

# Revision Lectures on ODEs (CB3)

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"In Michaelmas and Hilary, we teach you physics (or mathematics). In Trinity, we teach you how to pass exams!"

Anonymous tutor.

[direct]

## S1. Methods of integration (1st-order ODEs)

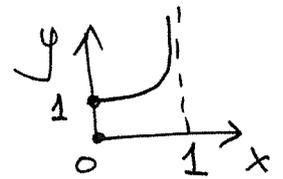
Ex. 1.  $y' = y^2$ ,  $y(0) = 1$  (1 initial condition for a 1st-order ODE)

$\frac{dy}{dx}$  Separate variables:

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + C \Rightarrow y = -\frac{1}{x+C} = \frac{1}{1-x}$$

$y(0) = -\frac{1}{C} = 1 \Rightarrow C = -1$



blows up at  $x=1$

NB:  $y=0$  also a solution, but does not satisfy  $y(0)=1!$

Ex. 2

$$y' = xy^2$$

$$\int \frac{dy}{y^2} = \int x dx$$

$$-\frac{1}{y} = \frac{x^2}{2} + C \Rightarrow y = -\frac{1}{\frac{x^2}{2} + C} = \frac{1}{1 - \frac{x^2}{2}}$$

$$y(0) = -\frac{1}{C} = 1 \Rightarrow C = -1$$

Q: What if the IC were  $y(0)=0$ ?

Such equations are called separable.

Sometimes the fact that an equation is separable is disguised and can be extracted by change of variables:

Ex. 3

$$y' = \sqrt{4x+2y-1}, \quad y(0) = 1$$

↑  
this is begging to be called  $z$ !

$$z = 4x + 2y - 1$$

$$z' = 4 + 2y' \Rightarrow y' = \frac{1}{2}z' - 2$$

$$\frac{1}{2}z' - 2 = \sqrt{z}$$

NB: This works because the change of variables was linear in  $x$ !

$$z' = 2\sqrt{z} + 4$$

$$\int \frac{dz}{2\sqrt{z} + 4} = \int dx = x + C$$

$$\parallel u = \sqrt{z}$$

So, to solve ODEs, you need to know your integrals!

$$\int \frac{du^2}{2u+4} = \int \frac{2udu}{2u+4} = \int \frac{u^{\overset{+2-2}{2}} du}{u+2} =$$

$$= \int du \left(1 - \frac{2}{u+2}\right) = u - 2 \ln|u+2| =$$

$$= \sqrt{z} - 2 \ln|\sqrt{z} + 2| =$$

$$= \sqrt{4x+2y-1} - 2 \ln|2 + \sqrt{4x+2y-1}|$$

$$y(0) = 1, \text{ so } C = 1 - 2 \ln 3$$

Answer:  $x + 1 - 2 \ln 3 = \sqrt{4x+2y-1} - 2 \ln(2 + \sqrt{4x+2y-1})$

↑ always write out the final answer!

Separable ~~and~~ ODEs (and, ultimately, all other integrable ODEs) are particular (particularly simple) cases of equations that are in fact just full differentials.

$$\begin{array}{c}
 \cancel{\text{ODE}} \quad P(x,y)dx + Q(x,y)dy = 0 \\
 \underbrace{\hspace{2cm}} \quad \underbrace{\hspace{2cm}} \\
 \parallel \quad \parallel \\
 \frac{\partial \Phi}{\partial x} \quad \frac{\partial \Phi}{\partial y}
 \end{array}$$

Then  $d\Phi(x,y) = 0$

and the solution is simply  $\Phi(x,y) = C$ .

For separable eqns,  $P = P(x)$  and  $Q = Q(y)$  (or reducible to that).

More generally, just check if  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ .

Ex.4  $2xy dx + (x^2 - y^2) dy = 0$

$$\frac{\partial}{\partial y} (\dots) = 2x \quad \frac{\partial}{\partial x} (\dots) = 2x$$

Hurray! There must be some function  $\Phi$  s.t.   
 treat y as const

$$\frac{\partial \Phi}{\partial x} = 2xy \Rightarrow \Phi = x^2y + C(y)$$

$$\frac{\partial \Phi}{\partial y} = x^2 - y^2 \Rightarrow x^2 + C'(y) = x^2 - y^2$$

$$\begin{array}{c}
 \text{treat } x \text{ as const} \\
 C(y) = -\frac{y^3}{3} + \text{const}
 \end{array}$$

So, solution is  $\Phi = x^2y - \frac{y^3}{3} = C_1$

Sometimes this effect can be achieved by first multiplying the whole thing by some "integrating factor":

Ex. 5 Let's reverse engineer such a question from Ex. 4:

$$\frac{1}{x} \cdot | 2xy dx + (x^2 - y^2) dy = 0$$

$$2y dx + (x - \frac{y^2}{x}) dy = 0$$

The integrating factor is obviously  $x!$

We can hide the path to solution further:

$$y' = - \frac{2y}{x - \frac{y^2}{x}} = \frac{2}{\frac{y}{x} - \frac{x}{y}}$$

you could find it systematically by trying an arbitrary function of  $x$ :  
 $\mu(x)$   
s.t.

~~Integration~~

$$2y\mu(x) dx + (x - \frac{y^2}{x})\mu(x) dy = 0$$

$$\frac{\partial}{\partial y}(\dots) = 2\mu \quad \frac{\partial}{\partial x}(\dots) = \mu(1 + \frac{y^2}{x^2}) + (x - \frac{y^2}{x})\mu'$$

We need 
$$2\mu = \mu' + \mu \frac{y^2}{x^2} + (x - \frac{y^2}{x})\mu'$$
  
$$(x - \frac{y^2}{x})\mu' = \mu(1 - \frac{y^2}{x^2})$$

$\mu = x$  does it!

(generally speaking such eqns for  $\mu$  are not easier to solve than the original equation, but here you only need one particular solution that works!)

Interestingly, our eqn has ended up being in particular form that is called homogeneous:

$$y' = f\left(\frac{y}{x}\right)$$

Homogeneous means that the equation does not change when you rescale  $x$  and  $y$  by the same amount:  $x \rightarrow \lambda x, y \rightarrow \lambda y$

There are various generalised versions of homogeneity, dealing with rescaling of  $x$  and  $y$  by different powers of  $\lambda$  - these were treated in the lectures and homework (revisit them!). ~~unintentionally~~ In all of those cases, there is a method for integrating these equations, so ~~it~~ being able to spot homogeneity is useful!

Ex.6

Let's do the simple case at hand:

The winning substitution is

$$z = \frac{y}{x}, \text{ or } y = xz(x)$$

Then  $y' = z + xz'$

So  $z + xz' = \frac{2}{z - \frac{1}{z}}$

$$xz' = \frac{2}{z - \frac{1}{z}} - z = \frac{2z}{z^2 - 1} - z = \frac{2z - z^3 + z}{z^2 - 1}$$

$$= -\frac{z^3 - 3z}{z^2 - 1} = -\frac{z(z^2 - 3)}{z^2 - 1}$$

~~scribbles~~

$$\int \frac{z^2 - 1}{z(z^2 - 3)} dz = -\int \frac{dx}{x}$$

~~scribbles~~

$$\frac{z^2-1}{z(z^2-3)} = \frac{A}{z} + \frac{Bz+C}{z^2-3}$$

$$Az^2 - 3A + Bz^2 + Cz = z^2 - 1$$

$$C=0 \quad A=\frac{1}{3} \quad A+B=1 \quad \text{so } B=\frac{2}{3}$$

$$-\int \frac{dx}{x} = \frac{1}{3} \int \frac{dz}{z} + \frac{2}{3} \int \frac{z dz}{z^2-3} =$$

$$\left( -\ln|x| + C_1 \right) = \frac{1}{3} \ln|z| + \frac{1}{3} \ln|z^2-3|$$

$$\frac{C_2}{x} = [z(z^2-3)]^{1/3}$$

$$\frac{C_3}{x^3} = z(z^2-3) = \frac{y}{x} \left( \frac{y^2}{x^2} - 3 \right)$$

$$C_3 = y^3 - 3x^2y$$

$$\text{or } x^2y - \frac{y^3}{3} = C_4 \quad \text{same solution as before.}$$

Note that in principle you should worry about whether the eqn is well defined (e.g., at  $z=1$  ours is not) and about dividing by 0 (as you would at  $z=\pm\sqrt{3}$ ). I refer you to my notes for how to handle this - although I very much doubt exam problems will focus on such matters.

