Particle motion in strong magnetic fields

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1. Introduction

This is the first lecture on magnetized plasmas. The behavior of plasmas in the presence of strong magnetic fields is interesting because there is a great number of situations in which the plasma is either confined by external magnetic fields (magnetic confinement fusion, plasma thrusters...) or it generates its own strong magnetic field (plasma dynamos, galaxy cluster plasmas...).

To understand the behavior of plasmas imbedded in strong magnetic fields, we first study the motion of charged particles in strong magnetic fields. We start by solving the simple problem of a particle in constant and uniform electromagnetic fields. We then use what we learn from this simple problem to solve the more general problem in a useful limit.

2. Constant and uniform E and B fields

We first solve the motion of a charged particle under the action of $\mathbf{E}$ and $\mathbf{B}$ fields that do not depend on time and do not vary spatially. The equations of motion are

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} \tag{2.1}$$

and

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{Ze}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2.2}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the particle, Ze and m are the charge and mass of the particle, and e is the proton charge.

The velocity $\mathbf{v}$ appears by itself in equation (2.2). Thus, we can use this equation to solve for $\mathbf{v}(t)$ and then use (2.1) to find $\mathbf{r}(t)$. There are many different ways to solve (2.2). We choose to solve it by using the unit vector in the direction of the magnetic field, $\hat{\mathbf{b}} = \mathbf{B}/B$, to split the velocity into the component parallel to the magnetic field, $v_\| = \mathbf{v} \cdot \hat{\mathbf{b}}$, and the components perpendicular to the magnetic field, $\mathbf{v}_\perp = \mathbf{v} - v_\| \hat{\mathbf{b}}$. The component of (2.2) parallel to the magnetic field is

$$\frac{\mathrm{d}v_\|}{\mathrm{d}t} = \frac{Ze}{m} E_\|, \tag{2.3}$$

where $E_\| = \mathbf{E} \cdot \hat{\mathbf{b}}$. This equation can be directly integrated in time to give

$$v_\|(t) = v_{\|0} + \frac{Ze}{m} E_\| t, \tag{2.4}$$

where $v_{\|0}$ is a constant of integration.

The components of (2.2) perpendicular to the magnetic field are

$$\frac{\mathrm{d}\mathbf{v}_\perp}{\mathrm{d}t} - \Omega \mathbf{v}_\perp \times \hat{\mathbf{b}} = \frac{Ze}{m} \mathbf{E}_\perp, \tag{2.5}$$

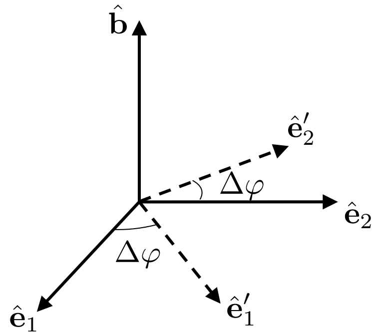

FIGURE 1. Local orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$ for the velocity. Note that the vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are defined up to a rotation. In the sketch we give two possible bases, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{e}}_1', \hat{\mathbf{e}}_2', \hat{\mathbf{b}}\}$, rotated an angle $\Delta\varphi$ with respect to each other.

where $\Omega = ZeB/m$ is the gyrofrequency, and $\mathbf{E}_\perp = \mathbf{E} - E_\| \hat{\mathbf{b}}$. This is an inhomogeneous linear system of equations with constant coefficients that one can solve using the usual, well-known techniques. With the relative velocity

$$\mathbf{w} = \mathbf{v} - \mathbf{v}_E, \tag{2.6}$$

where

$$\mathbf{v}_E = \frac{1}{B}\mathbf{E} \times \hat{\mathbf{b}} \tag{2.7}$$

is known as $\mathbf{E} \times \mathbf{B}$ drift, we rewrite equation (2.5) as

$$\frac{\mathrm{d}\mathbf{w}_\perp}{\mathrm{d}t} - \Omega \mathbf{w}_\perp \times \hat{\mathbf{b}} = 0. \tag{2.8}$$

This system of equations is linear and homogeneous. Its solution is

$$\mathbf{w}_\perp(t) = w_\perp[\cos(\Omega t + \alpha)\,\hat{\mathbf{e}}_1 - \sin(\Omega t + \alpha)\,\hat{\mathbf{e}}_2], \tag{2.9}$$

leading to

$$\mathbf{v}_\perp(t) = \mathbf{v}_E + w_\perp[\cos(\Omega t + \alpha)\,\hat{\mathbf{e}}_1 - \sin(\Omega t + \alpha)\,\hat{\mathbf{e}}_2], \tag{2.10}$$

where $w_\perp$ and α are constants of integration, and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are two unit vectors perpendicular to each other and perpendicular to $\hat{\mathbf{b}}$ that satisfy $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$ (see figure 1).

