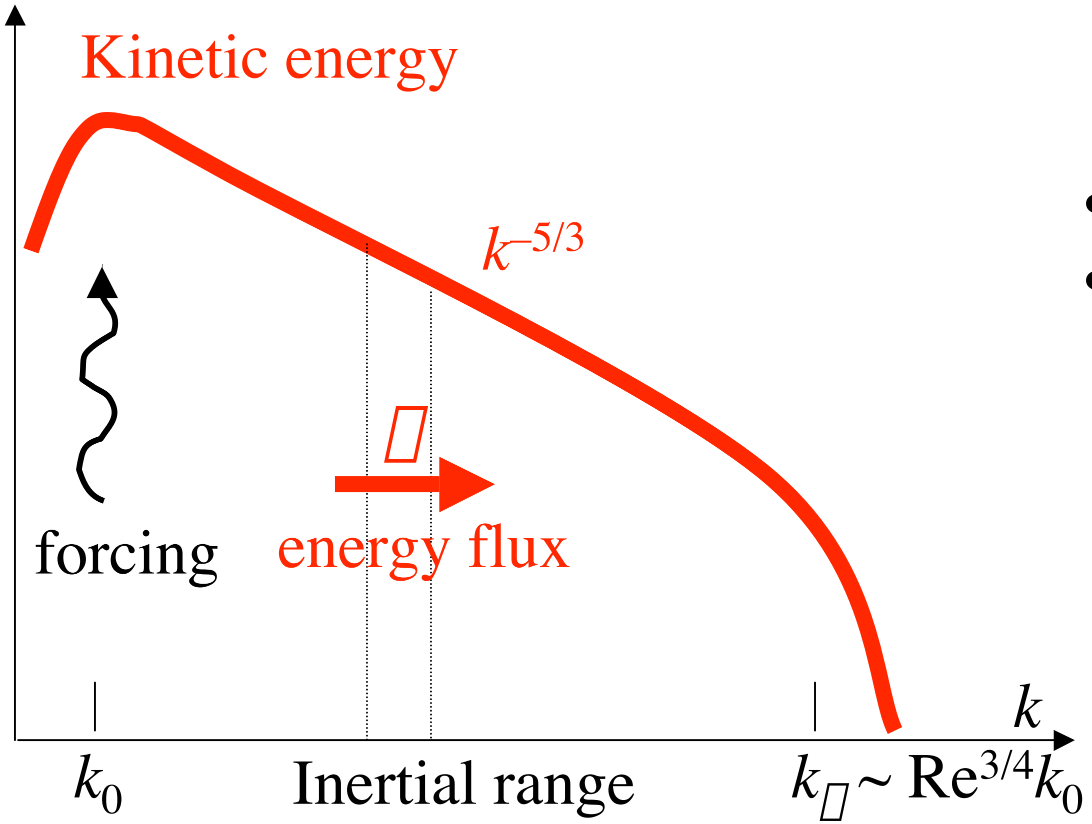
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0$$
$$\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B}$$

Strong uniform mean field $\boldsymbol{B}_0$ imposed

$B_0 > u_{\mathrm{rms}}$



Kolmogorov Turbulence



- Scale invariance
- Locality in k space

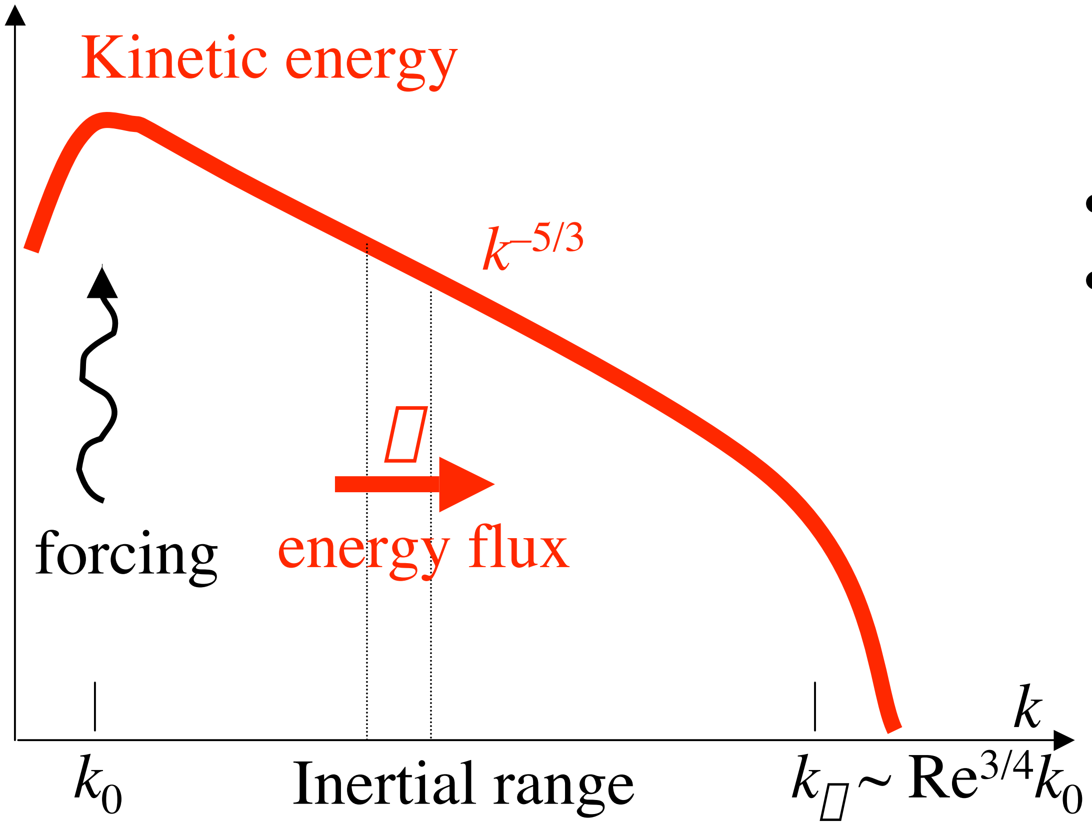
$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \mathrm{const}$$

Energy at scale l

Cascade time (rate of transfer)

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$

Kolmogorov Turbulence



- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \text{const}$$

Energy at scale l — Cascade time (rate of transfer)

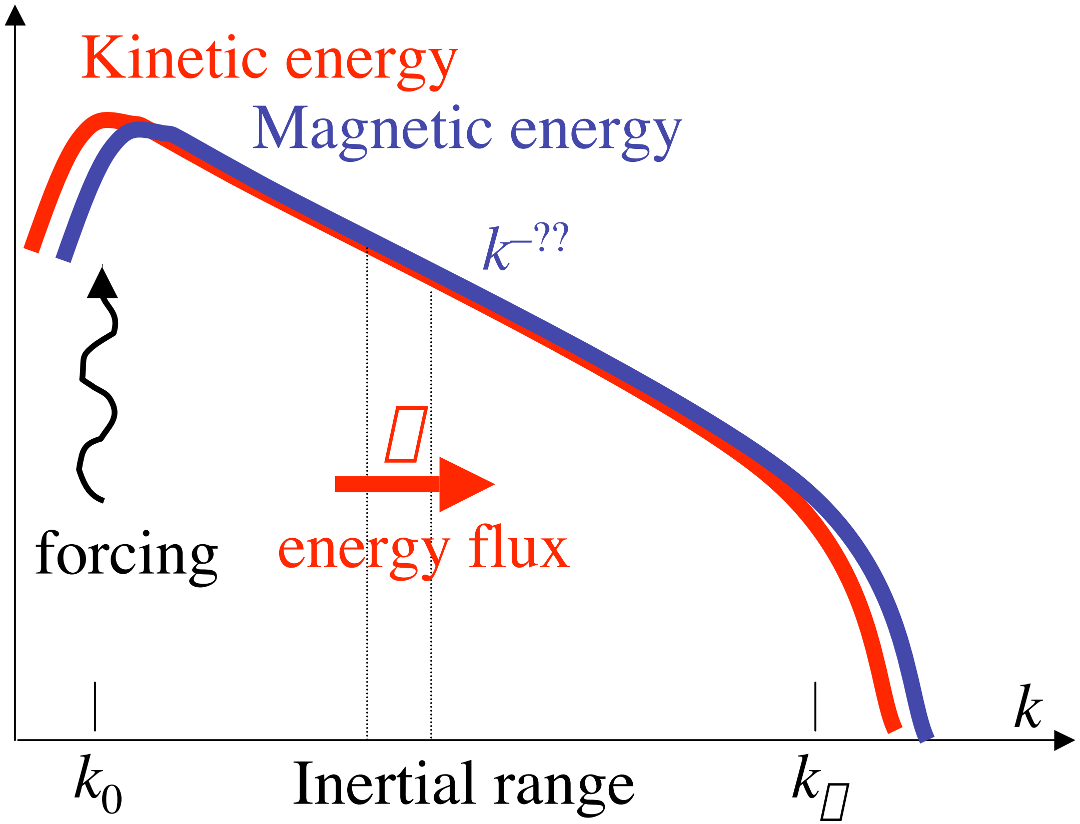
Only one time scale available at each l: the eddy-turnover time

$$\tau_l \sim \tau_{\text{eddy}} \sim l/u_l \longrightarrow u_l \sim \varepsilon^{1/3}\, l^{1/3}$$

$$\boxed{E(k) \sim \varepsilon^{2/3}\, k^{-5/3}} \quad \textbf{K41}$$

Kolmogorov spectrum fixed by dimensional analysis

MHD Turbulence à la Kolmogorov



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$\varepsilon \sim u_l^2 \tau_l^{-1}$ = const