Just like in the case separable equations, the fact that an equation is homogeneous can be hidden behind a change of variables. Especially simple case is when the required change of variables is linear.

Ex. 7  $y' = \frac{y+2}{2x+y-4}$  ↙ ↘ everything would be great ~~but~~ except for these constants

Try  $x = x_0 + \zeta$  ~~zeta~~  
 $y = y_0 + \eta$  ~~eta~~

and pick  $x_0$  and  $y_0$  to eliminate constants:

$$\frac{dy}{dx} = \frac{d\eta}{d\zeta} = \frac{y_0 + \eta + 2}{2x_0 + 2\zeta + y_0 + \eta - 4} = \frac{\eta}{2\zeta + \eta} = \frac{\eta/3}{2 + \eta/3}$$

if  $y_0 + 2 = 0 \Rightarrow y_0 = -2$   
 and  $2x_0 + y_0 - 4 = 0 \Rightarrow x_0 = 3$

For completeness, let us solve this.

$$\zeta = \frac{\eta}{3} \quad \text{or} \quad \eta = 3\zeta \quad (3)$$

$$\eta' = 3 + 3\zeta' = \frac{3}{2 + \zeta}$$

$$3\zeta' = \frac{3}{2 + \zeta} - 3 = \frac{3 - 2\zeta - 3\zeta^2}{2 + \zeta} = -\frac{\zeta^2 + \zeta}{\zeta + 2} =$$

$$= -\zeta \frac{\zeta + 1}{\zeta + 2}$$

NB: there are cases when we can't solve for  $x_0, y_0$ . See lectures for what to do then

For one, let me be careful about dividing by 0 before separating variables:

$$\zeta = 0 \text{ means } \frac{\eta}{\zeta} = \frac{y-y_0}{x-x_0} = \frac{y+2}{x-3} = 0$$

So  $y = -2$  is a solution

$$\zeta = -1 \quad \frac{y+2}{x-3} = -1 \Rightarrow y = 1-x \text{ is a sol.}$$

Otherwise,

$$-\int \frac{d\zeta}{\zeta} = \int \frac{\zeta+2}{\zeta(\zeta+1)} d\zeta = 2 \int \frac{d\zeta}{\zeta} - \int \frac{d\zeta}{\zeta+1}$$

$$-\ln|\zeta| + C = 2 \ln|\zeta| - \ln|\zeta+1|$$

$$-\ln|\zeta| + C = 2 \ln\left|\frac{\eta}{\zeta}\right| - \ln\left|\frac{\eta}{\zeta} + 1\right|$$

$$-\ln|\zeta| + C = -\ln|\eta+3| + \ln|\zeta|$$

$$\ln \eta^2 = -2 \ln|\zeta|$$

So  $\eta^2 = C_1(\eta+3)$

$$(y+2)^2 = C_1(y+2+x-3) = C_1(y+x-1)$$

Note that this solution includes  $y = -2$  (for  $C_1 = 0$ ) but w/o  $y = 1-x$ , which should be kept in mind separately. (alternatively, you could rewrite this with  $C_2 = \frac{1}{C_1}$  and then  $y = 1-x$  corresponds to  $C_2 = 0$  but  $y = -2$  should be kept in mind separately.)

Solution!

A largely important class of equations is linear equations, homogeneous and inhomogeneous

When they are 1-order, they can always be

↑  
homogeneous here refers to invariance wrt rescaling of  $y$ . ~~the other way around.~~

solved, by a standard method, which is a 1D version of the matrix method that linear algebra allows us to use for  $n$ th-order equations. If you understood it in 1D, you'll understand it in  $n$ -D!