Combining (2.4) and (2.10), we obtain

$$\mathbf{v}(t) = \left(v_{\|0} + \frac{Ze}{m} E_\| t \right) \hat{\mathbf{b}} + \mathbf{v}_E + w_\perp[\cos(\Omega t + \alpha)\,\hat{\mathbf{e}}_1 - \sin(\Omega t + \alpha)\,\hat{\mathbf{e}}_2]. \tag{2.11}$$

Integrating this equation in time once, we obtain the position,

$$\mathbf{r}(t) = \mathbf{r}_0 + \left(v_{\|0} t + \frac{Ze}{2m} E_\| t^2 \right) \hat{\mathbf{b}} + \mathbf{v}_E t + \rho[\sin(\Omega t + \alpha)\,\hat{\mathbf{e}}_1 + \cos(\Omega t + \alpha)\,\hat{\mathbf{e}}_2], \tag{2.12}$$

where $\mathbf{r}_0$ is a constant of integration, and

$$\rho = \frac{w_\perp}{\Omega} \tag{2.13}$$

is the gyroradius or Larmor radius.

The motion described by equations (2.11) and (2.12) is known as Larmor motion. The particle can move freely along the magnetic field (it can be accelerated or decelerated by the parallel electric field), but its motion in the plane perpendicular to the magnetic

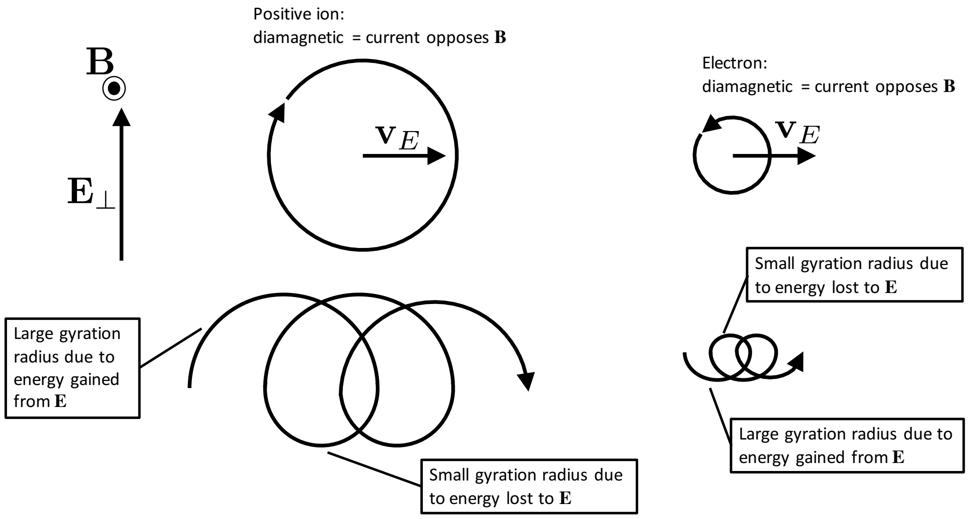

FIGURE 2. Sketch of the Larmor motion in the plane perpendicular to $\mathbf{B}$. The magnetic field $\mathbf{B}$ is pointing to the reader. The motion is diamagnetic because it produces a current that tends to oppose the background magnetic field $\mathbf{B}$. The $\mathbf{E}\times\mathbf{B}$ drift is the result of the particle gaining kinetic energy in the region where $Ze\mathbf{E}\cdot\mathbf{v} > 0$ and losing kinetic energy in the region where $Ze\mathbf{E}\cdot\mathbf{v} < 0$. The part of the orbit where the kinetic energy is high has a larger radius of gyration than the part of the orbit with less kinetic energy, leading to the $\mathbf{E}\times\mathbf{B}$ drift.

field is the composition of a constant drift, $\mathbf{v}_E$, and a circular motion. See Figure 2 for a sketch of the motion in the perpendicular plane and a simple physical picture to explain the $\mathbf{E}\times\mathbf{B}$ drift.