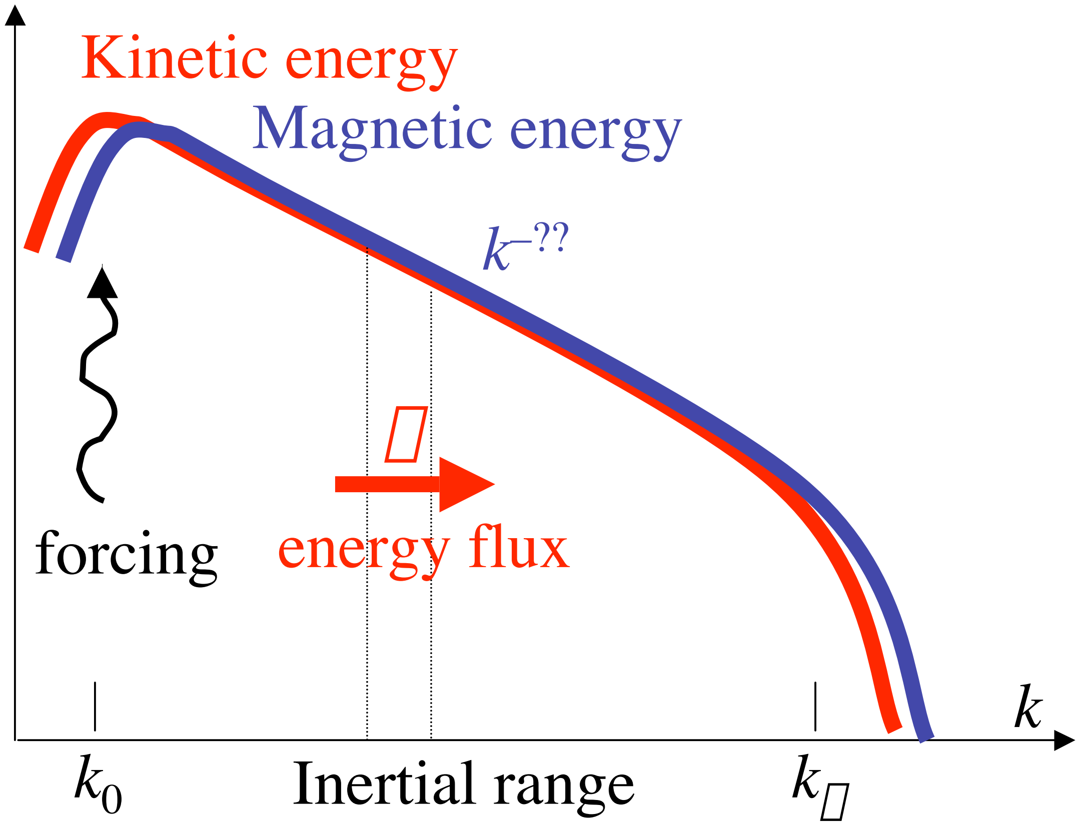
Energy at scale l

Cascade time (rate of transfer)

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$

$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$

MHD Turbulence à la Kolmogorov



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$\varepsilon \sim u_l^2\, \tau_l^{-1} = \text{const}$

Energy at scale l

Cascade time (rate of transfer)

Two time scales available:

turnover time: $\tau_{\text{eddy}} \sim l_\perp / u_l$

Alfvén time: $\tau_A \sim l_{\|} / v_A$

$v_A = B_0/(4\pi\rho)^{1/2}$

$\tau_l \sim ?$

Cannot fix scalings solely by dimensional analysis!

Interaction of Alfvén Wave Packets — I

Assume weak interactions: $\tau_{\rm eddy} >> \tau_{\rm A}$

- Wave packet passes through another: $\delta t \sim \frac{l_{\|}}{v_{\rm A}} \sim \tau_{\rm A}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Interaction of Alfvén Wave Packets — I

Assume weak interactions: $\tau_{\rm eddy} >> \tau_{\rm A}$

- Wave packet passes through another: $\delta t \sim \frac{l_{\|}}{v_{\rm A}} \sim \tau_{\rm A}$

- Its amplitude gets a kick: $\delta u_l \sim \frac{u_l^2}{l}\delta t \sim u_l \frac{u_l}{l}\frac{l_{\|}}{v_{\rm A}} \sim u_l \frac{\tau_{\rm A}}{\tau_{\rm eddy}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Interaction of Alfvén Wave Packets — I

Assume weak interactions: $\tau_{\text{eddy}} >> \tau_{\text{A}}$

- Wave packet passes through another: $\delta t \sim \frac{l_{\|}}{v_{\text{A}}} \sim \tau_{\text{A}}$

- Its amplitude gets a kick: $\delta u_l \sim \frac{u_l^2}{l}\delta t \sim u_l \frac{u_l}{l}\frac{l_{\|}}{v_{\text{A}}} \sim u_l \frac{\tau_{\text{A}}}{\tau_{\text{eddy}}}$

- Sum of kicks over time *t*: $\sum^t \delta u_l \sim u_l \frac{\tau_{\text{A}}}{\tau_{\text{eddy}}}\sqrt{\frac{t}{\tau_{\text{A}}}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Interaction of Alfvén Wave Packets — I

Assume weak interactions: $\tau_{\mathrm{eddy}} >> \tau_{\mathrm{A}}$

- Wave packet passes through another: $\delta t \sim \frac{l_{\|}}{v_{\mathrm{A}}} \sim \tau_{\mathrm{A}}$

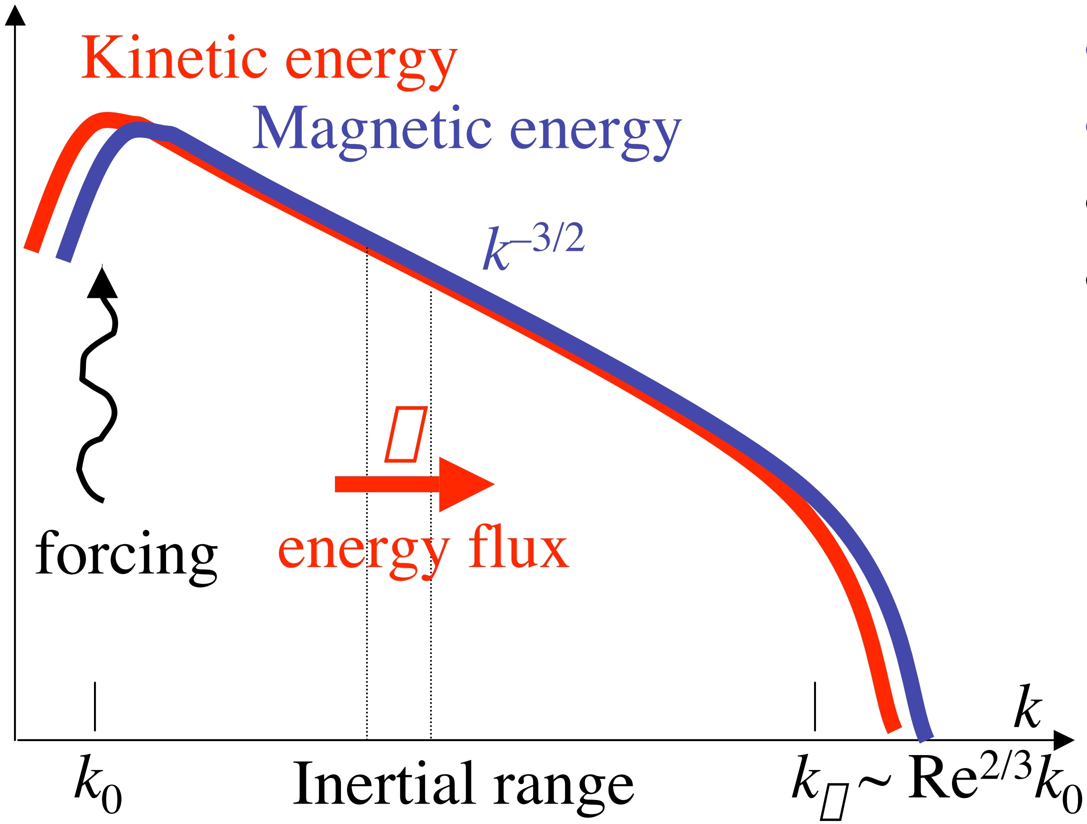
- Its amplitude gets a kick: $\delta u_l \sim \frac{u_l^2}{l}\delta t \sim u_l \frac{u_l}{l}\frac{l_{\|}}{v_{\mathrm{A}}} \sim u_l \frac{\tau_{\mathrm{A}}}{\tau_{\mathrm{eddy}}}$

- Sum of kicks over time *t*: $\sum^{t} \delta u_l \sim u_l \frac{\tau_{\mathrm{A}}}{\tau_{\mathrm{eddy}}}\sqrt{\frac{t}{\tau_{\mathrm{A}}}}$

- Cascade time: $t \sim \tau_l \;\Leftrightarrow\; \sum^{t} \delta u_l \sim u_l \;\Rightarrow\; \boxed{\tau_l \sim \frac{\tau_{\mathrm{eddy}}^2}{\tau_{\mathrm{A}}}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Iroshnikov-Kraichnan Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l — Cascade time (rate of transfer)

Additional physical assumptions:

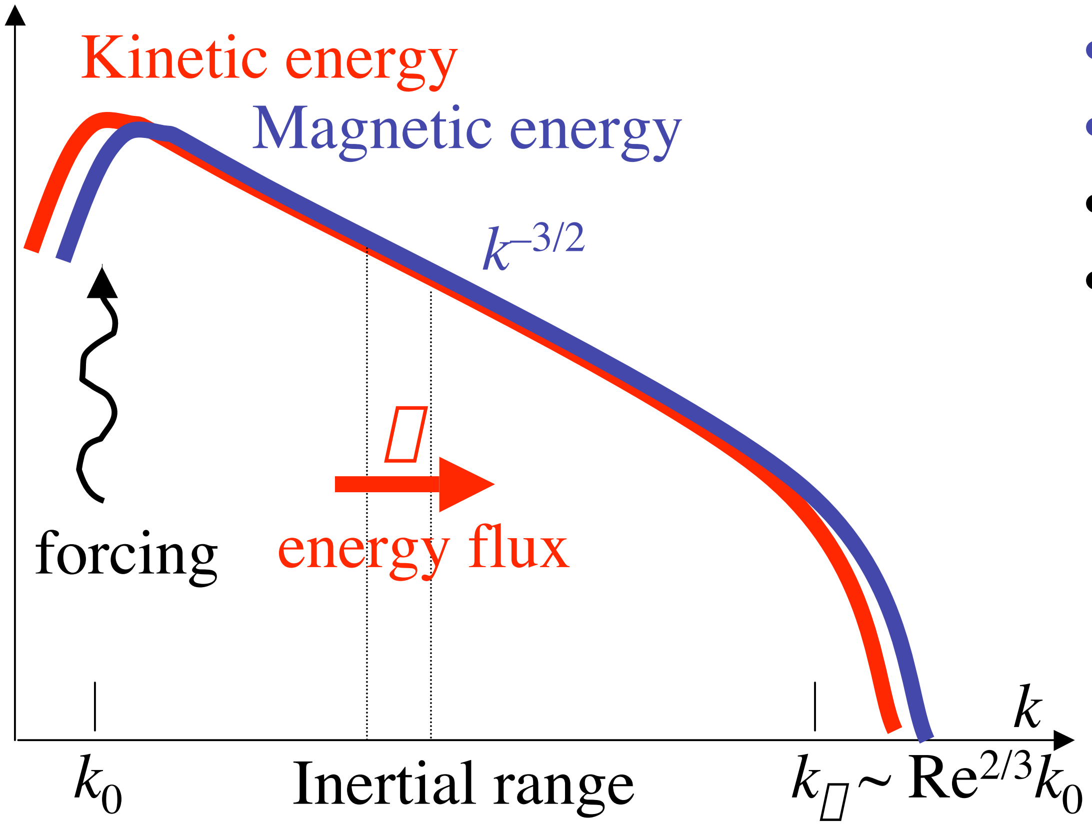
- **weak interactions: $\tau_{\mathrm{eddy}} \gg \tau_{\mathrm{A}}$**

$$\tau_l \sim \frac{\tau_{\mathrm{eddy}}^2}{\tau_{\mathrm{A}}} \sim \frac{l_\perp^2}{u_l^2} \frac{v_{\mathrm{A}}}{l_\parallel}$$

$$\boxed{u_l \sim (\epsilon v_{\mathrm{A}})^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}}$$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Iroshnikov-Kraichnan Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \mathrm{const}$$

