
Collisional Plasma Physics
Problem Set II

Due: Friday 11 June 2021

2.1 (25 points) Consider a spatially uniform plasma composed of one ion species with charge e and mass m_i and electrons with charge $-e$ and mass m_e. Assume that the ion average velocity is zero, $\mathbf{u}_i = 0$. The plasma is sufficiently close to a stationary Maxwellian that we can write the electron distribution function as

$$f_e(\mathbf{v}, t) \simeq f_{Me}(v) + f_{e1}(\mathbf{v}, t), \tag{1}$$

where $f_{e1} \ll f_{Me}$ and $f_{Me}(v)$ is a steady-state, stationary Maxwellian,

$$f_{Me}(v) = n_e \left(\frac{m_e}{2\pi T_e} \right)^{3/2} \exp\left(-\frac{m_e v^2}{2T_e} \right). \tag{2}$$

Expanding in $\sqrt{m_e/m_i} \ll 1$, the time evolution equation for f_{e1} is

$$\frac{\partial f_{e1}}{\partial t} = C_{ee}^{(\ell)}[f_{e1}] + \mathcal{L}_{ei}[f_{e1}]. \tag{3}$$

In what follows, you will determine the time evolution of $f_{e1}(\mathbf{v}, t)$ for the initial condition

$$f_{e1}(\mathbf{v}, t = 0) = \frac{m_e \mathbf{v} \cdot \mathbf{u}_e}{T_e} f_{Me}. \tag{4}$$

(a) Argue that f_{e1} can be written as

$$f_{e1}(\mathbf{v}, t) = \frac{m_e \mathbf{v} \cdot \mathbf{u}_e}{T_e} F_e(v, t) f_{Me}(v), \tag{5}$$

where the function $F_e(v, t)$ is only a function of the magnitude of $\mathbf{v}$, $v = |\mathbf{v}|$.

(b) To solve equation (3), we decompose $F_e(v, t)$ into the series

$$F_e(v, t) = \sum_{n=0}^{\infty} A_n \exp(-\overline{\nu}_n t) G_n(v), \tag{6}$$

where A_n and $\overline{\nu}_n$ are constants, and $G_n(v)$ are functions of v. Show that $\overline{\nu}_n$ and $G_n(v)$ are determined by the eigenvalue problem

$$\frac{3}{2} \int_0^{\pi} \left(C_{ee}^{(\ell)} \left[v G_n f_{Me} \cos\alpha \right] + \mathcal{L}_{ei} \left[v G_n f_{Me} \cos\alpha \right] \right) \cos\alpha \sin\alpha \, \mathrm{d}\alpha = -\overline{\nu}_n v G_n f_{Me}, \tag{7}$$

where α is the angle between $\mathbf{v}$ and $\mathbf{u}_e$.

(c) Show that the functions $G_n(v)$ satisfy the orthogonality property

$$\int_0^{\infty} G_n(v) G_m(v) v^4 f_{Me}(v) \, \mathrm{d}v = 0 \tag{8}$$

for $n \neq m$.

(d) Show that the eigenvalues $\overline{\nu}_n$ are extrema of the functional

$$\Theta[G] = \frac{\mathcal{N}[G]}{\mathcal{D}[G]}. \tag{9}$$

where

$$\mathcal{N}[G] = -\int vG\cos\alpha C_{ee}^{(\ell)}\left[vGf_{Me}\cos\alpha\right]\,\mathrm{d}^3v - \int vG\cos\alpha\mathcal{L}_{ei}\left[vGf_{Me}\cos\alpha\right]\,\mathrm{d}^3v \tag{10}$$

and

$$\mathcal{D}[G] = \frac{1}{3}\int v^2G^2f_{Me}\,\mathrm{d}^3v. \tag{11}$$

In other words, show that $\Theta[G_n] = \overline{\nu}_n$ and that $\Theta[G_n + \delta G] - \Theta[G_n] = 0$ to first order in $\delta G \ll G_n$.