Ex. 8  $xy' + (x+1)y = 3x^2e^{-x}$

Step 1: solve homogeneous eqn:

$$xy' + (x+1)y = 0$$

NB: what if  $x=0$  or  $y=0$ ?  
what if  $x=-1$ ?

$$\int \frac{dy}{y} = -\int \frac{x+1}{x} dx$$

$$\ln|y| = -x - \ln|x| + C$$

$$y = C_1 \frac{e^{-x}}{x}$$

Step 2: Weaponize the constant:

$$y = f(x) \frac{e^{-x}}{x} \leftarrow \text{look for solution in this form,}$$

$$x \left[ \psi' \frac{e^{-x}}{x} = \psi \frac{e^{-x}}{x} - \psi \frac{e^{-x}}{x^2} \right] + (x+1) \psi \frac{e^{-x}}{x} =$$

$$= 3x^2 e^{-x}$$

$$\psi' - \cancel{\psi} - \cancel{\psi} \frac{1}{x} + \cancel{\psi} + \frac{\psi}{x} = 3x^2$$

this cancellation is inevitable because  $\frac{e^{-x}}{x}$  was a solution of the hom. equation, so only the  $\psi'$  term can survive in the lhs (all terms  $\psi$  don't know that  $\psi$  is not a constant!)

$$\psi' = 3x^2$$

$$\psi = x^3 + C$$

NB: homogeneous solution always recovered because you can add it to any solution of the <sup>inhom.</sup> equation

Solution:

$$y = \frac{x^3 + C}{x} e^{-x} = x^2 e^{-x} + \frac{C}{x} e^{-x}$$

Note that if you could guess that this is a particular solution of the equation (a bit hard in this case, but not impossible), then you can use

$$y = y_{\text{homogeneous}} + y_{\text{particular}}$$

$\uparrow$  "CF"                       $\uparrow$  "PI"

This is because the CF already has all the constants needed to satisfy ~~the~~ initial conditions.

Other solvable cases include Bernoulli equation, which are reducible to linear and those

Riccati equations for which you can guess a particular solution and which are then reducible to Bernoulli eqns. See lectures...

$$y' = a(x)y^2 + b(x)y + c(x)$$

These are in fact 2-order <sup>(linear)</sup> equations in disguise

$$y'' + a(x)y' + b(x)y = f(x)$$

[these reduce to Riccati via  $z = y'/y$ ]

These are unsolvable except when  $a$  and  $b$  are constant or when one of the solutions can be guessed.

Since the equation is linear and 2-order, its solution is

$$y = C_1 y_1(x) + C_2 y_2(x) + y_{PI}(x)$$

$\swarrow$  constants of integration  
 $\uparrow$  2 linearly independent solutions of the homogeneous eqn.  
 $\uparrow$  particular solution of the inhom. eqn.

NB: All of this generalises to n-D equation

You can prove  $y_1$  and  $y_2$  will always exist and you can prove that  $y$  constructed as above is the unique solution, but that does not tell you how to get  $y_1$  and  $y_2$ . So you have to guess at least one of them.

Ex. 9.  $x(x+1)^2 y'' + 2(x+1)y' - 2y = (x+1)^3 e^x$

One solution of the hom. equation is

$$y_1 = x+1$$

Note that when the coefficients are polynomial, one can actually find such solutions systematically:

try  $y = x^n + a_{n-1}x^{n-1} + \dots$

and then substitute <sup>into eqn</sup> and work out  $n, a_{n-1}, \dots$

Each time the coefficient of the highest power of  $x$  vanishes, then both the coefficient ~~of~~ of all the lower powers by repeating the process such that all coefficients in front of powers of  $x$  vanish separately.

In any event, in the case at hand, it's easy to guess.

Now look for solutions in the form

$$y = \psi(x)(x+1)$$

$$y' = (x+1)\psi' + \psi$$

$$y'' = (x+1)\psi'' + 2\psi'$$

$$x(x+1)^2 [(x+1)\psi'' + 2\psi'] + 2(x+1) [(x+1)\psi' + \psi]$$

$$- 2(x+1)\psi = (x+1)^3 e^x$$

$$x(x+1)^3 \psi'' + 2x(x+1)^2 \psi' + 2(x+1)^2 \psi' + \cancel{2(x+1)}\psi$$

$$- 2(x+1)\psi = (x+1)^3 e^x$$

↑ this cancellation inevitable!