Energy at scale l — Cascade time (rate of transfer)

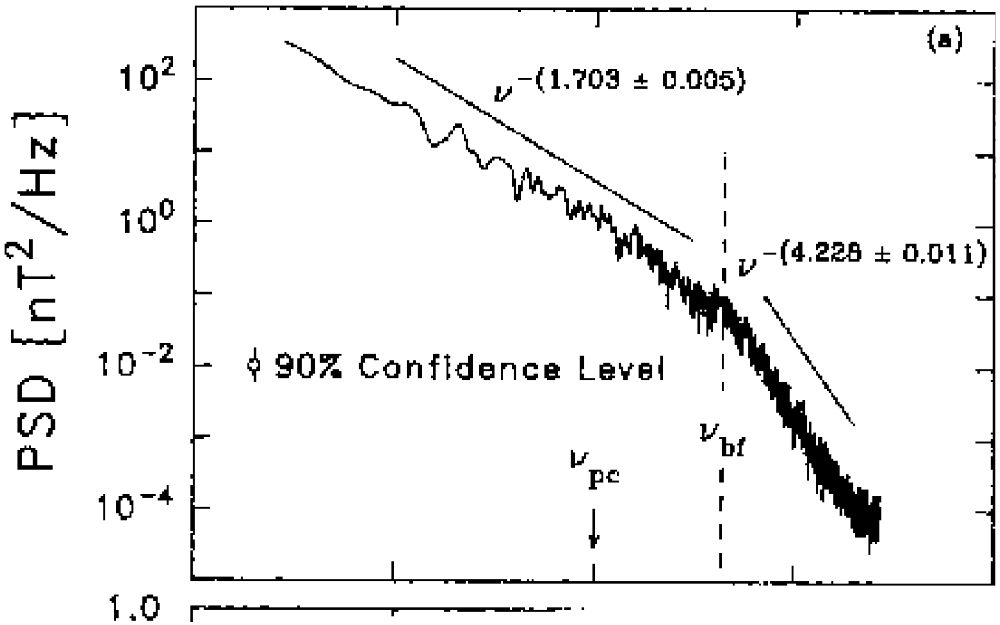
Additional physical assumptions:

- **weak interactions: $\tau_{eddy} >> \tau_A$**
- **isotropy: $l_{\parallel} \sim l_{\perp}$**

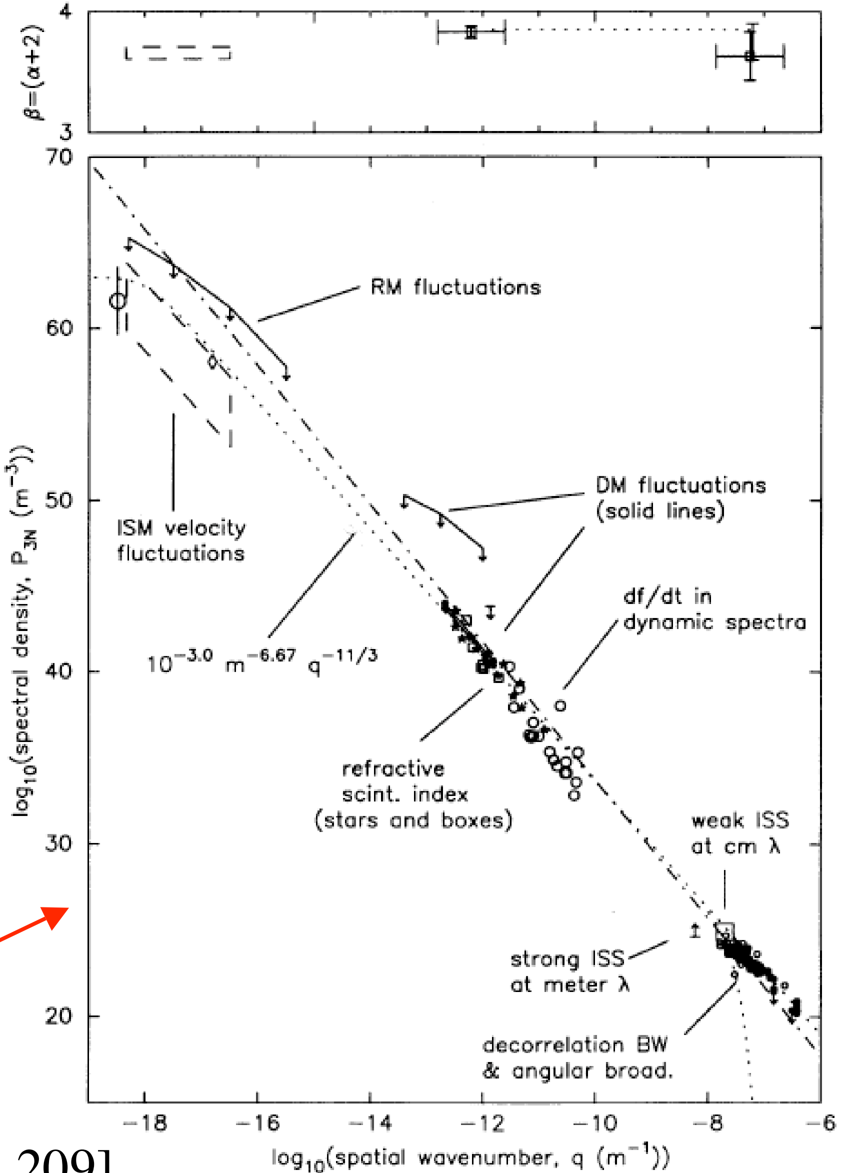
$$E(k) \sim (\varepsilon v_A)^{1/2} \, k^{-3/2}$$ **IK65**

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Observations: Spectrum is not $k^{-3/2}$?

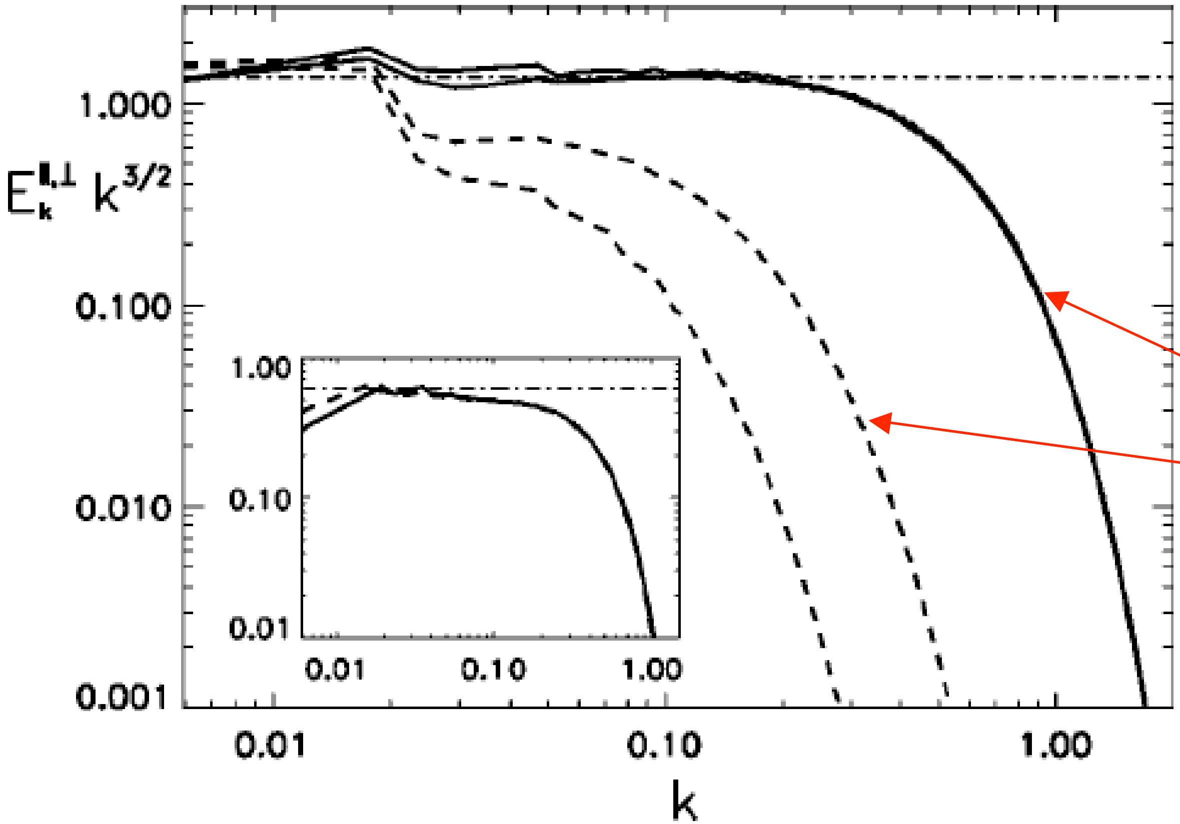


Solar wind
[Leamon *et al.* 1998]



Electron density in the ISM
[Armstrong *et al.* 1995, *ApJ* **443**, 209]

DNS: MHD Turbulence is Anisotropic!

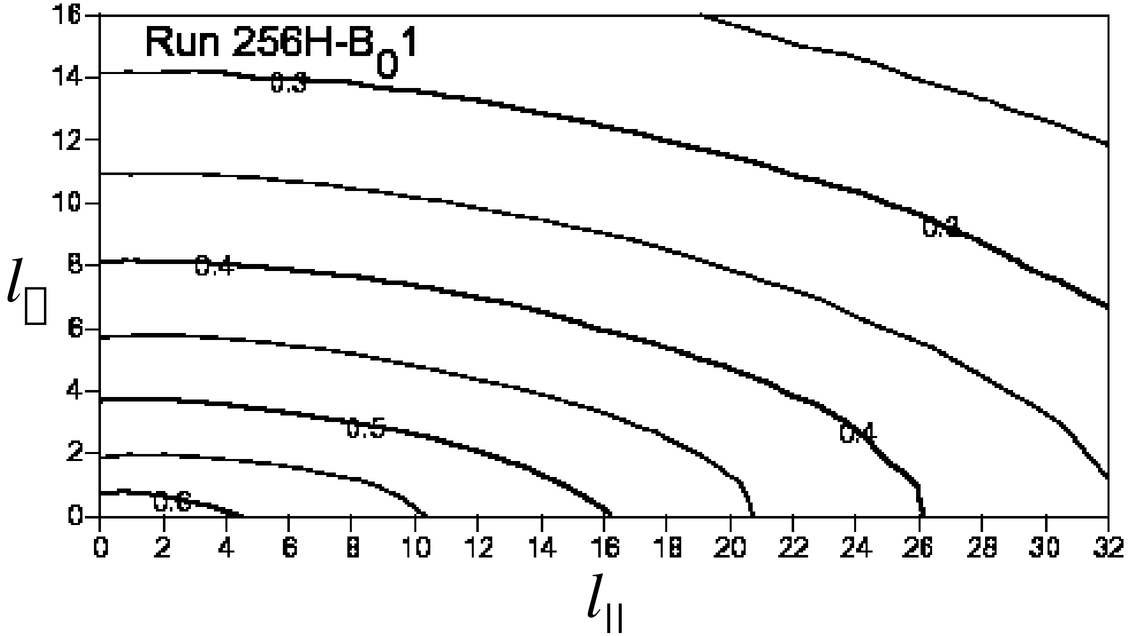


Müller *et al*. 2003, *PRE* **67**, 066302: perpendicular and parallel spectra

- **weak interactions: $\tau_{eddy} >> \tau_A$**
- **isotropy: $l_{\parallel} \sim l_{\perp}$**

→ $E(k) \sim (\varepsilon v_A)^{1/2} k^{-3/2}$ **IK65**

DNS: MHD Turbulence is Anisotropic!