(e) Evaluate $\Theta[G]$ for the trial function $G = a_0 + a_1L_1^{(3/2)}(x) + a_2L_2^{(3/2)}(x)$, where $L_p^{(\gamma)}(x)$ are modified Laguerre polynomials, $x = m_ev^2/2T_e$ and a_p are constants. Show that the extrema of $\Theta[G]$ with respect to the parameters a_0, a_1 and a_2 satisfy

$$\begin{pmatrix} 1 & \frac{3}{2} & \frac{15}{8} \\ \frac{3}{2} & \frac{13}{4} + \sqrt{2} & \frac{69}{16} + \frac{3\sqrt{2}}{4} \\ \frac{15}{8} & \frac{69}{16} + \frac{3\sqrt{2}}{4} & \frac{433}{64} + \frac{45\sqrt{2}}{16} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{\overline{\nu}_n}{\nu_{ee}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{35}{8} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}. \tag{12}$$

Using this expression, obtain an approximate value of the $\underline{\text{smallest decay rate}}$ $\overline{\nu}_0$ and an approximate form of $G_0(v)$. Normalize $G_0(v)$ such that

$$\frac{8}{3\sqrt{\pi}}\left(\frac{m_e}{2T_e}\right)^{5/2}\int_0^\infty G_0^2v^4\exp\left(-\frac{m_ev^2}{2T_e}\right)\,\mathrm{d}v = 1. \tag{13}$$

(f) Use the results in (c) and (e) to calculate the time evolution of f_{e1} for $t \gg \nu_{ee}^{-1} = \nu_{ei}^{-1}$.

2.2 (25 points) In this problem, we will reproduce part of Braginskii's calculation for ions. We consider a plasma with one ion species with charge e and mass m_i, and electrons with charge $-e$ and mass m_e. We will expand the equation in the small parameters

$$\frac{\rho_i}{L} \sim \frac{\lambda_{ii}}{L} \sim \sqrt{\frac{m_e}{m_i}} \ll 1 \sim \frac{|\mathbf{u}_i|}{v_{ti}}. \tag{14}$$

(a) Show that the ion Fokker-Planck kinetic equation in the coordinate $\mathbf{w} = \mathbf{v} - \mathbf{u}_i$ is

$$\begin{aligned}\Omega_i(\mathbf{w}\times\hat{\mathbf{b}})\cdot\nabla_wf_i - C_{ii}[f_i, f_i] = -\frac{\partial f_i}{\partial t} - (\mathbf{w} + \mathbf{u}_i)\cdot\nabla f_i + \mathbf{w}\cdot\nabla\mathbf{u}_i\cdot\nabla_wf_i \\ +\frac{\mathbf{F}_{ie} - \nabla p_i - \nabla\cdot\mathbf{\Pi}_i}{n_im_i}\cdot\nabla_wf_i + C_{ie}[f_i, f_e]. \end{aligned} \tag{15}$$

Here $\Omega_i = eB/m_i$ is the ion gyrofrequency and $\mathbf{u}_i = n_i^{-1}\int f_i\mathbf{v}\,\mathrm{d}^3v$ is the ion average velocity. [Hint: you will need to use the ion momentum conservation equation.]

(b) Expanding the distribution function as $f_i = f_{i0} + f_{i1} + \ldots$, with $f_{i1} \sim (\rho_i/L) f_{i0}$, show that equation (15) becomes to lowest order

$$\Omega_i (\mathbf{w} \times \hat{\mathbf{b}}) \cdot \nabla_w f_{i0} - C_{ii}[f_{i0}, f_{i0}] = 0. \tag{16}$$

Why are the electron-ion collisions negligible for the ions if they were important for the electrons?

(c) Show that the solution to equation (16) that satisfies the condition $\int f_{i0} \mathbf{w} \, \mathrm{d}^3 w = 0$ is the stationary Maxwellian

$$f_{i0} = f_{Mi} \equiv n_i \left(\frac{m_i}{2\pi T_i} \right)^{3/2} \exp\left(-\frac{m_i w^2}{2T_i} \right). \tag{17}$$

Justify the condition $\int f_{i0} \mathbf{w} \, \mathrm{d}^3 w = 0$.

(d) Show that to next order in $\rho_i/L \ll 1$, equation (15) becomes

$$\begin{aligned} \Omega_i (\mathbf{w} \times \hat{\mathbf{b}}) \cdot \nabla_w f_{i1} - C_{ii}^{(\ell)}[f_{i1}] = & -\frac{\partial f_{Mi}}{\partial t} - (\mathbf{w} + \mathbf{u}_i) \cdot \nabla f_{Mi} + \mathbf{w} \cdot \nabla \mathbf{u}_i \cdot \nabla_w f_{Mi} \\ & - \frac{\nabla p_i}{n_i m_i} \cdot \nabla_w f_{Mi} + \frac{m_e \nu_{ei}}{m_i} \left(\frac{T_e}{T_i} - 1 \right) \left(\frac{m_i w^2}{T_i} - 3 \right) f_{Mi}. \end{aligned} \tag{18}$$

[Hint: due to the orderings in (14), the term $C_{ie}[f_{Mi}, f_{Me}]$ enters at this order.] Note that we neglected the terms $\partial f_{Me}/\partial t$ and $\mathbf{w} \cdot \nabla \mathbf{u}_e \cdot \nabla f_{Me}$ in the equivalent electron equation, but we cannot neglect $\partial f_{Mi}/\partial t$ and $\mathbf{w} \cdot \nabla \mathbf{u}_i \cdot \nabla f_{Mi}$ in (18). Why?

