Lectures on Ordinary Differential Equations
(Oxford Physics Paper CP3)

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These are the notes for my lectures on Ordinary Differential Equations for 1st-year undergraduate physicists, taught since 2018 as part of Paper CP3 at Oxford. I will be grateful for any feedback from students, tutors or (critical) sympathisers. The latest, up-to-date version of this file is available at http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/ODE/2018/ODELectureNotes.pdf.

As typos, errors, and better ways of explaining things are found, I will update these notes in real time as lectures proceed. Please check back here for updates (the date tag of the version that you are reading appears above).

Below I give references to some relevant bits of various books. Having access to these books is not essential for being able to follow these lectures, but reading (these or other) books above and beyond your lecturer’s notes is essential for you to be able to claim that you have received an education.

The problem sets suggested for your tutorials are at the end of this document. They too will be updated in real time if errors are found or new/better questions arise.

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1. The Language of the Game

It might not be an exaggeration to say that the ability of physical theory to make predictions and, consequently, both what we proudly call our “understanding of the world” and our technological civilisation hinge on our ability to solve differential equations—or, at least, to write them down and make intelligent progress in describing their solutions in qualitative, asymptotic or numerical terms. Thus, what you are about to start learning may well be the most “practically” important bit of mathematics for you as physicists—so be excited!

I shall start by introducing some terms and notions that will form the language that we will speak and by stating some general results that enable us to speak it meaningfully. Then I shall move on to methods for solving and analysing various kinds of differential equations.

1.1. What is an ODE?

A differential equation is an equation in which the unknowns are functions of one or more variables and which contains both these functions and their derivatives. Physically, it is a relationship between quantities and their rates of change—in time, space or whatever other “independent variable” we care to introduce.

If more than one independent variable is involved, the differential equation is called a partial differential equation (PDE). Those will not concern me—you will encounter them next term, in CP4, and also, in a proper way, in the second-year Mathematical Methods (Effler 2009; Magorrian 2017; Lukas 2019).

It makes sense that this should happen after you have—hopefully—learned from me how to handle ordinary differential equations (ODEs), which are differential equations that involve functions of a single independent variable. They have the general form

$$F(x, y, y', ..., y^{(n)}) = 0,$$

where $x$ is the independent variable, $y$ is the function (or “dependent variable”\(^1\)) and $y'$, $y''$, ..., $y^{(n)}$ are its first, second, ..., $n$th derivatives.

The order of an ODE is the order of the highest derivative that it contains.

There can be more than one function involved: $y_1(x)$, $y_2(x)$, ..., in which case there would have to be more than one equation relating them and their derivatives (see §3.3).

I will denote the independent variable $x$, but also often $t$, when I particularly want to think of my ODE as describing the evolution of some physical quantity with time (or

\(^{1}\)We shall see in what follows that what is an “independent” and what is a “dependent” variable can be negotiable as we manipulate ODEs. If $y$ is a function of $x$, $x$ is a function of $y$. See §1.4.
just when I feel like it). A derivative is the instantaneous rate of change of a quantity with time (or the local rate of its change in space, or etc.). It is denoted, by convention,

\[
\frac{dy}{dt} \equiv \dot{y} \quad \text{or} \quad \frac{dy}{dx} \equiv y'.
\] (1.2)

The tall-fraction notation, introduced by Leibniz (Fig. 1), underscores the derivative’s meaning as the change in \( y \) per change in \( t \) (or \( x \)) and also signals that derivatives can (with caution) be handled as fractions, e.g.,

\[
\frac{dx}{dy} = \frac{1}{y'}
\] (1.3)

is the derivative of \( x \) with respect to \( y \).

**Example.** Perhaps the most famous and most important ODE in the world is

\[
\dot{y} = ay,
\] (1.4)

where \( a \) is some constant. It describes a situation in which the rate of growth (when \( a \) is real and positive) or decay (when \( a < 0 \)) of some quantity is proportional to the quantity itself, e.g., the growth of a population whose rate of increase is proportional to the number of individuals in it (i.e., a monogamous population; what is the ODE for a polygamous population?). This equation is **first-order**. It is also **linear**.

Generally, an ODE is **linear** if it has the form

\[
a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x),
\] (1.5)

or, to use Einstein’s convention of implied summation over repeated indices, here meant to run from 0 to \( n \),

\[
a_i(x)y^{(i)} = f(x).
\] (1.6)

When \( f(x) = 0 \), the linear ODE is called **homogeneous** [of which (1.4) is an example], otherwise it is **inhomogeneous**. Homogeneous linear ODEs have the important property that they do not change under an arbitrary rescaling \( y \rightarrow \lambda y \ \forall \lambda \neq 0 \).\(^2\) We shall see (§3) that linear ODEs have many nice properties that make them much easier to solve than most nonlinear ones—and they also turn out to be useful in the analysis of the latter (§§3.5.3, 7).

\(^2\)Nonlinear ODEs can also have this property, in which case they are easier to solve: see Q2.15.
An example of nonlinear ODE is

\[ \dot{y} = ay^2, \quad (1.7) \]

which describes the evolution of a radically polygamous (or rather “omnigamous”) population, whose rate of increase is proportional to the number of all possible pairings.

Let me return to (1.4). You can probably guess quite easily what its solution is:

\[ y = Ce^{at}, \quad (1.8) \]

where \( C \) is an arbitrary constant.\(^3\) The emergence of \( C \) highlights an important property of ODEs: typically, they have an infinite number solutions, parametrised by arbitrary constants.\(^4\) We shall see that the number of constants should be equal to the order of the equation, but some more work is needed to make this obvious.

The solution (1.8) has been written in terms of the familiar exponential function. But what is that? In fact, it is possible to introduce \( e^x \) by definition as the function whose derivative is equal to the function itself.

Exercise 1.1. Ponder or investigate how this can be done.

Thus, in a sense, (1.8) is the solution of (1.4) by definition. This illustrates a general principle: so called “elementary functions” can be thought of as nothing but solutions of some oft-encountered and useful ODEs. When an ODE cannot be solved in elementary functions—or, more generally, in quadratures, which means in terms of integrals of combinations of elementary functions—and most ODEs cannot be so solved!—one has four options:

(i) Introduce a new function, defined as the solution of the previously unsolved ODE, and study it in some way: e.g., work out a power series for it (see Q2.17), plot it, tabulate it, make an app for computing it. These, usually in application to second-order linear ODEs, are called special functions (a good textbook on them is Lebedev 1972).

(ii) Find approximate, or asymptotic, solutions (a graduate-level textbook: White 2010; a cult-status, wild romp: Bender & Orszag 1999; a short classic introduction: Erdélyi 2003; see also Tikhonov et al. 1985, §7).


(iv) Study solutions qualitatively (§§3.5 and 7).

There is another way in which (1.4) is special: it is resolved with respect to derivative, i.e., it is of the general form

\[ \dot{y} = f(t,y) \quad (1.9) \]

This type of ODE allows for a vivid way of thinking of the multiplicity of an ODE’s solutions, which we shall now discuss.

\(^3\)It is obvious that there should be a multiplicative constant in the solution of any homogeneous ODE because of the possibility of an arbitrary rescaling of \( y \).

\(^4\)In Q1.8, you get to play with an “inverse” problem: given an infinite, parametrised set of functions, find an ODE whose solutions they are.
1.2. Integral Curves and the Cauchy Problem


Let us suppose that the function \( f(t, y) \) is specified in some domain \( \mathcal{D} \) within the \( \mathbb{R}^2 \) plane \((t, y)\). Through each point of this domain, let us draw a line whose slope is \( f(t, y) \), i.e., whose direction vector is \((1, f)\) (if you like, you can normalise it by \( \sqrt{1 + f^2} \)). This is called the direction field of the ODE (1.9). Then solutions of (1.9) are curves \( y(t) \) that are at each point in \( \mathcal{D} \) tangent to its direction field (Fig. 2a). They are called integral curves. Solving an ODE means finding all its integral curves.

For example, the integral curves of (1.4), parametrised by \( C \) in (1.8), are graphs of the exponential function in the \((t, y)\) plane that intersect the \( y \) axis at \( y = C \) (Fig. 2b). We can pick a single one of these curves, i.e., make the solution unique if we append to (1.4) an additional requirement that \( y(t = 0) = y_0 \) for some given \( y_0 \). Then the desired curve (1.8) is the one with \( C = y_0 \).

More generally, an initial condition (IC)—which, if we were speaking in terms of a spatial variable \( x \), rather than time \( t \), we would call a boundary condition (BC)—is the statement that

\[
y(t_0) = y_0 \quad \text{for some } (t_0, y_0) \in \mathcal{D}.
\]

Equivalently, it is the statement that the integral curve that we seek must pass through the point \((t_0, y_0)\).

The problem of finding the solution of (1.9) that satisfies the initial condition (1.10) is called the initial-value problem, or Cauchy problem (Fig. 3).

Does the Cauchy problem always have a solution? If we can find one, is it the only one or are there others? In other words, is there an integral curve that passes through every point \((t_0, y_0)\) of \( \mathcal{D} \) and can these curves ever intersect?
1.3. Existence and Uniqueness

[Literature: Pontryagin (1962, §§1, 2.4, 20), Tikhonov et al. (1985, §2.2), Arnold (2006, §§2.1-2.3), Tenenbaum & Pollard (1986, §58), Coddington (1990, Ch. 5)]

The answer to these questions is given by the following existence and uniqueness theorem.

**Theorem 1.** Let \( f(t, y) \) and \( \partial f/\partial y \) exist and be continuous functions on some open domain\(^5\) \( \mathcal{D} \subset \mathbb{R}^2 \). Then

(a) \( \forall (t_0, y_0) \in \mathcal{D}, \exists \Delta t \) such that the Cauchy problem

\[
\dot{y} = f(t, y), \quad y(t_0) = y_0
\]

has a solution in the interval \( I = [t_0 - \Delta t, t_0 + \Delta t] \).\(^6\)

(b) This solution is unique, i.e., if \( y_1(t) \) and \( y_2(t) \) are solutions of (1.11) on the intervals \( I_1 \) and \( I_2 \), respectively, then \( y_1(t) = y_2(t) \ \forall t \in I_1 \cap I_2 \) (they are the same in the intersection of the intervals where they are solutions).

Thus, we are guaranteed that we can solve the initial-value problem for at least some time after (and before) \( t_0 \) and that if we have found a solution, we can rest assured that it is the only one. We are in business!

**Example.** Let me illustrate how some of this can be broken and, therefore, why one should not be dismissive about formulating mathematical results precisely and watching for all the stated conditions to be satisfied. Consider the following Cauchy problem:

\[
\dot{y} = y^{2/3}, \quad y(0) = 0
\]

Clearly, \( y(t) = 0 \) is a solution. But \( y(t) = (t/3)^3 \) is also a solution, as can be verified by direct substitution. The two integral curves intersect at \( t = 0 \) (Fig. 4)! What has gone

\(^5\) \( \mathcal{D} \) being an open domain means that around \( \forall \) point \( \in \mathcal{D} \), there is a circle of some radius such that all points within the circle are also \( \in \mathcal{D} \). Thus, if you think of \( \mathcal{D} \) as some bounded area, it cannot include its own boundary. Obviously, the whole space \( \mathbb{R}^2 \) is very much an open domain.

\(^6\) Formally, this means that \( \exists \) a function \( y(t) \) such that \( y(t_0) = y_0 \) and \( \forall t \in I, (t, y(t)) \in \mathcal{D}, \dot{y}(t) \) is continuous and \( \dot{y}(t) = f(t, y(t)) \).
Figure 4. Non-uniqueness: two solutions of (1.12), shown as bold lines, intersect at the origin.

Wrong? This is easy to see:

\[
f(t, y) = y^{2/3} \Rightarrow \frac{\partial f}{\partial y} = \frac{2}{3} y^{-1/3} = \infty \text{ at } y = 0. \tag{1.13}
\]

Thus, \( \partial f/\partial y \) does not exist at \( y = 0 \), the conditions of Theorem 1 are violated, and so the Cauchy problem is under no obligation to have a unique solution.

1.4. Parametric Solutions

Before I move on, let me generalise slightly what I have just done. Casting aside the bigoted language of “dependent” (\( y \)) and “independent” (\( x \) or \( t \)) variables, let me treat everybody equally and write an ODE in the so-called symmetric form:

\[
P(x, y)dx + Q(x, y)dy = 0. \tag{1.14}
\]

We shall see (§2) that ODEs always end up in this form before getting integrated. Clearly, any ODE of the form (1.9) can be recast in the form (1.14) \((Q = -1, P = f)\), but not necessarily vice versa because \(Q(x, y)\) may well vanish in some places, preventing one from dividing by it and simply recasting (1.14) as

\[
\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} \equiv f(x, y). \tag{1.15}
\]

To be precise, what we shall assume is that \(P\) and \(Q\) are continuous functions in some domain \(D \subset \mathbb{R}^2\) and that \(|P(x, y)| + |Q(x, y)| > 0 \forall (x, y) \in D\), i.e., that they do not vanish simultaneously. This means that in some parts of \(D\), where \(Q \neq 0\), our equation (1.14) can be written as (1.15), an ODE resolved with respect to \(dy/dx\), and in others, where \(P \neq 0\), it can be resolved with respect to \(dx/dy\):

\[
\frac{dx}{dy} = -\frac{Q(x, y)}{P(x, y)}. \tag{1.16}
\]

The symmetric form (1.14) is a compact way of writing the two equations (1.15) and (1.16) together.

Solutions of (1.14) can in general be thought of as determined \textit{parametrically} in terms of an auxiliary variable \(t\): namely, \((x(t), y(t))\) is a solution of (1.14) if it is specified in some interval \(\mathcal{I} \subset \mathbb{R}\), and, \(\forall t \in \mathcal{I}\), \(x(t)\) and \(y(t)\) are continuously differentiable, \((x(t), y(t)) \in D\), and

\[
P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt} = 0. \tag{1.17}
\]
Figure 5. Integral curves of (1.14), an ODE in symmetric form. The direction field is \((-Q, P)\). The initial condition is \((x_0, y_0)\).

One may be able to write such a parametric solution as \(y \text{ vs. } x\) or as \(x \text{ vs. } y\) in some sub-intervals of \(\mathcal{D}\), provided \(\dot{x} \neq 0\) or \(\dot{y} \neq 0\) in these sub-intervals. For example, if \(\dot{x} \neq 0\) in some sub-interval, then \(x = x(t)\) can be inverted there to give \(t = t(x)\). Then \(y = y(t(x))\) is the \(y \text{ vs. } x\) solution.

Geometrically, we can think of \((\dot{x}, \dot{y})\) as components of a velocity field on \(\mathcal{D} \subset \mathbb{R}^2\), or, formally, \(\mathbb{R}^3\) if the third component is set to 0. Then (1.14) says

\[
\begin{bmatrix}
  -Q \\
  P \\
  0
\end{bmatrix}
\times
\begin{bmatrix}
  \dot{x} \\
  \dot{y} \\
  0
\end{bmatrix}
= 0,
\]

i.e., the velocity field \((\dot{x}, \dot{y})\) is everywhere parallel to the vector field \((-Q, P)\). This means that \((-Q, P)\) is the direction field of our equation and the integral curves in \(\mathcal{D}\) are everywhere tangent to this direction field (cf. §1.2, where \(Q = -1\) and \(P = f\)). The realm of possibilities here is slightly richer than for (1.9) because now the integral curves can be vertical (Fig. 5). This was not allowed for (1.9) lest we had \(y' = \infty\) in some places.

The Cauchy problem in this set up is the problem of finding an integral curve that passes through some given point \((x_0, y_0)\) \(\in \mathcal{D}\).

The existence and uniqueness result still holds because it is local—for any point \((x_0, y_0)\), we can always find a neighbourhood where either \(P \neq 0\) or \(Q \neq 0\), so (1.14) can be written as either (1.16) or (1.15) and we are back to Theorem 1.

**Example.** Let me illustrate two points: how an ODE in the form (1.9) is usefully recast as (1.14) and how the existence theorem, in general, only works locally, rather than on the entire \(\mathcal{D}\). Let us go back to (1.7), the equation of omnigamous procreation, and set up a Cauchy problem for it:

\[
\frac{dy}{dt} = ay^2, \quad y(t_0) = y_0.
\]

(1.19)

This can be rearranged to read

\[
\frac{dy}{y^2} = a \, dt,
\]

(1.20)

which has the symmetric form (1.14) with \(P = a\) and \(Q = -1/y^2\). Anticipating the integration techniques to be properly discussed in §2, we integrate (1.20):

\[
\int_{y_0}^{y} \, \frac{dy}{y^2} = a \int_{t_0}^{t} \, dt \quad \Rightarrow \quad y = \frac{a^{-1}}{t_c - t}, \quad t_c = t_0 + \frac{a^{-1}}{y_0}.
\]

(1.21)

This solution blows up at a finite time \(t_c\) and does not exist for \(t \geq t_c\) (Fig. 6), even though the
equation itself is perfectly well defined on the entire plane \((t, y)\). At whatever point \((t_0, y_0)\) on
the plane we set the initial condition, the solution will always exist in some neighbourhood of
this point, but not globally.

In physical terms, we have just encountered the interesting phenomenon of *explosion*—defined
as a quantity becoming infinite in finite time. In the context of omnipresent populations, this
suggests that some regulating effect, which is not in the simple model (1.19) will switch on
before the time \(t_c\) comes—perhaps Victorian morality or contraception or exhaustion of the
food supply... (cf. Q3.12).

It is clear that we are ready to start solving things. Thus, before I shower you with
more generalities, let me teach you some methods for actually solving ODEs.

2. Some Methods for Solving First-Order ODEs

“Solving” an ODE in a narrow, literal sense, is “integrating” it—and “integration”
means integration, literally.\(^7\) So here is the first, most basic method.

2.1. *Equations in Full Differentials*

Consider a first-order ODE in symmetric form (1.14). It is easy to integrate if the
left-hand side is a full differential of some function, viz., if \(\exists \Phi(x, y)\) such that
\[
\forall (x, y) \in \mathcal{D}, \quad P(x, y) = \frac{\partial \Phi}{\partial x} \quad \text{and} \quad Q(x, y) = \frac{\partial \Phi}{\partial y}.
\]

Then (1.14) is called an *equation in full differentials* and becomes
\[
d\Phi(x, y) = 0 \quad \Rightarrow \quad \Phi(x, y) = C,
\]
where \(C\) is an arbitrary *constant of integration*. The level sets of \(\Phi(x, y)\) are the integral
curves of (1.14). The equation has been “integrated”.

How do we check in general whether the relations (2.1) hold for any function \(\Phi\)?
The necessary condition for that is easy to obtain: if these relations do hold, then,

---

\(^7\)Keep in mind, however, that in learning how to “integrate” ODEs, we are dealing with a small
minority of them. Here is an example of a ludicrously simple equation that nevertheless cannot
be solved in quadratures: \(y' = y^2 + x\), a Riccati equation (cf. §2.6).
differentiating the first of them with respect to $y$ and the second with respect to $x$, we find that these derivatives must be equal:

$$ \forall (x, y) \in \mathcal{D}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}. \quad (2.3) $$

The fact that this is a necessary condition means that you can always use a failure of (2.3) to rule out the existence of $\Phi$. In the event, if $\mathcal{D}$ is simply connected (meaning that any two points within it can be connected by a continuous curve all of which lies in $\mathcal{D}$), one can prove that (2.3) is also the sufficient condition for (2.1).

**Example.** Q1.2(a).

**Example.** Consider

$$ 2xy \, dx + (x^2 - y^2) \, dy = 0. \quad (2.4) $$

It is instantly clear that (2.3) is satisfied, so we are in business. It must be the case that

$$ \frac{d\Phi}{dx} = 2xy \quad \Rightarrow \quad \Phi = x^2y + \phi(y), \quad (2.5) $$

where $\phi(y)$ is the integration “constant”, as far as $x$ dependence is concerned, but really a function of $y$. We find it from

$$ \frac{d\Phi}{dy} = x^2 - y^2 \quad \Rightarrow \quad \phi' = y^2 \quad \Rightarrow \quad \phi(y) = -\frac{y^3}{3} + C. \quad (2.6) $$

Thus, the solution is

$$ \Phi(x, y) = x^2y - \frac{y^3}{3} = C \quad (2.7) $$

(different $C$ here than in the previous line).

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### 2.1.1. Integrating Factor

[Literature: Tenenbaum & Pollard (1986, §10)]

Suppose now that (2.3) is not satisfied. Do we give up? Not necessarily. There is sometimes a miraculous cure called an integrating factor: it is a function $\Lambda(x, y) \neq 0$ such that

$$ \Lambda(x, y) [P(x, y) \, dx + Q(x, y) \, dy] = 0 \quad (2.8) $$

is an equation in full differentials, i.e.,

$$ \frac{\partial \Lambda P}{\partial y} = \frac{\partial \Lambda Q}{\partial x}, \quad \text{or} \quad Q \frac{\partial \Lambda}{\partial x} - P \frac{\partial \Lambda}{\partial y} = \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \Lambda. \quad (2.9) $$

Generally speaking, (2.9) is a PDE (first-order) and is not necessarily any easier to solve than the original ODE (1.14), but remember that you do not require generality here—all you need is a particular solution of (2.9). If you can deduce/intuit/guess it, you are done.\(^8\) Good luck!

In what follows, you will see an endless procession of examples of integrating factors being used (see, e.g., §2.4.1—perhaps the most famous—and certainly the most examinable!—such example).

**Example.** Q1.2(b).

\(^8\)Sometimes it is useful to look for a solution in the form $\Lambda = \Lambda(x)$ or $\Lambda = \Lambda(y)$.

**Example.** Q1.5(a).
**Example.** This one illustrates the merits of opportunism. Consider
\[(x^2 + y^2 + y) dx - x dy = 0. \quad (2.10)\]
Spot that \(y dx - x dy = -x^2 d(y/x)\), so the above is
\[(x^2 + y^2) dx - x^2 d(y/x) = 0. \quad (2.11)\]
The integrating factor is \(\Lambda = 1/(x^2 + y^2)\). Indeed, multiplying by it gives us an equation in full differentials:
\[
dx - \frac{d(y/x)}{1 + (y/x)^2} = 0 \quad \Rightarrow \quad x - \arctan(y/x) = C. \quad (2.12)\]
Q1.5(b) is a more sophisticated example of this kind of skulduggery.

---

2.2. Separable Equations

[A particularly simple and very useful special case of an ODE in full differentials is
\[
\left[ P(x)dx + Q(y)dy = 0 \right]. \quad (2.13)
\]
This is known as a separable ODE, referring to the possibility (or, indeed, the manifest reality) of \(x\) and \(y\) entering separately. It very clearly does satisfy (2.3) and can be integrated instantly:
\[
\int dx P(x) + \int dy Q(y) = C. \quad (2.14)
\]
If we are solving a Cauchy problem and thus are looking for a solution that passes through some point \((x_0, y_0)\), the answer obviously is
\[
\int_{x_0}^{x} dx P(x) + \int_{y_0}^{y} dy Q(y) = 0. \quad (2.15)
\]
Most cases of successful integration of ODEs culminate in the equation under scrutiny being reduced to (2.13) via various manipulations involving rearranging things and/or changing variables.

**Example.** Q1.2(c)

2.2.1. Separable Equations and Composite Solutions

Here is a simple example of rearranging things. Suppose our equation is
\[
\left[ P_1(x)Q_1(y)dx + P_2(x)Q_2(y)dy = 0 \right]. \quad (2.16)
\]
Let us divide it through by \(Q_1(y)P_2(x)\) (an example of integrating factor):
\[
\frac{P_1(x)}{P_2(x)} dx + \frac{Q_2(y)}{Q_1(y)} dy = 0. \quad (2.17)
\]
The variables are separated, the equation is begging to be integrated.
You must be careful, however, about the possibility that \(Q_1(y) = 0\) at some \(y = y_0\) or/and \(P_2(x) = 0\) at some \(x = x_0\), in which case the reduction of (2.16) to (2.17) is only allowed when \(y \neq y_0, x \neq x_0\). Besides the solutions of (2.17), there are then other
integral curves of (2.18): a family of cubics and the axis $y = 0$. Examples of composite solutions are shown as bold lines.

solutions of (2.16): clearly, if $P_2(x_0) = 0$, the line $x = x_0$ is an integral curve, and so is the line $y = y_0$ if $Q_1(y_0) = 0$. There can be a number of such lines, one for each of the zeros of $P_2$ or $Q_1$.

If the integral curves of (2.17) touch (i.e., intersect and have the same slope at the point of intersection as) the lines $x = x_0$ and $y = y_0$, one can construct composite solutions by gluing together at the points where they touch bits of those integral curves and the lines $x = x_0$ and $y = y_0$.

Example. Let us consider (1.12) again:

$$\frac{dy}{dx} = \frac{y^2}{3}.$$  \hfill (2.18)

First, $y = 0$ is a solution. Assuming $y \neq 0$, we can separate variables and integrate our equation:

$$\int \frac{dy}{y^{2/3}} = \int dx \quad \Rightarrow \quad y = \left(\frac{x}{3} + C\right)^3. \hfill (2.19)$$

These integral curves touch the line $y = 0$ at a multitude of points $x = -3C$ and have the same slope $y' = 0$ there. Any number of composite solutions can now be constructed by gluing together cubics and bits of the $x$ axis.

The moral of the story is: do not lose these extra solutions when manipulating ODEs (example: Q1.4).

Example. Consider

$$xy' + y = y^2 \quad \Rightarrow \quad x \frac{dy}{dx} = y(y - 1) \quad \Rightarrow \quad xdy = y(y - 1)dx. \hfill (2.20)$$

This equation is easily separable and, therefore, integrable:

$$\int \frac{dy}{y(y - 1)} = \int \frac{dx}{x} \quad \Rightarrow \quad \ln |x| + C = \int \frac{dy}{y - 1} - \int \frac{dy}{y} = \ln \left| \frac{y - 1}{y} \right| \Rightarrow \quad \frac{y - 1}{y} = Cx \quad \Rightarrow \quad y = \frac{1}{\frac{1}{1 + Cx}}. \hfill (2.21)$$

Note that, here and henceforth, I change the integration constant at each step in some opportune way, but keep calling it $C$.

Is (2.21) it, have I solved (2.20)? Not quite: to separate variables, I used the integrating factor $1/xy(y-1)$, so should worry about dividing by 0.

First consider $x = 0$. This is a solution of the last equation in in (2.20), but not of the
first. Indeed, the last step in (2.20), taking our equation from the form resolved with respect to derivative, (1.9), to the symmetric form (1.14), made our equation a little more general: the line \( x = 0 \) was not a solution of it in its original form but is a solution of the symmetric version (cf. the discussion at the end of §1.4).

Now consider \( y = 0 \) and \( y = 1 \). Both of these lines are solutions of our equation. One of them, \( y = 1 \), is, in fact, already covered by (2.21) with \( C = 0 \), whereas \( y = 0 \) is new. Thus, the full set of solutions is

\[
y = \frac{1}{1 + Cx}, \quad y = 0. \tag{2.22}
\]

### 2.2.2. Reduction to Separable Form by Linear Change of Variables

[Literature: Tikhonov et al. (1985, §2.1.1), Yeomans (2014, §II.1’), Binney (2002, §5.4)]

Another simple case of an ODE that can be made separable is

\[
\frac{dy}{dx} = f(ax + by), \tag{2.23}
\]

where \( a \) and \( b \) are constant coefficients. The change of (dependent) variable

\[
z(x) = ax + by(x) \tag{2.24}
\]

does the trick:

\[
\frac{dz}{dx} = a + b \frac{dy}{dx} = a + bf(z) \quad \Rightarrow \quad \int \frac{dz}{a + bf(z)} = \int dx = x + C. \tag{2.25}
\]

Here again keep in mind that \( \forall z_0 \) such that \( a + bf(z_0) = 0 \), \( z = z_0 \) is a solution and so, besides the integral curves (2.25), (2.23) has solutions (integral curves) that are the lines

\[
y(x) = \frac{a}{b} x + \frac{z_0}{b}. \tag{2.26}
\]

**Example.** Q1.2(d).

**Example.** Consider

\[
(x + 2y)y' = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + 2y} \equiv f(x + 2y). \tag{2.27}
\]

Clearly, we must assume \( x + 2y \neq 0 \) in order for the equation to make sense. Following (2.24), let \( z = x + 2y \), whence

\[
\frac{dz}{dx} = 1 + 2 \frac{dy}{dx} = 1 + \frac{2}{z}. \tag{2.28}
\]

This is a separable equation:

\[
\int \frac{dz}{1 + \frac{2}{z}} = \int dx \quad \Rightarrow \quad \not x + C = \int \frac{(z + 2 - 2)}{2 + z} dz = \int dz - 2 \int \frac{dz}{2 + z} = z - 2 \ln |2 + z| = \not x + 2y - 2 \ln |2 + x + 2y| \]

\[
\Rightarrow \quad 2 + x + 2y = C e^y. \tag{2.29}
\]

This may not be the only solution because we have to consider separately the case when an attempt to separate variables would have led to dividing by 0, viz.,

\[
1 + \frac{2}{z} = 0 \quad \Rightarrow \quad z = -2 \quad \Rightarrow \quad x + 2y = -2. \tag{2.30}
\]

This solution is, in fact, already captured by (2.29) with \( C = 0 \). The point of the last argument is to show that \( C = 0 \) is allowed in (2.29).
2.3. Homogeneous Equations


An ODE in the symmetric form (1.14) is called **homogeneous** if \( \forall \lambda \neq 0, \)

\[
P(\lambda x, \lambda y) = Q(\lambda x, \lambda y),
\]

i.e., \( P \) and \( Q \) scale in the same way when their arguments are scaled.\(^{10}\) Equivalently, (1.14) does not change under the rescaling \( x \rightarrow \lambda x, y \rightarrow \lambda y. \)

Since (2.31) works for \( \forall \lambda, \) it should work, e.g., for \( \lambda = 1/x. \) Assuming \( Q \neq 0 \) and \( x \neq 0, \) we have

\[
\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)} = -\frac{P(\lambda x, \lambda y)}{Q(\lambda x, \lambda y)} \bigg|_{\lambda = 1/x} = -\frac{P(1, y/x)}{Q(1, y/x)}.
\]

Thus, a homogeneous ODE can be reduced (with all the usual caveats about dividing by zero) to

\[
\frac{dy}{dx} = f \left( \frac{y}{x} \right),
\]

which is sometimes quoted as the definition of a homogeneous first-order equation.

This equation is solved by the ansatz

\[
y(x) = x z(x).
\]

This gives us

\[
\frac{dy}{dx} = z + x \frac{dz}{dx} = f(z) \quad \Rightarrow \quad x \frac{dz}{dx} = [f(z) - z] \frac{dx}{dz} \Rightarrow \int \frac{dz}{f(z) - z} = \int \frac{dx}{x} = \ln |x| + C,
\]

assuming \( f(z) - z \neq 0. \) If the equation \( f(z_0) = z_0 \) has any roots, these give rise to additional solutions, which are the lines

\[
y(x) = z_0 x.
\]

**Example.** Q1.2(e). A slight generalisation of this scheme is explored in Q1.7.

**Example.** Consider

\[
xy' = \sqrt{x^2 - y^2} + y.
\]

Notice that this equation stays the same under the rescaling \( x \rightarrow \lambda x, y \rightarrow \lambda y \) and so is homogeneous. It is rendered into the form (2.33) if it is divided through by \( x \) (assuming, of course, \( x \neq 0): \)

\[
\frac{dy}{dx} = \sqrt{1 - \frac{y^2}{x^2}} + \frac{y}{x} \equiv f \left( \frac{y}{x} \right).
\]

\(^9\)With respect to \( x \) and \( y \) simultaneously, so not quite the same as the homogeneity of linear equations introduced after (1.5), but you can, hopefully, see the connection.

\(^{10}\)This is sometimes written so: \( \forall \lambda \neq 0 \ \exists \alpha \) such that \( P(\lambda x, \lambda y) = \lambda^\alpha P(x, y) \) and \( Q(\lambda x, \lambda y) = \lambda^\alpha Q(x, y), \) with the same \( \alpha \) for \( P \) and for \( Q. \)
Making the ansatz (2.34), we get a separable equation:

\[
\frac{dy}{dx} = f + x \frac{dz}{dx} = \sqrt{1-z^2} + f \Rightarrow \int \frac{dz}{\sqrt{1-z^2}} = \int \frac{dx}{x} \\
\Rightarrow \arcsin z = \ln |x| + C \Rightarrow y = x \sin \ln |Cx|, \quad (2.39)
\]

assuming \(1-z^2 \neq 0\). As usual, follow mathematical hygiene by considering separately the case

\[
1-z^2 = 0 \Rightarrow z = \pm 1 \Rightarrow y = \pm x. \quad (2.40)
\]

These too are solutions, additional to (2.39), and we now have a complete set.

2.3.1. Reduction to Homogeneous Form by Linear Change of Variables

[Literature: Yeomans (2014, §§II.2′-2′′), Tenenbaum & Pollard (1986, §8)]

As usual, there are classes of ODEs that can be reduced to (2.33) via simple manipulations. Consider, for example,

\[
\frac{dy}{dx} = f \left( \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \right). \quad (2.41)
\]

We can reduce this to (2.33) if we can somehow get rid of the constants \(c_1\) and \(c_2\). Let us do it by changing variables:

\[
x = x_0 + \xi, \quad y = y_0 + \eta, \quad (2.42)
\]

where \(x_0\) and \(y_0\) are some constants. Our aim will be accomplished if these constants satisfy

\[
\begin{align*}
\{ & a_1 x_0 + b_1 y_0 + c_1 = 0, \\
& a_2 x_0 + b_2 y_0 + c_2 = 0.
\}
\end{align*} \quad (2.43)
\]

These are two lines on the \(\mathbb{R}^2\) plane of points \((x_0, y_0)\). If they intersect, i.e., if \(a_1 b_2 \neq a_2 b_1\), then there is a solution \((x_0, y_0)\) and (2.41) becomes

\[
\frac{dy}{dx} = \frac{d\eta}{d\xi} = f \left( \frac{a_1 \xi + b_1 \eta}{a_2 \xi + b_2 \eta} \right) = f \left( \frac{a_1 + b_1 \eta/\xi}{a_2 + b_2 \eta/\xi} \right) \equiv g \left( \frac{\eta}{\xi} \right), \quad (2.44)
\]

assuming \(\xi \neq 0\). We are back to (2.33).

**Example.** Q.1.2(f).

If \(a_1 b_2 = a_2 b_1\), the two lines (2.43) are parallel and there is no solution \((x_0, y_0)\) for which we can pull this trick. However, if the two lines are parallel, then \(\exists k\) such that \(\forall(x, y), a_2 x + b_2 y = k(a_1 x + b_1 y)\), so (2.41) becomes

\[
\frac{dy}{dx} = f \left( \frac{a_1 x + b_1 y + c_1}{k(a_1 x + b_1 y) + c_2} \right) \equiv g(a_1 x + b_1 y). \quad (2.45)
\]

This is (2.23), which we already know how to solve. Victory.