So we now have a 1st-order equation for  $\psi$ :

~~$x(x+1)\psi'' + 2(x+1)\psi' = (x+1)^2 e^x$~~

$$x(x+1)\psi'' + 2(x+1)\psi' = (x+1)^2 e^x$$

$$x\psi'' + 2\psi' = e^x$$

This is a linear equation:  $p = \psi'$

~~Homogeneous version:~~ Homogeneous version:

$$xp' + 2p = 0$$

$$p' = -\frac{2}{x}p$$

$$\int \frac{dp}{p} = -\int \frac{2}{x} dx$$

$$\ln|p| = -2\ln|x| + C$$

$$p = \frac{C}{x^2}$$

To find solution of inhomogeneous equation, unleash the constant:

$$p = \frac{\varphi(x)}{x^2}$$

$$p' = -\frac{2\varphi}{x^3} + \frac{\varphi'}{x^2}$$

$$\frac{1}{x}\varphi' - \frac{2}{x^2}\varphi + \frac{2\varphi}{x^2} = e^x$$

$$\varphi' = xe^x$$

$$\varphi = \int dx xe^x = \int de^x x = xe^x + e^x + C_1$$

$$= (x+1)e^x + C_1$$



## §2. Solving ODE's via Linear Algebra.

Linear ODEs are a branch of Linear Algebra  
— especially ones with constant coefficients.

Ex. 10. 
$$\begin{cases} \dot{x} = 3x + 2y + 4e^{5t} \\ \dot{y} = x + 2y \end{cases}$$

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}}_{\text{matrix } A} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + 4 \begin{pmatrix} e^{5t} \\ 0 \end{pmatrix}$$

Step 1: Find eigenvalues and eigen vector of  $A$ :

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - 2 = 0$$

$$6 - 5\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - 5\lambda + 4 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{25-16}}{2} = 1, 4$$

$$\lambda_1 = 1: \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \cdot \vec{v}_1 = 0 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 4: \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \cdot \vec{v}_2 = 0$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(need not be  
normalized because  
it will be  
multiplied a const)

So, the homogeneous solution (CF) is

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix}_{CF} &= C_1 \vec{v}_1 e^{\lambda_1 t} + C_2 \vec{v}_2 e^{\lambda_2 t} \\ &= C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} \end{aligned}$$

Step 2: Find PI. Since the inhomogeneous term is  $\propto e^{st}$  and  $s \neq \lambda_1, \lambda_2$ , it should be possible to find the solution in the

form  $\begin{pmatrix} x \\ y \end{pmatrix}_{PI} = \begin{pmatrix} A \\ B \end{pmatrix} e^{st}$

More general method is, as usual, to weaponize the constants:  $C_1 \rightarrow \phi_1(t)$ ,  $C_2 \rightarrow \phi_2(t)$  etc.

Try it:

$$5 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} = - \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2A + 2B &= -4 & -6B + 2B &= -4 \Rightarrow B = 1 \\ A - 3B &= 0 & \text{---} & \text{---} & A &= 3 \end{aligned}$$

NB: Why wouldn't this work if we had  $e^{4t}$  or  $e^t$  instead of  $e^{st}$ ? Look in the lectures how to deal with such a situation.

Solution:  $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{st}$

use these to enforce initial conditions.

Here are two further examples, both harder:

$$\begin{cases} \dot{x} = 2y - x \\ \dot{y} = 4y - 3x + \frac{e^{3t}}{e^{2t} + 1} \end{cases}$$

Here you will need to vary constants.