(e) Using the continuity equation and the thermal energy equation for ions, simplify equation (18) to

$$\begin{aligned} \Omega_i (\mathbf{w} \times \hat{\mathbf{b}}) \cdot \nabla_w f_{i1} - C_{ii}^{(\ell)}[f_{i1}] = - \Bigg[& \left(\frac{m_i w^2}{2T_i} - \frac{5}{2} \right) \mathbf{w} \cdot \nabla \ln T_i \\ & + \frac{m_i}{T_i} \left(\mathbf{w}\mathbf{w} - \frac{w^2}{3} \mathbf{I} \right) : \nabla \mathbf{u}_i \Bigg] f_{Mi}. \end{aligned} \tag{19}$$

This equation is the one that Braginskii inverted to obtain f_{i1} and with it, calculate the ion heat flux and viscosity.

(f) Show that the equation for the gyrophase independent piece of f_{i1} is

$$\begin{aligned} C_{ii}^{(\ell)}[\langle f_{i1} \rangle_\varphi] = \Bigg[& \left(\frac{m_i w^2}{2T_i} - \frac{5}{2} \right) w \cos\alpha \, \hat{\mathbf{b}} \cdot \nabla \ln T_i \\ & + \frac{m_i w^2}{2T_i} (3\cos^2\alpha - 1) \left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathbf{I}}{3} \right) : \nabla \mathbf{u}_i \Bigg] f_{Mi}, \end{aligned} \tag{20}$$

where α is the angle between the velocity $\mathbf{w}$ and the direction of the magnetic field $\hat{\mathbf{b}}$, i.e. $\alpha = \arccos(\mathbf{w} \cdot \hat{\mathbf{b}}/w)$.

2.3 (25 points) In this problem, we solve equation (20) using a model collision operator. We use the modified Brownian motion operator (see problem 1.1)

$$C_{ii}[f_i] = \nu_{ii}\nabla_w \cdot \left(\frac{\Theta_i}{m_i}\nabla_w f_i + \mathbf{w} f_i\right), \tag{21}$$

where $\Theta_i = (\int f_i \, \mathrm{d}^3w)^{-1} \int f_i m_i w^2/3 \, \mathrm{d}^3w$ and $\mathbf{w} = \mathbf{v} - \mathbf{u}_i$.

(a) Show that the linearization of the collision operator in (21) around the Maxwellian in (17) is

$$\begin{aligned} C_{ii}^{(\ell)}[f_{i1}] = {} & \frac{\nu_{ii}T_i}{m_i}\nabla_w \cdot \left[f_{Mi}\nabla_w\left(\frac{f_{i1}}{f_{Mi}}\right)\right] \\ & + \frac{\nu_{ii}}{n_i}\left(\frac{m_i w^2}{T_i} - 3\right) f_{Mi}\int f_{i1}(\mathbf{w}')\left(\frac{m_i(w')^2}{3T_i} - 1\right)\mathrm{d}^3w'. \end{aligned} \tag{22}$$

(b) Show that in spherical coordinates,

$$\begin{aligned} \nabla_w \cdot \left[f_{Mi}\nabla_w\left(\frac{f_{i1}}{f_{Mi}}\right)\right] = {} & \frac{1}{w^2}\frac{\partial}{\partial w}\left[w^2 f_{Mi}\frac{\partial}{\partial w}\left(\frac{f_{i1}}{f_{Mi}}\right)\right] \\ & + \frac{f_{Mi}}{w^2\sin\alpha}\frac{\partial}{\partial\alpha}\left[\sin\alpha\frac{\partial}{\partial\alpha}\left(\frac{f_{i1}}{f_{Mi}}\right)\right] + \frac{f_{Mi}}{w^2\sin^2\alpha}\frac{\partial^2}{\partial\varphi^2}\left(\frac{f_{i1}}{f_{Mi}}\right), \end{aligned} \tag{23}$$

where α is the angle between $\mathbf{w}$ and the magnetic field, and φ is the gyrophase.