**Example.** Consider

\[
y' = \frac{y + 2}{2x + y - 4}. \quad (2.46)
\]

Chaging variables according to (2.42), we get

\[
\frac{dy}{dx} = \frac{d\eta}{d\xi} = \frac{y_0 + 2 + \eta}{2x_0 + y_0 - 4 + 2\xi + \eta} = \frac{\eta}{2\xi + \eta} = \frac{\eta/\xi}{2 + \eta/\xi} \equiv g \left( \frac{\eta}{\xi} \right), \quad (2.47)
\]
providing \( y_0 + 2 = 0 \) and \( 2x_0 + y_0 - 4 = 0 \), or \( x_0 = 3 \) and \( y_0 = -2 \). We now have a homogeneous equation, which is solved in the usual way. You can check that you now know how to do it: the full set of solutions is

\[
(y + 2)^2 = C(x + y - 1), \quad y = 1 - x. \tag{2.48}
\]

What if instead of (2.46), we are dealing with

\[
y' = \frac{2x + y + 2}{2x + y - 4}. \tag{2.49}
\]

To change it into a homogeneous equation via (2.42), we would require \( 2x_0 + y_0 + 2 = 0 \) and \( 2x_0 + y_0 - 4 = 0 \), which obviously cannot be satisfied simultaneously (the two lines are parallel). This is the case where (2.45) works: the equation is of the form

\[
y' = g(2x + y) \quad \text{and is solved using the method described in } \S \text{2.2.2. Check your answer:}
\]

\[
(2x + y - 4)^2 = Ce^{x-y}. \tag{2.50}
\]

2.4. Linear Equations


Linear equations (1.5) have special status because, in the vicinity of any given point, any equation can be expanded and turned into a linear one (“linearised”; see §3.5.3). I will spend quite a lot of time on their general properties in subsequent chapters (§§4–6), but let us have a preview here by considering the Cauchy problem for the simplest, first-order linear ODE:

\[
dy - [a(t)y + f(t)]dt = 0, \quad y(0) = y_0. \tag{2.51}
\]

I am setting the initial condition at \( t_0 = 0 \) without loss of generality as I can always declare the time at which the initial condition is set to be zero by definition.

The Cauchy problem (2.51) can be solved completely in two equivalent ways, which are both quite instructive, so I will go through both.

2.4.1. Solution via Integrating Factor

[Literature: Yeomans (2014, §II.3), Binney (2002, §2.3)]

Let us recast (2.51) in the symmetric form (1.14),

\[
dy - [a(t)y + f(t)]dt = 0, \tag{2.52}
\]

and look for an integrating factor. In general, it must satisfy (2.9), but we shall look for (and easily find) one that depends just on \( t \): \( \Lambda = \Lambda(t) \):

\[
\dot{\Lambda} = -a(t)\Lambda \quad \Rightarrow \quad \int \frac{d\Lambda}{\Lambda} = -\int dt a(t) + C \quad \Rightarrow \quad \Lambda(t) = \exp \left[ -\int_0^t dt' a(t') \right]. \tag{2.53}
\]

The integration is from zero for future convenience—since I only need a solution, I can integrate from wherever I like. Now multiplying (2.51) by \( \Lambda(t) \), we see that

\[
\frac{d}{dt} \Lambda y = \frac{d}{dt} \Lambda y - \frac{d\Lambda}{dt} \Lambda y = \frac{d}{dt} \Lambda y + a(t)\Lambda y \tag{2.54}
\]

and so our general linear equation (2.4) reduces to something that we can integrate directly:

\[
\frac{d}{dt} \Lambda(t)y = \Lambda(t)f(t) \quad \Rightarrow \quad y(t) = \frac{1}{\Lambda(t)} \left[ C + \int_0^t dt' \Lambda(t')f(t') \right]. \tag{2.55}
\]
Figure 8. Joseph-Louis Lagrange (1736-1813), born Giuseppe Lodovico Lagrangia, great Italian-French mathematician, astronomer, Euler’s (Fig. 22) protegé, Fourier’s teacher, senator and count of the Napoleonic Empire.

Since \( y(0) = C \), the solution of the Cauchy problem (2.51) is achieved by setting \( C = y_0 \). So, finally,

\[
y(t) = y_0 e^\int_0^t dt' a(t') + \int_0^t dt' e^\int_0^{t'} dt'' a(t'') f(t') .
\] (2.56)

2.4.2. Solution via Variation of Constant

[Literature: Pontryagin (1962, §2B)]

Here is another way to obtain this result. Consider first the homogeneous version of (2.51):

\[
\dot{y} = a(t)y .
\] (2.57)

It is separable:

\[
\int \frac{dy}{y} = \int dt a(t) \quad \Rightarrow \quad y(t) = C \exp \left[ \int_0^t dt' a(t') \right] = C \mathcal{E}(t) ,
\] (2.58)

where \( C \) is the integration constant and I have introduced the notation \( \mathcal{E}(t) \) for the exponential function to make further algebra look more compact\(^{11}\) (also, I rather like gothic letters). Once we know this solution of the associated homogeneous equation, there is a nifty trick called the method of variation of constants (invented, apparently, by Lagrange [Fig. 8]). It consists in turning \( C \) into a function \( \psi(t) \) and looking for the solution of the inhomogeneous equation (2.51) in the form

\[
y(t) = \psi(t) \mathcal{E}(t) .
\] (2.59)

Substituting this into (2.51), we get

\[
\dot{\psi} \mathcal{E}(t) + \psi(t) \dot{\mathcal{E}}(t) = a(t) \psi(t) \mathcal{E}(t) + f(t) \quad \Rightarrow \quad \dot{\psi} = \frac{f(t)}{\mathcal{E}(t)} .
\] (2.60)

Integrating this and then substituting into (2.59) gives us

\[
y(t) = \mathcal{E}(t) \left[ C + \int_0^t dt' \frac{f(t')}{\mathcal{E}(t')} \right] .
\] (2.61)

The initial condition is enforced by \( C = y_0 \) and we again have our solution (2.56).

\(^{11}\)Obviously, \( \mathcal{E}(t) = 1/\Lambda(t) \) from (2.53).
2.4.3. $CF + PI$

When we study the linear equations of arbitrary order in §4.4, this will be a rather obvious particular case. Still, let me flag a key insight here. The solution (2.61) consists of the solution of the associated homogeneous equation (2.57) plus a certain particular solution of the inhomogeneous equation (2.51). In fact, I claim that if I somehow found (deduced, dreamed, intuited, guessed, stole) a particular solution of (2.51)—let us call it $y_{PI}(t)$, where “PT” = “particular integral”—then the general solution of (2.51) is

$$y(t) = y_{CF}(t) + y_{PI}(t),$$

(2.62)

where $y_{CF}(t)$, known as the “complementary function” (hence “CF”), is the general solution of the associated homogeneous equation (2.57). That (2.62) is the general solution follows from the fact that (2.62) is clearly a solution, but $y_{CF}(t)$ already has a constant in it, which can be used to accommodate any initial condition—and so the resulting solution is the solution because the solution of the Cauchy problem is unique.

Note, by the way, that we have thus explicitly shown that, for a linear equation, the existence and uniqueness is global, rather than local—indeed, our solution is valid at all $t$, not just in the vicinity of $t = 0$. This property, as well as the prescription (2.62), will turn out to carry over to linear ODEs and systems of ODEs of any order (see §4).

**Example.** Q1.2(g).

**Example.** Consider

$$xy' + (x + 1)y = 3x^2 e^{-x}.$$  

(2.63)

The associated homogeneous equation is (assuming $x \neq 0$)

$$\frac{dy}{dx} = -\frac{x + 1}{x} y \Rightarrow \int \frac{dy}{y} = -\int \left(1 + \frac{1}{x}\right) \Rightarrow y = C e^{-x}.$$  

(2.64)

To find the solution of the inhomogeneous equation (2.63), weaponise the constant: $C \to \psi(x)$. Then

$$y' = \psi' \frac{e^{-x}}{x} - \psi \frac{e^{-x}}{x} = -\frac{x + 1}{x} y + 3x e^{-x} \Rightarrow \psi' = 3x^2 \Rightarrow \psi = x^3 + C.$$  

(2.65)

Therefore, the solution of (2.63) is

$$y = (x^3 + C) \frac{e^{-x}}{x}.$$  

(2.66)

2.5. Bernoulli Equations


A few further tricks exist. Consider Bernoulli’s equation (Fig. 9a)

$$y' + a(x)y = b(x)y^n,$$  

(2.67)

where $n \neq 0,1$, so the equation is nonlinear. If $n > 1$, $y(x) = 0$ is clearly a solution. Putting it to one side, divide (2.67) by $y^n$:

$$b(x) - a(x)y^{1-n} = \frac{y'}{y^n} = \frac{d}{dx} \frac{y^{1-n}}{1-n}.$$  

(2.68)
Letting $z = y^{1-n}$, we turn this into

$$z' + (1 - n)a(x)z = (1 - n)b(x).$$

(2.69)

This is a linear equation: follow standard operating procedure (§2.4).

**Example.** Q1.2(h).

### 2.6. Riccati Equations

[Literature: Bender & Orszag (1999, §1.6)]

Now consider *Riccati’s equation* (Fig. 9b)

$$y' = a(x)y^2 + b(x)y + c(x).$$

(2.70)

Equations of this form are generally impossible to solve in quadratures. There is, however, one loophole. If you can guess some particular solution of (2.70), let us call it $y_0(x)$, then the ansatz

$$y(x) = z(x) + y_0(x)$$

(2.71)

turns your Riccati equation into a Bernoulli equation (2.67) for $z(x)$. Watch:

$$y' = z' + y_0' = az^2 + 2azy_0 + bzy_0 + c + b'y_0 + c' \Rightarrow z' = [2a(x)y_0(x) + b(x)]z + a(x)z^2.$$  

(2.72)

You have an opportunity to try your hand at this method in Q1.9 (example). In §4.6, it will turn into something very useful indeed.

### 2.7. Equations Unresolved With Respect to Derivative

[Literature: Arnold (2006, §8.5)]

Let us go back to the most general form of ODE (1.1) and consider its first-order version:

$$F(x, y, y') = 0.$$  

(2.73)

What can you do about this equation? Well, obviously, if you can resolve (2.73) with respect to $y'$, this gets you back to familiar territory (1.9):

$$y' = f(x, y).$$

(2.74)

Note, however, that (2.73) can have more than one solution of the form (2.74)—e.g., if (2.73) is quadratic in $y'$—so solving (2.73) may amount to solving several ODEs of the form (2.74).
2.7.1. Cauchy Problem

What, you might wonder, about existence, uniqueness and all that? Do integral curves of (2.73) intersect? Yes, they generally do intersect—basically because (2.73) generally corresponds to more than one equation of the form (2.74). A generalisation of the Cauchy problem for this case is to specify

\[ y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = p_0. \]  

(2.75)

Obviously, not every set \((x_0, y_0, p_0)\) is legitimate as these numbers must satisfy \(F(x_0, y_0, p_0) = 0.\) Since, for any given \((x_0, y_0)\), this equation generally has more than one root \(p_0\), the point of (2.75) is to specify which root we are interested in. In other words, we ask for an integral curve of (2.73) that passes through \((x_0, y_0)\) and has a slope \(p_0\) there. Existence and uniqueness of the solution of (2.73) with initial condition (2.75) turn out to be guaranteed, analogously to Theorem 1, provided \(F, \partial F/\partial y, \partial F/\partial y'\) are continuous in some neighbourhood of \((x_0, y_0, p_0)\) and

\[ \frac{\partial F}{\partial y'}(x_0, y_0, p_0) \neq 0. \]  

(2.76)

Example. Consider

\[ xy'(xy' + y) = 2y^2. \]  

(2.77)

This is quadratic in \(xy'\). Solving it gives us two equations:

\[ xy' = y \quad \text{or} \quad xy' = -2y. \]  

(2.78)

Their solutions are

\[ y = Cx \quad \text{and} \quad y = \frac{C}{x^2}, \]  

(2.79)

respectively. If we ask for a solution with \(y(1) = 1\), we get \(C = 1\) in both cases, so both solutions pass through the point \((1, 1)\). To decide which of them to pick, we must specify \(y'(1) = 1\) or \(y'(1) = -2\).

Note that places where the condition (2.76) is broken are interesting. A curve in the \((x, y)\) plane that satisfies

\[ F(x, y, p) = 0, \quad \frac{\partial F}{\partial p}(x, y, p) = 0 \]  

(2.80)

is called the \(p\)-discriminant curve of (2.73). If this curve is also a solution (an integral curve) of this equation, then it is a rather special solution, in every point of which uniqueness is violated. This means that at its every point, this solution will be tangent to another integral curve of the equation. You will find a cute example of this in Q1.10 and others in Q1.11.

2.7.2. Solution via Introduction of Parameter

How do we solve equations like (2.73) if we cannot resolve them with respect to \(y'\)—or get something horrible/intractable if we do? Some useful advice on this is as follows.

First suppose that you can resolve (2.73) with respect to \(y\), rather than \(y'\):

\[ y = f(x, y'). \]  

(2.81)

Then let \(p = y'\) and rewrite (2.81) as

\[ dy = p\,dx \quad \text{and} \quad y = f(x, p). \]  

(2.82)

This implies

\[ p\,dx = dy = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial p} \, dp \Rightarrow \left( \frac{\partial f}{\partial x} - p \right) \, dx + \frac{\partial f}{\partial p} \, dp = 0. \]  

(2.83)
The resulting ODE has the symmetric form (1.14). Suppose we can integrate it and get the solution in the form
\[ p = p(x, C), \]
where \( C \) is an integration constant. Then the solution of (2.81) is
\[ y = f(x, p(x, C)). \]
This scheme is called the *method of introduction of parameter*.

**Example.** A useful and pretty class of ODEs that are solvable by this method is introduced in Q1.10; see also Q1.11.

**Example.** Consider
\[ y = xy' - (y')^2 = xp - p^2, \]  
where, as per the algorithm laid out above, a parameter was introduced: \( p = y' \). Then
\[ p dx = dy = x dp + p dx - 2p dp \Rightarrow (x - 2p) dp = 0. \]
There are two ways in which this can be satisfied:
\[ p = \frac{x}{2} \Rightarrow y = \frac{x^2}{4}, \]
\[ dp = 0 \Rightarrow p = C \Rightarrow y = C(x - C). \]
In both cases, to get to the answer, the solution for \( p \) was substituted into (2.86).

**Exercise 2.1.** Work out how to solve (2.73) if it can be resolved with respect to \( x \), i.e., how to solve an ODE that has the form
\[ x = f(y, y'). \]  

**Example.** Q1.11(iii).

A more general/abstract version of the above is as follows. Consider the general first-order ODE (2.73). The equation
\[ F(x, y, p) = 0 \]
defines a 2D surface in the 3D space \( \mathbb{R}^3 \) of points \( (x, y, p) \). Suppose the function \( F \) is such that this surface can be smoothly parametrised, i.e., it is described by some continuously differentiable functions
\[ x = \xi(u, v), \quad y = \eta(u, v), \quad p = \zeta(u, v), \]  
where \( (u, v) \) are auxiliary variables parametrising the surface (finding those and working out in what ranges of their values they describe which bits of the surface is the key nontrivial task here).\(^{12}\) Then
\[ dy = p dx \Leftrightarrow \frac{\partial \eta}{\partial u} du + \frac{\partial \eta}{\partial v} dv = \zeta \left( \frac{\partial \xi}{\partial y} dy + \frac{\partial \xi}{\partial v} dv \right), \]
leading to an ODE in the symmetric form (1.14):
\[ \left( \frac{\partial \eta}{\partial u} - \zeta \frac{\partial \xi}{\partial u} \right) du + \left( \frac{\partial \eta}{\partial v} - \zeta \frac{\partial \xi}{\partial v} \right) dv = 0. \]
Suppose we have found its solution
\[ v = v(u, C). \]

\(^{12}\)In order for this approach to work, at every point (and its neighbourhood) at least one of the mappings \( (u, v) \rightarrow (x, y) \), \( (u, v) \rightarrow (y, p) \) or \( (u, v) \rightarrow (x, p) \) must be one-to-one. Formally, this is expressed as the requirement that the Jacobians of these mappings do not all vanish simultaneously: \( [\partial(x, y)/\partial(u, v)]^2 + [\partial(y, p)/\partial(u, v)]^2 + [\partial(x, p)/\partial(u, v)]^2 > 0 \).
Then the original ODE (2.73) has the parametric solution
\[ x(u) = \xi(u, v(u, C)), \quad y(u) = \eta(u, v(u, C)). \] (2.96)
Obviously, the devil is in the detail—you have an opportunity to practice this detail in Q1.12 (example).

### 3. The Language of the Game: Going to Higher Order

Hopefully you now feel sufficiently empowered actually to solve things that you might tolerate a resumption of conceptual/general discussion. It is time for us to discuss how to handle ODEs of order higher than first. Let me start with an example of a second-order ODE, which is perhaps the second most important [after (1.4)] ODE in the world—and possibly the most important one in physics.

#### 3.1. Mathematical Pendulum

Consider a pendulum of length \(l\), with a point mass \(m\) at the end, subject to acceleration of gravity \(g\) (Fig. 10). The mass executes a trajectory that is parametrised by the angle \(\theta\) and follows the arc of the circle of radius \(l\). The velocity, always tangent to this circle, is \(l\dot{\theta}\) and the acceleration is \(l\ddot{\theta}\). Newton’s (Fig. 11) Second Law reads
\[ m\ddot{\theta} = -mg \sin \theta \quad \Rightarrow \quad \ddot{\theta} = -\omega_0^2 \sin \theta, \quad \omega_0 = \sqrt{\frac{g}{l}}. \] (3.1)
This nonlinear, second-order ODE is the equation of mathematical pendulum (“mathematical” because the rod is massless, the mass at the end of it is a point mass, and there is no friction of any kind). It describes the pendulum’s oscillations around the equilibrium \(\theta = 0\). If we further assume that these oscillations are small, \(\theta \ll 1\), then we can approximate \(\sin \theta \approx \theta\) and turn our ODE into a linear one:
\[ \ddot{\theta} = -\omega_0^2 \theta. \] (3.2)
This is the famous equation of a simple harmonic oscillator (SHO), which describes not just a gently perturbed pendulum but also countless other physical systems (or bits of physical systems) ranging from a mass attached to a spring to an elementary excitation in an (imaginary) box filled with radiation. Note that the linear ODE emerged as a result of the nonlinear ODE being expanded in the vicinity of its equilibrium point (the steady
The solution of \((3.1)\) is (verified by substitution)\(^{13}\)
\[
\theta(t) = A \cos(\omega_0 t + \phi),
\]  
where \(A\) and \(\phi\) are arbitrary constants (now there are two of them, as there must be, for a second-order ODE—we shall shortly see why). We can fix these constants by setting two initial conditions:
\[
\theta(0) = \theta_0 \quad \text{and} \quad \dot{\theta}(0) = \Omega_0.
\]  
the intial angular displacement and the initial angular velocity, respectively.

3.2. Laws of Motion

The equation of the pendulum is, of course, a particular case of the much more general differential equation that expresses Newton’s Second Law and, therefore, describes everything in the (classical) world:
\[
\ddot{r} = f(t, r, \dot{r}).
\]  
This is why ODEs are so dominant in physics: the laws of motion are ODEs. Since they are second-order ODEs, we better learn how to handle those.

Perhaps you already know from mechanics that, in order to solve \((3.5)\) and thus predict the future of the world, you need to know the initial position \(r(0) = r_0\) and the initial velocity \(\dot{r}(0) = v_0\). It would appear that there must be some way of formulating a Cauchy-style problem for second- (and, presumably, higher-) order ODEs and one must also hope that an existence and uniqueness theorem is available. This better be true, or what will become of the determinism of the natural world and of our ability to predict its fate?

3.3. All ODEs Are (Systems of) First-Order ODEs


First of all, let me point out that all ODEs are, in fact, first-order ODEs, or systems

\(^{13}\)Cosine is a familiar elementary function, but again, in fact, you might think of it as, by definition, the function that is the solution of \((3.2)\), rendered familiar and, therefore, “elementary”, by heavy use. You get to “derive” the cosine from a position of pretend ignorance in Q2.17(c).
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thereof. Indeed, (3.5) is nothing but

\[
\begin{align*}
\dot{r} &= v, \\
\dot{v} &= f(t, r, v).
\end{align*}
\] (3.6)

Thus, instead of having 1 or 3 or \( n \) second-order equations for the \( n \)-dimensional vector \( r \), we have 2 or 6 or \( 2n \) first-order equations for the \( 2n \)-dimensional vector \((r, v)\) (a second order ODE that lives in \( \mathbb{R}^{n+1} \) is a first-order ODE that lives in \( \mathbb{R}^{2n+1} \), the +1 being time). For example, (3.2) is

\[
\begin{align*}
\dot{\theta} &= \Omega, \\
\dot{\Omega} &= -\omega_0^2 \theta \\
\Leftrightarrow \quad \frac{d}{dt} \begin{pmatrix} \theta \\ \Omega \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} \theta \\ \Omega \end{pmatrix},
\end{align*}
\] (3.7)

an example of a system of coupled, linear, homogeneous, first-order ODEs. More generally,

\[
F(t, y, \dot{y}, \ddot{y}, \ldots, y^{(n)}) = 0
\] (3.8)
can be recast as this system of \( n \) first-order ODEs:

\[
\begin{align*}
F(t, y, p_1, \ldots, p_{n-1}, \dot{p}_{n-1}) &= 0, \\
\dot{y} &= p_1, \\
\dot{p}_1 &= p_2, \\
&\vdots \\
\dot{p}_{n-2} &= p_{n-1}.
\end{align*}
\] (3.9)

Thus, we can talk of \( n \) as the order of the original ODE or as the order of the corresponding system of first-order ODEs (\( n = \) the number of these ODEs).

If we restrict our attention to ODEs that are resolved with respect to the highest derivative,

\[
y^{(n)} = f(t, y, \dot{y}, \ldots, y^{(n-1)}),
\] (3.10)

we see that both (3.10) and many other systems of ODEs that do not necessarily reduce to a single higher-order equation can be written in the general form

\[
y = f(t, y),
\] (3.11)

where \( y \) is an \( n \)-dimensional vector function mapping some subset of \( \mathbb{R} \) to some subset of \( \mathbb{R}^n \) and \( f \) is an \( n \)-dimensional “velocity field”—it is called the phase velocity, while \( y \) is the “phase” of the physical system described by (3.11) (as in “phases of the moon”, i.e., states of the system). Note that since \( y \) includes physical velocities, as in (3.6), \( f \) includes the physical forces providing acceleration.

Naturally, the Cauchy (initial-value) problem for (3.11) is the problem of finding the solution that satisfies

\[
y(t_0) = y_0.
\] (3.12)

3.4. Existence and Uniqueness

[Literature: Pontryagin (1962, §§3, 21), Tikhonov et al. (1985, §§2.4, 2.6-2.7), Tenenbaum & Pollard (1986, §62), Coddington (1990, Ch. 6)]

It turns out, thank God, that the 1D existence and uniqueness Theorem 1 can be generalised to the \( n \)-dimensional case.\(^{14}\)

\(^{14}\)Note that this theorem officially confirms that the general solution of an \( n \)-th-order ODE must have \( n \) integration constants because \( n \) initial conditions are needed to pin down the solution uniquely [cf. the argument around (4.31)].
Figure 12. (a) Rudolf Otto Sigismund Lipschitz (1832-1903), German mathematician, student of Dirichlet. (b) No, this is not Hercule Poirot, this is Charles Émile Picard FRS (1856-1941), a brilliant French mathematician who proved the existence and uniqueness theorem with the Lipschitz continuity condition (3.13). He had a French academic career par excellence (ENS, chair at Sorbonne, Académie des sciences, seat 1 in Académie française, etc.), was a great lecturer (according to Hadamard), and married Hermite’s daughter Marie, with whom he had three children, all killed in World War I.

Theorem 2. Let all $f_i(t, y)$ and $\partial f_i/\partial y_j$ ($i, j = 1, \ldots, n$) exist and be continuous on some open domain $\mathcal{D} \subset \mathbb{R}^{n+1}$. Then

(a) $\forall (t_0, y_0) \in \mathcal{D}$, $\exists \Delta t$ such that the Cauchy problem (3.11–3.12) has a solution in the interval $\mathcal{I} = [t_0 - \Delta t, t_0 + \Delta t]$.

(b) This solution is unique, i.e., if $y_1(t)$ and $y_2(t)$ are solutions of (3.11–3.12) on the intervals $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, then $y_1(t) = y_2(t) \forall t \in \mathcal{I}_1 \cap \mathcal{I}_2$.

In fact, it turns out that this theorem can be made somewhat more general (meaning that its assumptions can be weakened): $f$ need not have all these derivatives but must only satisfy the so-called Lipschitz continuity condition (Fig. 12): $\exists L \in \mathbb{R}$ such that

$\forall (t, y_1) \text{ and } (t, y_2) \in \mathcal{D},$

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

(3.13)

(any $f$ that has all the first derivatives is Lipschitz-continuous, but not vice versa).

I shall not go any further into these matters, but this does not mean that you should not: in fact, I urge those of you who wish to be mathematically competent to read about the existence and uniqueness theorems and, indeed, work your way through their proofs (Pontryagin 1962 is a good undergraduate textbook on this subject; Arnold 2006 takes you to the next level of enlightenment). There are things that every physicist ought to have done at least once (although possibly not more than once) in her life—and this is one of them.

Thus, to sum up, ODEs in the form (3.11) are equations that express the rate of change of a vector quantity $y(t)$ (the “phase” of the system) as a function of $t$ and $y$ itself—and they have unique solutions if the “phase” $y_0$ where the system finds itself at some time $t_0$ is specified.

You can hopefully now see why this is an important subject: evolution of systems with given initial conditions is what we study in physics.

The rest of this chapter is an introduction to the ideas that we shall explore further
in §7. If you are impatient to learn how to solve higher-order linear equations, jump to §4 now and come back later.

3.5. Phase Space and Phase Portrait

Integral curves live in an \((n+1)\)-dimensional space, so, for (systems of) ODEs of order \(n > 1\), drawing these curves becomes challenging, pending the invention of a 3D iPad. It also turns out that, unless the equations are linear and have constant coefficients, finding their solutions in elementary functions or even quadratures is usually impossible.

Example. Consider the linear, homogeneous, second-order equation, which looks like a rather simple ODE:

\[
y'' + a(x)y' + b(x)y = 0.
\]

(3.14)

Let \(z = y'/y\). Then \(y' = zy, y'' = zy' + z'y,\) and, for \(y \neq 0\), the equation becomes

\[
z' + z^2 + a(x)z + b(x) = 0.
\]

(3.15)

This is a Riccati equation (2.70) and so, unless we can guess one of its solutions, it is unsolvable in quadratures. The case in which (3.14) is solvable will be discussed in §4.6. It is basically the same story as in §2.6.

These days, one’s reflex in such situations is to stick the recalcitrant equation into a computer and ask for solutions, but it tends to be the case that, if you do not understand what you are doing—and, specifically, if you do not have a good idea as to what to expect—you will get rubbish and be none the wiser. Therefore, what one needs to study is various methods of solving ODEs approximately and/or of analysing their behaviour qualitatively, i.e., methods for establishing certain key properties of their solutions without having to find these solutions precisely and/or in their entirety.

The objective of such an analysis is to understand how a physical system described by (3.11) might travel around its phase space, i.e., the space that contains all values of \(y\) accessible to the system (it is usually a subspace of \(\mathbb{R}^n\)). What parts of the phase space the system visits how often and where it ends up long-term generally must depend on where it started from (\(y_0\) at \(t = t_0\)), but in fact, often does not—or does in such a way that various initial conditions can be classified into broad categories with qualitatively similar evolution.

Let us specialise from (3.11) to a class of ODEs called autonomous. These are the systems whose phase velocity depends only on their position in phase space:\[15\]

\[
\dot{y} = f(y).
\]

(3.16)

Physically, these are systems in which forces depend only on positions and velocities (or analogous phase-space variables), but not explicitly on time—this can be interpreted as absence of external meddling with the system (hence “autonomous”). We may then set ourselves the task of sketching all possible trajectories of the system in phase space. Such a sketch is called the phase portrait of the system. Note that this is not the same thing as sketching the integral curves: the latter are curves in \(\mathbb{R}^{n+1}\) (including time) whereas the phase trajectories are curves in \(\mathbb{R}^n\) parametrised by \(t\) (they are projections of the integral curves onto the phase space).

In view of (3.16), these trajectories are everywhere tangent to the system’s phase velocity \(f\). Obviously, the actual sketching is only a (relatively) easy task when \(n = 2\). Note that the task is not thankless for an autonomous system because the phase portrait is independent of time for it, and so only needs to be produced once.

\[15\]In purely formalistic terms, all systems of ODEs can be reduced to autonomous systems. Indeed, consider a generic non-autonomous system (3.11) with \(y \in \mathbb{R}^n\). Define \(\eta \in \mathbb{R}^{n+1}\) to have the components \(\eta_i = y_i\) for \(i = 1, \ldots, n\) and \(\eta_{n+1} = t\). Then (3.11) can be written as \(\dot{\eta}_i = f_i(y, \eta_{n+1}) \equiv g_i(\eta)\) for \(i = 1, \ldots, n\) and \(\dot{\eta}_{n+1} = 1 \equiv g_{n+1}(\eta)\), or \(\dot{\eta} = g(\eta)\), an autonomous system in \(\mathbb{R}^{n+1}\). In practice, this is not terribly helpful, because increasing the number of dimensions, even from 2 to 3, makes things vastly more complicated.
Figure 13. Oscillations of a pendulum around the $\theta = 0$ equilibrium (a): (b) the direction field, (c) the phase portrait.

3.5.1. Linear Pendulum

Let me illustrate what I mean by all this and why such an approach is useful on the example of the SHO (3.7). For simplicity of exposition, I will rescale time and angular velocity to get rid of $\omega_0$ and rename variables: $\omega_0 t \rightarrow t$, $y = \Omega/\omega_0$, $x = \theta$. Then the system that we are dealing with is just

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x. \end{cases}$$

(3.17)

The right-hand side of this equation, $(y, -x)$, defines a velocity field in phase space $\mathbb{R}^2$. It is sketched in Fig. 13(b). The phase trajectories are tangent to this field. These trajectories (Fig. 13c) are circles—this is obvious if one observes that the velocity $(y, -x)$ is always perpendicular to the radius vector $(x, y)$ or, indeed, simply recalls the solution (3.3):

$$x = A \cos(t + \phi), \quad y = \dot{x} = -A \sin(t + \phi) \quad \Rightarrow \quad x^2 + y^2 = A^2.$$  

(3.18)

Thus, the phase portrait of the SHO is a family of concentric circles parametrised by the integration constant $A$ (Fig. 13c). If you pick a point on the $(x, y)$ plane, you fix $A$ and $\phi$, i.e., which circle you are on and where on it. The phase portrait now tells you where you would go from there: around the circle, clockwise.

There is another way to see this, which is very illuminating physically: multiply the first equation in (3.17) by $x$, the second by $y$, and add them:

$$\frac{d}{dt} \left( \frac{x^2}{2} + \frac{y^2}{2} \right) = 0 \quad \Rightarrow \quad E(x, y) = \frac{x^2}{2} + \frac{y^2}{2} = C.$$  

(3.19)

This integral is the energy (potential + kinetic), which, unsurprisingly, is conserved. The solutions lie on circles because they are constrained by the conservation of $E$.

3.5.2. Nonlinear Pendulum

Let me illustrate what I mean by all this and why such an approach is useful on the example of the SHO (3.7). For simplicity of exposition, I will rescale time and angular velocity to get rid of $\omega_0$ and rename variables: $\omega_0 t \rightarrow t$, $y = \Omega/\omega_0$, $x = \theta$. Then the system that we are dealing with is just

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x. \end{cases}$$

(3.20)

The easiest way to draw the phase portrait is by the same method (which, however, does not necessarily work for all systems) as we used to obtain (3.19). Indeed, the system (3.20) also has a conserved quantity and so can be integrated: multiply the first equation by $\sin x$, the second
Figure 14. Oscillations of a pendulum around the $\theta = \pi$ equilibrium (a): (b) the direction field, (c) the phase portrait.

by $y$ and add them, to get

$$\frac{d}{dt} \left( -\cos x + \frac{y^2}{2} \right) = 0 \quad \Rightarrow \quad \mathcal{E}(x, y) = 1 - \cos x + \frac{y^2}{2} = C.$$  \hspace{1cm} (3.21)

I have adjusted the constants in such a way as to make sure that $\mathcal{E}(x, y)$ returns to the old expression (3.19) when $x \ll 1$.

The phase trajectories of the system are level sets of $\mathcal{E}(x, y)$. You already know from your Calculus course how to sketch them. The idea is first to find “stationary points” (extrema) of $\mathcal{E}$, then Taylor-expand $\mathcal{E}$ around them to determine their nature, and then “fill in the blanks”, i.e., use common sense to work out how these local phase portraits fit together.

Stationary points are not called “stationary points” by accident. Indeed, the system (3.20) can be written equivalently as [cf. (7.35)]

$$\dot{x} = \frac{\partial \mathcal{E}}{\partial y}, \quad \dot{y} = -\frac{\partial \mathcal{E}}{\partial x}.$$  \hspace{1cm} (3.22)

Thus the extrema of $\mathcal{E}$ are indeed the stationary points (or fixed points, or equilibria) of the pendulum. They are

$$y = 0 \quad \text{and} \quad \sin x = 0 \quad \Rightarrow \quad x = \pi n, \quad n \in \mathbb{Z}. \hspace{1cm} (3.23)$$

For $n = 2k$, this is the obvious equilibrium at $x = 0$ and its identical copies at $x = 2\pi k$.\footnote{Remember that $x = \theta$, the angle, and so the phase space of the pendulum is, in fact, not really $\mathbb{R}^2$ but the surface of a cylinder: $-\infty < y < \infty$, $0 \leq x < 2\pi$, with the ends, 0 and $2\pi$ glued together.}

Clearly, $(x, y) = (0, 0)$ is a minimum of $\mathcal{E}$:

$$\left. \frac{\partial^2 \mathcal{E}}{\partial x^2} \right|_{(0,0)} = 1, \quad \left. \frac{\partial^2 \mathcal{E}}{\partial y^2} \right|_{(0,0)} = 1.$$  \hspace{1cm} (3.24)

In the vicinity of $(0,0)$, the phase portrait is the family of concentric circles already found in §3.5.1 and depicted in Fig. 13(c).

For $n = 2k + 1$, (3.23) is the upright equilibrium of the pendulum $(x = \pi$, Fig. 14a) and its periodic copies. This equilibrium, $(x, y) = (\pi, 0)$, is a saddle point:

$$\left. \frac{\partial^2 \mathcal{E}}{\partial x^2} \right|_{(\pi,0)} = -1, \quad \left. \frac{\partial^2 \mathcal{E}}{\partial y^2} \right|_{(\pi,0)} = 1.$$  \hspace{1cm} (3.25)

Let us look at the phase portrait in its vicinity. To do that, we expand (3.20) around this equilibrium point:

$$x = \pi + \delta x, \quad y = 0 + \delta y \quad \Rightarrow \quad \left\{ \begin{array}{l} \delta \dot{x} = \delta y, \\ \delta \dot{y} = \delta x. \end{array} \right.$$  \hspace{1cm} (3.26)
The phase velocity field \((\delta y, \delta x)\) is sketched in Fig. 14(b) and the resulting phase portrait in Fig. 14(c). The phase trajectories are hyperbolae, which is easily confirmed by expanding the energy (3.21) around \((\pi,0)\):

\[
E \approx 2 - \frac{\delta x^2}{2} + \frac{\delta y^2}{2} = C. \tag{3.27}
\]

Finally, this is all pieced together in Fig. 15. The phase portrait consists of

—closed trajectories around the minima of energy—they have \(\mathcal{E} < 2\) and describe a pendulum oscillating around its usual downward equilibrium,

—open trajectories—they have \(\mathcal{E} > 2\) and describe a pendulum executing full 360° rotations,

—the separatrices at \(\mathcal{E} = 2\).