Cho *et al*. 2002, *ApJ* **564**, 291: contours of velocity correlation functions

- **weak interactions: $\tau_{eddy} \gg \tau_A$**
- **isotropy: $l_{||} \sim l_\perp$**

$\longrightarrow$ $E(k) \sim (\varepsilon v_A)^{1/2}\, k^{-3/2}$ **IK65**

Interaction of Alfvén Wave Packets — II

Elsasser fields $\mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

Interaction of Alfvén Wave Packets — II

Elsasser fields $\mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \boxed{\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm}} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

Only counterpropagating waves interact:

$$\omega(\mathbf{k}) = \pm k_{\|} v_{\mathrm{A}}$$

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) \longrightarrow k_{\|1} - k_{\|2} = k_{\|3}$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \longrightarrow k_{\|1} + k_{\|2} = k_{\|3}$$

$$\longrightarrow \quad k_{\|2} = 0, \quad k_{\|1} = k_{\|3}$$

Interaction of Alfvén Wave Packets — II

Elsasser fields $\mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_A \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \boxed{\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm}} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

Only counterpropagating waves interact:

$$\omega(\mathbf{k}) = \pm k_{\|} v_A$$

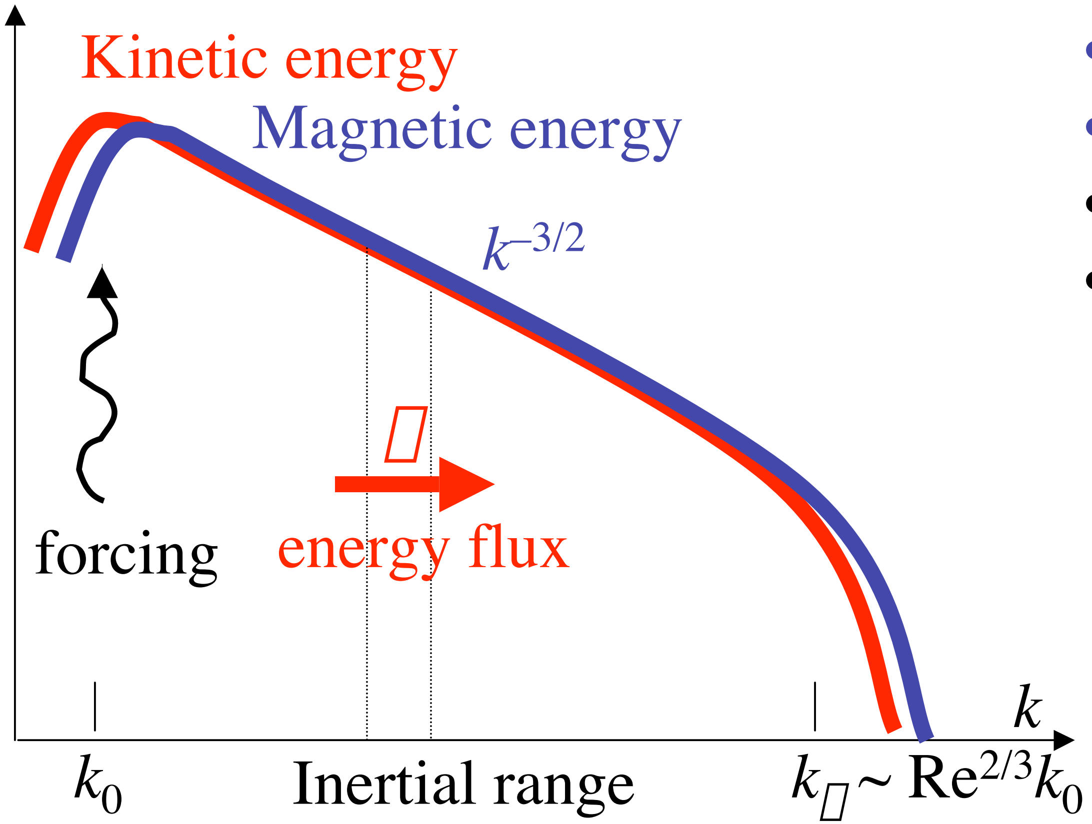
$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) \longrightarrow k_{\|1} - k_{\|2} = k_{\|3}$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \longrightarrow k_{\|1} + k_{\|2} = k_{\|3}$$

$$\longrightarrow \quad k_{\|2} = 0, \quad k_{\|1} = k_{\|3}$$

- **Alfvén waves interact via $k_{\|} = 0$ modes**
- **No cascade in $k_{\|}$**

Iroshnikov-Kraichnan Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\epsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

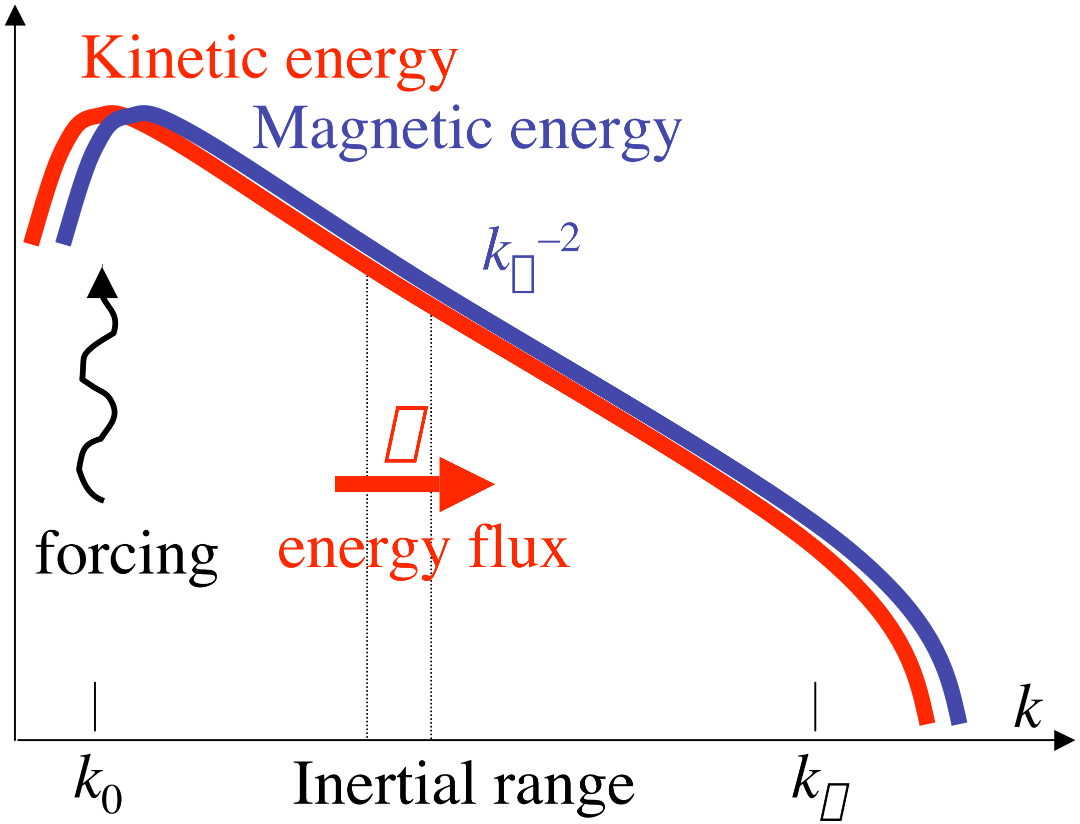
Additional physical assumptions:

- **weak interactions: $\tau_{\text{eddy}} \gg \tau_A$**

$$\tau_l \sim \frac{\tau_{\text{eddy}}^2}{\tau_A} \sim \frac{l_\perp^2}{u_l^2} \frac{v_A}{l_\parallel}$$

$$u_l \sim (\epsilon v_A)^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}$$

Weak MHD Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

→ $\varepsilon \sim u_l^2\, \tau_l^{-1} = \mathrm{const}$

Energy at scale l (points to u_l^2); Cascade time (rate of transfer) (points to τ_l^{-1})

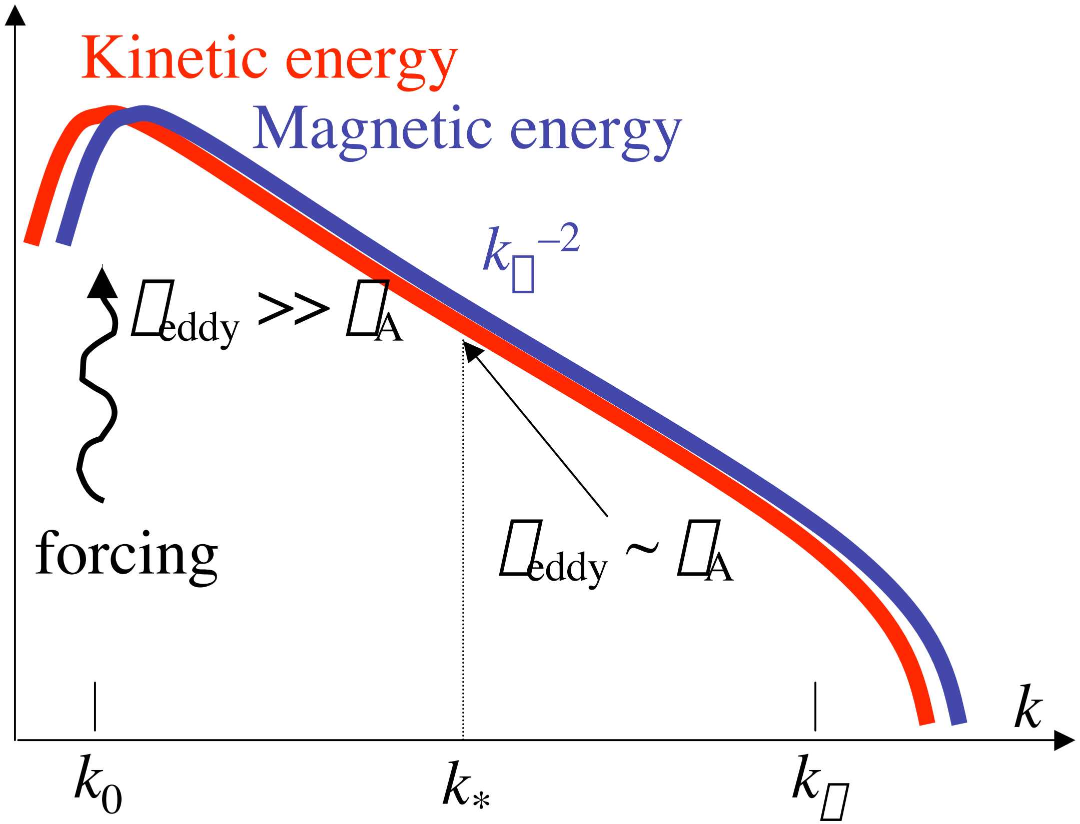
Additional physical assumptions:

- **weak interactions: $\tau_{eddy} \gg \tau_A$**
- **extreme anisotropy: $l_{||} \sim l_0$**
  (no cascade in $k_{||}$)

→ $E(k_{\perp}) \sim (\varepsilon k_{||} v_A)^{1/2}\, k_{\perp}^{-2}$

[e.g., Galtier *et al*. 2000, *JPP* **63**, 447; Lithwick & Goldreich 2003, *ApJ* **582**, 1220]

Weak MHD Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$\varepsilon \sim u_l^2\, \tau_l^{-1} = \text{const}$

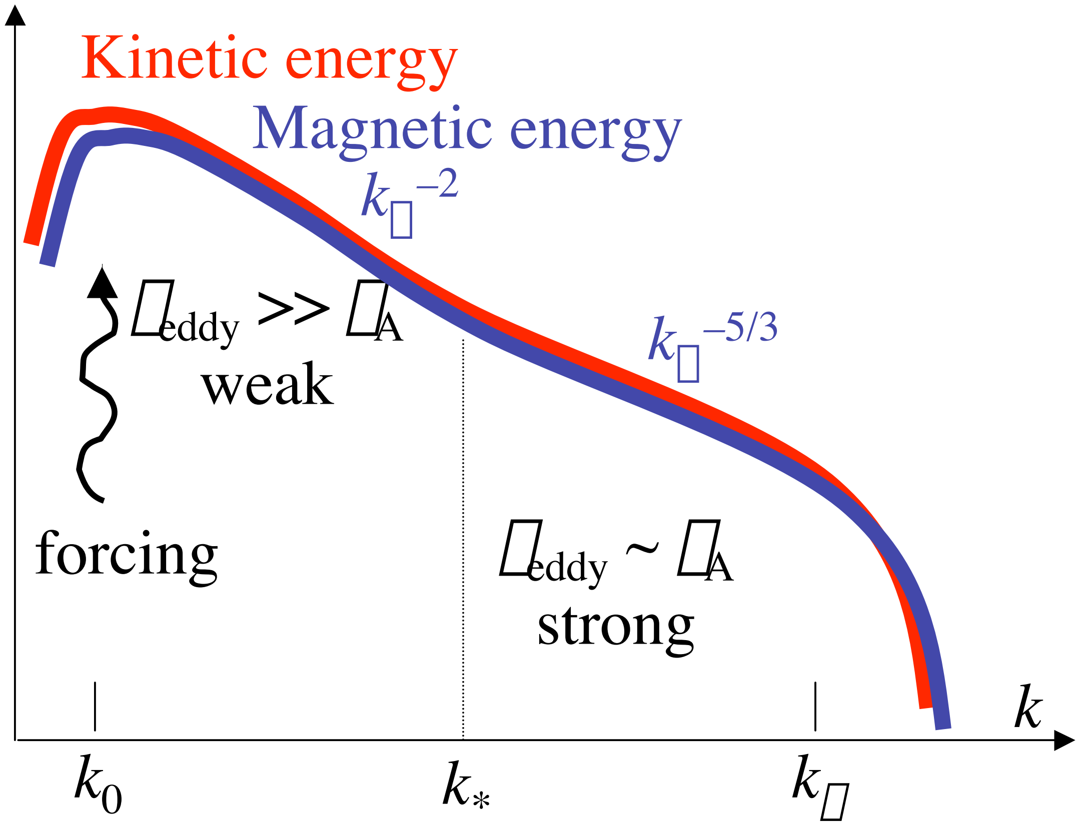
Energy at scale l — Cascade time (rate of transfer)

Weak interaction condition breaks down:

$$\frac{\tau_{\rm A}}{\tau_{\rm eddy}} \sim \frac{u_0}{v_{\rm A}}\left(\frac{l_0}{l_\perp}\right)^{1/2} \sim 1 \quad \text{when} \quad l_\perp \sim l_0\left(\frac{u_0}{v_{\rm A}}\right)^2 \equiv l_*$$

[e.g., Galtier *et al*. 2000, *JPP* **63**, 447; Lithwick & Goldreich 2003, *ApJ* **582**, 1220]

Goldreich-Sridhar Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

Additional physical assumptions:

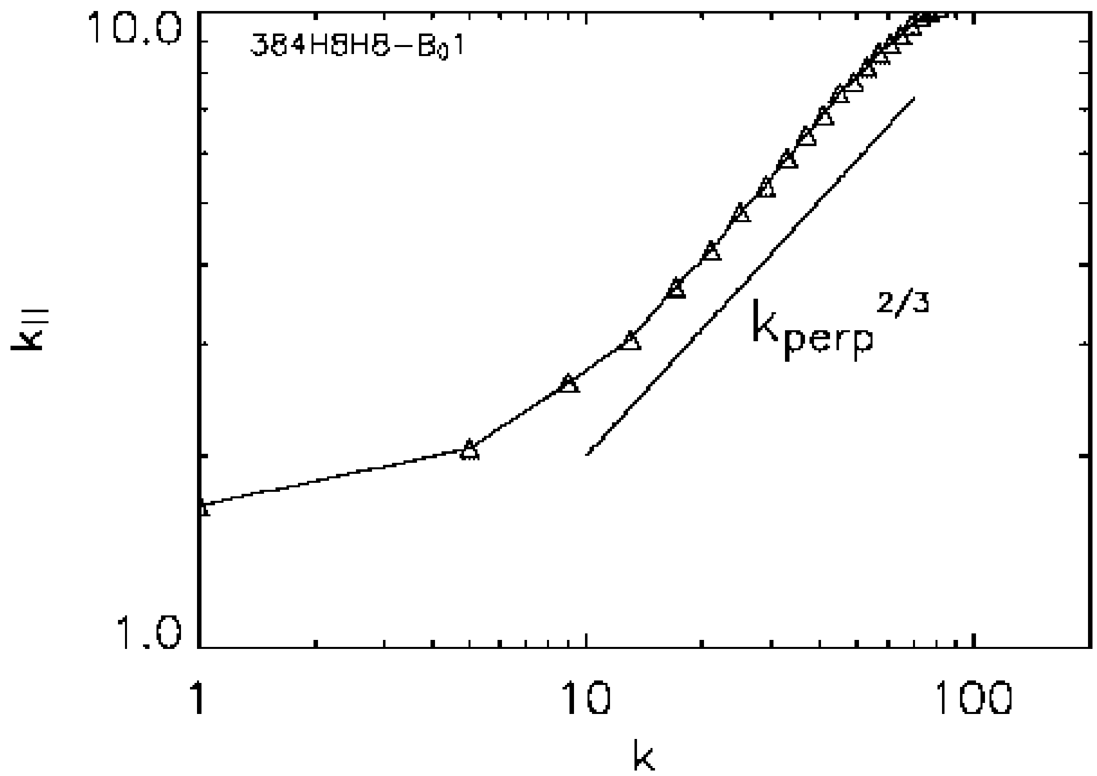
- **strong interactions: $\tau_{eddy} \sim \tau_A$** (critical balance)

$$E(k_\perp) \sim \varepsilon^{2/3} \, k_\perp^{-5/3}$$ **GS95**

$$k_{||} \sim \varepsilon^{1/3} v_A^{-1} \, k_\perp^{2/3}$$

[Goldreich & Sridhar 1995, *ApJ* **438**, 763]

Goldreich-Sridhar Turbulence: DNS



$B_0 \sim u_{\mathrm{rms}}$

Cho *et al*. 2003, *ApJ* **595**, 812:

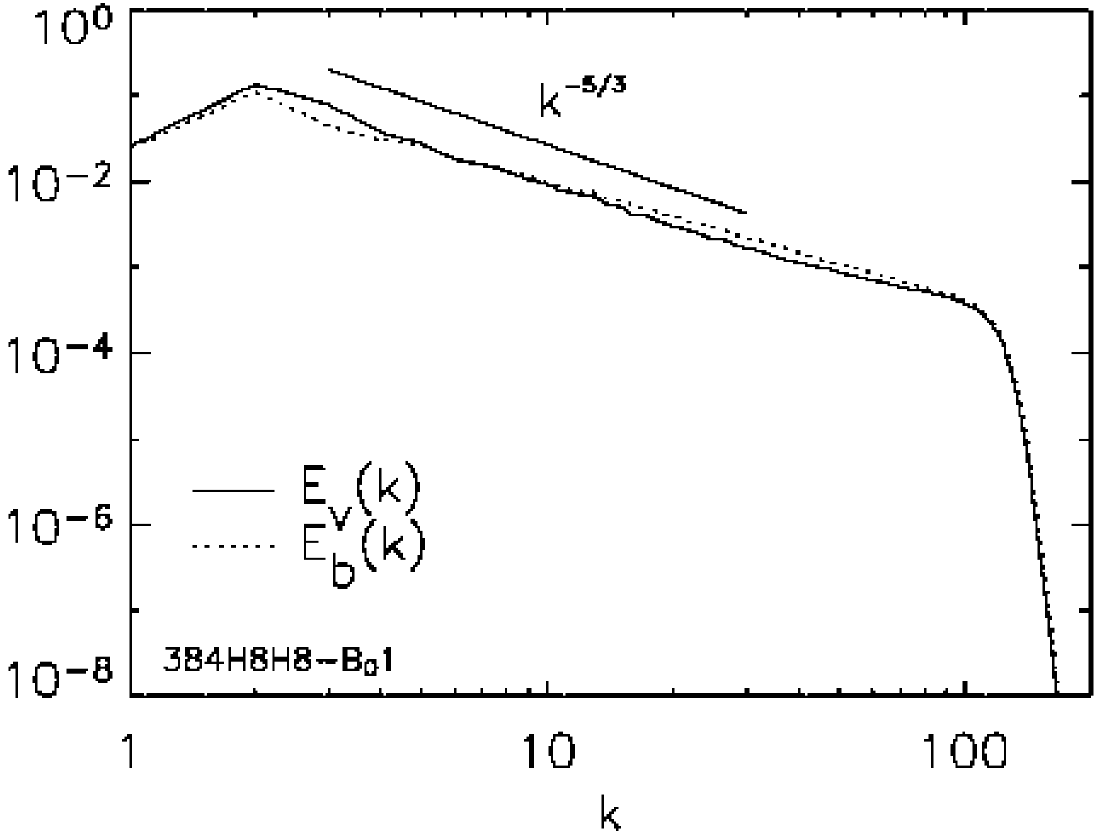
$l_{\|} \sim l_{\perp}^{2/3}$

- **strong interactions:** $\tau_{\mathrm{eddy}} \sim \tau_{A}$ ⟶ $E(k_{\perp}) \sim \varepsilon^{2/3}\, k_{\perp}^{-5/3}$ **GS95**