Answer: 
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t} (\arctan e^t + C_1) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t (C_2 - \ln \sqrt{e^{2t} + 1})$$

$$\begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 2x - y + 15e^t \sqrt{t} \end{cases}$$

Answer: 
$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t \left[ \begin{pmatrix} 2 \\ 2 \end{pmatrix} (C_1 + C_2 t - 4t^{5/2}) + \begin{pmatrix} C_2 \\ 10t^{3/2} \end{pmatrix} \right]$$

Here you will need to use

Jordan's Scheme (extracurricular) and also vary constants

2-order equations are a particular case of this, but their handling can be streamlined.

Ex. 11 
$$y'' + 2y' - 3y = x^2 e^{\alpha x}$$

In principle,  $p = y'$ , so the above is

$$\begin{cases} y' = p \\ p' = -2p + 3y + x^2 e^{\alpha x} \end{cases} \quad \text{and we know how to handle it.}$$

Doing so is equivalent <sup>try</sup> the following procedure  
(if you are unclear why that is, you need to work this out for yourself):

- look for homogeneous slus (CF) in the form

$$y_{1,2} = e^{\lambda_{1,2}x}$$

Try  $y = e^{\lambda x}$ :  $\lambda^2 + 2\lambda - 3 = 0$

$$\lambda = \frac{-2 \pm \sqrt{4+12}}{2} = 1, -3$$

So  $y_{CF} = C_1 e^x + C_2 e^{-3x}$

- Now we need a particular solution.

In general, you can do it via the "buy one get one free" method, i.e., by ~~varying~~ varying constants (see discussion of the 2-order equation with non-constant coefficients - it's the same calculation, but easier!)

However, when the inhomogeneity is <sup>in</sup> a nice form like  $x^2 e^{\alpha x}$  (more generally, polynomial times an exponential), you can look for solution in the form

$$\begin{matrix} \text{(polynomial of the same order)} \cdot e^{\alpha x} \\ \uparrow \\ Ax^2 + Bx + C \end{matrix} \text{ but only if } \alpha \neq \lambda_{1,2}$$

If  $\alpha = \lambda_1$  or  $\lambda_2$ , then multiply the whole thing by an extra power of  $x$ .

let's do it: let  $\alpha=1$ , so try

$$y_{PI} = x(Ax^2 + Bx + C)e^x$$

$$y' = (3Ax^2 + 2Bx + C + Ax^3 + Bx^2 + Cx)e^x$$
$$= [Ax^3 + (3A+B)x^2 + (2B+C)x + C]e^x$$

$$y'' = [Ax^3 + (6A+B)x^2 + (6A+4B+C)x + 2B+2C]e^x$$

Sub. into the equation and equate coefficients

in front of powers of  $x$ : this works automatically (why?)

$$x^3: A + 2A - 3A = 0$$

$$x^2: 6A + B + 6A + 2B - 3B = 1$$

$$x^1: 6A + 4B + C + 4B + 2C - 3C = 0$$

$$x^0: 2B + 2C + 2C = 0$$

⇓

$$A = \frac{1}{12} \quad B = -\frac{1}{16} \quad C = \frac{1}{32}$$

$$\text{So, } y = \left( C_1 + \frac{x^3}{12} - \frac{x^2}{16} + \frac{x}{32} \right) e^x + C_2 e^{-3x}$$

You can experiment and see how this works

when  $\alpha = -3$  or when  $\alpha \neq 1, -3$ , e.g.  $\alpha = 2$ .

↑  
no extra  $x$   
in this case!

The fact that the eqn is linear makes many useful tricks possible:

1) Superposition principle

2) "Complexification":

$$y'' + 2y' - 3y = x^2 \cos x = \operatorname{Re} [x^2 e^{ix}]$$

So, solve  $z'' + 2z' - 3z = x^2 e^{ix}$

and then

(same as before,  
with  $\lambda = i$ )

$$y = \operatorname{Re} z$$

This is always simpler than dealing with proliferating sin's and cos's.

A particular case of such equations is the oscillator. I won't return to it because we covered it in such detail that I can't bear going back. You should study it though because it is always in the exam!

3)  $x^2 y'' + xy' + y = \text{whatever}$

turns into a linear eqn with constant coefficients via  $x = e^t$

(another example of a "homogeneous" eqn, where you can rescale  $y$  and  $x$  by the same factor).