Picking a point \((x_0, y_0)\) and following the phase trajectory on which it lies will tell you what the pendulum will do if it starts at that point: it will oscillate if you start with \(\mathcal{E} < 2\) and rotate if you start with \(\mathcal{E} > 2\).

Thus, we have developed a pretty good picture of the behaviour of our system even though we did not solve the nonlinear equations (3.20) explicitly, i.e., obtained \(x = x(t), y = y(t)\).\(^\text{17}\) We have also been able to classify all possible initial conditions into broad groups that had different subsequent qualitative behaviour.

Of course, we were greatly helped by the existence of \(\mathcal{E}(x,y)\) and our ability to write an explicit expression (3.21) for it. In practice, however, all we really needed in order to sketch the global phase portrait were the positions of the fixed points and the phase portraits in their close vicinity. In §7.1, I will classify all possible fixed points for second-order ODEs.

This was the most elementary example of qualitative solution of a system of coupled ODEs.

3.5.3. Local Linear Analysis


Let me spell out a key general lesson that you are supposed to learn from this example. In order to analyse qualitatively an autonomous system of ODEs (3.16), first find its fixed points:

\[
f(y_0) = 0. \tag{3.28}
\]

\(^{17}\)It is, in this case, in fact, possible to do it in special functions, called elliptic functions.
Then expand the system around them:

\[ y = y_0 + \delta y \Rightarrow \delta \dot{y}_i = f_i(y_0 + \delta y) \approx f_i(y_0) + \frac{\partial f_i}{\partial y_j} \bigg|_{y_0} \delta y_j. \] (3.29)

This gives you a linear, homogeneous system of ODEs with constant coefficients that describes the behaviour of the original system in the vicinity of each fixed point:

\[ \delta \dot{y} = A \cdot \delta y, \quad A_{ij} = \frac{\partial f_i}{\partial y_j} \bigg|_{y_0}. \] (3.30)

After you learn how to handle such linear systems (§6), you will be able to describe life in the vicinity of fixed points and hence piece together global phase portraits for nonlinear systems of which they are local approximations (§7).

3.5.4. Damped Pendulum (Introduction to Dissipative Systems)


Before I wrap up this extended introduction, let me show you what sort of thing happens when energy is not conserved and so we cannot obtain the phase portrait of the system by plotting level sets \( E(x, y) = C \).

Let us add friction to the range of experiences available to our idealised pendulum. This is a force that opposes the pendulum’s motion and is proportional to its velocity (there is no friction on things standing still), so (3.1) is now written so:

\[ \ddot{\theta} = -\omega_0^2 \sin \theta - \gamma \dot{\theta}. \] (3.31)

Recasting it as a system of first-order ODEs in dimensionless variables, analogous to (3.20), gives

\[ \begin{cases} \dot{x} = y, \\ \dot{y} = -\sin x - \gamma y. \end{cases} \] (3.32)

Let us see how the energy (3.21) evolves for such a system:

\[ \dot{\mathcal{E}} = \frac{\partial \mathcal{E}}{\partial x} \dot{x} + \frac{\partial \mathcal{E}}{\partial y} \dot{y} = -\gamma y^2 \leq 0. \] (3.33)

Thus, the pendulum’s energy declines with time (naturally, since there is damping and no forcing). This means that the system is constantly slipping off the conservative trajectories (3.21) towards smaller values of \( \mathcal{E} \) (the pendulum’s oscillations are losing amplitude).

For example, in the vicinity of the fixed point \((x_0, y_0) = (0, 0)\) (which is still a fixed point),

\[ \begin{cases} \dot{x} = y, \\ \dot{y} = -x - \gamma y. \end{cases} \] (3.34)
The phase velocity of this system is not perpendicular to the radius vector: indeed,

$$\begin{pmatrix} -x - \gamma y \\ x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -\gamma y^2 < 0,$$

so the trajectories are ever tending inwards (Fig. 16a). The resulting phase portrait is a clockwise spiral wrapping onto the origin (Fig. 16b). We shall learn how to investigate stability and categorise the system’s behavior in such situations when we pick up this story again in §7. For now, noticing that the phase portrait near \((x_0, y_0) = (\pi, 0)\) is distorted but not qualitatively changed (still a saddle), we can adjust the phase portrait of Fig. 15 to include the effect of damping, as shown in Fig. 17: the effect is that all trajectories eventually wrap onto the still-life equilibrium of a downward hanging, energetically exhausted, unmoving pendulum.

As before, without obtaining an explicit solution of the full nonlinear problem, we have been able to deduce a qualitative phase portrait that gives us a pretty good idea of the system’s behaviour. The eventual fate of a damped pendulum is independent of initial conditions, although the details of getting there do depend on them:

— if the pendulum starts with \(E \leq 2\), it spirals straight into the \((0, 0)\) equilibrium;
— if it starts with \(E > 2\), it rotates a few times, getting slower as it loses energy, and then spirals in.

The key ingredient in developing this understanding was again the local analysis (§3.5.3) of the linearised system around the fixed points.

The damped pendulum is an example of a dissipative system, which, when left to its own devices, decays into a boring, quiescent state. This opens the door to a class of questions as to what will happen if we “drive” the system, i.e., continuously put energy into it. This can be done by introducing a force, making the system inhomogeneous—and, if the force is time-dependent, also non-autonomous. We shall study the forced oscillator in §5.3.

Let me observe, in anticipation of the developments in §7, that a fixed point is not the only possible type of attractor—a set towards which a dissipative system’s trajectories converge regardless of initial conditions. Other possibilities are a limit cycle (§7.3) and a strange attractor—harder beasts to handle but not outside our intellectual range!

Hopefully this extended introduction has given you an idea of the general narrative arc of this subject. We shall now return to learning some practical methods for solving (linear) ODEs.
4. Linear ODEs: General Principles

[Literature: Pontryagin (1962, §17), Binney (2002, §2), Coddington (1990, Ch. 3)]

There is little one can do about integrating higher-order (higher than first) nonlinear ODEs explicitly. Most useful methods here are qualitative (§7). So we are now going to spend some time with linear ODEs, because we can achieve something with them and because this is rather useful—both by itself (there are many physical systems described by them) and as a door to qualitative analysis of nonlinear equations, a key component of which is local linear analysis (§3.5.3).

As I have explained above (§3.3), all ODEs are (systems of) first-order ODEs. Thus, in our treatment of linear ODEs, it is formally sufficient to consider systems of the form

$$\dot{y} = A(t) \cdot y + f(t), \quad (4.1)$$

where $y$ and $f$ are vectors that live in $\mathbb{R}^n$ (which can be extended to $\mathbb{C}^n$ without any complications) and $A$ is an $n \times n$ matrix, in general time-dependent and with elements in $\mathbb{R}$ (or $\mathbb{C}$). In most books, the general theory of (4.1) is presented separately from the general theory of $n$-th-order linear ODEs (1.5). While this leads to a number of useful practical prescriptions, conceptually it is mostly a waste of time to deal with the latter type of ODEs separately. You will, however, find it instructive to work out in detail how what I am about to do here applies specifically to (1.5) (and, in particular, to the case with constant coefficients: see §§5.1.3, 5.2.2, and 6). Also, because of its multitudinous applications and relative lack of algebraic cumbersomeness, the second-order linear ODE,

$$y'' + a(x)y' + b(x)y = f(x), \quad (4.2)$$

is particularly important and popular. In many cases below, (4.2) will serve me as a default example of the application of general principles (and in §5, I will study to death its version with constant coefficients).

4.1. Existence and Uniqueness Theorem for Linear Equations


A radical strengthening of Theorem 2 is possible for linear ODEs: the existence and uniqueness are guaranteed not just locally (in a neighbourhood of the point where initial conditions are set), but also globally.

**Theorem 3.** If all $A_{ij}(t)$ and $f_i(t)$ ($i, j = 1, \ldots, n$) are continuous functions on the interval $\mathcal{I} = (t_1, t_2)$, then $\forall t_0 \in \mathcal{I}$ and $\forall y_0 \in \mathbb{R}^n$, the Cauchy problem

$$\dot{y} = A(t) \cdot y + f(t), \quad y(t_0) = y_0 \quad (4.3)$$

has a unique solution on the entire interval $\mathcal{I}$.

For linear ODEs, existence will be ascertained explicitly in what follows—by constructing a solution. Uniqueness will be needed to give us certainty that a solution that we construct for any given Cauchy problem (4.3) is the solution of it.
4.2. Superposition Principle

[Literature: Arnold (2006, §27.3), Yeomans (2014, §III.0)]

Consider first the homogeneous version of (4.1):

$$\dot{y} = A(t) \cdot y.$$  \hspace{1cm} (4.4)

It is very easy to prove, but momentously significant, that if \(y_1(t)\) and \(y_2(t)\) are solutions of (4.4), then \(\forall\) constants \(C_1\) and \(C_2\),

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$  \hspace{1cm} (4.5)

is also a solution of (4.4)—this can be verified by direct substitution. This result is called the superposition principle—the fundamental property that makes linear ODEs special.

Its mathematical meaning is quite transparent. Stated more generally, it says that any linear combination of solutions of (4.4) is also a solution of (4.4)—the consequence of the linearity of the operator \(d/dt - A(t)\). In the language of linear algebra, this means that solutions of (4.4) form a linear vector subspace of the space of \(\mathbb{R}^n\)-valued functions of a single variable. Note that extending all this to the case of \(A(t), f(t)\) and \(y(t)\) being complex presents no difficulty (because a complex number is a kind of linear combination of real numbers, once \(i\) has been properly defined).

4.2.1. Superposition Principle for Inhomogeneous Equations

[Literature: Tikhonov et al. (1985, §3.6.2), Yeomans (2014, §III.3)]

The superposition principle for homogeneous ODEs has two immediate corollaries that apply to inhomogeneous ODEs.

1) If \(y_1(t)\) is a solution of

$$\dot{y} = A(t) \cdot y + f_1(t)$$  \hspace{1cm} (4.6)

and \(y_2(t)\) is a solution of

$$\dot{y} = A(t) \cdot y + f_2(t),$$  \hspace{1cm} (4.7)

then, \(\forall\) \(C_1\) and \(C_2\),

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$  \hspace{1cm} (4.8)

is a solution of

$$\dot{y} = A(t) \cdot y + C_1 f_1(t) + C_2 f_2(t).$$  \hspace{1cm} (4.9)

This property, instantly verifiable by direct substitution, gives us a strategy for solving inhomogeneous ODEs whose \(f\)’s are linear superpositions of more elementary/easier-to-handle functions.

2) A corollary of the above (for \(f_1 = 0, f_2 = f\)) is the statement that if \(y_{PI}(t)\) is some particular solution of (4.1) (a “particular integral”, PI) and \(y_{CF}(t)\) is the general solution of the associated homogeneous equation (4.4) (the “complementary function”, CF), then

$$y(t) = y_{CF}(t) + y_{PI}(t)$$  \hspace{1cm} (4.10)

is the general solution of the inhomogeneous equation (4.1).\(^{18}\) The proof is simply the observation that, having found some \(y_{PI}(t)\), we can make the ansatz (4.10) and discover that \(y_{CF}(t)\) satisfies (4.4), i.e., such an ansatz reduces an inhomogeneous equation to

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\(^{18}\) We have already met this principle in the context of first-order linear equations: see (2.62).
a homogeneous one. It is perfectly alright that $y_{p1}(t)$ can be chosen in many different ways because the difference between any two particular solutions of the inhomogeneous equation (4.1) is a solution of its homogeneous counterpart (4.4) and so is included in $y_{CF}(t)$.

Thus, if we learn how to find general solutions to homogeneous linear ODEs, we might have a fighting chance of solving inhomogeneous ODEs as well.

### 4.3. General Solution of Homogeneous Equations

[Literature: Pontryagin (1962, §17), Tikhonov et al. (1985, §3.6.3)]

Since solutions of (4.4) form a linear vector space, we can capture them all if we are able to find a basis in their space. This basis will turn out to be a set of $n$ linearly independent solutions

$$\{y_1(t), \ldots, y_n(t)\}, \quad (4.11)$$

where $n$ is the order of the ODE. Such a set is called the fundamental system of solutions of (4.4). I will often use the shorter notation $\{y_i(t)\}$ to designate the set (4.11) (and similarly for other sets of objects indexed by $i = 1, \ldots, n$). Solutions of an ODEs are linearly dependent if $\exists C_1, \ldots, C_n$, not all of which are zero, such that

$$\forall t \in \mathcal{I}, \quad C_1 y_1(t) + \cdots + C_n y_n(t) \equiv C_i y_i(t) = 0 \quad (4.12)$$

(I will use Einstein’s convention of summation over repeated indices). The solutions $\{y_i(t)\}$ are independent otherwise, i.e., if (4.12) implies that all $C_i = 0$.

I am now going to demonstrate

1) that a fundamental system can always be constructed (i.e., it exists);

2) how, given a fundamental system, any Cauchy problem (4.3) can be solved in practice.

#### 4.3.1. (In)dependent Somewhere—(In)dependent Everywhere

I will need the following two lemmas.

**Lemma 1.** If $\exists t_0 \in \mathcal{I}$ such that the set of vectors $\{y_i(t_0)\}$ is linearly independent, then the set of solutions $\{y_i(t)\}$ is linearly independent.

**Proof.** Suppose the solutions $\{y_i(t)\}$ are linearly dependent. Then $\exists \{C_i\}$ such that (4.12) is satisfied. Since it is satisfied for all $t$, it is satisfied for $t_0$ and so $\{y_i(t_0)\}$ are linearly dependent. Contradiction. Q.E.D.

**Lemma 2.** If $\exists t_0 \in \mathcal{I}$ such that the set of vectors $\{y_i(t_0)\}$ is linearly dependent, then the set of solutions $\{y_i(t)\}$ is linearly dependent.

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19This says that $\{y_i(t)\}$ are linearly dependent as functions, i.e., the same set of constants $\{C_i\}$ gets them all to sum up to zero at every $t$. This is not the same thing as vectors $\{y_i(t)\}$ being linearly dependent at every $t$, because for the latter, it is enough to have a different set of constants $\{C_i(t)\}$ at every $t$. This distinction is important for understanding why the two Lemmas that are coming up are both true and non-trivial. In principle, the vectors $\{y_i(t)\}$ being linearly dependent at some particular $t$ does not automatically imply that the functions $\{y_i(t)\}$ are also dependent in the sense of the definition (4.12), but Lemma 2 will say that if these functions are solutions of (4.4), then they are.
Proof. If \( \{y_i(t_0)\} \) are linearly dependent, then \( \exists \{C_i\} \) such that
\[
C_i y_i(t_0) = 0. \tag{4.13}
\]
Consider the function \( y(t) = C_i y_i(t) \). Since \( y_i(t) \) are all solutions of (4.4), so is \( y(t) \). In view of (4.13), \( y(t_0) = 0 \). But identical 0 is a solution of (4.4) satisfying this initial condition. Since the solution of any Cauchy problem for (4.4) is unique, this implies \( y(t) = 0 \) identically. Therefore, \( C_i y_i(t) = 0 \) \( \forall t \). The solutions are dependent. Q.E.D.

Thus, if the vectors \( \{y_i(t)\} \) are (in)dependent for any \( t = t_0 \), they are (in)dependent for all \( t \) and so are the solutions that they represent.

4.3.2. The Fundamental Matrix and the Wronskian

Let me introduce some notation that will make everything slightly more compact and also push us further into speaking of linear ODEs in the language of linear algebra. If I denote \( y^j_i(t) \) the \( j \)-th vector component of the solution \( y_i(t) \), then the fundamental system \( \{y_i(t)\} \) can be packed into a single matrix with components \( y^j_i(t) \), i.e., with the solution \( y_i(t) \) serving as the \( i \)-th column of the matrix:
\[
Y(t) = (y_1(t) \ldots y_n(t)), \quad \text{or} \quad Y_{ji}(t) = y^j_i(t). \tag{4.14}
\]
I shall call this the fundamental matrix. In view of the two lemmas proven in §4.3.1, the determinant of this matrix,
\[
W(t) = \det Y(t), \tag{4.15}
\]
called the Wronskian (Fig. 18b), does not vanish anywhere if \( \{y_i(t)\} \) is a fundamental system. Conversely, if \( \{y_i(t)\} \) is just some set of \( n \) solutions of (4.4), checking whether it is a fundamental system amounts to checking that \( W(t) \neq 0 \) at some \( t \).

The Wronskian turns out to be easy to compute:
\[
W(t) = W(t_0) \exp \left[ \int_{t_0}^t dt' \text{tr} A(t') \right], \tag{4.16}
\]
where \( \text{tr} A = A_{ii} \), the trace of the matrix. This is Liouville’s formula (Fig. 18a). Its proof is
based on showing that $\dot{W} = \text{tr} A(t)W$. I leave it to you as an exercise, which you can complete either by figuring out on your own how to differentiate a determinant or by looking it up in a book, e.g., Pontryagin (1962, §17) or Arnold (2006, §§16.4, 27.6).

Note that the formula (4.16) makes it explicit that if $W(t)$ is (or is not) zero at any $t = t_0$, then it is (or is not) zero at all $t$ and so linear (in)dependence of a set of solutions is equivalent to linear (in)dependence of these solutions at any one point in time.
4.3.3. How to Construct a Fundamental System

We are now ready for the following result.

**Corollary 1.** A fundamental system of solutions of (4.4), i.e., a set of \( n \) linearly independent solutions (4.11) with \( n \) equal to the order of the ODE, always exists.

**Proof.** Take any basis set of vectors \( \{v_i\} \) in \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)). By Theorem 3, there is a unique solution \( y_i(t) \) for each of \( n \) Cauchy problems, set at some \( t_0 \),

\[
\dot{y}_i = A(t) \cdot y_i, \quad y_i(t_0) = v_i, \quad i = 1, \ldots, n. \tag{4.17}
\]

Since the vectors \( \{y_i(t_0)\} \) are linearly independent, so are the solutions \( \{y_i(t)\} \), by Lemma 1. There will always be \( n \) of them because any smaller system can be amended with more linearly independent solutions until we have \( n \), while no larger system can be independent because you cannot have more than \( n \) linearly independent vectors in \( \mathbb{R}^n \) and if vectors \( \{y_i(t)\} \) are linearly dependent for any \( t \), the corresponding set of solutions is also dependent (by Lemma 2).

4.3.4. How to Construct the Solution

Now let me show that the fundamental system (4.11) truly is a basis in the space of solutions.

**Corollary 2.** If \( \{y_i(t)\} \) is a fundamental system of solutions of (4.4), then, for any Cauchy problem

\[
\dot{y} = A(t) \cdot y, \quad y(t_0) = y_0, \tag{4.18}
\]

\( \exists \{C_i\} \) such that

\[
y(t) = C_i y_i(t) \tag{4.19}
\]

is the solution.

**Proof.** I will prove this by showing explicitly how to find the set of constants \( \{C_i\} \) such that any given initial condition is satisfied (this solution is then the solution by the uniqueness Theorem 3). The linear combination (4.19) is, using the notation (4.14),

\[
y(t) = Y(t) \cdot C, \quad \text{where } C = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}. \tag{4.20}
\]

In order to have \( y(t_0) = y_0 \), we must have

\[
C = Y^{-1}(t_0) \cdot y_0. \tag{4.21}
\]

The inverse \( Y^{-1}(t_0) \) exists \( \forall t_0 \) because the columns of the matrix are linearly independent and so the Wronskian \( \det Y \neq 0 \). Finally, the (unique) solution of the Cauchy problem (4.18) is

\[
\boxed{y(t) = Y(t) \cdot Y^{-1}(t_0) \cdot y_0}. \tag{4.22}
\]

The interpretation of this formula is that the initial condition \( y_0 \) gets decomposed—by multiplication by \( Y^{-1}(t_0) \)—into bits corresponding to each member of the fundamental system (“projected” onto the basis “vectors” of the space of solutions) and then—by multiplication by \( Y(t) \)—each of these bits is forwarded in time along its respective basis “vector” \( y_i(t) \).
Example. Let me apply all this to the homogeneous version of the second-order linear ODE (4.2):
\[ y'' + a(x)y' + b(x)y = 0. \]  (4.23)

Formally, this reduces to (4.4) as follows:
\[
\begin{cases}
  y' = p, \\
  p' = -a(x)p - b(x)y,
\end{cases}
\text{or } \frac{d}{dx} \left( \frac{y}{p} \right) = A(x) \cdot \left( \frac{y}{p} \right),
\text{ where } A(x) = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix},
\]  (4.24)

but we do not actually need to write it like this explicitly. We know from Corollary 1 that the fundamental system for (4.23) must consist of two \((n = 2)\) linearly independent solutions: \(\{y_1(x), y_2(x)\}\). To make sure they are linearly independent, we must check that their Wronskian does not vanish anywhere (see §4.3.2), viz.,
\[
Y(x) = \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}, \quad W(x) = \det Y(x) = y_1y'_2 - y_2y'_1 \neq 0.
\]  (4.25)

If this is true, then the general solution of (4.23) is (4.19):
\[
y(x) = C_1y_1(x) + C_2y_2(x).
\]  (4.26)

The Cauchy problem (4.18) is to find a solution that satisfies
\[
y(x_0) = y_0, \quad y'(x_0) = p_0
\]  (4.27)
for some given \(y_0\) and \(p_0\). In order for that to happen, the two constants \(C_1\) and \(C_2\) must satisfy
\[
\begin{cases}
  C_1y_1(x_0) + C_2y_2(x_0) = y_0, \\
  C_1y'_1(x_0) + C_2y'_2(x_0) = p_0
\end{cases}, \quad \text{or } Y(x_0) \cdot \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}.
\]  (4.28)

The desired solution is then obtained by resolving these two simultaneous equations for \(C_1\) and \(C_2\): this is (4.21). It is possible to do this because \(Y \neq 0\), so the inverse of \(Y(x_0)\) exists:
\[
\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{W(x_0)} \begin{pmatrix} y'_2(x_0) & -y_2(x_0) \\ -y'_1(x_0) & y_1(x_0) \end{pmatrix} \cdot \begin{pmatrix} y_0 \\ p_0 \end{pmatrix}.
\]  (4.29)

It should be obvious to you how to apply the same construction to the \(n\)-th-order linear ODE
\[
y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.
\]  (4.30)

Here is an elegant argument (which I learned from Binney 2002, §2.1.3) that concisely illustrates the whole story. Consider the \(n\)-th-order linear ODE (4.30). Let us imagine that we know \(y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)\). Then from (4.30) we can calculate \(y^{(n)}(x_0)\). Taking a derivative of (4.30), we can then calculate \(y^{(n+1)}(x_0)\). If we keep going, we can determine all derivatives of \(y\) at \(x_0\), provided all \(\{a_i(x)\}\) are as differentiable as they need to be. Knowing all the derivatives \(y^{(k)}(x_0)\), we can reconstruct the whole solution as a Taylor series (cf. Q2.17):
\[
y(x) = y(x_0) + \sum_{k=1}^{\infty} (x - x_0)^k \frac{y^{(k)}(x_0)}{k!}.
\]  (4.31)

Thus, if we have \(n\) initial conditions that fix the values of \(y(x_0), \ldots, y^{(n-1)}(x_0)\), we can find the (unique) solution (4.31) to (4.30). Clearly, to fix the values of these derivatives, we need to have \(n\) relationships between them (which do not have to be explicitly their values), amounting to having \(n\) constants \(\{C_i\}\), on which the general solution of (4.30) then depends. If we now let \(\{y_i(x)\}\) be \(n\) particular solutions of (4.30) such that for the \(i\)-th solution all constants except the \(i\)-th are zero, then the general solution is \(y(x) = C_i y_i(x)\). In the dark ages before André Lukas, when Oxford physics students did not know linear algebra, this was all one could tell them.
4.4. General Solution of Inhomogeneous Equations

Now that we know how to find all solutions of the homogeneous linear ODE (4.4), we are ready to go back to the inhomogeneous one (4.1). According to the general principle expressed by (4.10), once we have the fundamental system of solutions \( Y(t) \) for (4.4), all we need is a particular solution of (4.1), \( y_{\text{PI}}(t) \), and then its general solution is

\[
y(t) = Y(t) \cdot C + y_{\text{PI}}(t).
\]

(4.32)

The Cauchy problem \( y(t_0) = y_0 \) is then solved as in (4.21): the constants are

\[
C = Y^{-1}(t_0) \cdot [y_0 - y_{\text{PI}}(t_0)].
\]

(4.33)

This is great, but how do we find \( y_{\text{PI}}(t) \) if we cannot just guess it? Here is a rather clever method, called the method of variation of constants, which we have already encountered in the context of a single first-order linear ODE (§2.4.2).

The idea is to weaponise the constants by turning them into functions, \( C_i \rightarrow \psi_i(t) \) (same trick as in §2.4.2), and look for the solution of (4.1) in the form

\[
y(t) = \psi_i(t) y_i(t) = Y(t) \cdot \psi(t), \quad \text{where} \quad \psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}.
\]

(4.34)

Substituting this into (4.1), we get

\[
\dot{\psi} = \dot{Y} \cdot \psi + Y \cdot \dot{\psi} = A \cdot \psi + f \quad \Rightarrow \quad Y \cdot \dot{\psi} = f,
\]

(4.35)

because, \( Y \) being a matrix composed of solutions of (4.4), \( \dot{Y} = A \cdot Y \). Since \( \det Y \neq 0 \), (4.35) is easy to turn into an explicit expression for \( \dot{\psi} \), which we can then promptly integrate:

\[
\dot{\psi} = Y^{-1}(t) \cdot f(t) \quad \Rightarrow \quad \psi(t) = C + \int_{t_0}^{t} dt' Y^{-1}(t') \cdot f(t'),
\]

(4.36)

where \( C \) are integration constants, as usual. Putting this back into (4.34), we get the desired general solution of (4.1):

\[
y(t) = Y(t) \cdot \left[ C + \int_{t_0}^{t} dt' Y^{-1}(t') \cdot f(t') \right].
\]

(4.37)

The first term is \( y_{\text{CF}} \) and the second is \( y_{\text{PI}} \), which we have now found explicitly and in general (assuming that we know the fundamental system). Since I have picked the limits of integration in such a way as to make \( y_{\text{PI}}(t_0) = 0 \), the initial condition \( y(t_0) = y_0 \) is enforced by (4.21) and so the solution of the Cauchy problem (4.3) is

\[
y(t) = Y(t) \cdot \left[ Y^{-1}(t_0) \cdot y_0 + \int_{t_0}^{t} dt' Y^{-1}(t') \cdot f(t') \right],
\]

(4.38)

which is the generalisation of (4.22) to inhomogeneous ODEs. Recalling our interpretation of (4.22), it is easy to see the meaning of (4.38): the “force” \( f(t') \) can be thought of as a sequence of kicks to the system, each of these kicks gets decomposed into basis “vectors” \( y_i(t') \) of the space of solutions—by multiplication by \( Y^{-1}(t') \),—and then they all get forwarded in time along these basis “vectors” by multiplication by \( Y(t) \).
Figure 19. (a) George Green (1793-1841), an English baker and miller, who, in a triumph of mind (and inherited wealth) over matter, extricated himself and became a celebrated mathematical physicist and a Cambridge fellow. (b) Paul Adrien Maurice Dirac FRS (1902-1984), great English physicist, one of the founding fathers of quantum mechanics, protagonist of many stories of strange behaviour and author of many famous \textit{bons mots}, allegedly unpremediated. (c) Oliver Heaviside FRS (1850-1925), English mathematician, electrical engineer and autodidact, Maxwell’s apostle, bicycling enthusiast, inventor of a lot of very usefully non-rigorous (at the time) notation, including differential operators and vector calculus.

4.5. Green’s Function

[Literature: Binney (2002, §6)]

This is a good place for the following consequential digression.

Let us define a matrix $G(t, t_0)$, called \textit{Green’s function} (Fig. 19a), to be the solution of the following Cauchy problem

$$\frac{\partial}{\partial t} G(t, t_0) = A(t) \cdot G(t, t_0), \quad G(t_0, t_0) = 1.$$  

(4.39)

Then the solution of the Cauchy problem (4.3) is

$$y(t) = G(t, t_0) \cdot y_0 + \int_{t_0}^{t} dt' G(t, t') \cdot f(t').$$  

(4.40)

Indeed, checking this directly, we get $y(t_0) = y_0$ and

$$\dot{y}(t) = A(t) \cdot G(t, t_0) \cdot y_0 + G(t, t) \cdot f(t) + \int_{t_0}^{t} dt' A(t) \cdot G(t, t') \cdot f(t') = A(t) \cdot y(t) + f(t), \quad \text{Q.E.D.}$$  

(4.41)

The meaning and purpose of Green’s function (which is also sometimes called the \textit{response function}) is to quantify the response of the linear system to initial ($y_0$) and/or external ($f$) disturbances.

In fact, it is sometimes convenient (and conceptually appealing) to treat these two types of externally prescribed influences on equal grounds, interpreting the initial condition as a kind of instantaneous burst at $t = t_0$ and the external force as a protracted sequence of bursts, or kicks,\footnote{See discussion after formula (4.38).} at later (or, if you wish, also earlier) times—all this occurring within the overall history of evolution of $y$ that stretches from $t = -\infty$ to $t = +\infty$. One then defines Green’s function as the solution of the ODE

$$\frac{\partial}{\partial t} G(t, t_0) = A(t) \cdot G(t, t_0) + 1 \delta(t - t_0),$$  

(4.42)
Dirac’s delta function = limit of a sequence of collapsing functions with constant area underneath.

\[ \delta(t - t_0) \]

where \( \delta(t - t_0) \) is Dirac’s delta function (Fig. 19b), a spiky “function” (in a generalised sense that is made precise in the theory of distributions) defined to be zero everywhere except at \( t = t_0 \) and having the property that

\[ \forall f(t), \quad \int_{-\infty}^{+\infty} dt \delta(t - t_0)f(t) = f(t_0). \]  

(4.43)

Such a function can be formally constructed as the limit of a sequence of functions peaked at \( t_0 \), with ever decreasing width, increasing height and constant unit area underneath, e.g.,

\[ \delta(t - t_0) = \lim_{\epsilon \to +0} \frac{e^{-(t-t_0)^2/\epsilon^2}}{\sqrt{\pi\epsilon}} = \lim_{\epsilon \to +0} \frac{e^{-|t-t_0|/\epsilon}}{2\epsilon} = \lim_{\epsilon \to +0} \frac{\epsilon/\pi}{(t-t_0)^2 + \epsilon^2}. \]  

(4.44)

To (4.42), we attach a further stipulation that \( G(-\infty, t_0) = 0 \). This implies that \( G(t, t_0) = 0 \) for all \( t < t_0 \) because nothing happens in (4.42) until the unit burst at \( t = t_0 \), so zero is a perfectly good—and unique—solution until that moment. With Green’s function defined in this new way, the solution (4.40) becomes

\[ y(t) = \int_{-\infty}^{+\infty} dt' G(t, t') \left[ f(t')H(t' - t_0) + \delta(t' - t_0)y_0 \right]. \]  

(4.45)

Here \( H(t - t_0) \) is Heaviside’s step function (Fig. 19c), defined to be

\[ H(t - t_0) = \begin{cases} 1 & \text{if } t > t_0, \\ 0 & \text{if } t < t_0. \end{cases} \]  

(4.46)

and used in (4.45) to restrict the application of the external force only to \( t > t_0 \). You might find it amusing to convince yourself that, for all practical purposes, \( H'(t - t_0) = \delta(t - t_0) \).

It is not hard to check that (4.45) is indeed the solution of the Cauchy problem (4.3). Thus, this solution is now expressed as a convolution over the entire timeline between external disturbances and the system’s response, the latter encoded in \( G(t, t_0) \) and entirely independent of those external disturbances. Obviously, we can easily read off from (4.38) an explicit expression for the Green’s function in terms of the fundamental matrix:

\[ G(t, t_0) = Y(t) \cdot Y^{-1}(t_0)H(t - t_0). \]  

(4.47)

While these ideas might appear somewhat superfluous at this stage, you will encounter them many times again, e.g., in the Mathematical Methods course (Lukas 2019, §§5-8), when you deal with boundary-value problems for ODEs or with partial differential equations (and indeed already when you study the wave equation next term, there will be some tiptoeing around these notions).

**Exercise 4.1.** What is the Green’s function for the linear, first-order ODE that we solved in §2.4?

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21 Another example is (5.53).

22 What it is at \( t = t_0 \) is a moot point.
Thus, if we can find the fundamental system \( \{ y_i(t) \} \) (equivalently, the fundamental matrix \( Y(t) \) or the Green’s function \( G(t, t_0) \)) for the homogeneous linear ODE (4.4), we now know how to solve any Cauchy problem for any inhomogeneous ODE (4.1). But how do we find \( \{ y_i(t) \} \)? For a general linear ODE with time-dependent coefficients, this rather depends on what those coefficients are. When they are constant in time, everything can be solved completely generally—I shall do that in §6 (and, for a simple sub-case, in §5). But some progress can also be made with the case of time-dependent coefficients if we focus on our favorite example (4.2).

4.6. Buy One Get One Free

[Literature: Binney (2002, §2.4)]

So, consider the Cauchy problem for the second-order linear ODE (4.2):

\[
\begin{align*}
y'' + a(x)y' + b(x)y &= f(x), \\
y(x_0) &= y_0, \\
y'(x_0) &= p_0.
\end{align*}
\] (4.48)

Suppose that, by some feat of luck or inspiration, we have found one particular solution of the associated homogeneous equation (4.23). Let us call this solution \( y_1(x) \). It turns out that a kind of “buy one get one free” deal is available: with \( y_1(x) \) in hand, we can easily find another, linearly independent, solution \( y_2(x) \) of (4.23) and, therefore, the general solution of (4.48).

This is again done by the method of variation of constants, viz., by looking for a general solution in the form

\[
y(x) = \psi(x) y_1(x) \tag{4.49}
\]

Let us see what this ansatz does for us. Substituting (4.49),

\[
y' = \psi' y_1 + \psi y_1' \quad \text{and} \quad y'' = \psi'' y_1 + 2\psi' y_1' + \psi y_1''
\] (4.50)

into (4.48), we get

\[
\psi'' y_1 + 2\psi' y_1' + \psi y_1'' + a\psi' y_1 + a\psi y_1' + b\psi y_1 = f \tag{4.51}
\]

where the cancellations are due to \( y_1 \) being a solution of the homogeneous equation (4.23). As a result, we are left with a first-order ODE for \( \psi' \):

\[
\psi'' + \left( 2 \frac{y_1'}{y_1} + a \right) \psi' = \frac{f}{y_1} \tag{4.52}
\]

(assuming \( y_1 \neq 0 \)), i.e., the highly beneficial effect of the ansatz (4.49) is to lower the order of our original equation (4.48). We already know how to solve any first-order linear ODE, of which (4.52) is a perfectly ordinary example: using (2.61), we get

\[
\psi'(x) = \mathcal{E}(x) \left[ C_1 + \int_{x_0}^x \frac{f(x')}{y_1(x') \mathcal{E}(x')} \, dx' \right], \tag{4.53}
\]

where

\[
\mathcal{E}(x) = \exp \left\{ - \int_{x_0}^x \frac{d x'}{y_1(x')} \left[ 2 \frac{y_1'(x')}{y_1(x')} + a(x') \right] \right\} = \left[ \frac{y_1(x_0)}{y_1(x)} \right]^2 \exp \left[ - \int_{x_0}^x d x' a(x') \right]. \tag{4.54}
\]

Integrating (4.53) and putting it back into (4.49), we arrive at the general solution of
our inhomogeneous ODE, complete with two constants:

\[
y(x) = y_1(x) \left[ C_2 + C_1 \int_{x_0}^{x} dx' E(x') + \int_{x_0}^{x} dx' \frac{f(x')}{y_1(x'')E(x'')} \right].
\] (4.55)

The first term (one with \(C_2\)) is our already known solution \(y_1(x)\) of the associated homogeneous equation (4.23), the second (with \(C_1\)) is another, linearly independent, solution of this equation [cf. (4.26)], and the third (with \(f\)) is a particular solution of the inhomogeneous equation (4.48), so the overall solution has the structure (4.32), or (4.37). Finally, the constants are found by enforcing the initial conditions:

\[
y(x_0) = y_0 \quad \Rightarrow \quad y_0 = C_2 y_1(x_0) \quad \Rightarrow \quad C_2 = \frac{y_0}{y_1(x_0)},
\] (4.56)

\[
y'(x_0) = p_0 \quad \Rightarrow \quad p_0 = C_2 y_1'(x_0) + C_1 y_1(x_0) \quad \Rightarrow \quad C_1 = \frac{p_0 y_1(x_0) - y_0 y'_1(x_0)}{y_1^2(x_0)}.
\] (4.57)

The Cauchy problem is gloriously solved.