  (critical balance)

$k_{\|} \sim \varepsilon^{1/3} v_A^{-1}\, k_{\perp}^{2/3}$

Goldreich-Sridhar Turbulence: DNS



$B_0 \sim u_{rms}$

Cho *et al*. 2003, *ApJ* **595**, 812:

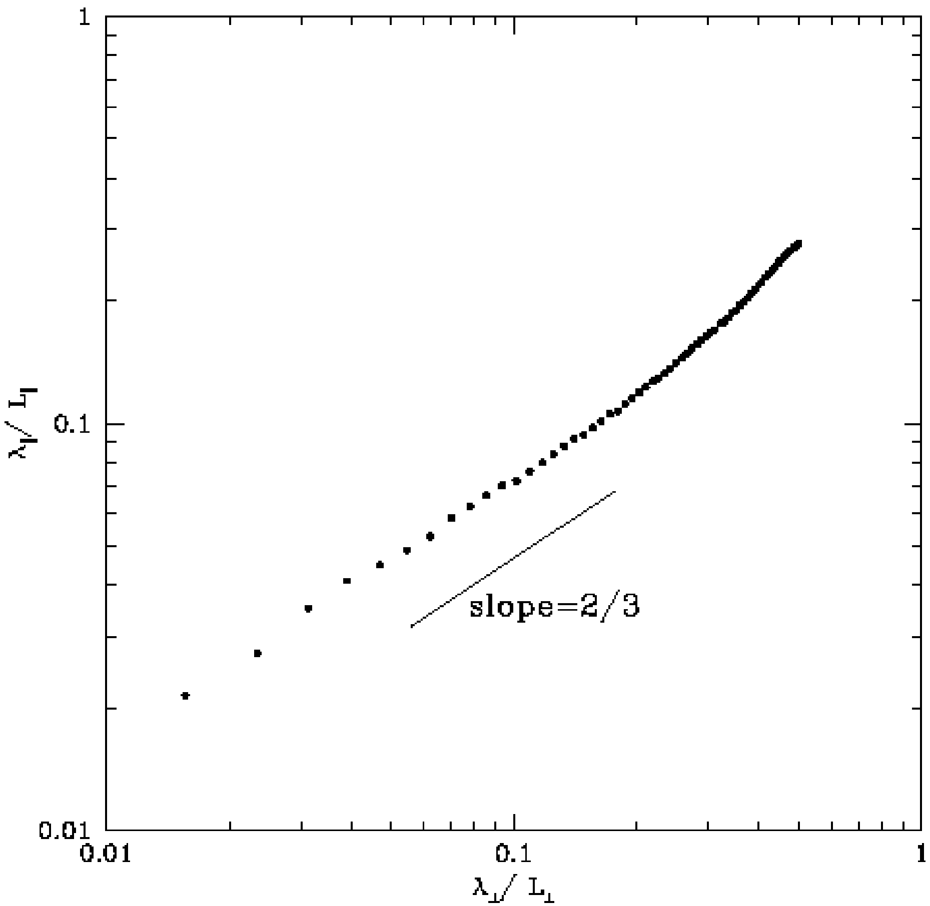
$E \sim k^{-5/3}$

• **strong interactions:** $\tau_{eddy} \sim \tau_A$ ⟶ $E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$ **GS95**

(critical balance)

$k_{||} \sim \varepsilon^{1/3} v_A^{-1} k_\perp^{2/3}$

Goldreich-Sridhar Turbulence: DNS



$B_0 = 100\ u_{rms}$

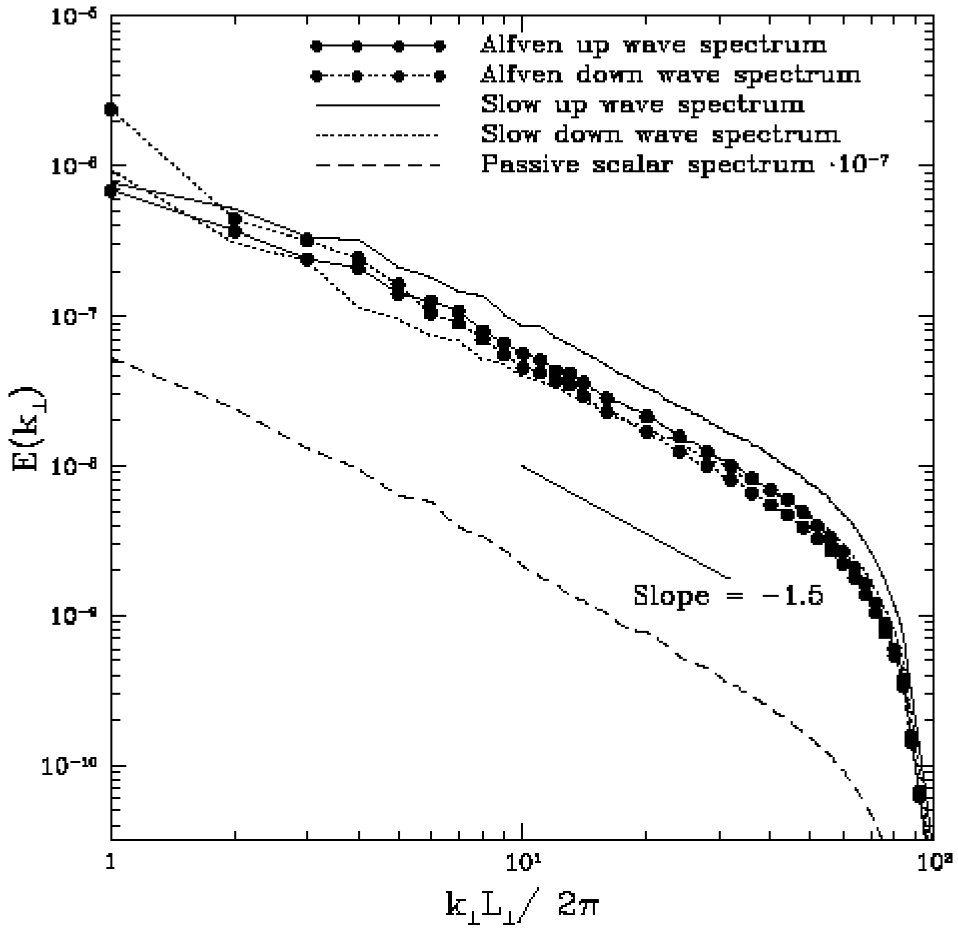
Maron & Goldreich 2001, *ApJ* **554**, 1175:

$l_{||} \sim l_\perp^{2/3}$

- **strong interactions: $\tau_{eddy} \sim \tau_A$** (critical balance) ⟶ $E(k_\perp) \sim \varepsilon^{2/3}\, k_\perp^{-5/3}$ **GS95**

$k_{||} \sim \varepsilon^{1/3} v_A^{-1}\, k_\perp^{2/3}$

Goldreich-Sridhar Turbulence: DNS



$B_0 = 100\ u_{rms}$

Maron & Goldreich 2001, *ApJ* **554**, 1175:

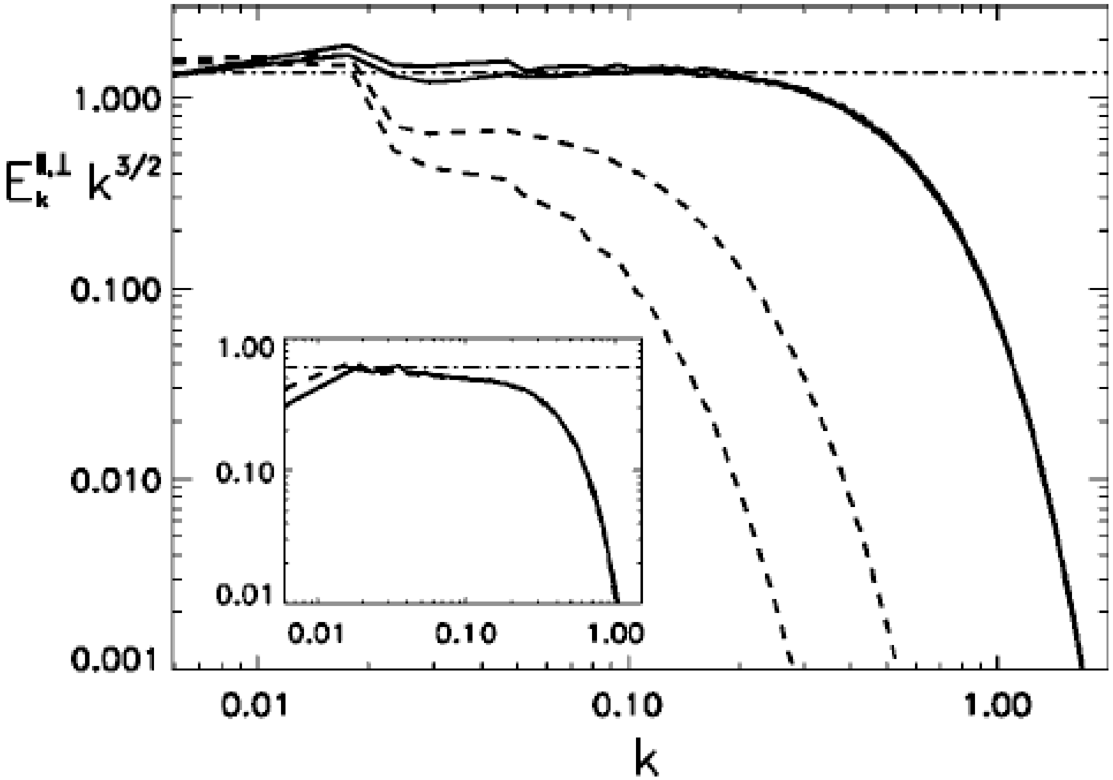
spectra closer to $k_\perp^{-3/2}$

Spectra do not fit!