3. Magnetized particles in general electromagnetic fields

The Larmor motion is analytically solvable because $\mathbf{E}$ and $\mathbf{B}$ are assumed constant in time and uniform in space. This solution can be extended to more general electromagnetic fields that are approximately constant and uniform in the time and length scales of the Larmor motion, Ω^{-1} and ρ. We consider a system of size L with a characteristic frequency

$$\omega \sim \frac{v_t}{L}, \tag{3.1}$$

where v_t is the thermal speed of the particles. The size and frequency of the system manifest themselves as the sizes of the derivatives of the electromagnetic fields,

$$\nabla\mathbf{E} \sim \frac{\mathbf{E}}{L}, \quad \nabla\mathbf{B} \sim \frac{\mathbf{B}}{L}, \quad \frac{\partial\mathbf{E}}{\partial t} \sim \omega\mathbf{E}, \quad \frac{\partial\mathbf{B}}{\partial t} \sim \omega\mathbf{B}. \tag{3.2}$$

If the magnetic field is sufficiently strong,

$$\rho_* = \frac{\rho}{L} = \frac{mv_t}{ZeBL} \ll 1, \quad \frac{\omega}{\Omega} = \frac{m\omega}{ZeB} \sim \rho_* \ll 1. \tag{3.3}$$

Note that we have defined the useful small parameter $\rho_* \ll 1$. In the limit described by (3.3), the particles must follow trajectories similar to the ones in equations (2.11) and (2.12) because the particle does not see appreciable changes in the electromagnetic fields in the characteristic time scales of its motion. We proceed to calculate the differences between the Larmor motion in (2.11) and (2.12) and the motion of a particle satisfying (3.3) in a general electromagnetic field.

Before starting the calculation, we need to order the electric field with respect to the magnetic field. The electric field along the magnetic field must be such that it balances the inertia of the particles in the plasma, that is, $ZeE_\| \sim m\omega v_t \sim mv_t^2/L$. This estimate gives

$$E_\| \sim \frac{mv_t^2}{ZeL}. \tag{3.4}$$

Conversely, the perpendicular magnetic field must be able to compete with the magnetic force, giving

$$\mathbf{E}_\perp \sim v_t B \sim \frac{1}{\rho_*}\frac{mv_t^2}{ZeL} \gg E_\|. \tag{3.5}$$

The perpendicular electric field can be much larger than the parallel electric field because in the directions perpendicular to the magnetic field, the strong magnetic field can balance large forces. We must consider equation (3.5) as an upper limit for the perpendicular electric field. In many applications, the perpendicular electric field is of the order of $E_\|$.

The equations that we need to solve are

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{v} \tag{3.6}$$

and

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{Ze}{m}\left[\mathbf{E}(\mathbf{r},t) + \mathbf{v}\times\mathbf{B}(\mathbf{r},t)\right]. \tag{3.7}$$

Note that in these equations, unlike in (2.1) and (2.2), $\mathbf{E}$ and $\mathbf{B}$ depend on time and space.

To solve equations (3.6) and (3.7) under the assumptions (3.2), (3.3), (3.4) and (3.5), we first propose a set of coordinates to separate the two time scales of the problem, Ω^{-1} and $\omega^{-1} \sim L/v_t$, to lowest order in ρ_*. This calculation can be organized in many different ways, but the final results are always the same (see, for example, Hazeltine & Meiss 2003; Goldston & Rutherford 1995; Cary & Brizard 2009).

3.1. *New phase space coordinates*

We use the Larmor motion in (2.11) and (2.12) to propose a new set of convenient phase space coordinates that separate the fast time scale Ω^{-1} from the slow time scale $\omega^{-1} \sim L/v_t$.