It is a straightforward but useful exercise for you to work out how this maps onto the general scheme laid out in §4.4.

**Exercise 4.2.** Use Liouville’s formula (4.16) to find the second linearly independent solution of a second-order ODE on the basis of knowing one of its solutions. Convince yourself that this is essentially the same procedure as laid out above.

**Exercise 4.3.** Work out the Green’s function for (4.48).

**Exercise 4.4.** Show that the “buy one get one free” scheme introduced here is, in fact, the same deal as was available for Riccati equations (§2.6).

You might be wondering how the scheme presented for a second-order ODE generalises to \(n\)-th-order ODEs of the type (4.30). The basic answer is that if you know some set of \(m\) linearly independent solutions, you can lower the order of the ODE to \(n - m\). I leave it to your curiosity to investigate how this is done.

There is, alas, no general method for finding \(y_1(x)\)—and indeed it is generally impossible to do, as you must have realised if you still remember the discussion around (3.14). Everything becomes rather simple, however, if we specialise to the case of constant coefficients—we shall do that in §5.

While there is no general method, there are some empirical recommendations that apply to equations whose coefficients are polynomials. Here they are:

- Try

  \[y = e^{\lambda x}.
  \] (4.58)

  Plug this into your equation and see if it works for some \(\lambda\). If it does, hurrah. **Example.** Q2.9(b) (and all of §5!).

- Try

  \[y = x^n.
  \] (4.59)

  Plug this into your equation and set to zero the coefficient of the highest power of \(x\). Hence get the value of \(n\). Then check if \(y = x^n\) is a solution. **Example.** Q2.9(a) (and §5.1.4).

- If (4.59) is not a solution, but \(n\) is a positive integer, try putting the polynomial

  \[y = x^n + \alpha_1 x^{n-1} + \alpha_2 x^{n-2} + \cdots + \alpha_n
  \] (4.60)

  into the equation and finding the coefficients \(\{\alpha_i\}\) such that it is a solution. **Example.** Q2.8.
Consider
\[ 2x(x + 2)y'' + (2 - x)y' + y = 0. \] (4.61)

Try \( y = x^n \). The term containing the highest power of \( x \), which is \( x^n \), is
\[ 2x^2 n(n - 1)x^{n-2} - xnx^{n-1} + x^n = 0 \Rightarrow (2n - 1)(n - 1) = 0. \] (4.62)

Let us try \( n = 1/2 \): success, \( y(x) = \sqrt{x} \) is a solution! Let us then look for a general solution in the form (4.49):
\[ y(x) = \psi(x)\sqrt{x} \Rightarrow \psi' = C_1 \left( \frac{1}{2\sqrt{x}} + \frac{1}{x^{3/2}} \right) \Rightarrow \psi = C_1 \left( \sqrt{x} - \frac{2}{\sqrt{x}} \right) + C_2. \] (4.64)

The solution of (4.61) is, therefore,
\[ y = \psi(x)\sqrt{x} = C_1(x - 2) + C_2\sqrt{x}. \] (4.65)

The two solutions are quite obviously linearly independent and we have two constants ready to meet our every desire as far as initial conditions are concerned.

What if, in (4.62), we had chosen the root \( n = 1 \) instead of \( n = 1/2 \)? Trying \( y(x) = x \) in (4.61) leads to disappointment: it is not a solution. But we do not despair and try the prescription (4.60): \( y(x) = x + \alpha \). Putting this into (4.61) shows that it is a solution if \( \alpha = -2 \). Of course it is—this is the other solution in (4.65). Pretending you do not know what the second solution is, you can now look for a general solution in the form \( y(x) = \psi(x)(x - 2) \), follow the same algorithm as above and see if you can get (4.65) again.

Another useful exercise is to turn (4.61) into an inhomogeneous equation by putting some \( f(x) \) into its right-hand side and then to solve it by variation of the constant \( C_1 \) in (4.64) [or just use (4.53)]. Try, e.g., \( f(x) = 1 \). It is, of course, ridiculously easy to guess what the particular integral is, but see if you can recover it by the standard method that I have described.

5. Second-Order Linear ODE with Constant Coefficients

While, in principle, I could have moved straight on to §6, the second-order linear ODE with constant coefficients is both ubiquitous and important all across physics, so I will discuss it in particular detail. This section is one long worked example. One way for you not to get too bored with it might be to do it all entirely on your own and then check back that you have caught all the interesting features.

5.1. Homogeneous Equation

Let us start with the homogeneous equation
\[ ay + a\dot{y} + by = 0, \] (5.1)
where \( a \) and \( b \) are (real) constants. We saw in §4.6 that if we managed to find one particular solution, we would be basically sorted. Let me take a “rabbit out of hat” approach and say that the winning strategy is to seek a solution in the form (4.58):
\[ y(t) = C e^{\lambda t} \] (5.2)
(you might already have realised that this is the inevitable choice for linear ODEs with...
constant coefficients; in §6, we shall arrive at it naturally). Substituting (5.2) into (5.1), we find the condition (known as the auxiliary equation) for (5.2) to be a solution:

\[ \lambda^2 + a\lambda + b = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \] (5.3)

This is excellent news: unless \( a^2 = 4b \), we have two different values of \( \lambda \) and so two different solutions of (5.1). They are very obviously linearly independent, so we have our general solution (4.26):

\[ y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \] (5.4)

There are two interesting cases.

- **\( a^2 > 4b \)**: then \( \lambda_1 \) and \( \lambda_2 \) are both real and the solution is

  \[ y(t) = e^{-at/2} \left( C_1 e^{t\sqrt{a^2/4-b}} + C_2 e^{-t\sqrt{a^2/4-b}} \right). \] (5.5)

Thus, our equation (5.1) has growing and/or decaying solutions depending on the signs of \( \lambda_1 \) and \( \lambda_2 \). The initial conditions,

\[ y(0) = y_0 \quad \text{and} \quad y'(0) = v_0, \] (5.6)

determine what mixture of the two solutions emerges as a result. If we wait long enough, we will be left with the fastest growing (or, failing growth, slowest decaying) solution, with everything else exponentially small in comparison (cf. §7.1.1).

- **\( a^2 < 4b \)**: in this case, \( \lambda_1 \) and \( \lambda_2 \) are complex:

  \[ \lambda_{1,2} = -\frac{a}{2} \pm i \sqrt{b - \frac{a^2}{4}}. \] (5.7)

The solution is

\[ y = e^{-at/2} \left( C_1 e^{i\Omega t} + C_2 e^{-i\Omega t} \right), \] (5.8)

where the constants \( C_1 \) and \( C_2 \) are, in general, complex. However, one often wants a real solution, representing some real physical quantity. The solution (5.8) can be recast in two equivalent forms in which the constants can, if we wish, be purely real:

\[ y = e^{-at/2} \left( A \cos \Omega t + B \sin \Omega t \right) = e^{-at/2} \alpha \cos(\Omega t - \phi), \] (5.9)

where \( A = C_1 + C_2 = \alpha \cos \phi \) and \( B = i(C_1 - C_2) = \alpha \sin \phi \) are both real if the initial conditions (5.6) are real.

**Exercise 5.1.** Find \( A, B, \alpha \) and \( \phi \) in terms of \( y_0 \) and \( v_0 \).

The solution (5.9) describes oscillations of frequency \( \Omega \), phase-shifted by \( \phi \), and with the amplitude \( \alpha e^{-at/2} \), which either exponentially grows (\( a < 0 \)) or decays (\( a > 0 \)).
5.1.1. Damped Oscillator


When \( a \equiv \gamma > 0 \) and \( b = \omega_0^2 > 0 \), (5.1), or

\[
\ddot{y} + \gamma \dot{y} + \omega_0^2 y = 0,
\]

(5.10)
describes small oscillations of a damped pendulum (damped SHO) with “own” (or “natural”) frequency \( \omega_0^2 \) and friction coefficient \( \gamma \)—we have encountered its nonlinear version in §3.5.4. The solution (5.9) describes explicitly what we already deduced “graphically” in working out its phase portrait (Fig. 16): oscillations at the frequency

\[
\Omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}
\]

(5.11)
with superimposed exponential damping at the rate \( \gamma/2 \) (Fig. 21a). Note that in the physically interesting limit \( \gamma \ll \omega_0 \), \( \Omega \approx \omega_0 \).

In general, when \( \omega_0 > \gamma/2 \), the oscillator is called underdamped (cf. §7.1.6). When \( \omega_0 < \gamma/2 \), it is called overdamped and is described by (5.5).

It is interesting to look at the severe-overdamping limit, \( \gamma \gg \omega_0 \):

\[
y(t) = C_1 \exp \left[ -\frac{\gamma t}{2} \left( 1 - \sqrt{1 - \frac{4\omega_0^2}{\gamma^2}} \right) \right] + C_2 \exp \left[ -\frac{\gamma t}{2} \left( 1 + \sqrt{1 - \frac{4\omega_0^2}{\gamma^2}} \right) \right] \\
\approx C_1 e^{-(\omega_0^2/\gamma)t} + C_2 e^{-(\gamma-\omega_0^2/\gamma)t}.
\]

(5.12)
Thus, we have two modes: one damped quickly (at the rate \( \approx \gamma \gg \omega_0 \)), the other slowly (at the rate \( \approx \omega_0^2/\gamma \ll \omega_0 \)). What is the physics of these two solutions?

To work out the physics, one usually must look at what kind of perturbations are subject to

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23This controls the restoring force: gravity for a pendulum, Hookean tension for a spring, etc.
which of the damping rates. Let us impose the initial conditions (5.6):
\begin{align}
C_1 + C_2 &= y_0, \quad (5.13) \\
-\frac{\omega_0^2}{\gamma} C_1 - \left(\gamma - \frac{\omega_0^2}{\gamma}\right) C_2 &= v_0. \quad (5.14)
\end{align}
If \(v_0 = 0\), i.e., if there is only an initial displacement,
\[C_2 \approx -\frac{\omega_0^2}{\gamma^2} C_1 \ll C_1 \Rightarrow C_1 \approx y_0, \quad y(t) \approx y_0 e^{-(\omega_0^2/\gamma)t}. \quad (5.15)\]
The solution decays slowly. In contrast, if \(y_0 = 0\), i.e., if the oscillator is given an initial push but no displacement,
\[C_1 = -C_2 \Rightarrow C_2 \approx -\frac{v_0}{\gamma}, \quad y(t) \approx \frac{v_0}{\gamma} \left(1 - e^{-\gamma t}\right) e^{-(\omega_0^2/\gamma)t}. \quad (5.16)\]
The initial velocity first decays quickly, \(\dot{y}(t) \approx v_0 e^{-\gamma t}\), while the displacement grows until reaching a peak \(\approx v_0/\gamma\) at \(t \approx \gamma^{-1} \ln(\gamma^2/\omega_0^2)\). After this time, we are back to the regime in which the residual displacement decays at the slow rate. This is all illustrated in Fig. 21(b).

All this makes straightforward sense: in a highly frictional/viscous environment, an initial displacement of a pendulum will relax very slowly, whereas any motion initially imparted to it will stop very quickly.

Another way to see this is as follows. The equation
\[\ddot{y} + \gamma \dot{y} + \omega_0^2 y = 0 \quad (5.17)\]
has two possible dominant balances: either between the restoring force and friction,
\[\gamma \dot{y} \approx -\omega_0^2 y, \quad (5.18)\]
giving the slowly damped displacement (5.15), or between friction and inertia,
\[\ddot{y} \approx -\gamma \dot{y}, \quad (5.19)\]
giving the quickly damped velocity (5.16).

All this might appear rather elementary, but it is worth understanding it clearly early on in your career: you will be surprised how relevant these things prove to be later on!

5.1.2. Homogeneous Equation: Degenerate Case

[Literature: Pontryagin (1962, §8), Yeomans (2014, §III.1), Binney (2002, §2.5)]

Now consider the special case of \(a^2 = 4b\) (an oscillator with \(\gamma = 2\omega_0\) is called critically damped). In this case, the two solutions of the auxiliary equation (5.3) are degenerate (meaning they are the same):
\[\lambda_1 = \lambda_2 = -\frac{a}{2}. \quad (5.20)\]
This means that we only have one solution available to us:
\[y_1(t) = Ce^{-at/2}. \quad (5.21)\]
To find the second one, we must invoke the “buy one get one free” deal of §4.6. Namely, we seek the solution of (5.1) in the form
\[y(t) = \psi(t)e^{-at/2}. \quad (5.22)\]
Substituting this back into (5.1) and recalling that \(b = a^2/4\), we find
\[\ddot{\psi} = 0 \Rightarrow \dot{\psi}(t) = C_1 t + C_2 \Rightarrow y(t) = (C_1 t + C_2)e^{-at/2}. \quad (5.23)\]

\[\text{In systems with several different damping rates, the eventual approach to equilibrium always happens at the slowest of these rates, and along the corresponding eigendirection (cf. §7.1.1).} \]
A. A. Schekochihin

Figure 22. Leonhard Euler (1707-1783) may well have been the greatest mathematician of all time—and was certainly the most prolific. He was also, it appears, the best paid professor in the Russian history (in St. Petersburg)—possibly the best-spent money in the Russian history.

Obviously, you do not need to repeat this calculation every time. If $\lambda_1 = \lambda_2 = \lambda$, just remember that one solution is $e^{\lambda t}$ and the other is $te^{\lambda t}$ (they are, clearly, linearly independent—if this is not obvious, prove it by calculating their Wronskian). In §§5.2.2 and 6.3, we shall see that there is a certain systematicity to the appearance of powers of $t$ in front of exponentials.

5.1.3. Above and Beyond: n-th-Order Homogeneous Equation

[Literature: Pontryagin (1962, §§7-8), Tikhonov et al. (1985, §§3.1, 3.5), Bender & Orszag (1999, §1.4)]

Exercise 5.2. Consider an $n$th-order ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

(5.24)

where $\{a_i\}$ are (real) constant coefficients. Let $\lambda_1, \ldots, \lambda_n$ be the roots of the auxiliary equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

(5.25)

What is the general solution of (5.24)? How does one handle the case when some roots are complex? What about the case when some are degenerate? You may well have to resort to reading books in order to do this exercise in full generality.\(^{25}\)

5.1.4. Scale-Invariant (Euler’s) Equation


Having worked out the general method for solving (5.1), we have, in fact, also solved Euler’s equation (Fig. 22)

$$x^2y'' + axy' + by = 0.$$

(5.26)

This equation has the property of scale invariance: it does not change under rescaling of $x$ by an arbitrary factor: $x \rightarrow ax$. Therefore, it makes sense to seek solutions as power laws (which are scale-free functions):

$$y(x) = Cx^\lambda.$$

(5.27)

Substituting this back into (5.26) gives a polynomial auxiliary equation,

$$\lambda(\lambda - 1) + a\lambda + b = 0,$$

(5.28)

\(^{25}\)If you wish to race ahead, you might derive some inspiration for this exercise from §6.3, but it is also possible to solve it without matrix algebra.
which has two solutions $\lambda_1$ and $\lambda_2$. If $\lambda_1 \neq \lambda_2$, they give us the fundamental system for (5.26), so the general solution is
\[ y(x) = C_1x^{\lambda_1} + C_2x^{\lambda_2}. \] (5.29)
If $\lambda_1 = \lambda_2 = \lambda$, this gives us only one solution, $x^\lambda$; the second one is $x^\lambda \ln x$ (check).
In fact, we did not need a separate theory for Euler’s equation because it is reduced to (5.1) by a simple change of variables:
\[ x = e^{t} \Rightarrow xy' = \dot{y}, \quad x^2y'' = \ddot{y} = \dot{y} + (a - 1)\dot{y} + by = 0 \] (5.30)
and then the standard operating procedure applies. It is not hard to see that the same change of variables will reduce the $n$-th-order scale-invariant equation for $y(x)$
\[ x^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_1xy' + a_0y = 0 \] (5.31)
to an $n$-th-order equation with constant coefficients for $y(t)$. If you have time on your hands, you can try and work out what those coefficients are.

5.2. Inhomogeneous Equation
Inevitably, we must now consider the inhomogeneous equation
\[ \ddot{y} + a\dot{y} + by = f(t). \] (5.32)
According to the general scheme explained in §4.4, since we already know how to solve the homogeneous version of (5.32), all we need is to work out a particular solution, $y_{PI}(t)$. Then
\[ y(t) = C_1e^{\lambda_1t} + C_2e^{\lambda_2t} + y_{PI}(t) \quad \text{or} \quad y(t) = (C_1t + C_2)e^{\lambda t} + y_{PI}(t) \] (5.33)
if $\lambda_1 \neq \lambda_2$ or $\lambda_1 = \lambda_2 = \lambda$, respectively. There are two methods for finding $y_{PI}(t)$:
— for certain simple forms of $f(t)$, it is easy to guess;
— for general $f(t)$, it can be determined via the method of variation of constants: the answer is contained in (4.37) or in (4.55).

**Exercise 5.3.** Given $a$ and $b$, work out the explicit formula for $y_{PI}(t)$ for general $f(t)$. Identify special cases requiring separate consideration.

Leaving the general consideration to you as an exercise, I will instead go through a few simple special cases of $f(t)$ for which the task of guessing $y_{PI}(t)$ is particularly easy.

5.2.1. Some Tips for Finding Particular Solutions

**Polynomial force.** Suppose $f(t)$ is a polynomial of degree $m$:
\[ f(t) = F_0 + F_1t + \cdots + F_mt^m. \] (5.34)
It is then not hard to guess that
\[ y_{PI}(t) = \alpha_0 + \alpha_1t + \cdots + \alpha_mt^m \] (5.35)
will work. Indeed, substituting this into (5.32) will produce powers of $t$ up to $t^m$ in the left-hand side. Collect the coefficient in front of each power $k = 0, \ldots, m$ and equate it to $F_k$. This will give you $m$ equations for $m$ unknowns $\{\alpha_i\}$.
In particular, if $m = 0$, i.e., $f(t) = F_0 = \text{const}$, then $y_{PI} = F_0/b$. Physically, recalling
our oscillator (§5.1.1), we interpret this solution as the constant displacement resulting from the balance between the restoring force $by = \omega_0^2 y$ and the constant applied force $f = F_0$ (Fig. 23).

**Exponential force.** Now consider

$$f(t) = Fe^{\mu t}. \quad (5.36)$$

The obvious solution to try is

$$y_{P1}(t) = Ae^{\mu t}. \quad (5.37)$$

Substituting this back into (5.32) gives us a relationship between $A$ and $\mu$:

$$A \left( \mu^2 + a\mu + b \right) = F \quad \Rightarrow \quad A = \frac{F}{\mu^2 + a\mu + b}. \quad (5.38)$$

This is fine unless $\mu^2 + a\mu + b = 0$, i.e., unless $\mu$ coincides with $\lambda_1$ or $\lambda_2$, the solutions of the auxiliary equation (5.3). Indeed, in this case (5.37) is one of the solutions of the associated homogeneous equation, so cannot work as a particular solution of the inhomogeneous one! What should we do?—What we always do, try variation of constants: supposing $\mu = \lambda_1$, we seek a solution in the form

$$y(t) = \psi(t)e^{\lambda_1 t}. \quad (5.39)$$

This is obviously a particular case of the calculation that we already went through in §4.6, but let me do this simple case from scratch, lest you complain that I never show you any worked examples: since

$$\dot{y} = (\dot{\psi} + \lambda_1 \psi)e^{\lambda_1 t} \quad \text{and} \quad \ddot{y} = (\ddot{\psi} + 2\lambda_1 \dot{\psi} + \lambda_1^2 \psi)e^{\lambda_1 t},$$

our equation becomes, after we use the fact that $\lambda_1^2 + a\lambda_1 + b = 0$,

$$\ddot{\psi} + (a + 2\lambda_1)\dot{\psi} = F. \quad (5.41)$$

We only need one particular solution. If $a + 2\lambda_1 \neq 0$, this one works:

$$\ddot{\psi} = \frac{F}{a + 2\lambda_1} \quad \Rightarrow \quad \dot{\psi} = \frac{Ft}{a + 2\lambda_1} \quad \Rightarrow \quad y_{P1}(t) = \frac{Fte^{\lambda_1 t}}{a + 2\lambda_1}. \quad (5.42)$$

Just as in §5.1.2, multiplication by $t$ has done the trick.

What if $a + 2\lambda_1 = 0$? This is the degenerate case discussed in §5.1.2, in which $\lambda_1 = \lambda_2 = \lambda = a/2$, so both $e^\lambda t$ and $te^\lambda t$ are solutions of the homogeneous equation (5.1) and cannot, therefore, be particular solutions of the inhomogeneous one. In this case, (5.41) is, in fact, even easier to solve:

$$\ddot{\psi} = F \quad \Rightarrow \quad \dot{\psi} = \frac{F}{2} t^2 \quad \Rightarrow \quad y_{P1}(t) = \frac{F}{2} t^2 e^{\lambda t}. \quad (5.43)$$
Composite force. Obviously, if \( f(t) \) is a linear combination of more elementary functions, such as (5.34) or (5.36), we can use the superposition principle (4.8) to construct \( y_{P1}(t) \) as the same linear combination of particular solutions of the kind derived above. For example,

\[
f(t) = A \cos \omega t = \frac{A}{2} \left( e^{i\omega t} + e^{-i\omega t} \right)
\]

is just a linear combination of two exponentials (5.36), with \( \mu = \pm i\omega \). This situation describes the physically important case of an oscillator subjected to a periodic force, so I will analyse it to death in §5.3.

Example. Q2.4, Q2.5.

5.2.2. Above and Beyond: Quasipolynomials and n-th-Order Inhomogeneous Equation

[Literature: Pontryagin (1962, §10), Tikhonov et al. (1985, §§3.1, 3.5), Arnold (2006, §26)]

Let me mention a general result that summarises and generalises the examples that I showed you in §5.2.1.

Consider the \( n \)-th-order ODE

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y + a_0 y = (F_0 + F_1 t + \cdots + F_m t^m) e^{\mu t}.
\]

(5.45)

This equation has a particular solution

\[
y_{P1}(t) = t^k (a_0 + a_1 t + \cdots + a_m t^m) e^{\mu t},
\]

(5.46)

where \( k = 0 \) if \( \mu \) is not a root of the auxiliary equation (5.25) and \( k \) is equal to the degree of degeneracy of \( \mu \) if it is a root of (5.25) (i.e., \( k \) is equal to the number of times that \( \mu \) occurs as a root). The constants \( \{a_i\} \) can be found by substituting (5.46) into (5.45) and equating coefficients.

Functions of the form \((a_0 + a_1 t + \cdots + a_m t^m) e^{\mu t}\) are called quasipolynomials. The reason (5.46) works is that derivatives of quasipolynomials of degree \( m \) are also quasipolynomials of degree \( m \). I leave it to your own researches (or literature searches) to work out the complete proof.

If you have done Exercise 5.2, you now know the general solution of (5.45), an \( n \)-th-order inhomogeneous ODE with constant coefficients and a quasipolynomial force.

5.3. Forced Oscillator


We are now going to spend some time with the equation of the forced oscillator:

\[
\ddot{y} + \gamma \dot{y} + \omega_0^2 y = F \cos \omega t.
\]

(5.47)

To reiterate, the first term is inertia, the second friction, the third the Hookean (or gravitational, or whatever) restoring force and the right-hand side is the external periodic driving force.\(^{26}\) We already know the solution of the associated homogeneous equation (see §5.1). Let us find the particular solution. As I explained after (5.44), the force in

\[^{26}\text{This is an equation of more general interest than it might superficially appear. When you poke an oscillator with a (perhaps periodic) force of arbitrary functional form } f(t), \text{ this force is usually representable (via Fourier transform) as a sum of periodic functions of different frequencies. Many complicated physical systems can be described, at least in the linear approximation, as superpositions of SHOs (which may be damped, forced, coupled). Thus, what we are studying here is a fundamental building block of many more complicated situations.}\]
(5.47) is a linear combination of two complex exponentials, so the particular solution will also be some linear combination of these complex exponentials, or, equivalently, of sines and cosines. The unimaginative (and laborious) way of finding it is to write

\[ y_{P1}(t) = A \cos \omega t + B \sin \omega t, \quad (5.48) \]

substitute this into (5.47) and work out the coefficients \( A \) and \( B \). There is, however, a much quicker way. Notice that \( \cos \omega t = \text{Re} e^{i\omega t} \), the coefficients in (5.47) are real and so the operation of taking the real part commutes with the differential operator in the left-hand side of (5.47). So we “complexify” (which, in this context, is a synonym of “simplify”) our equation by introducing a complex function \( z(t) \) that satisfies

\[ \ddot{z} + \gamma \dot{z} + \omega_0^2 z = Fe^{i\omega t}. \quad (5.49) \]

Then, clearly, \( y = \text{Re} z \), so all we need to do is solve (5.49) and then take the real part. As we did for the exponential force (5.36), let us look for a solution in the form

\[ z(t) = Ce^{i\omega t}. \quad (5.50) \]

This gives us, upon substitution of (5.50) into (5.49),

\[ C = \frac{F}{\omega_0^2 - \omega^2 + i\omega\gamma} = \frac{F(\omega_0^2 - \omega^2 - i\omega\gamma)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} = \frac{F(\cos \phi - i \sin \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}, \quad (5.51) \]

whence

\[ z(t) = \frac{Fe^{i(\omega t - \phi)}}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}, \quad \text{where} \quad \tan \phi = \frac{\omega\gamma}{\omega_0^2 - \omega^2}. \quad (5.52) \]

Taking the real part gives us

\[ y_{P1}(t) = A \cos(\omega t - \phi), \quad \text{where} \quad A = \frac{F}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}. \quad (5.53) \]

Finally, the general solution of (5.47), with two constants eager to accommodate initial conditions, is

\[ y(t) = e^{-\gamma t/2} [C_1 \cos \Omega t + C_2 \sin \Omega t] + A \cos(\omega t - \phi), \quad (5.54) \]

where \( \Omega \) is given by (5.11).

**Exercise 5.4.** Let \( y(0) = y_0 \) and \( \dot{y}(0) = v_0 \). Work out \( C_1 \) and \( C_2 \).

The part of the solution (5.54) containing the integration constants is transient: it decays away after \( t \gg \gamma^{-1} \) and with it all memory of the initial conditions is lost. What does matter in the long-time limit is the interaction between the external forcing and the linear response properties of the system, the outcome of which is represented here by the last term in (5.54), called the stationary, or sometimes steady-state, solution. This part of the solution endures as long as the force is applied and describes forced oscillations. They are independent of the initial state of the oscillator, they have the amplitude \( A \) that is proportional but not equal to the amplitude of the force and they are phase-shifted by \( \phi \) with respect to it.

This is the simplest example of a generic feature of forced dissipative systems: their long-time behaviour is usually independent of initial conditions because dissipation washes away the system’s memory.
5.3.1. Resonance

Let us examine how the amplitude of forced oscillations, given in (5.53), depends on the parameters of the problem (Fig. 24). The most interesting feature is that, when $\gamma \ll \omega_0$ (which is a “natural” physical limit for an oscillator—assuming we do want it to oscillate), $A(\omega)$ has a sharp peak around $\omega_0$, i.e., when the frequency of the driver is close to the “own” frequency of the oscillator. This phenomenon is called resonance. The precise calculation, achieved by setting $A'(\omega) = 0$ places the peak at

$$\omega_{\text{res}} = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \approx \omega_0.$$  \hspace{1cm} (5.55)

This resonant frequency is just slightly below the own frequency of the undamped oscillator. The peak amplitude is

$$A_{\text{res}} \approx A(\omega_0) = \frac{F}{\omega_0 \gamma}. \hspace{1cm} (5.56)$$

The phase at the resonance, (5.52) with $\omega = \omega_{\text{res}}$ given by (5.55), satisfies

$$\tan \phi_{\text{res}} = \frac{2 \omega_0}{\gamma}. \hspace{1cm} (5.57)$$

The width of the peak (or, as one sometimes says, “of the resonance”) is $\sim \gamma$. This is manifest in the functional dependence $A(\omega)$, but can be made more quantitative in the following way, popular in experimental circles. When $|\omega - \omega_0| \sim \gamma \ll \omega_0$, $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$ and (5.53) becomes

$$A(\omega) \approx \frac{F}{2\omega_0 \sqrt{(\omega_0 - \omega)^2 + \gamma^2/4}} \hspace{1cm} (5.58)$$
Figure 25. Resonance in the case of $\gamma = 0$: oscillations grow secularly.

This implies that the amplitude drops to half of its maximum value (5.56) when

$$\frac{A(\omega)}{A_{\text{res}}} \approx \frac{\gamma}{2\sqrt{(\omega_0 - \omega)^2 + \gamma^2/4}} = \frac{1}{2} \Rightarrow |\omega - \omega_0| = \frac{\sqrt{3}}{2} \gamma.$$  (5.59)

Thus, the so-called full width at half maximum (FWHM) is $\sqrt{3}\gamma$.

Other interesting features of the $A(\omega)$ dependence are

— the amplitude at zero frequency is

$$A(0) = \frac{F}{\omega_0^2},$$  (5.60)

which is just the force balance between a constant driving force and the restoring force, $\omega_0^2 y = F$ (Fig. 23);

— the amplitude decays at high frequencies,

$$A(\omega) \approx \frac{F}{\omega^2}, \quad \text{when} \quad \omega \gg \omega_0 \gg \gamma,$$  (5.61)

because, when the force oscillates too fast, the system has no time to respond, so the force averages out and nothing much happens.\(^{27}\)

As $\gamma \to +0$, the resonance becomes infinitely sharp, $A_{\text{res}} \to \infty$ and $\phi_{\text{res}} \to \pi/2$.

It is instructive to examine the case when $\gamma = 0$ literally (which is, of course, a mathematical abstraction) and $\omega = \omega_0$. The calculation of the particular solution in this case changes because $e^{i\omega_0 t}$ is itself one of the solutions of the homogeneous counterpart of (5.49). The answer is (5.42) with $a = 0$ and $\lambda_1 = i\omega_0$:

$$y_{\text{pt}}(t) = \text{Re} \left( \frac{Ft e^{i\omega_0 t}}{2i\omega_0} \right) = \frac{tF}{2\omega_0} \sin \omega_0 t.$$  (5.62)

The full solution of (5.47) is then

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{tF}{2\omega_0} \sin \omega_0 t.$$  (5.63)

This solution has no damping away of the initial conditions and no stationary state; the amplitude of forced oscillations grows secularly with time (Fig. 25).

Another instructive exercise is to work out the solution of (5.47) with $\gamma = 0$ and $\omega \neq \omega_0$ and then show how (5.63) is recovered in the limit $\omega \to \omega_0$ (see Q2.6).

\(^{27}\)The case of a rapidly oscillating force becomes much more interesting if the force is made to depend on the displacement: see §5.4.
5.3.2. Energy Budget of Forced Oscillator

In §3.5.4, we already saw how the energy of a (damped) pendulum would behave and how understanding its behaviour could help one work out the system’s phase portrait (Fig. 16b). Let us now investigate how the oscillating force enters the energy budget.

Multiplying (5.47) by \( \dot{y} \), we find that the energy, kinetic \( \frac{\dot{y}^2}{2} \) plus potential \( \frac{\omega_0^2 y^2}{2} \), evolves according to

\[
\frac{d\mathcal{E}}{dt} = \frac{d}{dt} \left( \frac{\dot{y}^2}{2} + \frac{\omega_0^2 y^2}{2} \right) = -\gamma \dot{y}^2 + \dot{y} F \cos \omega t. \tag{5.64}
\]

The first term on the right-hand side is the dissipation (work done per unit time by the oscillator against the friction force), the second term is the power injected by the driving force (work done per unit time by the driver on the oscillator).

At \( \gamma t \gg 1 \), we have stationary forced oscillations:

\[
y = A \cos(\omega t - \phi), \quad \dot{y} = -\omega A \sin(\omega t - \phi). \tag{5.65}
\]

Using these expressions, we can find the stored energy in the oscillator averaged over the period \( T = \frac{2\pi}{\omega} \), \( \langle \mathcal{E} \rangle = \frac{1}{T} \int_0^T dt \mathcal{E}(t) \): since the mean square of a cosine or a sine over a period is \( \frac{1}{2} \),

\[
\langle \mathcal{E} \rangle = \langle \dot{y}^2 \rangle + \frac{\omega_0^2 y^2}{2} = \frac{(\omega^2 + \omega_0^2)A^2}{4} = \frac{(\omega^2 + \omega_0^2)F^2}{4 \left[ (\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2 \right]} \tag{5.66}
\]

\[
= \frac{F^2}{2\gamma^2} \quad \text{when} \quad \omega = \omega_0.
\]

The dissipated power is

\[
\mathcal{D} = \gamma \langle \dot{y}^2 \rangle = \frac{\gamma \omega^2 A^2}{2} = \frac{\gamma \omega^2 F^2}{2 \left[ (\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2 \right]} \left( = \frac{F^2}{2\gamma} \quad \text{when} \quad \omega = \omega_0 \right). \tag{5.67}
\]

The injected power is

\[
\mathcal{P} = \langle \dot{y} F \cos \omega t \rangle = -\frac{\omega F^2 (\sin \omega t \cos \omega t \cos \phi - \cos^2 \omega t \sin \phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}
\]

\[
= \frac{\gamma \omega^2 F^2}{2 \left[ (\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2 \right]} \left( = \frac{F^2}{2\gamma} \quad \text{when} \quad \omega = \omega_0 \right) \tag{5.68}
\]

because, from (5.51), \( \sin \phi = \omega \gamma / \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2} \). Thankfully, \( \mathcal{P} = \mathcal{D} \), as it ought to be in steady state—that this ought to be the case follows from averaging (\langle 5.64 \rangle):

\[
\mathcal{P} - \mathcal{D} = \frac{d}{dt} \langle \mathcal{E} \rangle = 0. \tag{5.69}
\]

To measure how well an oscillator can store energy, one sometimes defines the so-called quality factor (or “Q-factor”):

\[
Q = 2\pi \frac{\text{stored energy}}{\text{energy lost per period}} \bigg|_{\text{at resonance}} = \frac{2\pi \langle \mathcal{E} \rangle}{(2\pi/\omega) \mathcal{D}} \bigg|_{\omega = \omega_0} = \frac{\omega_0}{\gamma}. \tag{5.70}
\]

Note that the quality factor can be defined in the same way for a damped oscillator (§5.1.1), using the (slowly damped) energy in the homogeneous solution, and it is equal to the same thing (I leave this to you as an exercise; see Q2.2).