- **strong interactions:** $\tau_{eddy} \sim \tau_A$ ⟶ $E(k_\perp) \sim \varepsilon^{2/3}\ k_\perp^{-5/3}$ **GS95**

  (critical balance) $k_{||} \sim \varepsilon^{1/3} v_A^{-1}\ k_\perp^{2/3}$

Goldreich-Sridhar Turbulence: DNS



$B_0 \approx 5, 10 u_{rms}$

Müller *et al*. 2003, *PRE* **67**, 066302:

spectra closer to $k_\perp^{-3/2}$

Spectra do not fit!

- **strong interactions:** $\tau_{eddy} \sim \tau_A$ ⟶ $E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$ **GS95**

  (critical balance) $k_{||} \sim \varepsilon^{1/3} v_A^{-1} k_\perp^{2/3}$

Issues With $k_{||} = 0$ ("Mean Modes")

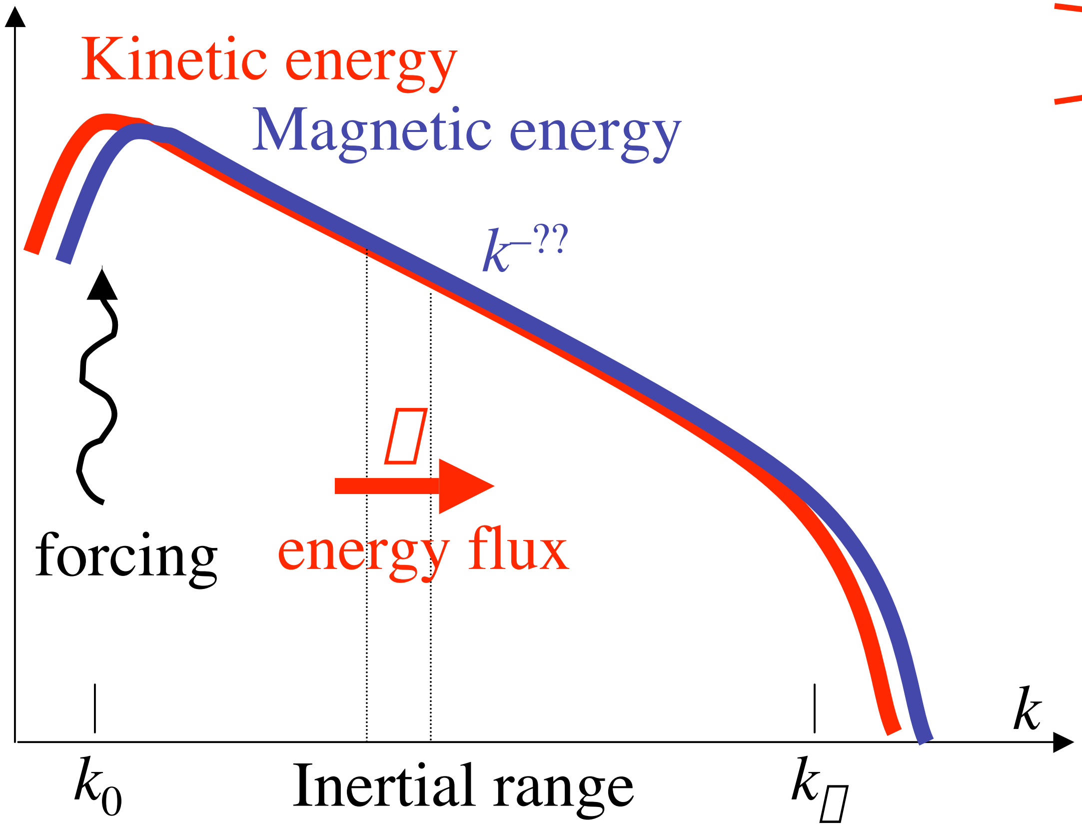
- **Very important: they mediate interaction between Alfvén waves**
  (in the weak-interaction limit, Alfvén waves are passive with respect to $k_{||} = 0$ modes)

- **They are not Alfvén waves themselves, rather, they are 2D MHD:**
  liable to form long-lived low-$k_{\perp}$ structures
  some evidence of $k^{-3/2}$ spectrum
  [2D MHD: Kinney *et al*. 1995, *PoP* **2**, 3623; Biskamp & Swartz 2001, *PoP* **8**, 3282]
  [3D RMHD: Kinney & McWilliams 1998, *PRE* **57**, 7111]

- **In simulations with strong B_0, do these modes get mixed up with the Alfvénic spectrum?**
  (I think this is certainly true in Müller *et al*. simulations)

- **A numerical effect only?**
  (existence of such modes depends on periodic boundaries)

Isotropic MHD Turbulence: No Mean Field



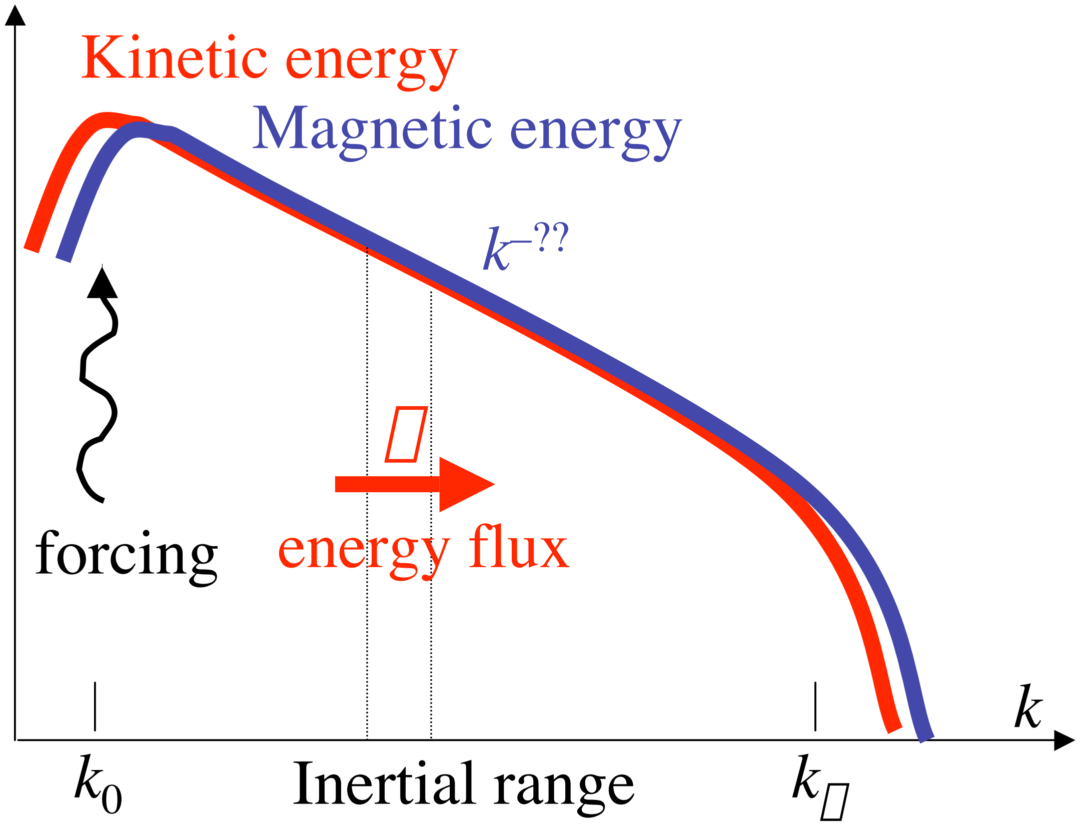
- ~~**Strong mean field B_0**~~
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

Isotropic MHD Turbulence: No Mean Field



- **Strong *large-scale* field?**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

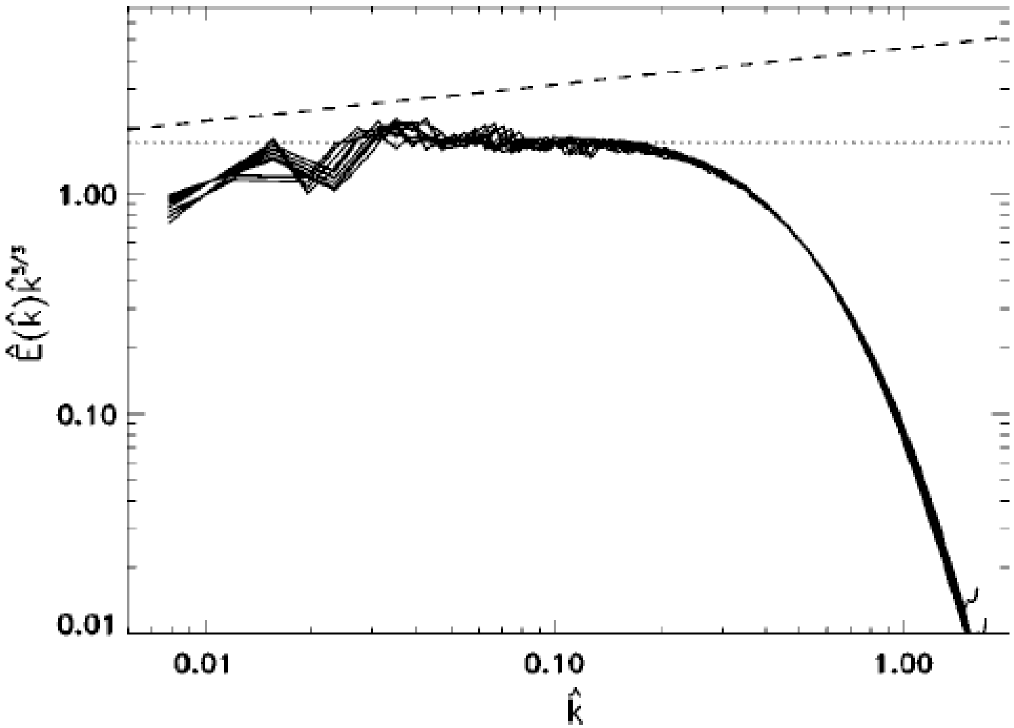
$$\varepsilon \sim u_l^2 \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

etc…

Isotropic MHD Turbulence: DNS (Decaying)



Biskamp & Müller 2000, *PoP* **7**, 4889:

$E(k) \sim k^{-5/3}$ claimed

- **Strong *large-scale* field?**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in *k* space