(c) Solve equation (20) using the linearized model collision operator in (22). Try a solution of the form

$$\langle f_{i1}\rangle_\varphi = \left[C\left(\frac{m_i w^2}{2T_i} - \frac{5}{2}\right) w\cos\alpha + K\frac{m_i w^2}{2T_i}(3\cos^2\alpha - 1)\right] f_{Mi}, \tag{24}$$

where C and K are constants that you have to determine. Check that the solution satisfies the constraint $\int \langle f_{i1}\rangle_\varphi w\cos\alpha \, \mathrm{d}^3w = 0$.

(d) Use the solution of part (c) to show that the ion parallel heat flux is

$$\mathbf{q}_{i\|} = \int \frac{1}{2}m_i w^2\, \mathbf{w}\, \langle f_i\rangle_\varphi \, \mathrm{d}^3w = -\frac{5n_iT_i}{6m_i\nu_{ii}}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla T_i. \tag{25}$$

(e) Use the solution in part (c) to show that the ion parallel viscosity is

$$\mathbf{\Pi}_{i\|} = \int m_i\left(\mathbf{w}\mathbf{w} - \frac{w^2}{3}\mathbf{I}\right)\langle f_i\rangle_\varphi \, \mathrm{d}^3w = -\frac{n_iT_i}{2\nu_{ii}}\left(3\hat{\mathbf{b}} \cdot \nabla\mathbf{u}_i \cdot \hat{\mathbf{b}} - \nabla \cdot \mathbf{u}_i\right)\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{\mathbf{I}}{3}\right). \tag{26}$$

2.4 (25 points) Consider a steady state, collisional, magnetized plasma composed of ions with charge e and mass m_i, and electrons with charge $-e$ and mass m_e immersed in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ and bounded by two plates at $z = 0$ and $z = L$. The electron density n_e, the ion and electron average velocities $\mathbf{u}_i$ and $\mathbf{u}_e$ and the electron and ion temperatures T_i and T_e only depend on z. The plasma is sufficiently collisional that Braginskii equations apply. The ion and electron average velocities are of the same order, but they are smaller than the ion thermal speed,

$$|\mathbf{u}_i| \sim |\mathbf{u}_e| \ll v_{ti}. \tag{27}$$

(a) Show that for

$$\sqrt{\frac{m_i}{m_e}}\lambda_{ee} \ll L, \tag{28}$$

the electron thermal energy equation simply gives that the ion and electron temperatures are equal to each other, that is, $T_i \simeq T_e = T$, where T is the plasma temperature.

(b) Sum the ion and electron thermal energy equations to obtain an equation for the plasma temperature T. Use quasineutrality $n_e = n_i$ to simplify the equation. Show that for

$$L \ll \sqrt{\frac{m_i}{m_e}}\frac{v_{ti}}{|\mathbf{u}_e|}\lambda_{ee}, \tag{29}$$

this equation for T simply becomes

$$\nabla \cdot (q_{e\|}\hat{\mathbf{b}}) \simeq 0. \tag{30}$$

(c) Considering $\ln \Lambda_{ee} \simeq 17$ a constant, show that equation (30) is

$$\frac{\partial}{\partial z}\left(KT^{5/2}\frac{\partial T}{\partial z}\right) = 0, \tag{31}$$

where K is a positive constant. Determine the constant K and explain why the plasma density does not enter in this equation.

(d) Find the plasma temperature profile $T(z)$ and the parallel electron heat flux $q_{e\|}(z)$ assuming that $T(z = 0) = T_b$ and $T(z = L) = T_t$, with $T_b > T_t$. Sketch $T(z)$.

(e) Using the parallel component of the total momentum conservation equation and the orderings above, show that the electron density satisfies $n_e = p_0/T$, where p_0 is a constant.

(f) Using the parallel component of the electron momentum equation and the results above, show that the parallel current $J_\| = en_e(u_{i\|} - u_{e\|})$ is

$$J_\| = T^{3/2}\left(C_1 E_\| + C_2\frac{\partial T}{\partial z}\right), \tag{32}$$

where C_1 and C_2 are positive constants. Determine C_1 and C_2. Is $J_\|$ consistent with our assumption (27)?

(g) Argue that $J_\|$ is constant, and hence use the results in part (f) and part (d) to calculate $E_\|$ as a function of T, $J_\|$ and $q_{e\|}$. Sketch $E_\|(z)$ for $J_\| > 0$ and $J_\| < 0$.

(h) Calculate the potential difference between the plates, $V = \int_0^L E_\| \, \mathrm{d}z$, and show that it is equal to

$$V = \frac{7(T_b^2 - T_t^2)L}{4C_1(T_b^{7/2} - T_t^{7/2})}J_\| + \frac{C_2}{C_1}(T_b - T_t). \tag{33}$$

Take the limit $T_t \to T_b$ in this equation. Does the result in this limit make physical sense?