**Exercise 5.5.** Work out the steady-state solution for a forced, overdamped oscillator (\( \gamma \gg \omega_0 \)).
Figure 26. Pendulum whose point of suspension rapidly oscillates (a) vertically, (b) horizontally (Q3.9).

see §5.1.1). How long does one have to wait for the initial conditions to be forgotten? Show that $P \propto \omega^{-2}$ when $\gamma \ll \omega$ and $P \propto \gamma^{-1}$ when $\gamma \gg \omega$ and explain why this is, physically.

I reckon I have spent quite enough time belabouring the second-order linear ODE with constant coefficients! Let us move on to more sophisticated material.

5.4. (Nonlinear) Digression: Rapidly Oscillating Force


To take a break from the tedium of linear equations and linear oscillations, let me show you an interesting piece of physics that is applicable to both linear and nonlinear systems. Let us consider a 1D system whose behaviour is described by the following ODE

$$\ddot{x} = -U'(x) + f(x, t).$$

(5.71)

Physically, $x$ is some displacement variable, $U(x)$ is the potential energy, and $f(x, t)$ is an additional force that is allowed to depend on time. For the forced linear oscillator (in this case, without damping), $U(x) = \omega_0^2 x^2/2$ and $f(x, t) = F \cos \omega t$. For the nonlinear pendulum considered in §3.5.2, $U(x) = \omega_0^2(1 - \cos x)$.

I would like to study a situation in which the force $f$ oscillates extremely rapidly, i.e., its frequency $\omega$ is much larger than any other frequencies appearing in the equation (e.g., $\omega \gg \omega_0$). We already saw in §5.3.1 that when the force $f$ had a constant amplitude, this was not a terribly interesting scenario: the amplitude of forced oscillations became very small because the effect of the force largely averaged out [see (5.61)]. Let us complicate matters by assuming that the oscillating force also depends on the displacement $x$:

$$f(x, t) = F(x) \cos \omega t.$$  

(5.72)

An example of such a situation is a mathematical pendulum (§3.1) whose point of suspension oscillates vertically as, say, $a \cos \omega t$ (Fig. 26a). This gives rise to an additional (non-inertial) vertical force $-ma\omega^2 \cos \omega t$, which, when projected on the direction of the motion of the mass, becomes $-ma\omega^2 \cos \omega t \sin \theta$. The equation (3.1) for the pendulum is then modified as follows:

$$ml\ddot{\theta} = -mg \sin \theta - ma\omega^2 \sin \theta \cos \omega t.$$  

(5.73)

In the notation of this chapter, $x = \theta$ and

$$\ddot{x} = -\omega_0^2 \sin x - \omega_1^2 \sin x \cos \omega t, \quad \text{where} \quad \omega_0^2 = \frac{g}{l}, \quad \omega_1^2 = \frac{a}{l} \omega^2.$$  

(5.74)

This is (5.71) with the force (5.72):

$$\ddot{x} = -U'(x) + F(x) \cos \omega t,$$  

where $U(x) = \omega_0^2(1 - \cos x)$ and $F(x) = -\omega_1^2 \sin x$.  

(5.75)

We are assuming $\omega \gg \omega_0, \omega_1$ (the latter means $a \ll l$).

Let us seek the solution to this equation as a sum of some slow average motion and a fast
oscillation on top of it. Formally, let
\[ x(t) = \bar{x}(t) + \delta x(t), \quad \text{where} \quad \bar{x}(t) = \langle x(t) \rangle = \frac{1}{T} \int_{t}^{t+T} dt' x(t'), \quad T = \frac{2\pi}{\omega} \quad \text{and} \quad \langle \delta x(t) \rangle = 0. \]

(5.76)

Assuming \( \delta x \ll \bar{x} \) (as can be verified \textit{a posteriori}), we can expand the right-hand side of our equation in \( \delta x \):
\[ \ddot{x} + \delta \ddot{x} = -U'(\bar{x}) - \delta x U''(\bar{x}) + F(\bar{x}) \cos \omega t + \delta x F'(\bar{x}) \cos \omega t. \]

(5.77)

Taking the average of this equation gives us
\[ \ddot{x} = -U'(\bar{x}) + F'(\bar{x}) (\delta x(t) \cos \omega t). \]

(5.78)

Subtracting (5.78) from (5.77) and neglecting all terms that contain \( \delta x \) (which is small) but keeping its time derivatives (which are large), we get:
\[ \delta \ddot{x} = F(\bar{x}) \cos \omega t \quad \Rightarrow \quad \delta x(t) = -\frac{F(\bar{x})}{\omega^2} \cos \omega t \quad \Rightarrow \quad \langle \delta x(t) \cos \omega t \rangle = -\frac{F(\bar{x})}{2\omega^2}. \]

(5.79)

Substituting the last expression into (5.78), we arrive at the following rather neat result:
\[ \ddot{x} = -U'(\bar{x}) - \frac{F'(\bar{x}) F(\bar{x})}{2\omega^2} = -U'_{\text{eff}}(\bar{x}), \quad \text{where} \quad U'_{\text{eff}}(\bar{x}) = U(\bar{x}) + \frac{F^2(\bar{x})}{4\omega^2}. \]

(5.80)

Thus, the effect of a rapidly oscillating force is to change the potential energy of the system to a new, “effective” form \( U'_{\text{eff}} \), which includes what is easily seen to be the average kinetic energy of the rapid oscillations:
\[ \frac{F^2(\bar{x})}{4\omega^2} = \frac{\langle \delta \dot{x}^2 \rangle}{2}. \]

(5.81)

The effect of the additional force is to push the system out of the regions of phase space where the rapid oscillations have larger amplitude.

Applying these general results to our example of a pendulum with an oscillating suspension point [see (5.75)], we find that
\[ U'_{\text{eff}} = \omega_0^2 (1 - \cos x) + \frac{\omega_0^2 \sin^2 x}{4l^2} = \frac{g}{l} (1 - \cos \theta) + \frac{a^2 \omega^2}{4l^2} \sin^2 \theta. \]

(5.82)

Equilibria of the system satisfy
\[ U'_{\text{eff}}(\theta_0) = \left( \frac{g}{l} + \frac{a^2 \omega^2}{2l^2} \cos \theta_0 \right) \sin \theta_0 = 0 \quad \Rightarrow \quad \theta_0 = 0, \pi, \text{arccos} \left( -\frac{2gl}{a^2 \omega^2} \right). \]

(5.83)

As in §3.5.2, an equilibrium is stable if it is a minimum of the energy, i.e., if
\[ U''_{\text{eff}}(\theta_0) = \left( \frac{g}{l} + \frac{a^2 \omega^2}{2l^2} \cos \theta_0 \right) \cos \theta_0 - \frac{a^2 \omega^2}{2l^2} \sin^2 \theta_0 > 0. \]

(5.84)

Thus, the downward equilibrium \( \theta_0 = 0 \) is always stable, but, for a wiggly pendulum, the upright equilibrium \( \theta_0 = \pi \) can also be stable, provided \( a^2 \omega^2 > 2gl \). There is also a “slanted” equilibrium, but it is always unstable. 29

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28 Check that, with this \( \delta x(t) \), all the terms that we have neglected to get it are indeed smaller than the term that we have retained. Note, by the way, that the solution that we have got for \( \delta x(t) \) is exactly the same solution as (5.61), but it is made consequential by the fact that \( F \) now depends on \( x \).

29 A slanted equilibrium turns out to be the stable one if the pendulum is wiggled (sufficiently vigorously) in the horizontal direction: see Fig. 26(b) and Q3.9.

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**Exercise 5.6.** Armed with the experience of §3.5.2, sketch the “effective” phase portrait [i.e., the phase portrait of the averaged motion \( \langle \bar{x}, \bar{\dot{x}} \rangle \)] for the wiggly pendulum discussed above. Investigate what effect friction will have.
Another example of an effective average force arising due to a fast-oscillating real force is the case of a charged particle subject to rapidly oscillating electric field that has some spatial variation:

$$\ddot{x} = \frac{q}{m} E(x) \cos \omega t \quad \Rightarrow \quad m\ddot{x} = -\frac{q^2 E^2(x)}{4m\omega^2}. \quad (5.85)$$

This effective force is called ponderomotive force. It pushes charged particles out of regions where the oscillating electric field is stronger.

6. Systems of Linear ODEs with Constant Coefficients


From the point of view of presenting a systematic, general theory, I could have skipped everything from §4.5 onward and jumped right to this point. By the end of §4.4, we learned how to solve the Cauchy problem for any (system of) linear ODEs, provided we knew their fundamental system of solutions, neatly packed into the fundamental matrix $Y(t)$. I am now going to show you how to find this matrix for the linear ODEs of arbitrary order $n$ but with constant coefficients:

$$\dot{y} = A \cdot y + f(t). \quad (6.1)$$

This is the point at which this course merges completely with linear algebra.

The basic idea of the method is ridiculously simple but extraordinarily powerful. Suppose we can find $n$ linearly independent eigenvectors $\{v_i\}$ of the matrix $A$, with $n$ eigenvalues $\{\lambda_i\}$:

$$A \cdot v_i = \lambda_i v_i. \quad (6.2)$$

If we now consider the homogeneous ODE associated to (6.1),

$$\dot{y} = A \cdot y, \quad (6.3)$$

and ask for its solutions in the form

$$y_i(t) = \xi_i(t)v_i, \quad (6.4)$$

we find that $\xi_i(t)$ must satisfy

$$\dot{\xi}_i = \lambda_i \xi_i \quad \Rightarrow \quad \xi_i(t) = C_i e^{\lambda_i t}. \quad (6.5)$$

Clearly, $\{y_i(t)\}$ are linearly independent because $\{v_i\}$ are, and (by assumption) there are $n$ of them. This gives us a fundamental system, so any solution of the homogeneous equation (6.3) can be written as$^{30}$

$$y(t) = \sum_{i=1}^{n} C_i v_i e^{\lambda_i t}, \quad (6.6)$$

any solution of the inhomogeneous equation (6.1) can be found by variation of constants

$^{30}$The solutions of the auxiliary equation (5.3), $\lambda_1$ and $\lambda_2$, which featured so prominently all through §5, are, of course, the eigenvalues $\{\lambda_i\}$, and the solution (5.4) is a particular case of (6.6), for a second-order ODE.
(see §4.4) and any Cauchy problem then duly solved by (4.38). The fundamental matrix is
\[
Y(t) = (v_1 e^{\lambda_1 t} \ldots v_n e^{\lambda_n t}) .
\]  

I am now going to go through the construction of this solution step by step and then also deal with the slightly tricky question of what to do if a basis made of \( n \) eigenvectors of \( A \) does not exist (§6.3).

6.1. Diagonalisable Systems With No Degenerate Eigenvalues


Let us look for right eigenvectors\(^{31}\) of the matrix \( A \):
\[
A \cdot v = \lambda v .
\]  
Let us suppose that the \( n \)-th-order polynomial equation
\[
\det(A - \lambda I) = 0
\]  
has \( n \) roots \( \lambda_1, \ldots, \lambda_n \) and they are all different, i.e., they are non-degenerate right eigenvalues of \( A \). Then their corresponding eigenvectors \( \{v_i\} \) form a basis (not necessarily orthogonal).

**Proof.** Let
\[
(A - \lambda_i I) \cdot v_i = 0, \quad i = 1, \ldots, n.
\]  
Suppose \( \{v_i\} \) are not linearly independent. Then it must be possible to write one of them as a linear combination of the rest:
\[
v_n = \sum_j \alpha_j v_j ,
\]  
where \( \alpha_j \neq 0 \) are some constants and the summation is over some subset of \( v_j \)'s that are independent (and \( j \neq n \)). Then, on the one hand,
\[
A \cdot v_n = \lambda_n v_n = \lambda_n \sum_j \alpha_j v_j ,
\]  
whereas on the other hand,
\[
A \cdot v_n = \sum_j \alpha_j A \cdot v_j = \sum_j \alpha_j \lambda_j v_j .
\]  
Therefore,
\[
\sum_j \alpha_j (\lambda_j - \lambda_n) v_j = 0 \quad \Rightarrow \quad \alpha_j (\lambda_j - \lambda_n) = 0 ,
\]  
because \( v_j \)'s are independent by assumption. Since \( \lambda_j \neq \lambda_n \), the above implies \( \alpha_j = 0 \) for all \( j \). Contradiction. Q.E.D.

Now pack the right eigenvectors into a matrix, as columns, and the eigenvalues into another matrix, as diagonal elements:
\[
R = (v_1 \ldots v_n) , \quad L = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ 0 & \ldots & \lambda_n \end{pmatrix} \equiv \text{diag} \{\lambda_i\} .
\]  

\(^{31}\)I assume nothing about the matrix, so, in general, have to allow that its right eigenvectors are not the same as its left eigenvectors (\( v \cdot A = \lambda v \)).
Then
\[ A \cdot R = (\lambda_1 v_1 \ldots \lambda_n v_n) = R \cdot L. \] (6.16)

Since \( \{v_i\} \) are independent, \( \det R \neq 0 \), so \( R \) has an inverse. Therefore,
\[ A = R \cdot L \cdot R^{-1}. \] (6.17)

Note that this implies
\[ R^{-1} \cdot A = L \cdot R^{-1}, \] (6.18)
so rows of the inverse matrix \( R^{-1} \) are left eigenvectors of \( A \), while its left eigenvalues are the same as its right eigenvalues:
\[ R^{-1} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \text{where} \quad u_i \cdot A = \lambda_i u_i. \] (6.19)

Let us now use (6.17) to rewrite our homogeneous equation (6.3) as
\[ \dot{y} = A \cdot y = R \cdot L \cdot R^{-1} \cdot y. \] (6.20)

Dotting this with \( R^{-1} \) on the left, we get
\[ \frac{d}{dt} R^{-1} \cdot y = L \cdot R^{-1} \cdot y \quad \Rightarrow \quad \dot{\xi} = L \cdot \xi, \quad \text{where} \quad \xi = R^{-1} \cdot y. \] (6.21)

But \( L \) is a diagonal matrix, so our system of ODEs breaks into \( n \) decoupled equations:
\[ \dot{\xi}_i = \lambda_i \xi_i \quad \Rightarrow \quad \xi_i(t) = C_i e^{\lambda_i t} \] (6.22)
(no summation over \( i \) here). What we have done, effectively, is diagonalised our system of ODEs by transforming it into coordinates defined by the eigenvectors of the matrix \( A \). Let us transform back into the original coordinates: the solution is
\[ y(t) = R \cdot \xi(t) = \sum_i C_i v_i e^{\lambda_i t}, \] (6.23)
just as was promised in (6.6).

Even if \( A \) is real, its eigenvalues are under no obligation to be real (unless \( A \) is Hermitian; see §6.2). Therefore, the solution (6.23) may well contain complex exponentials, complex eigenvectors and complex constants. There is nothing wrong with this, but it may be a bit unseemly if we are solving a physical system whose solution is supposed to be real. Just like in the case of a linear second-order ODE [see (5.8) and (5.9)], it is quite easy to recast the solution (6.23) in an explicitly real form.

Since \( \{\lambda_i\} \) are roots of a polynomial, the complex ones come in conjugate pairs. So suppose \( \lambda \) and \( \lambda^* \) are eigenvalues. Then the corresponding eigenvectors are also complex conjugates of each other: indeed, \( A \) being real,
\[ A \cdot v = \lambda v \quad \Rightarrow \quad A \cdot v^* = \lambda^* v^*. \] (6.24)

Then the corresponding bit of the general solution (6.23) is
\[ C_1 v e^{\lambda t} + C_2 v^* e^{\lambda^* t} = A \text{Re}(v e^{\lambda t}) + B \text{Im}(v e^{\lambda t}) \]
\[ = e^{\Gamma t} \left[ (A \cos \Omega t + B \sin \Omega t) \text{Re}(v) + (B \cos \Omega t - A \sin \Omega t) \text{Im}(v) \right], \] (6.25)
where \( \lambda = \Gamma + i\Omega \) and the constants \( A = C_1 + C_2 \) and \( B = i(C_1 - C_2) \) can be treated as real. The solutions \( \text{Re}(v e^{\lambda t}) \) and \( \text{Im}(v e^{\lambda t}) \) are real and linearly independent both of each other and of all the other solutions, so they can be used instead of \( v e^{\lambda t} \) and \( v^* e^{\lambda^* t} \). You get to practice this trick in Q3.3 (example).
For convenience of further manipulations, let me collect all the exponentials into a diagonal matrix:

\[ E(t) = \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \equiv \text{diag}\{e^{\lambda_i t}\}. \quad (6.26) \]

With this notation,

\[ \xi(t) = E(t) \cdot C \Rightarrow y(t) = R \cdot E(t) \cdot C, \quad (6.27) \]

where, as usual, \( C \) is the vector of integration constants. In the language of §4.3, we have found the fundamental matrix to be

\[ Y(t) = R \cdot E(t), \quad (6.28) \]

as promised in (6.7).

Thus, we have the general solution of (6.3). Note that we did not need to calculate \( R^{-1} \) explicitly in order to obtain (6.23) or (6.27). We will need it to find \( C \) necessary to enforce any given initial condition.

So, let us now solve the Cauchy problem, \( y(0) = y_0 \). Since \( E(0) = I \), we have

\[ R \cdot C = y_0 \Rightarrow C = R^{-1} \cdot y_0 \Rightarrow y(t) = R \cdot E(t) \cdot R^{-1} \cdot y_0, \quad (6.29) \]

which, in view of (6.28), is just (4.22), because \( Y(0) = R \). Let me reiterate the procedure that the formula (6.29) represents:

—take the initial disturbance \( y_0 \) and project it \( (R^{-1} \cdot y_0) \) onto the eigendirections of the matrix \( A \);

—evolve in time the part of the disturbance corresponding to each eigendirection with the exponential factor corresponding to its eigenvalue \( (E(t) \cdot R^{-1} \cdot y_0) \);

—project the result back onto the original coordinates \( (R \cdot E(t) \cdot R^{-1} \cdot y_0) \).

The physics of it all is now clear: in a physical system represented by \( A \), initial disturbances along each eigenvector \( v_i \) of \( A \) grow or decay at the rate \( \lambda_i \). Any arbitrary initial disturbance \( y_0 \) can be decomposed into these elementary ones and evolves accordingly.

**Exercise 6.1.** How does one solve the system

\[ \dot{y} = \alpha(t) A \cdot y, \quad (6.30) \]

where \( \alpha(t) > 0 \) is a known, continuous, scalar function and \( A \) a time-independent matrix?

### 6.1.1. General Solution of Inhomogeneous Equation

The solution of the Cauchy problem for the inhomogeneous equation (6.1) is found in the usual way, via variation of constants, etc. In fact, it is just an application of the general result (4.38) to the case where \( Y(t) \) is given by (6.28), \( t_0 = 0 \) and \( Y(0) = R \). Noticing that

\[ Y^{-1}(t') = E^{-1}(t') \cdot R^{-1} = E(-t') \cdot R^{-1} \quad \text{and} \quad Y(t) \cdot Y^{-1}(t') = R \cdot E(t-t') \cdot R^{-1} \quad (6.31) \]

because \( \forall t_1 \) and \( t_2, E(t_1)E(t_2) = E(t_1 + t_2) \), we get

\[ y(t) = R \cdot E(t) \cdot R^{-1} \cdot y_0 + \int_0^t dt' R \cdot E(t-t') \cdot R^{-1} \cdot f(t') \quad . \quad (6.32) \]
Examining (6.32) and comparing it with the solution (2.61) of the 1D problem, you should now be struck by the exhilarating realisation that we have just worked out is how to exponentiate a matrix:

\[ e^{At} = R \cdot E(t) \cdot R^{-1}. \]  

(6.33)

The trick was to diagonalise the matrix, exponentiate the eigenvalues, put them on the diagonal and transform it all back to the original coordinates. With this prescription, solving ODEs in \( n \) dimensions is no more difficult (conceptually) than solving them in one. Knowledge is power. Now that you have it, use it.

Exercise 6.2. What is the Green’s function (§4.5) for a system of linear ODEs with constant coefficients? Does it make sense as a generalisation of the 1D case (Exercise 4.3)?

Example. Consider the vector generalisation of the exponential force (5.36):

\[ f(t) = F e^{\mu t}, \]  

(6.34)

where \( F \) is a constant vector. Let us work out the particular solution for this case—the second term in (6.32):

\[
y_{PI}(t) = \int_0^t dt' R \cdot E(t - t') \cdot R^{-1} \cdot F e^{\mu t'} \\
= R \cdot \text{diag} \left\{ e^{\lambda_i t} \int_0^t dt' e^{(\mu - \lambda_i)t'} \right\} \cdot R^{-1} \cdot F \\
= R \cdot \text{diag} \left\{ \frac{e^{\mu t} - e^{\lambda_i t}}{\mu - \lambda_i} \right\} \cdot R^{-1} \cdot F \]  

(6.35)

This formula works provided \( \mu \neq \lambda_i \). If, for some \( i \), \( \mu = \lambda_i \), then that element of the diagonal matrix must be replaced by \( t e^{\lambda_i t} \). This follows either by doing the time integral correctly for this special case or by taking the limit \( \mu \to \lambda_i \):

\[
\frac{e^{\mu t} - e^{\lambda_i t}}{\mu - \lambda_i} = e^{\lambda_i t} \frac{e^{(\mu - \lambda_i)t} - 1}{\mu - \lambda_i} \to t e^{\lambda_i t} \quad \text{as} \quad \mu - \lambda_i \to 0. 
\]  

(6.36)

Example. Q3.4.

Sometimes, before solving the Cauchy problem, it is easier to find the general solution first, equipped with \( n \) constants:

\[ y(t) = \sum_i C_i v_i e^{\lambda_i t} + y_{PI}(t). \]  

(6.37)

Then, for the force in the form (6.34), one gets the particular solution as follows:

\[ y_{PI}(t) = u e^{\mu t} \quad \Rightarrow \quad \mu u = A \cdot u + F \quad \Rightarrow \quad u = -(A - \mu I)^{-1} \cdot F. \]  

(6.38)

Again, this only works if \( \mu \) is not an eigenvalue of \( A \) (otherwise the inverse does not exist).

Exercise 6.3. Work out what to do if \( \mu \) is an eigenvalue of \( A \).

Example. Consider

\[
\begin{aligned}
\dot{x} &= 3x + 2y + 4e^{5t}, \\
\dot{y} &= x + 2y.
\end{aligned} \]  

(6.39)

To solve this, consider the homogeneous equation first. It is in the form (6.3), where the matrix is

\[
A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = \lambda^2 - 5\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 1. 
\]  

(6.40)
The corresponding eigenvectors are

\[ \boldsymbol{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]

Therefore, the general solution of the homogeneous equation is

\[ \begin{pmatrix} x \\ y \end{pmatrix}_{\text{CF}} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t. \] (6.42)

This is the homogeneous bit of (6.37). Now let us look for the particular solution in the form

\[ \begin{pmatrix} x \\ y \end{pmatrix}_{\text{PI}} = \begin{pmatrix} A \\ B \end{pmatrix} e^{5t} \Rightarrow A = 3, \quad B = 1. \] (6.43)

The values of the constants are obtained by direct substitution of the trial solution into (6.39). Final answer:

\[ \begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{4t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t}. \] (6.44)

If such a simple method of finding \( \boldsymbol{y}_{\text{PI}} \) does not work, the universal panacea is to vary constants in (6.42): let \( C_1 \rightarrow \psi_1(t), \) \( C_2 \rightarrow \psi_2(t) \), substitute into (6.39) and solve for \( \psi_{1,2} \).

You can now have some fun generalising some of the other results of §§5.2.1 and 5.2.2 to the case of an \( n \)-dimensional system of ODEs. Note, however, that we have not yet learned how to deal with a situation in which not all \( \{\lambda_i\} \) are different.

### 6.2. Hermitian Systems

Suppose now that our matrix is Hermitian: \( \mathbf{A} = \mathbf{A}^\dagger(\equiv \mathbf{A}^* \mathbf{T}). \) If \( \{\mathbf{v}_i\} \) are its right eigenvectors, then

\[ \mathbf{A} \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i \Rightarrow \mathbf{v}_i^* \mathbf{A}^* = \mathbf{A}^* \cdot \mathbf{v}_i^* = \mathbf{A}^\dagger \cdot \mathbf{v}_i^* = \mathbf{v}_i^* \cdot \mathbf{A}. \] (6.45)

Thus, the left eigenvectors are \( \mathbf{u}_i = \mathbf{v}_i^* \) and, since the left eigenvalues must be the same as the right eigenvalues [see (6.19)], \( \lambda_i = \lambda_i^* \) are all real.\(^{32}\) Therefore, \( \mathbf{R}^{-1} = \mathbf{R}^\dagger \), a unitary matrix.\(^{33}\) Another very useful property is that, if \( \lambda_i \neq \lambda_j \), then the corresponding eigenvectors are orthogonal:

\[ \lambda_i \mathbf{v}_i \cdot \mathbf{v}_j^* = \mathbf{v}_j^* \cdot \mathbf{A} \cdot \mathbf{v}_i = \mathbf{u}_i \cdot \mathbf{v}_j^* \cdot \mathbf{v}_i \Rightarrow (\lambda_i - \lambda_j) \mathbf{v}_i \cdot \mathbf{v}_j^* = 0 \Rightarrow \mathbf{v}_i \cdot \mathbf{v}_j^* = 0. \] (6.46)

Therefore, if all \( \{\lambda_i\} \) are different, we are in the (simpler version of) the situation described in §6.1, when the matrix \( \mathbf{A} \) is diagonalisable and the solution of (6.1) is (6.32), with the added simplification that \( \mathbf{R}^{-1} = \mathbf{R}^\dagger \).

What if not all the eigenvalues of \( \mathbf{A} \) are different (i.e., some are degenerate)? For a Hermitian matrix, it is still guaranteed that there is an orthogonal basis \( \{\mathbf{v}_i\} \) in which \( \mathbf{A} \) is diagonal.

**Proof.** This is done by induction. First, when the dimension of \( \mathbf{A} \) is \( n = 1 \), \( \mathbf{A} \) is automatically diagonal.

Now consider an \( n \times n \) matrix \( \mathbf{A} \) with \( n > 1 \) and assume that, in fewer dimensions than \( n \), Hermitian matrices are diagonalisable. Suppose \( \lambda \) is one of the eigenvalues of \( \mathbf{A} \). Consider all vectors \( \mathbf{w} \) that satisfy

\[ \mathbf{A} \cdot \mathbf{w} = \lambda \mathbf{w}. \] (6.47)

---

\(^{32}\)This also follows directly from \( \lambda_i \mathbf{v}_i \cdot \mathbf{v}_i^* = \mathbf{v}_i^* \cdot \mathbf{A} \cdot \mathbf{v}_i = \lambda_i^* \mathbf{v}_i^* \cdot \mathbf{v}_i \).

\(^{33}\)If \( \mathbf{A} \) is real, then Hermitian means symmetric, \( \mathbf{A} = \mathbf{A}^T \), and \( \mathbf{u}_i = \mathbf{v}_i \), so \( \mathbf{R}^{-1} = \mathbf{R}^T \) is an orthogonal matrix, describing rotations in \( \mathbb{R}^n \).
They form a linear subspace, \( W \). All vectors \( \mathbf{v} \) orthogonal to \( W \) (i.e., all \( \mathbf{v} \) such that \( \forall \mathbf{w} \in W, \mathbf{w}^* \cdot \mathbf{v} = 0 \)) also form a linear subspace, \( V \), with \( \dim V < n \). Then, since \( A \) is Hermitian,

\[
\mathbf{w}^* \cdot A \cdot \mathbf{v} = (A \cdot \mathbf{w})^* \cdot \mathbf{v} = \lambda \mathbf{w}^* \cdot \mathbf{v} = 0 \quad \Rightarrow \quad A \cdot \mathbf{v} \in V.
\] (6.48)

Thus, \( V \) is invariant under \( A \), and, since \( \dim V < n \) and \( A \) is Hermitian, \( A \) restricted to \( V \) is diagonalisable. But \( A \) restricted to \( W \) is also diagonal, by assumption (6.47). Thus, \( A \) is diagonalisable over all space \( W + V \).

The practical way to construct the diagonalising basis is, for each eigenvalue \( \lambda \) whose degree of degeneracy is not unique (i.e., the number of times it is repeated) is \( m \), to realise that \( \dim W = m \) and pick any \( m \) orthogonal vectors in \( W \) (they are all eigenvectors with eigenvalue \( \lambda \) and their choice is not unique, but that is perfectly fine).

The diagonalisability of \( A \) means that the fundamental system of solutions of (6.3) still consists of the functions \( \mathbf{v}_i e^{\lambda_i t} \). Even if \( \lambda_i = \lambda_j \) for some \( i \neq j \), the corresponding solutions are linearly independent because they are all eigenvectors with eigenvalue \( \lambda \) and the basis vectors \( \mathbf{v}_i \) and \( \mathbf{v}_j \) are orthogonal. In other words, the solution of (6.1) is still (6.32). The “physical” difference with the non-degenerate case is that initial disturbances along the directions corresponding to a degenerate eigenvalue grow or decay at the same rate.

6.3. Non-Hermitian Systems

What if the matrix is non-Hermitian? As long as it is diagonalisable (which it definitely is if it has \( n \) different eigenvalues and which it might be if they are not all different), i.e., if there is a basis consisting of \( n \) linearly independent (but, not necessarily orthogonal) eigenvectors, then everything is fine and you can follow the procedure described in §6.1. But diagonalisability of non-Hermitian matrices is not guaranteed and situations when one must deal with non-diagonalisable matrices are quite common.

Here is a consequential example. Consider the \( n \)-th order linear ODE with constant coefficients

\[
y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y + a_0 y = f(t).
\] (6.49)

As I have explained before (§3.3), any such equation can be rebranded as a system of \( n \) coupled first-order ODEs. Indeed, letting \( p_i = y^{(i)} \) for \( i = 1, \ldots, n-1 \), we can rewrite (6.49) as follows

\[
\begin{cases}
p_{n-1} &= -a_{n-1} p_{n-1} - \cdots - a_1 p_1 - a_0 y + f(t), \\
p_{n-2} &= p_{n-1}, \\
\vdots \\
p_1 &= = p_2, \\
y &= p_1,
\end{cases}
\] (6.50)

or, in matrix form,

\[
\frac{d}{dt} \begin{pmatrix} p_{n-1} \\ p_{n-2} \\ \vdots \\ y \end{pmatrix} = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ p_{n-2} \\ \vdots \\ y \end{pmatrix} + \begin{pmatrix} f \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
\] (6.51)

The matrix here is very obviously not Hermitian, so nothing is guaranteed. It is not a hard exercise to show that the equation for its eigenvalues is simply the auxiliary equation (5.25):

\[
\det(A - \lambda I) = 0 \quad \Leftrightarrow \quad \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.
\] (6.52)

If this has \( n \) distinct roots, then \( A \) is diagonalisable by the scheme explained in §6.1, the general solution of the homogeneous equation associated to (6.49) is

\[
y(t) = C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t}
\] (6.53)

(cf. Exercise 5.2), and the solution of the inhomogeneous equation is obtained via variation of constants in the usual manner. If, on the other hand, any of the roots are degenerate, the matrix is not diagonalisable.
There are, of course, many other systems, not necessarily reducible to (6.49), in which $A$ is non-diagonalisable. This happens when for some eigenvalue $\lambda$ that occurs $m$ times, there are fewer than $m$ independent vectors that satisfy $A \cdot w = \lambda w$, i.e., the subspace $W$ of vectors that satisfy this equation has $\dim W < m$. It turns out that when one encounters such systems, one need not despair.

6.3.1. Solution by Triangulation

The reason there is hope is Schur’s triangulation theorem (Fig. 27).

**Theorem 4.** For any matrix $A$, there is a unitary transformation that converts it into an (upper) triangular form, viz., $\forall A \exists U$ such that $U^{-1} = U^\dagger$ and

$$U^\dagger \cdot A \cdot U = T = \begin{pmatrix} \lambda_1 & \text{stuff} \\ 0 & \lambda_n \end{pmatrix}. \quad (6.54)$$

The triangular matrix $T$ has stuff on and above the diagonal and zeros below. It is very easy to see that, if Schur’s theorem holds, the diagonal elements of $T$ are its eigenvalues and that they are also the eigenvalues of $A$:

$$\det(A - \lambda I) = \det(U \cdot T \cdot U^\dagger - \lambda U \cdot U^\dagger) = \det[U \cdot (T - \lambda I) \cdot U^\dagger] = \det(T - \lambda I) = (\lambda_1 - \lambda) \ldots (\lambda_n - \lambda). \quad (6.55)$$

I will prove Schur’s theorem in §6.3.2—and in the process show you how to find $U$—but first let me show you how to solve an arbitrary system of ODEs with constant coefficients once one has the triangulating unitary transformation in hand.

Consider the homogeneous equation (6.3) and assume that we have $U$ that satisfies (6.54), i.e.,

$$\dot{y} = A \cdot y = U \cdot T \cdot U^\dagger \cdot y. \quad (6.56)$$

Analogously to (6.21), we dot this with $U^\dagger$ on the left and get:

$$\frac{d}{dt} U^\dagger \cdot y = T \cdot U^\dagger \cdot y \Rightarrow \dot{\xi} = T \cdot \xi, \quad \text{where } \xi = U^\dagger \cdot y. \quad (6.57)$$

Let us call $T_{ij}$ the elements of $T$ above the diagonal and rewrite our system explicitly as follows

$$\frac{d}{dt} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & T_{12} & \cdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}. \quad (6.58)$$

34 In fact, you sort of already know how to do this, operationally: effectively, you have done this many times, when you applied the Gaussian elimination procedure to bring a matrix into an “upper echelon” form (Lukas 2017, §3.3).
Moving upwards and onwards, the (n-th line of (6.58) is
\[ \dot{\xi}_n = \lambda_n \xi_n \implies \xi_n(t) = C_n e^{\lambda_n t}. \] (6.59)
Then the (n-1)-st line of (6.58) is
\[ \dot{\xi}_{n-1} = \lambda_{n-1} \xi_{n-1} + T_{n-1,n} \xi_n(t). \] (6.60)
This is a first-order, inhomogeneous linear ODE and it is solved in the usual manner (§2.4.2), by variation of constants:
\[ \xi_{n-1}(t) = \psi(t) e^{\lambda_{n-1} t} \implies \dot{\psi} = T_{n-1,n} C_n e^{(\lambda_n - \lambda_{n-1}) t} \]
\[ \implies \psi(t) = C_{n-1} + C_n T_{n-1,n} \begin{cases} \frac{1}{\lambda_n - \lambda_{n-1}} & \text{if } \lambda_{n-1} \neq \lambda_n, \\ t & \text{if } \lambda_{n-1} = \lambda_n. \end{cases} \] (6.61)
Thus,
\[ \xi_{n-1}(t) = C_{n-1} e^{\lambda_{n-1} t} + C_n T_{n-1,n} e^{\lambda_n t} \begin{cases} \frac{1}{\lambda_n - \lambda_{n-1}} & \text{if } \lambda_{n-1} \neq \lambda_n, \\ t & \text{if } \lambda_{n-1} = \lambda_n. \end{cases} \] (6.62)
Moving upwards and onwards, the (n-2)-nd line of (6.58) is
\[ \dot{\xi}_{n-2} = \lambda_{n-2} \xi_{n-2} + T_{n-2,n-1} \xi_{n-1}(t) + T_{n-2,n} \xi_n(t). \] (6.63)
Again we have a first-order linear ODE with a known inhomogeneous part. Etc., etc. Once all \( \xi \)'s have been found in this manner, we can recover the solution of (6.3) by reversing the triangulating transformation,
\[ y(t) = U \cdot \xi(t). \] (6.64)
Observing how powers of \( t \) pop up in front of exponentials every time a degenerate eigenvalue is repeated, you can probably see now how to handle the last part of Exercise 5.2 (how did you handle it then? was that equivalent to what we are doing here?).