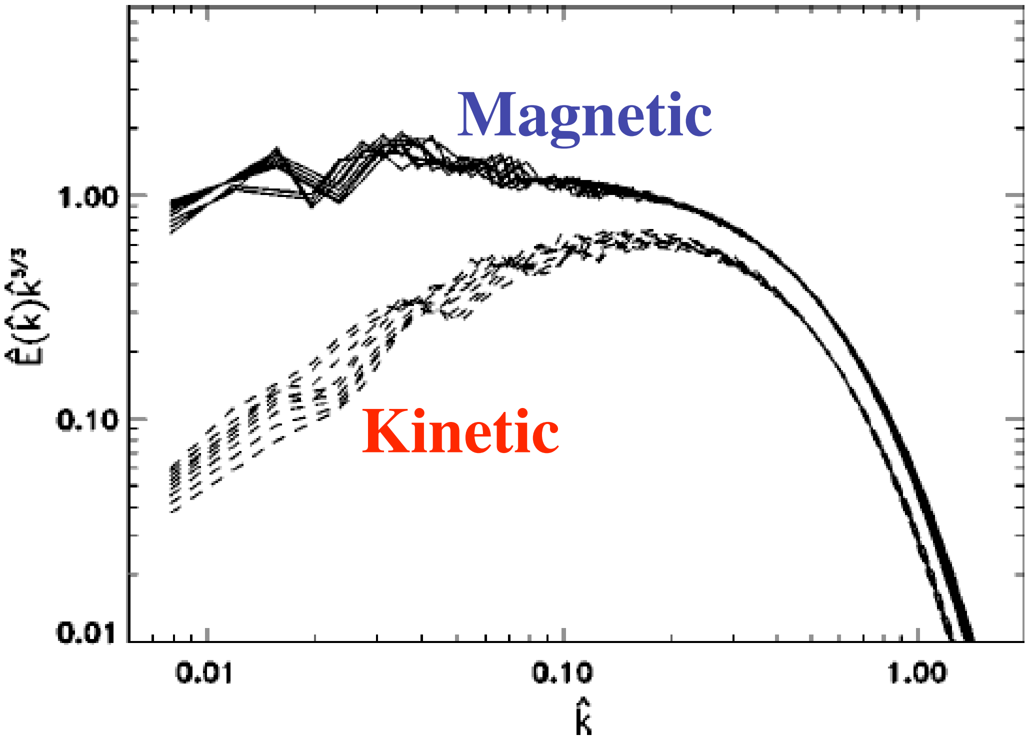
$\varepsilon \sim u_l^2 \tau_l^{-1} = \text{const}$

Energy at scale *l*

Cascade time (rate of transfer)

etc…

Isotropic MHD Turbulence: DNS (Decaying)



Biskamp & Müller 2000,
PoP **7**, 4889:

$E(k) \sim k^{-5/3}$ claimed

- **Strong *large-scale* field?**
- ~~**Alfvénic state: $u_l \sim B_l$**~~
- Scale invariance
- Locality in *k* space

→ $\varepsilon \sim u_l^2 \tau_l^{-1} = \text{const}$

Energy at scale *l*

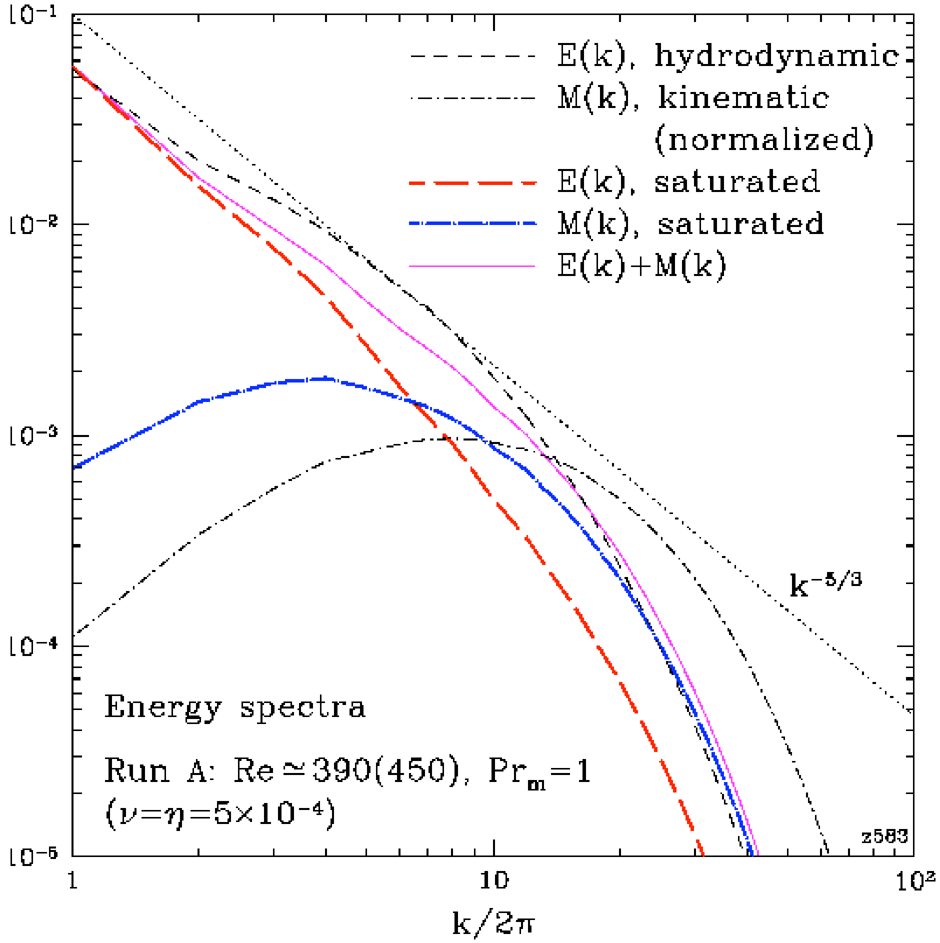
Cascade time (rate of transfer)

↓

etc…

NB: decay controlled by helicity conservation

Isotropic MHD Turbulence: DNS (Forced)



- ~~Strong *large-scale* field?~~
- ~~Alfvénic state: $u_l \sim B_l$~~
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

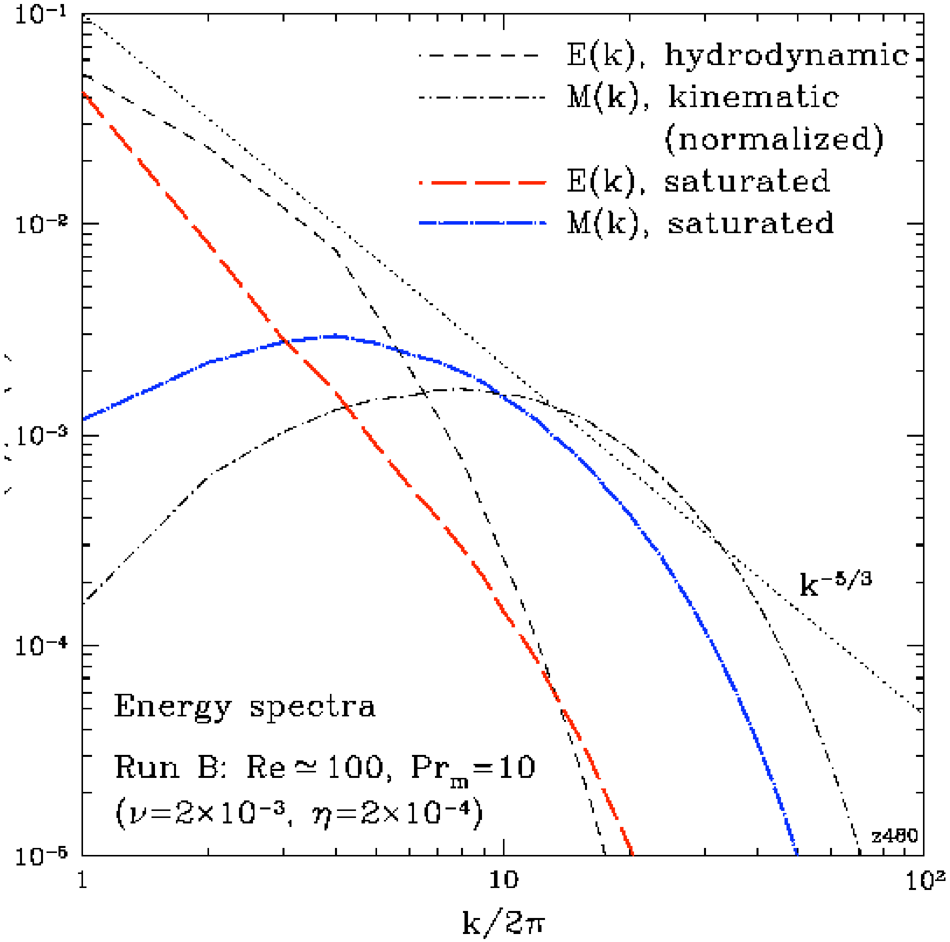
etc…

AAS *et al*. 2004, *ApJ* **612**, 276

See also Maron *et al*. 2004, *ApJ* **603**, 569

Haugen *et al*. 2004, *PRE* **70**, 016308

Isotropic MHD Turbulence: DNS (Forced)



- ~~**Strong *large-scale* field?**~~
- ~~**Alfvénic state: $u_l \sim B_l$**~~
- Scale invariance
- ~~Locality in *k* space~~

Low Pm issues

[AAS *et al*. 2004, astro-ph/0412594]

Large Pm: small-scale dynamo

[AAS *et al*. 2004, *ApJ* **612**, 276]

… outside the scope of this lecture

AAS *et al*. 2004, *ApJ* **612**, 276

See also Maron *et al*. 2004, *ApJ* **603**, 569

Haugen *et al*. 2004, *PRE* **70**, 016308

Some Questions for Discussion

- **Is there a universal scaling theory of MHD turbulence?**

  …or are there distinct regimes?

  — **Anisotropic**: strong B_0 (what does "strong" mean?)

  — **Decaying isotropic** (helicity conservation etc.)

  — **Forced isotropic** (small-scale dynamo, Pm, etc.)

  — Periodic or other boundary conditions (mean modes etc.)

- **Is GS95 theory correct for strong B_0?**

  Are the basic assumptions right?

  — Are interactions local in k space?

  — Are elementary objects Alfvén waves?

  What about $k_{||} = 0$ modes?

- **What else do we want to know?**

  Correlation functions?

  Diagnostics of structure?

Inspiring Quote

…a considerable mathematical effort towards a detailed understanding of the mechanism of turbulence is called for. The entire experience with the subject indicates that the purely analytical approach is beset with difficulties, which at this moment are still prohibitive. The reason for this is … that our intuitive relationship to the subject is still too loose -- not having succeeded at anything like deep mathematical penetration in any part of the subject, we are still quite disoriented as to the relevant factors, and as to the proper analytical machinery to be used.

Under these conditions there might be some hope to 'break the deadlock' by extensive, but well-planned, computational efforts. It must be admitted that the problems in question are too vast to be solved by a direct computational attack, that is, by an outright calculation of a representative family of special cases. There are, however, strong indications that one could name certain strategic points in this complex, where relevant information must be obtained by direct calculations. If this is properly done … there is a reasonable chance of effective real penetrations in this complex of problems and gradually developing a useful, intuitive relationship to it. This should, in the end, make an attack with analytical methods, that is truly more mathematical, possible.

John von Neumann 1949