Finally, extending the above procedure to the general inhomogeneous equation (6.1) presents no conceptual difficulty, so I leave you to your own devices with that (cf. §5.2.2).

6.3.2. Proof of Schur’s Triangulation Theorem

The proof of Schur’s triangulation theorem gives one some idea as to how to construct the unitary transformation \( U \).

Start by finding just one eigenvector of our matrix:
\[ A \cdot v_1 = \lambda_1 v_1. \] (6.65)
Consider an orthonormal basis \( \{ w_i \} \) where \( w_1 = v_1 \) and all the other vectors are chosen so as to satisfy \( w_i^* \cdot w_j = \delta_{ij} \). Then the \( n \times n \) matrix
\[ R = (v_1 \ w_2 \ldots \ w_n) \] (6.66)
in which \( w_i \)'s serve as columns is a unitary transformation, viz., \( R^{-1} = R^\dagger \). Apply this transformation to \( A \):
\[ R^\dagger \cdot A \cdot R = \begin{pmatrix} v_1^* \\ w_2^* \\ \vdots \\ w_n^* \end{pmatrix} \begin{pmatrix} \lambda_1 v_1 \cdot A \cdot w_2 & \ldots & v_1^* \cdot A \cdot w_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & A_{n-1} \end{pmatrix}, \] (6.67)
where the \((n-1) \times (n-1)\) matrix \( A_{n-1} \) has elements \( w_i^* \cdot A \cdot w_j \) with \( i, j = 2, \ldots, n \). In words, we have found a unitary transformation whereby all sub-diagonal elements in the first column of our transformed matrix are now zero.

For \( n = 2 \), this completes the process of triangulation. For \( n > 2 \), subject the matrix \( A_{n-1} \) to
the same treatment by finding one of its eigenvectors and hence a unitary transformation \( R_{n-1} \) that clears up the subdiagonal elements in the first column of \( A_{n-1} \), etc. Keep going until the matrix has been turned into a triangular one.

Let me argue formally by induction that this works. We have seen that for \( n = 2 \), any matrix can be triangulated by a unitary transformation. Now suppose any \((n-1) \times (n-1)\) matrix can be so triangulated. Then, in (6.67), \( \exists U_{n-1} \), a unitary transformation such that

\[
A_{n-1} = U_{n-1} \cdot T_{n-1} \cdot U_{n-1}^†,
\]

(6.68)

where \( T_{n-1} \) is an \((n-1) \times (n-1)\) triangular matrix. Therefore, (6.67) becomes

\[
A = R \cdot \begin{pmatrix} 1 & 0 \\ 0 & U_{n-1} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 & \text{stuff} \\ 0 & T_{n-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & U_{n-1}^† \end{pmatrix} \cdot R^† = U \cdot T \cdot U^†,
\]

where

\[
T = \begin{pmatrix} \lambda_1 & \text{stuff} \\ 0 & T_{n-1} \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & U_{n-1} \end{pmatrix}
\]

(6.69)

are, respectively, the desired \( n \times n \) triangular matrix and the unitary transformation that turns \( A \) into it. Q.E.D.

Note that

\[
U_{n-1} = R_{n-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & U_{n-2} \end{pmatrix}, \quad U_{n-2} = R_{n-2} \cdot \begin{pmatrix} 1 & 0 \\ 0 & U_{n-3} \end{pmatrix}, \ldots,
\]

(6.70)

where \( R_{n-1}, R_{n-2}, \ldots \) are unitary transformations constructed analogously to (6.66), but for the \((n-1), (n-2), \ldots\)-dimensional matrices that are still in need of triangulation at the corresponding steps of this procedure.

Note, finally, that the technique of solution by triangulation is perfectly applicable to systems whose eigenvalues are not degenerate and that are, in fact, diagonalisable. Obviously, the diagonalisation (6.17) is also a triangulation, because a diagonal matrix is triangular. However, there is no requirement that any particular triangulation be unique, and the (in general, non-unitary) diagonalising transformation \( R \) in (6.17) is under no obligation to be the same as the unitary triangulating transformation \( U \). Obviously, in solving a diagonalisable system of ODEs, if you go down the triangulation route, while the intermediate steps will be different, you will still get the same answer—to see this, just assume that no two eigenvalues are the same and examine the output of the scheme laid out in §6.3.1.

**Exercise 6.4.** Prove that if the matrix \( A \) is Hermitian, then any triangulating transformation is the diagonalising transformation, i.e., the triangular matrix \( T \) is unique and equal to the diagonal matrix \( L \) of the eigenvalues of \( A \).

**Exercise 6.5.** Consider the second-order homogeneous equation (5.1), turn it into a system of first-order equations, and solve it by the triangulation method. Identify \( A, U \) and \( T \). For what value of \( b \) is the triangular matrix diagonal? Consider specially the case \( b = a^2/4 \) and check that you have recovered the solution (5.23).

### 6.3.3. Solution via Jordan Form


There is, in fact, an (operationally) simpler method for solving ODEs with non-diagonalisable matrices. It is based on another magic result from linear algebra: *Jordan’s theorem* (Fig. 28).

Let me first introduce some new objects. Consider a degenerate eigenvalue \( \lambda_1 \) that repeats \( m \) times. Suppose it has at least one eigenvector \( v_1 \). Then the set of vectors \( \{v_1, \ldots, v_k\} \), where \( k \leq m \), is called a *Jordan chain* of \( \lambda_1 \) if they satisfy

\[
A \cdot v_1 = \lambda_1 v_1,
\]

\[
A \cdot v_2 = \lambda_1 v_2 + v_1,
\]

\[
\ldots
\]

\[
A \cdot v_k = \lambda_1 v_k + v_{k-1}.
\]

(6.72)
Figure 28. Marie Ennemond Camille Jordan (1838-1922), French mathematician; the only mildly curious factoids about him that I have been able to find are that he was (i) a nephew of Puvis de Chavannes, (ii) famous for eccentric notation in his lectures at Collège de France (where he was the successor of Liouville; see Fig. 28a), (iii) not to be confused with Pascual Jordan, who worked with Born and Heisenberg on matrix quantum mechanics and was a Nazi storm trooper.

The eigenvector $v_1$ is called the generator of the chain, and the rest of the vectors associated vectors, or generalised eigenvectors of $A$.

Exercise 6.6. Prove that the vectors $\{v_1, \ldots, v_k\}$ given by (6.72) are linearly independent.

If $\lambda_1$ is non-degenerate ($m = 1$), then no associated vectors exist. If $m > 1$ and $\lambda_1$ has $l < m$ independent eigenvectors, then it will have $l$ different Jordan chains generated by them and the total length of these chains will be $k_1 + \cdots + k_l = m$.

This means that, given $n$ eigenvalues $\{\lambda_i\}$, some of which may be degenerate, one can always construct a basis consisting of their eigenvectors and their associated vectors. If, as usual, we pack these basis vectors together as columns of a matrix $R$, the latter matrix will transform $A$ into the following Jordan form:

$$R^{-1} \cdot A \cdot R = J = \begin{pmatrix} J_1 & 0 & \cdots \\ 0 & J_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

(6.73)

where to each Jordan chain of eigenvalue $\lambda_i$ within the basis corresponds the Jordan block

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 \\ 1 & \lambda_i & 1 \\ 0 & \cdots & \lambda_i \end{pmatrix}$$

(6.74)

($\lambda_i$ on the diagonal, 1’s above the diagonal, 0’s everywhere else). That all of this can always be done is guaranteed by the following Jordan’s theorem.

Theorem 5. For any matrix $A$, there is always a basis in $\mathbb{C}^n$ consisting of Jordan chains, or, $\forall A \exists R$ such that $R^{-1} \cdot A \cdot R = J$.

While Jordan’s theorem is much harder to prove than Schur’s, it provides a much easier algorithm for actually solving ODEs with degenerate eigenvalues. Indeed, as usual, we have

$$\dot{y} = A \cdot y = R \cdot J \cdot R^{-1} \cdot y \ \Rightarrow \ \dot{\xi} = J \cdot \xi, \ \text{where} \ \xi = R^{-1} \cdot y.$$  

(6.75)

$^{35}$Some of which can be trivial $1 \times 1$ “matrices” containing just $\lambda_1$ as their sole element.
Let us consider any one Jordan block in $J$, say $J_1$. It generates the following set of equations:

$$
\begin{align*}
\dot{\xi}_1 &= \lambda_1 \xi_1 + \xi_2, \\
\dot{\xi}_{k-1} &= \lambda_1 \xi_{k-1} + \xi_k, \\
\dot{\xi}_k &= \lambda_1 \xi_k.
\end{align*}
$$

(6.76)

Solving them starting from the last, in the manner similar to what we did in §6.3.1, we get

$$
\xi_k(t) = C_k e^{\lambda_1 t},
$$

$$
\xi_{k-1}(t) = (C_{k-1} + C_k t) e^{\lambda_1 t},
$$

$$
\xi_{k-2}(t) = \left( C_{k-2} + C_{k-1} t + C_k \frac{t^2}{2} \right) e^{\lambda_1 t},
$$

$$
\cdots
$$

$$
\xi_1(t) = \left( C_1 + C_2 t + \cdots + C_k \frac{t^{k-1}}{(k-1)!} \right) e^{\lambda_1 t}.
$$

(6.77)

Since $y = R \cdot \xi$, the corresponding part of the solution of our original ODE is, in terms of the vectors of the Jordan chain $\{v_1, \ldots, v_k\}$,

$$
y(t) = \xi(t) v_1 + \cdots + \xi(t) v_k 
= e^{\lambda_1 t} \left[ C_1 v_1 + C_2 (tv_1 + v_2) + C_3 \left( \frac{t^2}{2} v_1 + tv_2 + v_3 \right) + \cdots 
+ C_k \left( \frac{t^{k-1}}{(k-1)!} v_1 + \cdots + tv_{k-1} + v_k \right) \right]
= C_1 y_1(t) + \cdots + C_k y_k(t).
$$

(6.78)

Doing this for each Jordan block and then adding up all the solutions, we get the general solution of our original system. Indeed, since the number of linearly independent solutions $y_i(t)$ and, accordingly, integration constants $C_i$ in (6.78) is equal to the length of the Jordan chain $(k)$, the sum of the lengths of the Jordan chains corresponding to each degenerate eigenvalue is equal to its degree of degeneracy $(m)$ and the sum of the degrees of degeneracy of all eigenvalues of $A$ is equal to $n$, the resulting solution will have $n$ constants multiplying $n$ independent solutions.

This is easy to use without having to remember these derivations: to solve the general homogeneous ODE (6.3),

—find all eigenvalues $\{\lambda_i\}$ of the matrix $A$;
—find all eigenvectors corresponding to each $\lambda_i$;
—if $\lambda_i$ is repeated $m$ times but only has $l < m$ eigenvectors, find $m - l$ associated vectors by constructing a Jordan chain (6.72) generated by each eigenvector;
—for each Jordan chain, write a piece of the general solution in the form (6.78);
—add them all up—this is the general solution of (6.3);
—enforce initial conditions via the $n$ available constants.

If you are faced with an inhomogeneous equation (6.1), you will need to find a particular solution. In principle, our general formula (4.38) tells you how to do this if you know the fundamental matrix $Y(t)$. You do know this matrix: the linearly independent solutions $y_i(t)$ that are its columns are just $e^{\lambda_i t}$ times the vector polynomial expressions multiplying the integration constants in (6.78). For a quasipolynomial force, i.e., for a force consisting of products of polynomials and exponentials, you can work out the answer explicitly (cf. §5.2.2). I shall leave you to it.

Example. Q3.7.
7. Qualitative Solution of Systems of Autonomous Nonlinear ODEs

I promised at the start of this course that, once we have got through all the necessary standard material, we could have some fun and learn how to solve complicated systems in an uncomplicated (=qualitative) way. One might call it an enlightened theory of sketching—fittingly, it has an element of art to it, as well as of science. What we shall be sketching will be phase portraits—the basic lay out of phase trajectories for autonomous ODEs

\[ \dot{y} = f(y) \] (7.1)

(the salient feature being the absence of explicit time dependence in the right-hand side).

Let me remind you that phase trajectories are the projections of the integral curves (§1.2) onto the phase space, which is the space where the solutions \( y(t) \) live (typically, it is all or part of \( \mathbb{R}^n \)). In other words, they are the “field lines” of the phase velocity \( f(y) \), i.e., they are the curves whose tangent vector at \( y \) is \( f(y) \). The usefulness of a phase portrait is that it tells you where the system will go from any given initial location in phase space—regardless of exactly \( \text{when} \) it might find itself at that location. It only makes sense to draw phase portraits for autonomous systems because if the phase velocity \( f \) changed with time, any one-time snapshot of its field lines would be transient and we could not trace the system’s time evolution by following those field lines.\(^{36}\)

Mathematically, this narrative amounts to saying that

1) if \( y(t) \) is a solution of (7.1), then \( \forall t_0 \ y(t + t_0) \) is also a solution, i.e., a time-shifted phase trajectory is the same trajectory;

2) if \( y_1(t) \) and \( y_2(t) \) are phase trajectories and \( y_1(t_1) = y_2(t_2) \) for some \( t_1 \) and \( t_2 \), then \( y_2(t) = y_1(t + t_1 - t_2) \), i.e., any two phase trajectories either do not intersect or coincide.

**Exercise 7.1.** These statements are perhaps obvious, but do practice your mathematical reasoning by proving them formally. Think also about how they fail for non-autonomous systems.

All phase trajectories can be classed into three types:

1) trajectories without self-intersections;

2) closed loops, or cycles, i.e., trajectories that bite their own tail;

3) fixed points (or stationary points, or equilibria)—these are phase trajectories consisting of a single point such that the system starting exactly at that point remains there forever. They are simply solutions of

\[ f(y_0) = 0. \] (7.2)

Note that fixed points do not have to be isolated—in principle, there can be whole continuous sets of them (see, e.g., §7.1.3).

**Exercise 7.2.** This classification is a theorem. Prove it. What you have to prove is that if a phase trajectory is not a solution of (7.2) and intersects itself, then it is closed loop, periodic forever.

Clearly, fixed points are special.

Mathematically, their intuitive specialness can be given precise meaning by the statement that, in the vicinity of any point in phase space that is not a fixed point, the

\(^{36}\text{Sometimes, for non-autonomous systems, one can do this in some average sense: usually, when the rate of non-autonomous change of } f \text{ is either very large (§5.4) or very small (§7.2).} \)
phase trajectories can be mapped onto a set of straight-line segments, whereas near a fixed point, they cannot be [this should be obvious from (7.3)]. This is the basis for the intuitive expectation that one might be able to sketch the phase portrait of a system if one works out what happens near all of its fixed points and then “connects” these local mini-portraits in some sensible way. This is not always quite as straightforward as it sounds, but it is the basic idea.

Physically, one is also obviously interested in the fixed points because they are the equilibria of the system. If a fixed point proves to be stable, i.e., if one can show that, starting in some vicinity of it, the system eventually ends up arbitrarily close to it, then such a fixed point represents long-term fate of the system, at least for some range of initial conditions (this is called the basin of attraction). Long-time behaviour is what one is often after, so this is clearly useful.

Since we are interested in the system’s behaviour near a particular point $y_0$, let us expand (7.1) in the vicinity of $y_0$, as promised in §3.5.3:

$$\delta \dot{y} = f(y_0 + \delta y) = f(y_0) + A \cdot \delta y + f_{nl}(y_0, \delta y), \quad A_{ij} = \left. \frac{\partial f_i}{\partial y_j} \right|_{y=y_0},$$

(7.3)

where $f_{nl}(y_0, \delta y)$ represents the nonlinear terms and is assumed to vanish quicker than linearly with $\delta y$:

$$\lim_{|y| \to 0} \frac{f_{nl}(y_0, \delta y)}{|\delta y|} = 0.$$  

(7.4)

If $y_0$ is not a fixed point, then (7.3) just describes locally constant motion [$\delta \dot{y} \approx f(y_0) = \text{const}$], but if it is, $f(y_0) = 0$ and we have an approximately linear system in $\delta y$. This is what we are now going to study, while returning from time to time to the question of whether our qualitative conclusions are likely to withstand the effect of small $f_{nl}$.

Life in $n$ dimensions is complicated and sketching it on a 2D sheet of paper requires more artistic skill and geometrical imagination than we can perhaps muster within the confines of this course. I will, therefore, start—and stay—with 2D systems. I hasten to say that 2D is rather special: basically because in 2D, a trajectory constrained never to cross itself is very constrained indeed. However, a physicist’s intuition is not built in a day, so let us see what a 2D world can teach us.

7.1. Classification of 2D Equilibria


You can view this section as a preparation for something bigger (not, I am afraid, to...
be fully consummated within the time limitations of this course) or just as a set of (very simple) examples of how one uses the analytical machinery developed in §6.

Entirely without loss of generality, $y_0 = 0$ and $\delta y = y$ in (7.3). We are specialising to the 2D system

$$\dot{y} = A \cdot y \iff \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{where} \quad y = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (7.5)$$

You know the drill: the first order of business is to find eigenvalues:

$$\det(A - \lambda I) = 0 \implies \lambda^2 - (a + d) \lambda + ad - bc = 0 \implies \lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}, \quad (7.6)$$

where $T = \text{tr}A$ and $D = \det A$. There are a number of possibilities, each of which has a name and a distinctive phase portrait.

7.1.1. **Nodes:** $T^2 > 4D > 0$

If we assume also $T > 0$, this corresponds to

$$\lambda_1 > \lambda_2 > 0. \quad (7.7)$$

With two different eigenvalues available, we are guaranteed that there will be two linearly independent eigenvectors $v_1$ and $v_2$, which, when packed into $R = (v_1 \ v_2)$, will transform our system into a diagonal form (see §6.1):

$$\begin{pmatrix} x \\ y \end{pmatrix} = R \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi v_1 + \eta v_2 \implies \begin{cases} \dot{\xi} = \lambda_1 \xi, \\ \dot{\eta} = \lambda_2 \eta \end{cases} \implies \begin{cases} \xi = \xi_0 e^{\lambda_1 t}, \\ \eta = \eta_0 e^{\lambda_2 t} \end{cases}. (7.8)$$
The phase trajectories in the plane \((\xi, \eta)\) are, therefore, the curves\(^{39}\)
\[
\frac{\xi}{\xi_0} = \left(\frac{\eta}{\eta_0}\right)^{\lambda_1/\lambda_2}.
\]  
(7.9)

Since \(\lambda_1/\lambda_2 > 1\), these are “generalised parabolae”, i.e., they are tangent at zero to the “slow” direction \(v_2\). Note that \(\xi = 0\) and \(\eta = 0\) are also phase trajectories, corresponding to the initial conditions \(\xi_0 = 0\) and \(\eta_0 = 0\), respectively. The corresponding phase portrait is sketched in Fig. 29(a)—the arrows, indicating the directions of the system’s travel, are based on the fact that both \(\xi\) and \(\eta\) must increase with time. This type of equilibrium is called an unstable node.

**Exercise 7.3.** The phase trajectories seem to intersect at zero. Is this a problem?

Since \(v_1\) and \(v_2\) are not necessarily orthogonal, this picture gets somewhat distorted when transformed back into the \((x, y)\) plane (Fig. 29b), but its basic feature remains the same: trajectories run away from the fixed point, hugging the slow eigendirection.

If we let \(T < 0\) and rearrange the eigenvalues to be
\[
\lambda_1 < \lambda_2 < 0,
\]  
(7.10)

we get the same story, but with direction of time reversed, i.e., the trajectories now all rush into the fixed point, again hugging the slow eigendirection (Figs. 29c,d). This is called a stable node. An example of this type of behaviour was the overdamped oscillator, where indeed it was the slow damping that dominated long-time dynamics (see §5.1.1 with \(T = \gamma\) and \(D = \omega_0^2\)).

7.1.2. **Saddles: \(T^2 > 0 > 4D\)**

In this case, the two eigenvalues have opposite signs:
\[
\lambda_1 > 0 > \lambda_2.
\]  
(7.11)

The same analysis applies as in §7.1.1, but (7.9) now becomes an equation for “generalised hyperbolae”:
\[
\frac{\xi}{\xi_0} = \left(\frac{\eta}{\eta_0}\right)^{-\lambda_1/|\lambda_2|}.
\]  
(7.12)

The trajectories approach the fixed point along the stable eigendirection \((v_2)\) and then veer off and away along the unstable one \((v_1)\): see Fig. 30. This type of equilibrium is

---

\(^{39}\)You can obtain (7.9) also by noticing that \(d\xi/d\eta = \dot{\xi}/\dot{\eta} = (\lambda_1/\lambda_2)\xi/\eta\) and integrating this equation.
called a **saddle**. We saw an example of it when we studied the “upright” equilibrium of the mathematical pendulum (§3.5.2, Fig. 14).

The two axes, $\xi = 0$ and $\eta = 0$, are also trajectories, the former leading the system straight into the fixed point, the latter out of it. These are called **separatrices**. The first separatrix, the $\eta$ axis, also known in the litterati circles as the **stable manifold**, is quite special in the sense that for a system starting off close to it, a small perturbation in the initial conditions that puts the system on the other side of the separatrix has the effect of dramatically altering subsequent evolution: the system will end up at $\xi = -\infty$ as opposed to $\xi = +\infty$, or vice versa. This is your first inkling of the phenomenon of **sensitive dependence on initial conditions**, which, transferred to more dimensions and more complicated systems, makes the world chaotic, but also opens the door to its statistical description (see Schekochihin 2019, §1).

### 7.1.3. Pathologies: $D = 0$

In this case, at least one of the eigenvalues is zero:

$$
\begin{align*}
\lambda_1 > \lambda_2 &= 0 \\
\lambda_1 < \lambda_2 &= 0
\end{align*}
$$

(7.13)

This means that

$$
\dot{\eta} = 0,
$$

(7.14)

so the straight lines $\eta = \text{const}$ are phase trajectories, which all run away from the $\xi = 0$ axis if $\lambda_1 > 0$ or towards it if $\lambda_1 < 0$ (Fig. 31). The entire line $\xi = 0$ consists of (non-isolated) fixed points. I will call these cases **stripes** (to match the stars in §7.1.5).

Obviously, if $\lambda_1 = 0$ as well, then every point in the plane is a fixed point of our linear system, but this is an intensely uninteresting case.

### 7.1.4. Effect of Nonlinearity

I have called the above case a pathology because it requires some rather special arrangements ($D = 0$). It is also a bit of an irrelevance because what, in fact, a zero eigenvalue implies is that the linearisation of the system in the corresponding eigendirection has proved trivial and so the dynamics in that direction is fully decided by the nonlinearity, which has so far been lurking invisibly on the sidelines. For example, if, in (7.3), $f_{\text{nl}}$ projected on the $\xi = 0$ axis proves to be a positive contribution at $|\eta| > 0$, say $\eta^2$, so

$$
\dot{\eta} \approx \eta^2,
$$

(7.15)

then we again only have a single fixed point (at zero) and this fixed point is unstable in the $\eta$ direction—a nonlinearly unstable saddle (if $\lambda_1 < 0$) or node (if $\lambda_1 > 0$), with the
departing trajectories hugging the slow (nonlinear) direction even more tightly than in the linear case.\footnote{An example of such a situation is the \( x = y = 0 \) equilibrium in Q3.12.}

This brings in the broader question: to what extent are the local phase portraits that we construct on the basis of linear theory likely to be good qualitative descriptions of what happens nonlinearly, even in the vicinity of the fixed point? The basic qualitative answer is that linear conclusions are fine as long as they do not depend on the eigenvalues being precisely equal to some special value: thus, nodes and saddles survive because a small perturbation of the trajectories by the nonlinear terms cannot knock them off the basic trends of exponential growth or decay set by robustly non-zero, non-equal eigenvalues—unlike in the pathological case of §7.1.3, where a zero eigenvalue gives the nonlinearity license to take over.\footnote{A fixed point with all \( \text{Re} \lambda \neq 0 \) is called hyperbolic (just a name, nothing to do with saddles, necessarily). One can show that local phase portraits around hyperbolic fixed points are robust. Mathematically, “robust” means topologically equivalent, which means that the phase portraits are topologically equivalent to their linear approximations in the vicinity of the fixed point.}

Mathematically, it is possible to prove qualitative robustness of the linear phase portraits by placing careful upper bounds on the nonlinearity’s contribution. All this can be made much more precise, but I hope this qualitative discussion makes enough sense to be believable at the first pass. If you want to see for yourself how to investigate the matter quantitatively, and, in particular, how to prove some of the statements about the effect of the nonlinearities that I have made above or will make below, a good undergraduate read is Glendinning (1994, §5.2), while the original source of biblical proportions is Andronov et al. (1973).

7.1.5. Pathologies: \( T^2 = 4D \)

This is another pathological case: the eigenvalues are equal,

\[
\lambda_1 = \lambda_2 \equiv \lambda \neq 0 ,
\]

and can be either positive or negative (we are not interested in the trivial case of them being zero).

In this case, it matters whether the matrix \( A \) has two linearly independent eigenvectors or only one. Suppose it has two. Then they span the plane and, in fact, any vector is an eigenvector, i.e., the system can hurtle towards (\( \lambda < 0 \)) or away from (\( \lambda > 0 \)) the fixed point along any straight line originating at zero (Fig. 32a). Indeed, the phase trajectories satisfy (7.9) with \( \lambda_1/\lambda_2 = 1 \):

\[
\frac{\xi}{\xi_0} = \frac{\eta}{\eta_0} ,
\]

which describes straight lines. Such a node is called a star (stable or unstable depending on the sign of \( \lambda \)).

Let us work out precisely for what kind of matrix \( A \) this scenario occurs. All vectors must be eigenvectors, so we must have

\[
\forall \begin{pmatrix} x \\ y \end{pmatrix} , \quad \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad \Rightarrow \quad b = c = 0 , \quad a = d = \lambda \quad \Rightarrow \quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} .
\]

Thus, stars are a bit boring. Are they also irrelevant, requiring \( b = c = 0 \) and thus potentially sensitive to nonlinear perturbations?

It turns out that it is possible to find classes of \( f_{\text{nl}} \) such that a star will stay a star (i.e., trajectories will still approach the fixed point at all angles) or will look more like a node (with trajectories hugging one direction) or a focus (§7.1.6).\footnote{In particular, a star stays a star if the nonlinearity is quadratic or higher-order.} Let us see now what happens in the arguably more generic situation when \( bc \neq 0 \), i.e., when \( A \) is allowed to have off-diagonal terms. It is then not diagonalisable, but, as we know from
Figure 32. (a) An unstable star in the basis \( \{ v_1, v_2 \} \) where \( A \) is diagonal \( (\lambda_1 = \lambda_2 > 0) \). In the original Cartesian basis, the picture is the same, with trajectories still approaching the fixed point at all angles. A stable case \( (\lambda_1 = \lambda_2 < 0) \) looks the same but with arrows reversed. (b) An unstable degenerate node in the triangulating basis \( \{ v, w \} \). (c) A stable degenerate node in the same basis—it is a mirror image of (b).

Schur’s theorem (§6.3.1), it can be turned into a triangular form by a unitary transformation. The matrix of this transformation is \( U = (v \ w) \), where \( v \) is the sole available eigenvector and \( w \) is any vector orthogonal to it (see §6.3.2). Then

\[
\begin{pmatrix} x \\ y \end{pmatrix} = U \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \xi v + \eta w \Rightarrow \begin{cases} \dot{\xi} = \lambda \xi + \alpha \eta, \\ \dot{\eta} = \lambda \eta, \end{cases}
\]

where we can always have \( \alpha > 0 \) (by reversing the sign of \( w \) if not). In order to sketch the phase portrait of this system, let us observe that

\[
\frac{d\eta}{d\xi} = \left( \frac{\xi}{\eta} + \frac{\alpha}{\lambda} \right)^{-1} \rightarrow \begin{cases} 0, & \text{when } \eta = 0, \xi \text{ finite,} \\ \frac{\lambda}{\alpha}, & \text{when } \xi = 0, \eta \text{ finite,} \\ \infty, & \text{when } \eta = -\frac{\lambda}{\alpha} \xi. \end{cases}
\]

This is enough to piece together a phase portrait: when \( \lambda > 0 \), phase trajectories run away from the fixed point along the one available eigendirection before executing a partial U-turn and taking off to infinity (Fig. 32b); when \( \lambda < 0 \), they do the same thing in reverse and in mirror reflection (Fig. 32c). These are unstable and stable degenerate (or improper) nodes. An example of this kind of behaviour was the critically damped oscillator (§5.1.2).

Like the stars, the degenerate nodes qualitatively survive the effect of quadratic or higher-order nonlinear terms, but for gentler nonlinearities, they are liable to change, roughly speaking in the direction of looking more like spirals converging into or coming out of foci (§7.1.6)—not, perhaps, surprising as they are a case that sits at the boundary in the \((T,D)\) space separating nodes and foci.

Note that the method that I have used to figure out the phase portraits in Figs. 32(b,c) is representative of the sort of thing you might generally have to resort to when in doubt: work out the phase-velocity field \( (\dot{\xi}, \dot{\eta}) \) [or \( (\dot{x}, \dot{y}) \)] on some special curves in phase space and try to make the phase portrait consistent with these scraps of information. The most obvious special curves are the axes of whatever coordinate system you choose to work in (here \( \xi = 0 \) and \( \eta = 0 \)) and the null-clines, which are the curves where \( \dot{\xi} = 0 \) or \( \dot{\eta} = 0 \) (equivalently, \( d\eta/d\xi = \infty \) or 0), i.e., where the “phase flow” is either vertical or horizontal.
7.1.6. **Foci: $4D > T^2 > 0$**

In this case, we have two different complex eigenvalues that are conjugates of each other:

$$\lambda_{1,2} = \Gamma \pm i\Omega,$$

where $\Gamma = \frac{T}{2}$, $\Omega = \sqrt{D - \frac{T^2}{4}}$.

(7.21)

An example of such a system is the underdamped oscillator (5.9), or, equivalently, the $\theta = 0$ equilibrium of the damped mathematical pendulum ($\S$3.5.4, Fig. 16). It is instantly clear what will happen qualitatively: the trajectories will wind around the fixed point with the period $\frac{2\pi}{\Omega}$ (this is the 2D representation of an oscillation) and spiral either in or out of it, depending on the sign of $\Gamma$ (Fig. 33). Fixed points of this type are called stable or unstable foci (or spirals). They turn out to be robust to any nonlinear meddling: close enough to the fixed point, a spiral stays a spiral, regardless of what the nonlinearity does (this makes intuitive sense).

In the interest of show-casing some nice mathematical tricks, let me demonstrate explicitly that phase trajectories are indeed spirals. Since the two eigenvalues are different, there will be two linearly independent eigenvectors, complex conjugates of each other: $v$ and $v^*$ [see (6.24)]. If we denote $v = u + i w$, where $u$ and $w$ are both real, we can make a matrix $S = (wu)$ and ask what it will do to $A$:

$$A = S \cdot \left( \begin{array}{cc} \Gamma & -\Omega \\ \Omega & \Gamma \end{array} \right) \cdot S^{-1}.$$  

(7.22)

If we apply this decomposition to our equation (7.5), we get

$$\dot{y} = S \cdot \left( \begin{array}{cc} \Gamma & -\Omega \\ \Omega & \Gamma \end{array} \right) \cdot S^{-1} \cdot y \quad \Rightarrow \quad \dot{\xi} = \left( \begin{array}{cc} \Gamma & -\Omega \\ \Omega & \Gamma \end{array} \right) \cdot \xi,$$

where $\xi = S^{-1} \cdot y \equiv \left( \begin{array}{c} \xi \\ \eta \end{array} \right)$.

(7.23)

Now let $\zeta = \xi + i\eta$. Then

$$\dot{\zeta} = \dot{\xi} + i\dot{\eta} = \Gamma \xi - \Omega \eta + i\Omega \xi + i\Gamma \eta = (\Gamma + i\Omega)\zeta \quad \Rightarrow \quad \zeta = r_0 e^{\Gamma t + i(\Omega t + \phi_0)},$$

(7.24)

where $r_0 = \sqrt{\xi_0^2 + \eta_0^2}$ and $\phi_0$ are the radius and phase of the initial condition. In the $(\xi, \eta)$ plane, this is an anticlockwise logarithmic spiral with period $2\pi/\Omega$ and radius $r = r_0 e^{\Gamma t} = r_0 e^{(\Gamma/\Omega)(\phi - \phi_0)}$. When transformed back into the $(x, y)$, this is still a spiral, if possibly somewhat

of the nonlinear and the linearised systems can be mapped onto each other by a continuous one-to-one transformation.

$^{42}$Q3.10(ii) is an example of the latter scenario.
Figure 34. (a) A centre \((\lambda_{1,2} = \pm i\Omega)\) in the basis \(\{v, w\}\), where \(A\) is transformed according to (7.22). (b) A centre in the original Cartesian basis. The basis \(\{e_1, e_2\}\) is one where the energy has the form (7.34); the basis vectors determine the directions of the semi-major and semi-minor axes of the elliptic trajectories.

Note that we have discovered the usefulness of the transformation to polar coordinates: for \((\xi, \eta) \rightarrow (r, \phi)\), our original equation (7.5) becomes just

\[
\begin{align*}
\dot{r} &= \Gamma r, \\
\dot{\phi} &= \Omega.
\end{align*}
\]

(7.25)

It is often a good idea to manipulate even your nonlinear system into these coordinates and think of the 2D dynamics in terms of distance from the origin and phase and of the nonlinearities as making the effective growth/damping rate \(\Gamma\) and frequency \(\Omega\) depend on \(r\) and \(\phi\) (see, e.g., Glendinning 1994, §5.2).

7.1.7. Centres: \(4D > T^2 = 0\)

In this case, the eigenvalues are purely imaginary:

\[
\lambda = \pm i\Omega.
\]

(7.26)

From the above analysis, it is instantly clear that phase trajectories will circle around the fixed point (Fig. 34), rather than spiralling in our out. These trajectories are ellipses, as I will prove in §7.1.8. A fixed point of this type is called a centre. An example is the \(\theta = 0\) equilibrium of an undamped pendulum (§3.5.1, Fig. 13).

In principle, this is again a pathological case and you might imagine (and would be correct in many cases) that a small nonlinearity that provides a damping or a growth of perturbations will turn a centre into a focus.

7.1.8. Conservative Systems

However, the case of centres is rather important because it describes a situation in which there is a conservation law: the closed trajectories are level sets of some function that, throughout the system’s evolution, stays the same as it was initially. We shall call this function energy. In many physical settings, energy conservation is a fundamental principle that governs both linear and nonlinear dynamics—in which case, the energy of the linearised system is an approximate version of the energy of the nonlinear system, so the nonlinearity conspires in making the system conservative. The phase trajectories must then stay on surfaces of constant energy. In such situations, a centre remains resilient to nonlinear perturbations.

We have seen an example of this in the nonlinear, undamped pendulum (§3.5.2, Fig. 15). A slight generalisation of this example is as follows. Consider a 1D system...
subject to a potential force:
\[
\dot{x} = -U'(x), \quad \text{or} \quad \begin{cases} \dot{x} = y \\ \dot{y} = -U''(x) \end{cases}
\] (7.27)

This conserves energy:
\[
E = \frac{y^2}{2} + U(x) \Rightarrow \dot{E} = 0.
\] (7.28)

At the fixed point, assumed to be at the origin, \(U'(0) = 0\), so, in its vicinity,
\[
\begin{cases} \dot{x} = y, \\ \dot{y} = -U''(0)x \end{cases} \quad \text{and} \quad E = \frac{y^2}{2} + U(0) + U''(0)x^2/2.
\] (7.29)

If \(U''(0) > 0\), the trajectories are ellipses \(y^2 + U''(0)x^2 = \text{const}\), and the energy conserved by the linear system is just the true energy (7.28) approximated near the fixed point. The phase trajectories of the full nonlinear system are level sets of \(E = \text{const}\) and so will also be closed curves, at least as long as \(E\) is sufficiently close to \(U(0)\). Thus, the qualitative nature of the centre is not destroyed by nonlinearity.

It is not hard to figure out what the conserved energy for the linearised system (7.5) is: it is clear that it should be quadratic in \(x\) and \(y\) and so we may let
\[
E = \alpha x^2 + 2\beta xy + \gamma y^2.
\] (7.30)

Since, \(\forall x\) and \(y\), we must have
\[
\dot{E} = 2\alpha x\dot{x} + 2\beta(\dot{x}y + xy) + \gamma y\dot{y} = 2(\alpha a + \beta c)x^2 + 2(\beta a + \beta d + \gamma c)xy + 2(\gamma b + \gamma d)y^2 = 0,
\] (7.31)

setting to zero the coefficients of \(x^2, y^2\) and \(xy\) gives us three linear equations for \(\alpha, \beta\) and \(\gamma\):
\[
\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \begin{pmatrix} c & 0 & 0 \\ a & d & c \\ 0 & b & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0.
\] (7.32)

These equations are readily seen to be solvable when \(T = a + d = 0\), as indeed we assumed when we embarked on studying the case of two imaginary eigenvalues. This gives us
\[
E = -\frac{1}{2}cx^2 + axy + \frac{1}{2}by^2 = y \cdot B \cdot y, \quad \text{where} \quad B = \frac{1}{2} \begin{pmatrix} -c & a \\ a & b \end{pmatrix}.
\] (7.33)

Since \(B\) is a symmetric matrix, it has two real eigenvalues \(\lambda_1\) and \(\lambda_2\) and two orthonormal eigenvectors \(e_1\) and \(e_2\), which, once combined into the orthogonal matrix \(R = (e_1 \, e_2)\), rotate our plane into the coordinates in which \(B\) is diagonal:
\[
E = y \cdot R \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot R^T \cdot y = \lambda_1 x^2 + \lambda_2 y^2, \quad \text{where} \quad \begin{pmatrix} x \\ y \end{pmatrix} = R \cdot \begin{pmatrix} X \\ Y \end{pmatrix}.
\] (7.34)

Since \(\lambda_1 \lambda_2 = \det B = -(bc + a^2)/4 = D/4 > 0\) by the founding assumption of this section, the eigenvalues have the same sign, which can be assumed positive (otherwise, redefine \(E \rightarrow -E\)). Thus, the integral curves of our equation (7.5) are ellipses \(E = \text{const}\) with semi-major and semi-minor axes \((E/\lambda_1)^{1/2}\) and \((E/\lambda_2)^{1/2}\) (assuming \(\lambda_1 \leq \lambda_2\)). Their orientation is given by \(e_1\) and \(e_2\), respectively. \(^{44}\)

\(^{43}\) Another way to get this result is to notice that, in view of (7.24), \(E = \xi^2 + \eta^2 = y \cdot (S^{-1})^T \cdot S^{-1} \cdot y\), so \(B = (S^{-1})^T \cdot S^{-1}\) (up to an arbitrary multiplicative constant because if \(E\) is conserved, so is any multiple of it). Working out the eigenvector \(v = u + iw\), hence \(S = (u \, u)\), and hence \(B\), should get you (7.33). Yet another, and the quickest, way of getting (7.33) is to find \(E = H(x, y)\) from (7.35).

\(^{44}\) A somewhat quicker, but less satisfying to linear-algebra aficionados, way to pin down these
This is it: we have covered all possible 2D equilibria. This is sometimes called the Poincaré classification. In Fig. 35, all the different cases (12 of them: saddles, stable/unstable nodes, degenerate nodes, stars, stripes, foci, and centres) are placed in their applicability regions of the \((T,D)\) plane.

The way forward now is to identify the fixed points of nonlinear systems, work out local phase portraits and then “connect the dots” (Fig. 38). This works surprisingly well for many (simple) systems, especially if one is willing to be assisted by some additional tricks like looking at the null-clines (see discussion at the end of §7.1.5). However, before we can do this with a modicum of confidence, we have to learn about one other possible feature of 2D systems that can also describe their long-time behaviour and that certainly affects the look of their phase portraits: limit cycles.

**Exercise 7.4. Strogatz’s Classification of Love Affairs.** It might amuse you to recast the classification of 2D equilibria in terms of the dynamics of a relationship between two lovers whose feelings for one another depend on the other’s response: see Strogatz (1994, §5.3 and its accompanying exercises). Thinking about romantic interpretations and implications of different parameter regimes might help you master the material. #oxlove

**Exercise 7.5.** Classify the 3D equilibria of linear ODEs (Arnold 2006, §21.1 lets you have a glimpse of the answer but leaves some space for creativity).

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7.2. (Nonlinear) Digression: Hamiltonian Systems and Adiabatic Invariants

[Literature: Landau & Lifshitz (1976, §49)]

This topic does not strictly belong here, but there is nothing wrong with taking a quick peek at more advanced material when the context is begging for it.\(^4\)

The context in question is that of conservative systems, whose phase portraits locally look like centres and remain robust to nonlinearity because there is a conserved energy—the centres are the minima of this energy and its level sets are the phase trajectories of the system. Such ellipses is to notice that the phase-space points at the ends of the major and minor axes must satisfy \(\mathbf{y} \cdot \dot{\mathbf{y}} = 0\) (the phase velocity at these points is perpendicular to the radius vector; see Fig. 34b). This gives \(\mathbf{y} \cdot \mathbf{A} \cdot \mathbf{y} = 0\), whence one gets a quadratic equation for \(y/x\), its two solutions giving the orientations of the axes of the ellipses. Their lengths are then given by \((x^2 + y^2)^{1/2}\).

\(^4\)A nice, carefully worked example (Fig. 38b) can be found in Strogatz (1994, §6.4) or in Glendinning (1994, §5.3, the same example).

\(^4\)I am grateful to Radek Grabarczyk for getting me to include this material.
systems of ODEs in mechanics are known as Hamiltonian systems (Fig. 36). They have the general form:

\[
\begin{cases}
\dot{x} = \frac{\partial H}{\partial y}, \\
\dot{y} = -\frac{\partial H}{\partial x},
\end{cases}
\]  

(7.35)

where \(H(x, y)\) is a function, known as the Hamiltonian, which turns out to be the energy of the system: indeed, defining

\[\mathcal{E}(t) = H(x(t), y(t)),\]  

(7.36)

where \((x(t), y(t))\) is the phase trajectory, we find immediately, using (7.35), that

\[\dot{\mathcal{E}} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} = 0.\]  

(7.37)

Going back to §3.5.2, you might recall that equations of motion for the undamped pendulum could indeed be written in the form (7.35) [see (3.22)]. The same is very obviously true for the slightly more general case (7.27) and also for the general 2D linearised system (7.5) with \(d = -a\) and the energy (7.33).

Now let me venture slightly outside my declared remit of studying only autonomous systems and show you how sometimes one can get somewhere with certain non-autonomous ones. Imagine that some parameter of our system, which I will call \(\Lambda(t)\), is changing very slowly. "Slowly" means that, for a system following closed trajectories around a centre, the parameter changes little over one period \(T\) of the motion: formally,

\[T \dot{\Lambda} \ll \Lambda.\]  

(7.38)

In the new, non-autonomous system that such a situation engenders, \(x\) and \(y\) still satisfy (7.35), but the Hamiltonian function now depends on time via \(\Lambda\), viz., \(H = H(x, y, \Lambda(t))\). A good example of such a situation is again our pendulum, whose length is now slowly changes: e.g., the rope on which the mass hangs could be slowly pulled up (Fig. 37a), i.e., the slow-changing parameter is \(\Lambda = l(t)\). Then

\[\begin{cases}
\dot{x} = y, \\
\dot{y} = -\omega_0^2(t)x
\end{cases} \implies H = \frac{y^2}{2} + \omega_0^2(t)\frac{x^2}{2}, \text{ where } \omega_0(t) = \sqrt{\frac{g}{l(t)}}.
\]  

(7.39)
Figure 37. (a) Pendulum with slowly changing length. (b) Conserved adibatic invariant (7.48) is the area of phase space enclosed by a closed phase trajectory (which is a level set of the Hamiltonian).

The energy is no longer conserved:

$$
\dot{E} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial y} \dot{y} + \frac{\partial H}{\partial \Lambda} \dot{\Lambda} = \frac{\partial H}{\partial \Lambda} \dot{\Lambda}.
$$

(7.40)

It turns out, however, that there is another quantity that is conserved—more robustly than energy, even if only approximately. This quantity is called the adiabatic invariant and is equal to the area of the region of the phase space enclosed by the trajectory followed by the system in each oscillation. I am going to derive this result now.

Let us average (7.40) over the period $T$ of oscillations:

$$
\langle \dot{E} \rangle = \dot{\Lambda} \frac{1}{T} \int_{t}^{t+T} dt' \frac{\partial H}{\partial \Lambda}(x(t'), y(t'), \Lambda).
$$

(7.41)

Here $\dot{\Lambda}$ has been pulled outside the averaging because, by the assumption (7.38), $\Lambda(t)$ changes little over one period. Let us change the integration variable in the time integral:

$$
\int_{t}^{t+T} dt' \frac{\partial H}{\partial \Lambda} = \oint dx \frac{\partial y}{\partial \Lambda}.
$$

(7.42)

where the first of the equations (7.35) has been used for $dx/dt'$. The integral is now over a closed phase trajectory described by (7.35), or, equivalently, over a level set of the Hamiltonian with a given fixed energy:

$$
H(x, y, \Lambda) = \mathcal{E} = \text{const.}
$$

(7.43)

If we treat $x$ as the independent variable (over which we integrate), then $y = y(x, \Lambda, \mathcal{E})$. Let us differentiate (7.43) with respect to $\Lambda$:

$$
\frac{\partial H}{\partial \Lambda} + \frac{\partial H}{\partial y} \frac{\partial y}{\partial \Lambda} = 0.
$$

(7.44)

From here, we can substitute $\partial H/\partial \Lambda$ into (7.42):

$$
\int_{t}^{t+T} dt' \frac{\partial H}{\partial \Lambda} = - \oint dx \frac{\partial y}{\partial \Lambda}.
$$

(7.45)

This goes into (7.41), which also contains the period of the motion $T$. We can calculate it by the same change-of-integration-variable trick as in (7.42):

$$
T = \int_{t}^{t+T} dt' = \int_{t'}^{t+T} \frac{dt'}{dx} dx = \oint dx \left( \frac{\partial H}{\partial y} \right)^{-1} = \oint dx \frac{\partial y}{\partial \mathcal{E}}.
$$

(7.46)
Figure 38. Simple examples of phase portraits. (a) An abstract system with an unstable focus and a saddle: \( \dot{x} = x - y \), \( \dot{y} = x^2 - 4 \) (as an exercise, check that this is indeed the phase portrait). (b) A Lotka–Volterra population model (“rabbits and sheep” competing for the same grass) with three nodes and a saddle: \( \dot{x} = x(3 - x - 2y) \), \( \dot{y} = y(2 - x - y) \); see Strogatz (1994, §6.4) or Glendinning (1994, §5.3). The one interesting qualitative feature here is that, depending on which side of the stable manifold (the separatrix going into the saddle; see §7.1.2) you start, you will end up in a different long-term equilibrium.

Using (7.45) and (7.46), we now deduce from (7.41):

\[
\langle \dot{E} \rangle = -\dot{\Lambda} \oint \frac{dy}{dx} \frac{\partial y}{\partial \Lambda} \oint dxdy \partial y \partial E \Rightarrow 0 = \oint dxdy \left( \frac{\partial y}{\partial E} \langle \dot{E} \rangle + \frac{\partial y}{\partial \Lambda} \dot{\Lambda} \right) = \left\langle \frac{d}{dt} \oint ydxdy \right\rangle.
\]  

Thus, we have a quantity, the adiabatic invariant, that does not change on average, even though both \( E \) and \( \Lambda \) do (i.e., it changes even more slowly than they do):

\[
\langle I \rangle = 0, \quad \text{where} \quad I = \oint ydxdy.
\]  

But \( I \) is nothing but the area inside the closed phase trajectory (Fig. 37b), as I promised.

Let us return to our example (7.39) of a linear oscillator with a slowly changing length, and ask how the amplitude of the oscillations will change as the mass is slowly pulled up. For any given value of energy, the phase trajectory is an ellipse:

\[
\frac{y^2}{2} + \omega_0^2 \frac{x^2}{2} = \epsilon,
\]

with semi-axes \( \sqrt{2\epsilon} \) and \( \sqrt{2\epsilon}/\omega_0 \). The adiabatic invariant (7.48) is the area of this ellipse:

\[
I = \frac{2\pi\epsilon}{\omega_0}.
\]

Since this must stay constant as the frequency increases (as \( l \) gets smaller, \( \omega_0 \) gets larger), the energy of the oscillations will increase proportionally to it.

Thus, armed with adiabatic invariance, we can trace the time evolution of the phase portrait of a non-autonomous system and thus predict the system’s behaviour.

**Exercise 7.6.** What will happen to the mean square amplitude of the oscillations and to their mean square velocity? Sketch how the phase portrait of the pendulum will change with time.

### 7.3. Limit Cycles and Auto-Oscillations

[Literature: Strogatz (1994, §7), Glendinning (1994, §5.8)]

Apart from fixed points, limit cycles are the only other non-boring type of phase
trjectories in 2D (see the classification before Exercise 7.2)—the closed trajectories. Physically, they represent a nonlinear phenomenon known as auto-oscillations, which are instances of periodic (but not usually sinusoidal) behaviour that can happen in dissipative systems at finite amplitude and are an interesting example of a natural tendency for things to go back and forth.

As I noted before, the 2D world is quite restrictive for phase trajectories because they must wonder on a plane without ever intersecting each other or themselves. Imagine, e.g., that we can identify a region $\mathcal{R}$ of phase space, called a trapping region, that trajectories can enter but not exit: this can be done by tracing a closed curve $\mathcal{C}$ in the $(x, y)$ plane and showing that everywhere on $\mathcal{C}$ the phase-velocity field $(\dot{x}, \dot{y})$ either points inwards or is tangent to $\mathcal{C}$. What kind of future does a trajectory have if it enters such a region? One obvious possibility is that there is a stable fixed point inside it, perhaps a node (§7.1.1) or a focus (§7.1.6), into which all trajectories eventually converge (Fig. 39a).

What, however, if there are no fixed points there or if there are but they are unstable? It is not hard to imagine how to construct the latter situation. Let us say that we have a nonlinear system whose linearisation around a fixed point identifies that fixed point as a stable node or focus and whose global nonlinear behaviour is such that phase trajectories from far away drift towards that fixed point (i.e., there is a trapping region $\mathcal{R}$ enclosing the fixed point). Now imagine that we change some parameter of the system in such a way as to render the fixed point unstable (say, turn it into an unstable focus) without necessarily changing very much the far-away nonlinear behaviour of phase trajectories, so the trapping region still exists. Then we can easily draw a small circle enclosing our unstable fixed point in such a way that all phase trajectories cross this circle in the outward direction. Thus, we now have an annular trapping region (Fig. 39b), with no fixed points inside and with phase trajectories rushing in from both outside and inside the annulus. It is perhaps intuitive that they will not be able to wonder around there aimlessly forever but instead eventually converge onto a closed trajectory, a limit cycle.

\[\text{I mean isolated closed trajectories, i.e., ones in whose immediate vicinity the phase trajectories are not closed. In conservative systems, the whole phase space can be foliated into level sets of a conserved energy, which are all closed trajectories (§7.1.7), but, having spent quite a lot of time on them (see §§3.5.2, 7.1.7, 7.2, Q3.8, Q3.9), I am not interested in them anymore.}\]

\[\text{The latter case corresponds to a situation in which some part of } \mathcal{C} \text{ coincides with one of the phase trajectories. This sometimes proves to be a useful trick in identifying uncrossable borders in phase space (try this in Q3.12).}\]
Figure 40. (a) Jules Henri Poincaré (1854-1912), great French mathematician and polymath, responsible for too many foundational results and insights to give him justice in a caption. Neither modern mathematics nor physics would have been the same without him. A case in point for Landau’s observation that ability in science is measured on a log scale. (b) Ivar Otto Bendixson (1861-1935), Swedish mathematician, who was also a politician interested in the plight of poor students. He is famous mostly for dotting the i’s and crossing the t’s on Poincaré’s theorem.

Figure 41. Poincaré–Bendixson theorem: trajectory $C$ wonders into trapping region $R$, never leaves, ends up on the limit cycle.

7.3.1. Poincaré–Bendixson Theorem

This narrative is formally confirmed by the following *Poincaré–Bendixson theorem* (Fig. 40).

**Theorem 6.** Suppose that

(a) the phase velocity field of the ODE

$$\dot{y} = f(y)$$

is continuously differentiable in some open domain $D \subset \mathbb{R}^2$;

(b) there exists a phase trajectory $C$ of (7.51) that enters and never leaves some closed, bounded region $R \subset D$, which contains no fixed points of (7.51).

Then $R$ contains at least one closed, periodic phase trajectory of (7.51) and the trajectory $C$ converges to it as $t \to \infty$ (Fig. 41).

You will find a readable proof of this result in, e.g., Glendinning (1994, §5.8).

While, formally speaking, you only need one phase trajectory trapped in $R$, the practical strategy for applying the Poincaré–Bendixson theorem is to go for overkill and find $R$, a trapping region, such that no trajectories can leave and some definitely enter. Practical tips for constructing $R$ are:
Figure 42. Phase portraits for Odell’s model of population dynamics (Q3.12): (a) when there is a stable focus, (b) when there is an unstable focus and a limit cycle (the bold curve).

—look for curves in phase space that phase trajectories can cross in one direction only;

—where opportune, complete the boundary of \( R \) with pieces of phase trajectories (which other phase trajectories cannot cross);

—more often than not, limit cycles enclose unstable fixed points,\(^{52}\) as envisioned above, in which case \( R \) will have “holes” surrounding those fixed points.

An excellent example of all this is Q3.12 (Fig. 42), dealing with a model of population dynamics. Models of population dynamics, sometimes referred to as Lotka–Volterra\(^{53}\) equations, are a prolific field of application of the theory of dynamical systems. They typically involve two species that multiply and die at some rates related to their number and to the availability of food, and whose numbers are coupled to each other either because they compete for the same food supply (e.g., Fig. 38b) or because one of them is the food supply of the other (in which case we are dealing with a predator-prey model; e.g., Fig. 42).

If you followed the advice of footnote 45, you have already encountered one such model. In that example, there were a number of fixed points but no limit cycles. In fact, limit cycles are an endemic feature of predator-prey models. This makes sense: what they describe is a basic sequence of events in which the predators eat the prey, get happy and numerous, eat ever more prey, drive it near extinction, the situation becomes unsustainable, food shortages ensue, predator numbers start dwindling, prey gets a respite, starts procreating with renewed vigour, its numbers recover, at which point predators perk up at the increased food supply and the cycle repeats itself. Regrettable perhaps, but such is the (limit) cycle of life.

\(^{52}\)That is because they are often “born” as some parameter is changed and a stable fixed point becomes unstable—this is called a Hopf bifurcation (see Strogatz 1994, §8.2 or, for a somewhat more detailed treatment, Glendinning 1994, §8.8). This happens, e.g., in Q3.12: Fig. 42 shows the phase portraits of the system considered there with a stable focus (a) and with an unstable focus and a limit cycle (b); a Hopf bifurcation happened between (a) and (b). Eberhard Hopf (1902-1983) was an Austrian-German-American mathematician, student of Schur (Fig. 27), a founding father of the bifurcation theory and the ergodic theory. In a feat of bad judgment (or even worse politics), he managed to move to Nazi Germany from the US in 1936, where he proceeded to have a successful professorial career all through the lifetime of the Third Reich—and ended up back in the US after the war!

\(^{53}\)Alfred James Lotka (1880-1949), American chemist and statistician. Vito Volterra (1860-1940), Italian mathematician best known for his theory of integral equations. He was a senator of the Italian Kingdom, an enthusiastic participant of World War I, and an opponent of Mussolini, who kicked him out of his job and, in the end, Italy (Volterra was philosophical about it: “Muovono gl’imperi, ma i teoremi d’Euclide conservano eterna giovinezza”).
7.3.2. Auto-Oscillations

Since fixation on oscillators has been a hallmark of our course, let me discuss the circumstances in which limit cycles make an appearance in this context. Physically, they represent a situation in which a system undergoes sustained oscillations with a fixed period and finite amplitude (so they are a nonlinear phenomenon). This comes about when there is an energy source and an energy sink and the system finds a regime in which it can periodically draw energy from the source, then dissipate it, then draw more, dissipate again, etc. Your heart is a nonlinear oscillator of this kind, as is your grandfather’s clock (if you thought that its pendulum was an example of the boring, linear oscillations that we discussed in §§3.5 and §5, do investigate the matter more thoroughly!).

The simplest, best known example of such a system is van der Pol’s oscillator (Fig. 43a):

\[ \ddot{x} + \gamma(x^2 - 1)\dot{x} + x = 0, \quad \gamma > 0. \]  

(7.52)

When \( x \ll 1 \), this looks like the usual linear oscillator (5.10) (time rescaled to make \( \omega_0 = 1 \)), but with the sign of friction reversed: now \( \gamma \) represents an instability, i.e., any small oscillation initially triggered in the system will grow exponentially at the rate \( \gamma \). This is the energy source in the system. Since small initial perturbations grow, they must eventually stop being small, \( x \) reaching order-unity amplitude. At that point the \( \gamma x^2 \dot{x} \) term switches on, representing nonlinear friction: finite displacements are dissipated by the system—this is the energy sink.

Intuitively, it is perhaps clear that this should lead to a limit cycle: the instability amplifies perturbations, they get to a finite amplitude, overshoot \( (x > 1) \), nonlinear friction kicks in, perturbations decay, \( x \) drops below unity, instability turns on again, etc. Mathematically, it is not an elementary task either to prove that there is a limit cycle or to work out its shape—there is, in fact, a sizeable body of literature dedicated solely to van der Pol’s oscillator, which has proved an inspiration to nonlinear dynamicists.

In order to show you a solvable example of a limit cycle (there will be another in §7.3.4), let me consider a modified version of van der Pol’s equation (Anosov 2016),
which, as far as I know, does not describe anything real, but is easy to analyse:

\[ \ddot{x} + \gamma(x^2 + \dot{x}^2 - 1)\dot{x} + x = 0. \]  

(7.53)

Recasting (7.53) as a 2D dynamical system, we make it into

\[
\begin{cases}
\dot{x} = y, \\
\dot{y} = -x + \gamma(1 - x^2 - y^2)y.
\end{cases}
\]  

(7.54)

Let us go to polar coordinates, \( x = r \cos \phi, \ y = r \sin \phi, \) or, equivalently, to make calculations even more compact, turn our \( \mathbb{R}^2 \) plane into \( \mathbb{C} \): \( z = x + iy = re^{i\phi}. \) Then

\[ \dot{z} = \dot{r}e^{i\phi} + i\dot{\phi}re^{i\phi} = y - ix + i\gamma(1 - r^2)r \sin \phi. \]  

(7.55)

Dividing through by \( e^{i\phi} \) and requiring the real and imaginary parts of the resulting equation to be satisfied separately, we get

\[
\begin{cases}
\dot{r} = \gamma r(1 - r^2) \sin^2 \phi, \\
\dot{\phi} = -1 + \gamma r(1 - r^2) \sin \phi \cos \phi.
\end{cases}
\]  

(7.56)

There is an unstable focus at the origin and a stable cycle at \( r = 1, \) which phase trajectories approach clockwise both from inside and outside (Fig. 44b). There is, of course, a trapping region: this can be, e.g., any annulus \( r_1 < r < r_2 \) with \( r_1 < 1 \) and \( r_2 > 1. \) All trajectories cross these circular boundaries inwards except on the \( x \) axis \( (\sin \phi = 0), \) but that is OK: they slip up or down the circle tangentially and then turn inwards anyway.

7.3.3. Liénard’s Theorem

[Literature: Strogatz (1994, §7.4)]

There is a rigorous mathematical result that guarantees a limit cycle for van der Pol’s equation

\footnote{Although it is not necessarily unphysical: there is nonlinear friction here that is due to finite displacement and finite velocity. Why not! The reason everyone prefers van der Pol’s equation (7.52) is that it is much harder and wonderfully nontrivial, with a whimsically shaped limit cycle (Fig. 44a; see also §7.3.4).}
The condition (a) of Theorem 7 implies that $g(x)$ be an odd function, $g(x) > 0$ for $x > 0$; (b) $f(x)$ is an even function; (c) the function $F(x) = \int_0^x dx' f(x')$, which is odd, has a single zero at $x > 0$, viz., $F(x_0) = 0$, $F(x) < 0$ at $0 < x < x_0$, $F(x) > 0$ and $F'(x) \geq 0$ at $x > x_0$, and $F(x \to \infty) \to \infty$. Then the system (7.57) has a unique, stable limit cycle enclosing the origin.

Let us use Liénard’s variables [see (7.58)] for (7.52), except also rescale everything with $\gamma$:

\[ g(x) = x, \quad f(x) = \gamma(x^2 - 1), \quad F(x) = \gamma x \left(\frac{x^2}{3} - 1\right), \quad (7.59) \]

so Theorem 7 applies: there is a unique, stable limit cycle (Fig. 44a). Hurrah.

7.3.4. Relaxation Oscillations

There is a parameter regime in which it is relatively easy to see that van der Pol’s equation (7.52) exhibits a limit cycle—and that limit cycle has quite pleasingly nontrivial and physically interesting behaviour. This is the case of $\gamma \gg 1$, a kind of “overdamped” nonlinear oscillator (cf. §5.1.1).

Let us use Liénard’s variables [see (7.58)] for (7.52), except also rescale everything with $\gamma$:

\[ y = \gamma^{-1}\dot{x} + F(x), \quad \text{where} \quad F(x) = x \left(\frac{x^2}{3} - 1\right) \quad \Rightarrow \quad \begin{cases} \dot{x} = \gamma[y - F(x)], \\ \dot{y} = -\gamma^{-1}x. \end{cases} \quad (7.60) \]

The salient property of this system (not unlike a regular overdamped oscillator in that respect; see §5.1.1) is the presence of two disparate time scales: generally speaking, $x$ changes very fast, at the rate $\dot{x} \sim \gamma \gg 1$, while $y$ changes slowly, at the rate $\dot{y} \sim \gamma^{-1} \ll 1$. The presence of a large coefficient $\gamma$ in the equation for $\dot{x}$ means that, in any generic place on the phase plane, the system will have a very large horizontal velocity towards a state in which the expression multiplied by $\gamma$ is small, i.e., towards the null-cline where $\dot{x} = 0$. This is the cubic curve $y = F(x)$; see Fig. 45(a). When $y > F(x)$, $\dot{x} > 0$ pushes the system right, towards the upward-pointing branch of the cubic; when $y < F(x)$, $\dot{x} < 0$ and the system moves left, towards the downward-pointing branch. Once the system’s trajectory hits this cubic, it is forced to stay in its vicinity, because were it to try and run away, it would be pushed right back by increased $\dot{x}$. In the vicinity of the cubic,
Figure 45. Overdamped van der Pol’s oscillator: (7.52) with $\gamma \gg 1$. (a) A typical trajectory’s journey onto the limit cycle in the Liénard’s phase plane [see (7.60)]. (b) Relaxation oscillations: $x$ vs. $t$. The long, slow bits taking $x$ from $\pm 2$ to $\pm 1$ correspond to the phase trajectory crawling along the cubic $y = F(y)$; they last for a time $\sim \gamma \gg 1$. The quick transitions are the mad dashes from the extrema of the cubic to its opposite brach; they take a time $\sim \gamma^{-1} \ll 1$.

$y - F(x) \sim \gamma^{-2}$, $\dot{x}$ and $\dot{y}$ are comparable and slow ($\sim \gamma^{-1}$). At $x > 0$ ($x < 0$), we have $\dot{y} < 0$ ($\dot{y} > 0$), so the system will crawl slowly down (up) along the cubic, until it reaches its minimum (maximum), can dash across to the other branch of the cubic and resume the crawling (Fig. 45a). This describes a cyclic auto-oscillation, consisting of slow crawls and mad dashes (or, if you like, gradual build-ups and fast releases).

It is possible to calculate approximately the period of these oscillations. Clearly, most of the time, the system is crawling (Fig. 45b), so all we need to know is the duration of a crawl. Since the crawl happens along the null-cline,

$$y \approx F(x) \Rightarrow \dot{y} \approx F'(x) \dot{x}. \quad (7.61)$$

But, since, also $\dot{y} = -\gamma^{-1} x$, we find

$$F'(x) \dot{x} \approx -\gamma^{-1} x \Rightarrow \frac{dt}{dx} = -\gamma F'(x) x = -\gamma x^2 - 1. \quad (7.62)$$

The crawl occurs between the maximum (or minimum) of the cubic at $x = \pm 1$ and the point that is the final destination of each dash, where $F'(x) = F(\mp 1) = \pm 2/3$, which is $x = \pm 2$. Therefore, the period is

$$T = 2\gamma \int_1^2 dx \frac{x^2 - 1}{x} = (3 - 2 \ln 2) \gamma. \quad (7.63)$$

It is always nice to be able to solve something.

One can solve a lot more than this. For example, the case of weak damping, $\gamma \ll 1$, is begging to be analysed by some suitable approximate method (see Strogatz 1994, §7.6). One could also ask what happens if you force these kinds of oscillators (they can go chaotic!). You are now fully (or, at least, somewhat) equipped both to ask such questions and to work through the answers to them contained in the literature. Enjoy.

Exercise 7.7. Sketch roughly the phase portrait of van der Pol’s oscillator (7.52) with $\gamma \gg 1$ in the usual phase plane $(x, \dot{x})$.

7.4. Outlook: Attractors, Chaos and Many Degrees of Freedom

Let me say a few words about where it all leads from here—as usual, all the really exciting stuff is for the future!

[to be finished up...]

IUCUNDI ACTI LABORES
(a) Pontryagin (b) Arnold

Figure 46. (a) Lev Pontryagin (1908-1988), eminent Soviet mathematician with an impressive story of achievement in the face of adversity; he was not, however, a nice guy and allegedly took some opportunities to behave disgracefully presented by the Soviet regime. (b) Vladimir Arnold (1937-2010), great Soviet/Russian mathematician, unbeloved by the Soviet government, student of Kolmogorov, author of cult texts, educator and superhero. Depending on how hard-core a mathematician you prove to be, you might continue from his ODE book to the sequels: Arnold (1997), Arnold (2004) and Arnold (1989), although perhaps not in your first year.

Sources and Further Reading

In preparing this course, I have drawn inspiration from a number of sources. The main ones were

—the previous set of Oxford Physics’ ODE lectures, for which very concise, prescription-style notes are available (Yeomans 2014)—this may well be all you need to pass your exam; another worthwhile read is the earlier version of the Oxford course by Binney (2002), written in his signature lucid and idiosyncratic style;\(^{56}\)

—the textbook by Romanko (2001), with an attendant gigantic collection of problems (Romanko et al. 2011; Ipatova et al. 2012), reflecting many decades of experience of teaching ODEs at MIPT (I took this course as an undergraduate there in the early 1990s, when it was taught by A. A. Abramov, whose lectures, as recorded by me then, have proved useful again, after all these years);\(^{57}\)

—the celebrated (if slightly dated) and very lucid lectures by the famous blind mathematician Pontryagin (1962, in English) (Fig. 46a)—this is a traditional exposition of the subject \textit{par excellence}; assuming you do not read Russian and assuming also that you are interested in learning the subject rather than in obtaining only the minimum training required for your exam, this is the book that I recommend to you as a more thorough undergraduate course than I can offer within the existing time constraints;

—the classic cult text by the great Arnold (2006) (Fig. 46b), which exists in English and which I recommend as further reading to those of you who have a taste for intellectual challenge; Arnold will not teach you how to solve ODEs, but he will teach you how to think about them and much else besides.

Two old, affordable (in Dover editions) and, on first inspection, entirely adequate

\(^{56}\)Both sets of notes have many examples of how to solve standard Oxford homework/collection/exam-style questions.

\(^{57}\)Other, somewhat older, Russian books that I have consulted were Fedoryuk (1985), an MIPT classic, and the rival course from MSU, Tikhonov \textit{et al.} (1985, in English), which is, I hate to admit, also quite good, and rather more concise.
undergraduate textbooks from this side of the Iron Curtain are Tenenbaum & Pollard (1986) (written in a cheerfully spoon-feeding style, with a huge number of worked examples, but of fairly limited scope overall) and Coddington (1990) (a more formal, although still undergraduate, text, focusing primarily on a thorough treatment of linear equations and the existence theorems; looks dull but solid to me). Oxford’s own official textbook (from the Mathematical Institute) appears to be Collins (2006), although it does not belong to the affordable category and, at any rate, seems to view ODEs as a trivial footnote to PDEs. A modern American textbook on both ODEs and Dynamical Systems, with careful explanations and lots of examples, is Perko (2001). Finally, Schaeffer & Cain (2016) is another recent (possibly the most recent) American textbook, which looks quite civilised to me—it quickly gets to much more advanced topics than my course ever could, but is written in a friendly style and may be suitable as further reading on the modern mathematical theory of ODEs.

Another direction for further reading is asymptotic methods. These are for those who like doing stuff. There are elements here of dark arts and black magic. Much of it has gone from the world, but it is still possible to learn from the masters of old: I have given several references in the text (see, e.g., §1.1 and Q2.17), but if you have to pick one, pick Bender & Orszag (1999), which is a wild, exuberant ride. It is a graduate-level book, supposedly, but, after you have done Complex Analysis (and maybe some Mathematical Methods), you will know everything you need to know to attack all of it—and quite a bit of it you can, in fact, attack right away (this said, I cannot promise you definite success as it depends on whether the Force is strong with you).

Finally, §7 was the tip of a large iceberg that is the theory of dynamical systems, chaos, and nonlinear oscillations. Some good books on these topics, in the order of increasing difficulty, are Strogatz (1994), Glendinning (1994) and Guckenheimer & Holmes (1983). The last of these is a classic, serious, graduate-level text that is not always easy to read even for graduates, whereas the other two are undergraduate lecture courses—both are good but in rather different ways: the first, American one (from MIT and Cornell), is cheerful, friendly, spoon-feeding (especially at first), eclectic, effusive, and mindful of a student audience of potentially variable mathematical ability; the second, British (from Cambridge and Warwick), is more relaxed about overstretching those who have not been properly introduced (to basic maths), but is unfailingly nice, polite, and gracefully informal in its style of exposition.

Acknowledgments

Before taking over the ODE lectures, I had taught this subject for 9 years (2009-17) to Merton undergraduates. I am grateful to them for teaching me how to teach it. This was greatly helped by my graduate teaching assistants Richard Fern, Adrian Menssen, Archie Bott and Glenn Wagner. Two generations of first-year Merton students (Radek Grabarczyk, Graham Mathews, Nicholas Mitchell, Alex Pett, Jan Siuta, Marcell Szakaly, Miles Testa, Robert Vickers, Lasse Wolf, Alex Christie, Andrei Cristea, Andrei Eftime, Megan Evans, William Isotta, Jeremi Litarowicz, Rishin Madan, Benedict Yorston) and

58 In fact, its §1 is a pretty good introduction to the main methods for solving ODEs and can be read as a short (33 pages), to-the-point, example-rich text for the present course.

59 Those who read Russian might also like Karlov & Kirichenko (2003). Note that both Perko (2001) and Schaeffer & Cain (2016), already mentioned above, also deal with the theory of dynamical systems—they perhaps think of it more as a brunch of the general theory of ODEs than of classical physics.
my GTAs Toby Adkins and Toby Swann have been the captive test group for some measurements of the effect this course has had, so I am greatly in their debt. A number of students who took the course asked useful questions, sent feedback, or pointed out various lapses in these notes—all of this very welcome indeed; not all of these students introduced themselves but those who did were Tejas Acharya, Josh Greensmith, Sam Gunatilleke, Tony He, Zaman Keinath-Esmail, and JJ Thio. I am also grateful to tutors Michael Barnes (Univ), Felix Parra (Worcester) and Christopher Ramsay (Merton) for comparing notes and sharing experience and to Adam Povey for useful feedback on the problem sets. I would like to thank John Magorrian, who made my task so much easier by arranging his lectures on Linear Algebra so as to absorb and/or deliver the necessary background material (Magorrian 2019) in good time before I needed it. Finally, I am very grateful to Hans Kraus for letting me do this my way.

REFERENCES


Problem Set 1: First-Order ODEs

MINIMAL SET

1.1. [JY+] Determine the order of the following differential equations and whether they are linear or nonlinear. Rewrite them as systems of 1st-order ODEs.

   (i) \( y'' + k^2 y = f(x) \), \hspace{1cm} (1.1)

   (ii) \( y''' + 2yy' = \sin x \), \hspace{1cm} (1.2)

   (iii) \( y' + y^2 = xy \). \hspace{1cm} (1.3)

1.2. [JB/JY+] Solve the following differential equations using the method stated.

   (a) Full differential (§2.1):

   \[
   y' = \frac{(3x^2 + 2xy + y^2) \sin x - 2(3x + y) \cos x}{2(x + y) \cos x}. \tag{1.4}
   \]

   (b) Integrating factor (§2.1.1):

   \[ y^2 dx - (xy + x^3) dy = 0. \tag{1.5} \]

   (c) Separable (§2.2):

   \[
   y' = \frac{xe^y}{1 + x^2}, \quad \text{at} \; x = 0, \tag{1.6}
   \]

   (d) Reducible to separable by change of variables (§2.2.2):

   \[
   y' = 2(2x + y)^2. \tag{1.8}
   \]
(e) Homogeneous (§2.3):

\[ 2y' = \frac{xy + y^2}{x^2}. \]  

(f) Reducible to homogeneous by change of variables (§2.3.1):

\[ y' = \frac{x + y - 1}{x - y - 2}. \]  

(g) Linear (§2.4):

(i) \[ y' + \frac{y}{x} = 3, \quad x = 0 \text{ at } y = 0, \]  

(ii) \[ y' + y \cos x = \sin 2x. \]  

(h) Bernoulli (§2.5):

\[ y' + y = xy^{2/3}. \]  

1.3. [JB/JY+] Solve the following 1st-order differential equations:

(i) \[ y' = \frac{x - y \cos x}{\sin x}, \]  

(ii) \[ (3x + x^2)y' = 5y - 8, \]  

(iii) \[ y' + \frac{2x}{y} = 3, \]  

(iv) \[ y' + \frac{y}{x} = 2x^{3/2}y^{1/2}, \]  

(v) \[ 2y' = \frac{y}{x} + \frac{y^3}{x^3}, \]  

(vi) \[ xy'y' - y^2 = (x + y)^2 e^{-y/x}, \]  

(vii) \[ x(x - 1)y' + y = x(x - 1)^2, \]  

(viii) \[ 2xy' - y = x^2, \]  

(ix) \[ y' = \cos(y + x), \quad y = \frac{\pi}{2} \text{ at } x = 0, \]  

(x) \[ y' = \frac{x - y}{x - y + 1}, \]  

(xi) \[ y' = \cos 2x - y \cot x, \quad y = \frac{1}{2} \text{ at } x = \frac{\pi}{2}, \]  

(xii) \[ y' + ky = y^n \sin x, \quad n \neq 1. \]  

Supplementary Problems

1.4. [JB+] By introducing a new variable \( z = 4y - x, \) or otherwise, find all solutions of the ODE

\[ y' - 16y^2 + 8xy = x^2. \]  

You should find this solution:

\[ y = \frac{x}{4} - \frac{1}{8} \tanh(2x + C). \]  

Are there any others? Hint: if you divide by zero too recklessly, you may lose some solutions.
1.5. Full Differentials. Solve the following equations, which can be reduced to equations in full differentials.

(a) \[2 \left( x - y^4 \right) \, dy = y \, dx, \quad (1.28)\]

Hint: look for an integrating factor (§2.1.1) in the form \( \Lambda(y) \).

(b) \[y \left( 2y \, dx - x \, dy \right) + x^2 \left( y \, dx + 2x \, dy \right) = 0. \quad (1.29)\]

Hint: identify within the equation combinations that are full differentials of some functions of \( x \) and \( y \) and then use those functions to introduce a change of variables that will allow you to separate (the new) variables and integrate.

1.6. Sometimes an ODE can be turned into a homogeneous one (§2.3) by the change of variables \( y = z^n \). Solve the following system by this method:

\[y \, dx + x \left( 2xy + 1 \right) \, dy = 0. \quad (1.30)\]

Hint: you will need to find \( n \) such that the equation does become homogeneous; once you have done this, remember that sometimes it is more convenient to look for a solution in the form \( x = x(y) \), rather than \( y = y(x) \).

1.7. Quasi-homogeneous Equations. (a) An ODE \( y' = f(x, y) \) is called quasi-homogeneous if \( \forall \lambda \) and some \( \alpha \neq 0, \beta \neq 0, \)

\[f(\lambda^\alpha x, \lambda^\beta y) = \lambda^{\beta - \alpha} f(x, y). \quad (1.31)\]

Show that the change of variables \( y = x^{\beta/\alpha} z \) reduces a quasi-homogeneous equation to a separable one.

(b) Solve the resulting equation in quadratures. Show that if \( \exists z_0 \) satisfying \( f(1, z_0) = \beta z_0/\alpha \), then \( y = z_0 x^{\beta/\alpha} \) is also a solution of the equation. Under what condition are composite solutions (§2.2.1) possible?

You can learn more about quasi-homogeneous equations from Arnold (2006, §§6.4-6.5).

Extracurricular Problems

1.8. Integral Curves and Orthogonal Curves. (a) If a family of curves on the plane \( (x, y) \) is specified in the form

\[f(x, y, C) = 0, \quad (1.32)\]

where \( C \) is a parameter, then one can find a first-order ODE for which these curves are integral curves by taking the differential of (1.32) and then substituting \( C = C(x, y) \) extracted from (1.32).

By this method, find the ODE whose integral curves are

\[y = \tan \left[ \ln(Cx) \right]. \quad (1.33)\]

(b) Show that the curves that are orthogonal to the integral curves of the ODE

\[F(x, y, y') = 0 \quad (1.34)\]

are integral curves of the ODE

\[F \left( x, y, -\frac{1}{y'} \right) = 0. \quad (1.35)\]
The explicit expression for these orthogonal curves can then be found by integrating (1.35).

(c) Find the curves orthogonal to the family (1.33). Check your answer:

\[ 3x^2 + 2y^3 + 6y = C. \quad (1.36) \]

1.9. Riccati equations. (a) By the method of §2.6, solve the following Riccati equation

\[ y' = y^2 - 2e^x y + e^{2x} + e^x. \quad (1.37) \]

(b) The equation in Q1.4 is also a Riccati equation. Solve it again, but now using the method of §2.6.

1.10. Lagrange’s and Clairaut’s Equations. (a) Lagrange’s equation (Fig. 8) is

\[ y = a(y') x + b(y'), \quad (1.38) \]

where \( a(p) \) and \( b(p) \) are some given continuously differentiable functions. As suggested in §2.7.2, introduce \( p = y' \) and rewrite this equation as a first-order ODE with respect to \( p \) and \( x \). Find the solution in quadratures, in the form \( x(p) \), assuming \( a(p) \neq p \).

(b) What if \( \exists p_0 \) such that \( a(p_0) = p_0 \)? Can you find a special solution of (1.38) that exists in this case?

(c) Now suppose that \( a(p) = p \forall p \). Then (1.38) is called Clairaut’s equation (Fig. 47):

\[ y = xy' + b(y'). \quad (1.39) \]

Find all possible solutions of this equation, commenting on the number of continuous derivatives that \( b(p) \) must have in order for these solutions to be valid.

(d) Sketch the integral curves of (1.39) if \( b(p) = p^2 \). They will turn out to be a family of lines hugging a certain envelope. Find the curve that describes this envelope.

(e) Does the solution of (1.39) with \( b(p) = p^2 \) that passes through the point \( (x_0, y_0) \) exist \( \forall (x_0, y_0) \in \mathbb{R}^2 \)? If not, for which \( (x_0, y_0) \) does it exist? Given some \( (x_0, y_0) \) for which
it does exist, is it unique? If not, is the solution unique in the vicinity of \((x_0, y_0)\) if we also specify \(y'(x_0) = p_0\)? How many legitimate options for \(p_0\) are there at each point? How far can such a solution be continued uniquely? Show graphically using the result of (d) how an infinite number of composite solutions (§2.2.1) can be constructed.

If you are intrigued by Clairaut’s equation, you will find a number of interesting facts, examples and applications in Arnold (2006, §8.5) and Tenenbaum & Pollard (1986, §61).

**1.11. Equations unresolved with respect to derivative.** Solve the following ODEs via introduction of parameter (§2.7.2):

\[
\begin{align*}
(i) \quad (y')^2 - 2xy' &= x^2 - 4y, \\
(ii) \quad y' &= e^{-xy'/y}, \\
(iii^*) \quad y' &= ye^{-xy'/y}, \\
(iv^*) \quad y' &= y^2e^{-xy'/y}.
\end{align*}
\]

Are \(p\)-discriminant curves [see (2.80) in §2.7.1] of these equations also their integral curves? What does this imply?

**1.12: Parametric Solutions.** Consider an equation formed by \(y'\) and \(x\) lying on the “folium of Descartes”:

\[
(y')^3 + x^3 = xy'.
\]

Try \(y' = p = xu\), work out a parametrisation of the curve \((x, y')\), and then find a parametric solution of (1.44) in the form \(x = x(u)\) and \(y = y(u)\). In what range of values of \(u\) (and, therefore, of \(x\) and \(y\)) is your solution valid?
Problem Set 2: Second-Order ODEs

MINIMAL SET

2.1. [JY] Homogeneous ODEs. Solve these equations:

(i) \( y'' + 2y' - 15y = 0 \) \hspace{1cm} (2.1)
(ii) \( y'' - 6y' + 9y = 0 \), \( y = 0 \), \( y' = 1 \) at \( x = 0 \) \hspace{1cm} (2.2)
(iii) \( y'' - 4y' + 13y = 0 \) \hspace{1cm} (2.3)
(iv) \( y''' + 7y'' - 7y' - 15y = 0 \). \hspace{1cm} (2.4)

In (iii), write the solution both in terms of complex exponentials and in terms of sines and cosines.

2.2. [JY] Damped Oscillator. A damped harmonic oscillator is displaced by a distance \( y_0 \) and released at time \( t = 0 \). Show that the subsequent motion is described by the differential equation

\[
m\ddot{y} + m\gamma \dot{y} + m\omega_0^2 y = 0 \quad \text{with} \quad y = y_0, \quad \dot{y} = 0 \quad \text{at} \quad t = 0,
\]

(2.5)

explaining the physical meaning of the parameters \( m \), \( \gamma \) and \( \omega_0 \).

(a) Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping. (iv) What happens for \( \gamma = 0 \)?

(b) For a lightly damped oscillator \( (\gamma \ll \omega_0) \), the quality factor, or \( Q \)-factor, is defined as

\[
Q = \frac{2\pi}{\text{energy lost per period of oscillation}} \frac{\text{energy stored}}{2}. \quad (2.6)
\]

Show that \( Q = \frac{\omega_0}{\gamma} \).

2.3. [JY+] Boundary-Value Problem. (a) Solve the ODE

\[
y'' + k^2 y = 0. \quad (2.7)
\]

(b) Consider the above equation on the interval \( x \in [0, L] \). Are there solutions that are not identically zero everywhere but have \( y(0) = y(L) = 0 \)? For which values of \( k \) do such solutions exist? Find these solutions and explain in terms of linear algebra the meaning of what you have done.

(c*) Can any function that is defined in the interval \([0, L]\) and vanishes at its ends be represented as a linear combination of solutions of (2.7) with different \( k \)'s? Work out how to do it.

The full theory of boundary-value problems is not covered in this course. Read up on the topic or wait till the Mathematical Methods course in your 2nd year (see Eßler 2009, §25 and Lukas 2019, §5)—although you will, in fact, need to solve (2.7) many times in CP4.

2.4. [JY] Inhomogeneous ODEs. Consider the equation

\[
y'' - 3y' + 2y = f(x). \quad (2.8)
\]
What is the general form (in terms of functions and constants) of its particular solution for

\[ f(x) = \begin{align*}
  & (i) \ x^2, \\
  & (ii) \ e^{4x}, \\
  & (iii) \ e^x, \\
  & (iv) \ \sinh x, \\
  & (v) \ \sin x, \\
  & (vi) \ x \sin x, \\
  & (vii) \ e^{2x} + \cos^2 x.
\end{align*} \tag{2.9}

If you dislike unfinished business, work out the constants.

2.5. [JB/JY+] Inhomogeneous ODEs. Solve these equations:

\[ \begin{align*}
  & (i) \quad 5y'' + 2y' + y = 2x + 3, \quad y = -1, \ y' = 0 \text{ at } x = 0, \tag{2.16} \\
  & (ii) \quad y'' - y' - 2y = e^{2x}, \tag{2.17} \\
  & (iii) \quad 4y'' - 4y' + y = 8e^{x/2}, \quad y = 0, \ y' = 1 \text{ at } x = 0, \tag{2.18} \\
  & (iv) \quad y'' + 3y' + 2y = xe^{-x}, \tag{2.19} \\
  & (v) \quad y'' - 4y' + 3y = 10 \cos x, \tag{2.20} \\
  & (vi) \quad y'' + 4y = x + \cos 2x, \quad y = 0 \text{ when } x = 0, \tag{2.21} \\
  & (vii) \quad y'' - 2y' + 2y = e^{x}(1 + \sin x), \quad y = 0 \text{ at } x = 0 \text{ and at } x = \frac{\pi}{2}, \tag{2.22} \\
  & (viii) \quad y'' + 2y' + y = 2e^{-x} + x^3 + 2 \cos x, \tag{2.23} \\
  & (ix) \quad y'' - 2y' + y = 3e^x, \quad y = 3, \ y' = 0 \text{ at } x = 0, \tag{2.24} \\
  & (x) \quad x^2y'' + xy' + y = x. \tag{2.25}
\end{align*} \]

2.6. [JB] Forced Oscillator. When a varying couple \(I \cos \omega t\) is applied to a torsional pendulum with natural period \(2\pi/\omega_0\) and moment of inertia \(I\), the angle of the pendulum obeys the equation of motion

\[ \ddot{\theta} + \omega_0^2 \theta = \cos \omega t. \tag{2.26} \]

The couple is first applied at time \(t = 0\) when the pendulum is at rest in equilibrium.

(a) Show that, in the subsequent motion, the root-mean-square angular displacement is

\[ \langle \theta^2 \rangle^{1/2} = \frac{1}{|\omega_0^2 - \omega^2|}, \tag{2.27} \]

where the average is taken over a time large compared to \(1/|\omega_0 - \omega|\).

(b) Discuss the motion as \(|\omega_0 - \omega| \to 0\).

2.7. [JY] Forced and Damped Oscillator. Consider the damped oscillator of Q2.2 subject to an oscillatory driving force:

\[ m \ddot{y} + m\gamma \dot{y} + m\omega_0^2 y = F \cos \omega t. \tag{2.28} \]

(i) Explain what is meant by the stationary solution of this equation, and calculate this solution for the displacement \(y(t)\) and the velocity \(\dot{y}(t)\).

(ii) Sketch the amplitude and phase of \(y(t)\) and \(\dot{y}(t)\) as a function of \(\omega\).

(iii) Determine the resonant frequency for both the displacement and the velocity.
(iv) Defining $\Delta \omega$ as the full width at half maximum of the resonance peak calculate $\Delta \omega/\omega_0$ to leading order in $\gamma/\omega_0$.

(v) For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as in Q2.2. How is $Q$ related to the width of the resonance peak?

2.8. [JB] Verify that $y = x + 1$ is a solution of

$$(x^2 - 1)y'' + (x + 1)y' - y = 0.$$  \hspace{1cm} (2.29)

Hence find the general solution of this equation. Check your answer:

$$y = C_1(x + 1) + C_2 \left( \frac{x + 1}{4} \ln \left| \frac{x - 1}{x + 1} \right| + \frac{1}{2} \right).$$  \hspace{1cm} (2.30)

2.9. [JY+] Consider the differential equation

$$x(x + 1)y'' + (2 - x^2)y' - (2 + x)y = (x + 1)^2.$$  \hspace{1cm} (2.31)

(a) One of its homogeneous solutions is $y_1(x) = 1/x$. Find the general solution.

(b) Now pretend that you do not know that $1/x$ is a homogeneous solution, but know the second homogeneous solution, $y_2(x)$, that you found in (a) (in fact, if you stare at the equation for a few seconds, or minutes, you will see that you could have guessed that solution). Use the knowledge of $y_2(x)$ to find both $y_1(x)$ and the general solution of the equation.

2.10. Nonlinear ODEs. All of this problem set so far has been a drill in solving linear, second-order ODEs. There are few general methods for solving nonlinear ones (and most of them cannot be integrated in quadratures anyway). Still, there are a few tricks, which you now have an opportunity to practice. All of them, naturally, are based on methods for lowering the order of the equation down to first.

(a)[JY] If bits of the equation can be manipulated into full derivatives of some expressions, the equation’s order can sometimes be lowered by direct integration. Practice this by solving the ODE

$$1 + yy'' + (y')^2 = 0.$$  \hspace{1cm} (2.32)

(b)[Ox] If the equation does not contain $y$, only its derivatives, the order is lowered by treating the lowest-order derivative that does appear as the new function. Practice this by solving the ODE

$$xy'' = y' + (y')^3.$$  \hspace{1cm} (2.33)

(c) If the equation does not contain $x$, you can lower its order by letting $p = y'$ and looking for solutions in the form $p = p(y)$. Note that then, by the chain rule, $y'' = pp'$. Solve by this method (plus other appropriate changes of variable) the ODE

$$y''(y - 1) + y'(y - 1)^2 = (y')^2.$$  \hspace{1cm} (2.34)

In Q2.15, you will learn two other methods for solving nonlinear ODEs.
2.11. [JB] Solve the differential equation
\[ y'' - 2y' + (k^2 + 1)y = e^x \sin^2 x \] (2.35)
for general values of the real parameter \( k \). Identify any special values of \( k \) for which your solution fails and solve the equation also for those values.

2.12. [Ox] Find the continuous solution with continuous first derivative of the equation
\[ y'' + 2y' + 2y = \sin x + f(x), \quad \text{where} \quad f(x) = \begin{cases} 0, & x \leq 0, \\ x^2, & x > 0, \end{cases} \] (2.36)
subject to \( y(-\pi/2) = y(\pi) = 0 \). Hint: obtain a general solution for each of the cases \( x < 0 \) and \( x > 0 \) and then determine any loose constants by making these solutions agree at \( x = 0 \).

2.13. [JB+] Oscillator with Modulated Force. A mass \( m \) is constrained to move in a straight line and is attached to a spring of strength \( m\omega_0^2 \) and a dashpot which produces a retarding force \( m\gamma v \), where \( v \) is the velocity of the mass and \( \gamma \ll \omega_0 \). An amplitude-modulated periodic force \( mA\cos \sigma t \sin \omega t \) with \( \sigma \ll \omega \) and \( \omega = \omega_0 \) is applied to the mass. Show that, in the long-time limit, the displacement is an amplitude-modulated wave
\[ y = -\frac{A \sin(\sigma t + \phi) \cos \omega t}{2\omega \sqrt{\sigma^2 + \gamma^2/4}}, \quad \tan \phi = \frac{\gamma}{2\sigma}. \] (2.37)

2.14: Consider the differential equation
\[ 9xy'' + (6 + x)y' + \lambda y = 0. \] (2.38)
There are several values of \( \lambda \) for which this can be solved via reduction to a 1st-order equation. For at least one of them, it is possible to find a solution that
(i) satisfies \( y(x \to \pm \infty) \to 0 \), but is not zero everywhere,
(ii) is continuous at \( x = 0 \).

Experiment with solutions corresponding to various tractable values of \( \lambda \) and find one for which the above two properties are satisfied. The solution you find may, but does not have to be,
\[ y = C e^{-x/9} \int_{-\infty}^{x} dz \frac{e^{z/9}}{|z|^{2/3}}. \] (2.39)

2.15. Homogeneous Nonlinear ODEs. Continuing from Q2.10, here are some more methods for lowering the order of nonlinear ODEs.

(a) An equation is called homogeneous with respect to \( y \) if it does not change under the rescaling \( y \to \lambda y \ \forall \lambda \neq 0 \) (i.e., every term contains the same number of powers of \( y \) or its derivatives). For such an equation, one can lower the order by introducing a new function \( z(x) \) via \( y' = y z \). Then \( y'' = y(z' + z^2) \), so the second derivative is expressible in terms of the first; powers of \( y \) will cancel because the equation is homogeneous.

Solve by this method the following ODE:
\[ xy'' + x(y')^2 = 3yy'. \] (2.40)
Check your answer: \( y^2 = Ax^4 + B \).
(b*) This equation can also be solved by the method that you practiced in Q2.10(a), if you can spot the full derivatives. Do it.

(c) A generalised version of (a) is an equation that does not change under the rescaling \( x \to \lambda x, \ y \to \lambda^\alpha y \ \forall \lambda \neq 0 \) and for some \( \alpha \) (the linear version of this with \( \alpha = 0 \) is Euler’s equation, §5.1.4; you encountered the first-order case in Q1.7). This is solved by letting \( x = e^t \) when \( x > 0 \), \( x = -e^t \) when \( x < 0 \), and \( y = z(t)e^{\alpha t} \). The result will be an equation for \( z \) in which \( t \) does not appear explicitly, so you can use the method practiced in Q2.10(c).

Consider the following ODE:

\[
x^2(y'' + 2yy') + 2xy^2 - 2y = 0, \quad x > 0
\]

and establish for what value of \( \alpha \) it has the invariance property described above. Then solve the equation by the proposed method.

---

**Extracurricular Problems**

2.16. Higher-Order Linear ODEs with Constant Coefficients. Solve the following ODEs

(i) \( y'''' - 6y''' + 8y'' + 6y' - 9y = 0 \),

(ii) \( y'''' + 4y'' + 4y = 0 \).

Hint: in (i), you will need to guess some roots of a 4th-order polynomial and then factorise it.

2.17. Series Solution of Linear ODEs. ODEs (usually second order) that cannot be solved in quadratures can sometimes be solved in terms of series. One can then give these solutions names and establish all their properties, thus expanding the library of functions that we know how to handle. Such functions are called special functions (a classic textbook on them is Lebedev 1972).

Consider Airy’s equation (Fig. 48a)

\[
y'' + xy = 0.
\]

This equation cannot, alas, be solved in quadratures. Seek its solution in the form of a power series

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

(a) Without worrying about convergence of this series or about the legitimacy of differentiating it term by term, work out what equations the coefficients \( c_n \) must satisfy in order for (2.45) to be a solution of (2.44).

(b) Find two linearly independent such series solutions (remember that if they are linearly independent at one point, they are linearly independent everywhere; see §4.3). Use d’Alembert’s Ratio Test (Fig. 48b) to show that both series converge. What is the general solution of (2.44)?

If you are having a difficulty with these derivations, you will find the solution in, e.g., Tikhonov et al. (1985, §3.8).

Certain linear combinations of the solutions that you have found are called Airy
functions, an example of special functions. Entire books are written about them (Vallée & Soares 2004).

(c) Find the general solution of (2.7) of Q2.3 (with $k = 1$ for simplicity) by the same method and convince yourself that the series that you have obtained are sines and cosines. These functions too were once special.

It is not always possible to find solutions in the form of a Taylor series (2.45). For example, the equation

$$x^2 y'' + xp(x)y' + q(x)y = 0,$$

where $p(x)$ and $q(x)$ are representable as convergent Taylor series, is only guaranteed to have one solution in the form of a Frobenius series (Fig. 48c)

$$y(x) = x^\alpha \sum_{n=0}^{\infty} c_n x^n,$$

where $\alpha$ is not, in general, a (positive) integer; you can always get the second solution via the “buy one get one free” scheme (§4.6). You will encounter the Frobenius method in the Mathematical Methods course (see Effler 2009, §23 or Lukas 2019, §5.3 for some examples; if you want to learn the general method, see, e.g., White 2010, §4 or Bender & Orszag 1999, §3; see also Coddington 1990, Ch. 4). However, no one stops you from getting intrigued now and at least convincing yourself that (2.47) works for (2.46) and that (2.45) in general does not. A good example of (2.46) to play with is Bessel’s equation (Fig. 48d):

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $\nu$ is a real number (interesting things happen depending on whether it is an integer). On Bessel functions too, there are entire books in the library (short and elementary: Bowman 2003; long and comprehensive: Watson 1944).
Problem Set 3: Systems of ODEs
(vacation work)

MINIMAL SET

3.1. [JY] Solve the coupled differential equations
\[
\begin{align*}
\dot{x} + ax - by &= f, \\
\dot{y} + ay - bx &= 0,
\end{align*}
\]
(3.1)
where \(a, b,\) and \(f\) are constants.

3.2. [JB] Solve the coupled differential equations
\[
\begin{align*}
y' + 2z' + 4y + 10z - 2 &= 0, \\
y' + z' + y - z + 3 &= 0,
\end{align*}
\]
(3.2)
where \(y = 0\) and \(z = -2\) at \(x = 0\).

3.3. Find the general, real solutions of the following homogeneous systems of ODEs
(i) \[
\begin{align*}
\dot{x} &= -2y + 2z, \\
\dot{y} &= x - y + z, \\
\dot{z} &= y - z,
\end{align*}
\]
(3.3)
(ii) \[
\begin{align*}
\dot{x} &= 4x - y - z, \\
\dot{y} &= x + 2y - z, \\
\dot{z} &= x - y + 2z.
\end{align*}
\]
(3.4)

3.4. Find the general, real solutions of the following inhomogeneous systems of ODEs
(i) \[
\begin{align*}
\dot{x} &= 4x + 3y - 3z, \\
\dot{y} &= -3x - 2y + 3z, \\
\dot{z} &= 3x + 3y - 2z + 2e^{-t},
\end{align*}
\]
(3.5)
(ii) \[
\begin{align*}
\dot{x} &= -5x + y - 2z + \cosh t, \\
\dot{y} &= -x + y + 2 \sinh t + \cosh t, \\
\dot{z} &= 6x - 2y + 2z - 2 \cosh t.
\end{align*}
\]
(3.6)

SUPPLEMENTARY PROBLEMS

3.5. [JB] Solve the coupled differential equations
\[
\begin{align*}
2y'' - 3y' + 2z' + 3y + z &= e^{2x}, \\
y'' - 3y' + z' + 2y - z &= 0.
\end{align*}
\]
(3.7)
Is it possible to have a solution to these equations for which \(y = z = 0\) at \(x = 0\)? What is wrong with this system?

3.6. Charged Particle in Electromagnetic Field. A particle of mass \(m\) and charge \(q\) is placed, initially at rest, in straight, constant electric \(E\) and magnetic \(B\) fields, which are neither perpendicular nor parallel to each other. It will experience the Lorentz force
\[
F = q \left( E + \frac{v \times B}{c} \right),
\]
(3.8)
where \(v\) is the particle’s velocity. Find \(v(t)\) and sketch the particle’s motion.
3.7. **Non-diagonalisable systems.** Find the general solutions of the following homogeneous systems of ODEs

(i) \[
\begin{align*}
\dot{x} &= 2x + y, \\
\dot{y} &= 4y - x,
\end{align*}
\]

(ii) \[
\begin{align*}
\dot{x} &= x - 2y, \\
\dot{y} &= -x - y - 2z, \\
\dot{z} &= y + z,
\end{align*}
\]

(iii) \[
\begin{align*}
\dot{x} &= 2x - 5y - 8z, \\
\dot{y} &= 7x - 11y - 17z, \\
\dot{z} &= -3x + 4y + 6z.
\end{align*}
\]

You can find solutions by triangulation (§6.3.1) or via Jordan’s basis (§6.3.3), whichever you prefer.

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**Extracurricular Problems**

In the following questions, especially Q3.8 and Q3.12, you might gain interesting insight (and a useful preview of the answer) by plotting phase portraits on the computer (using, e.g., Mathematica, Python or something of the sort).

3.8. **Duffing’s Oscillator.** Consider the following nonlinear oscillator:

\[
\ddot{x} + \gamma \dot{x} + x + \alpha x^2 + \beta x^3 = 0. \tag{3.12}
\]

With \(\alpha = 0\), this describes a (damped) oscillator whose restoring force strengthens (\(\beta > 0\)) or weakens (\(\beta < 0\)) with amplitude—these are called the cases of **hardening spring** or **softening spring**. When \(\alpha \neq 0\), there is another amplitude-dependent force, which pushes/pulls at the oscillator differently depending on the direction of the displacement.

Sketch the phase portraits of this system for the following cases:

(i) \(\gamma = 0, \ \alpha = 0, \ \beta > 0\), \tag{3.13}

(ii) \(\gamma = 0, \ \alpha = 0, \ \beta < 0\), \tag{3.14}

(iii) \(\gamma = 0, \ 0 < \alpha \ll \sqrt{|\beta|}, \ \beta < 0\). \tag{3.15}

Having done that, deduce what happens in each of these cases when \(0 < \gamma \ll 1\). The name of the game is to get a qualitatively adequate sketch with as little work as possible.

Sketch the functional form of the potential energy. If you think of (3.12) as describing a particle moving in this potential, what do your phase portraits say about this particle’s motion?

If you liked this so much that you want to play a bit more, look at the case when the sign of the \(x\) term is reversed (i.e., linearly, instead of a restoring force, there is an instability).

You can read more about this kind of nonlinear oscillators in Strogatz (1994, §§7.6, 12.5), Landau & Lifshitz (1976, §29) and Glendinning (1994, §7.2).

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60Georg Duffing (1861-1944), a German engineer, famous for his oscillator, which, with the sign of the \(x\) term reversed and with a periodic external force added, is one of the simplest known systems that can exhibit chaotic behaviour.
3.9. Wiggly Pendulum. Consider a nonlinear pendulum whose point of suspension rapidly oscillates in the horizontal direction as $a \cos \omega t$ (see Fig. 26b of §5.4), where $\omega \gg \sqrt{g/l}$ and $a \ll l$ ($l$ is the length of the pendulum, $g$ is the acceleration of gravity). Determine its equilibria and the conditions under which they are stable. Sketch the phase portrait for the pendulum’s motion averaged over the rapid oscillations.

3.10. How Nonlinearity Takes Over. Consider the nonlinear systems

\begin{align}
  (i) \quad \begin{cases} 
    \dot{x} = -y + \alpha x(x^2 + y^2), \\
    \dot{y} = x + \alpha y(x^2 + y^2), 
  \end{cases} & (3.16) \\
  (ii) \quad \begin{cases} 
    \dot{x} = -x - \frac{\alpha y}{\ln(x^2 + y^2)}, \\
    \dot{y} = -y + \frac{\alpha x}{\ln(x^2 + y^2)}, 
  \end{cases} & (3.17)
\end{align}

where $\alpha$ is a (real) parameter. Determine the nature of the fixed point at $(x, y) = 0$. Does the answer survive the restoration of the nonlinearity? Solve the nonlinear equations exactly and determine the nature of the fixed point depending on the value of $\alpha$.

*Hint.* Going to the complex plane in the vein of (7.25) (§7.1.6) might help shorten calculations.

3.11. Limit Cycles. Consider the nonlinear systems

\begin{align}
  (i) \quad \begin{cases} 
    \dot{x} = x - (x - y) \sqrt{x^2 + y^2}, \\
    \dot{y} = y - (x + y) \sqrt{x^2 + y^2}, 
  \end{cases} & (3.18) \\
  (ii) \quad \begin{cases} 
    \dot{x} = y + \frac{1}{4} x(1 - 2x^2 - 2y^2), \\
    \dot{y} = -x + \frac{1}{2} y(1 - x^2 - y^2). 
  \end{cases} & (3.19)
\end{align}

Sketch their phase portraits. Show, in particular, that there is a stable limit cycle in both cases. In (i), you should be able to derive the cycle explicitly. In (ii), you will need to construct a trapping region.

3.12. [Ox] Odell’s Predator-Prey Model. Consider the following system of nonlinear, coupled ODEs:

\begin{align}
  \begin{cases} 
    \dot{x} = x[x(1 - x) - y], \\
    \dot{y} = y(x - a), 
  \end{cases} & (3.20)
\end{align}

where $a > 0$ is a parameter (Odell 1980). This model describes a vegetarian species $x$, who are omnigamous (birth rate $\propto x^2$), limited by the availability of food (death rate $\propto x^3$), and are eaten by a predator species $y$ at the rate proportional to the latter’s population size; the predators are monogamous but procreate conditionally on the availability of prey, at a rate proportional to $x$, and have a death rate controlled by the parameter $a$. The population sizes $x$ and $y$ are normalised in some appropriate way ($x$ to its value corresponding to the equilibrium in the absence of predator).

Determine the equilibria (fixed points) of the system and their nature depending on the value of the parameter $a$. Sketch the phase portrait for each qualitatively different parameter regime. What is the condition for a limit cycle to exist? Interpret your qualitative solutions in terms of population dynamics.

A preview of some of the answers is Fig. 42 (§7.3.1). Before doing this problem, you may wish to practice sketching phase portraits by reproducing ones given in Fig. 38.

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61Sir William Wiggly FRS (1918-2018), a prominent British vacillator.