The time evolution of the position of the particle is mostly dominated by the slow time scale. Recalling that $E_\|$ and $\mathbf{E}_\perp$ are ordered as in (3.4) and (3.5), equation (2.12) becomes

$$\mathbf{r}(t) = \underbrace{\mathbf{r}_0}_{\sim L} + \left(\underbrace{v_{\|0}t}_{\sim v_t t} + \underbrace{\frac{Ze}{2m}E_\| t^2}_{\sim v_t^2 t^2/L}\right)\hat{\mathbf{b}} + \underbrace{\mathbf{v}_E t}_{\sim v_t t} + \underbrace{\rho[\sin(\Omega t+\alpha)\,\hat{\mathbf{e}}_1 + \cos(\Omega t+\alpha)\hat{\mathbf{e}}_2]}_{\sim \rho_* L \ll L}. \tag{3.8}$$

These estimates show that only a small piece of order $\rho_* L \ll L$ changes appreciably in the fast time scale $\Omega^{-1} \ll L/v_t$, and that it takes a time $t \sim L/v_t$ for the particle to move a distance L. Thus, we assume that $\mathbf{r}(t)$ is mostly dominated by the slow time scale, and we use $\mathbf{r} = (x, y, z)$ as the spatial coordinates for our equations.

For the velocity, equation (2.11) gives

$$\mathbf{v}(t) = \left(\underbrace{v_{\|0}}_{\sim v_t} + \underbrace{\frac{Ze}{m}E_\| t}_{\sim v_t^2 t/L}\right)\hat{\mathbf{b}} + \underbrace{\mathbf{v}_E}_{\sim v_t} + \underbrace{w_\perp[\cos(\Omega t+\alpha)\,\hat{\mathbf{e}}_1 - \sin(\Omega t+\alpha)\,\hat{\mathbf{e}}_2]}_{\sim v_t}. \tag{3.9}$$

The velocity has two different time scales. On the one hand, we have the fast gyration with a characteristic time scale Ω^{-1} (see the cosine and sine terms). On the other hand, it takes a time of the order of $L/v_t \gg \Omega^{-1}$ for the parallel velocity to change by an amount of order v_t. Since the fast time scale appears inside cosines and sines, it is convenient to describe it using a phase $\varphi(t) \sim \Omega t$ that we call gyrophase. In (2.11), we will replace

$$\Omega t + \alpha \rightarrow \varphi(t). \tag{3.10}$$

We expect the parallel velocity $v_{\|}(t)$ and the perpendicular velocity $w_{\perp}(t)$ to change in the slow time scale L/v_t. In (2.12), we will replace

$$v_{\|0} + \frac{Ze}{m} E_{\|} t \rightarrow v_{\|}(t) \tag{3.11}$$

and

$$w_{\perp} \rightarrow w_{\perp}(t). \tag{3.12}$$

Using the substitutions (3.10), (3.11) and (3.12) in (2.11), we find the desired form for $\mathbf{v}(t)$,

$$\mathbf{v}(t) = \underbrace{v_{\|}(t)\hat{\mathbf{b}}(\mathbf{r}(t), t)}_{\sim v_t} + \underbrace{\mathbf{v}_E(\mathbf{r}(t), t)}_{\sim v_t} + \underbrace{w_{\perp}(t)\, \hat{\mathbf{e}}_{\perp}(\mathbf{r}(t), \varphi(t), t)}_{\sim v_t}, \tag{3.13}$$

where we have made it explicit that many terms depend on the particle position because $\mathbf{E}$ and $\mathbf{B}$ are not uniform in space. We have also defined the useful function

$$\hat{\mathbf{e}}_{\perp}(\mathbf{r}, \varphi, t) = \cos\varphi\, \hat{\mathbf{e}}_1(\mathbf{r}, t) - \sin\varphi\, \hat{\mathbf{e}}_2(\mathbf{r}, t). \tag{3.14}$$

The vectors $\hat{\mathbf{e}}_1(\mathbf{r}, t)$ and $\hat{\mathbf{e}}_2(\mathbf{r}, t)$ are defined as for the Larmor motion, that is, they form a local, right-handed, orthonormal basis with $\hat{\mathbf{b}}(\mathbf{r}, t)$ at every position $\mathbf{r}$ and every time t (see figure 1). The plane formed by the vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is well defined (it is the plane perpendicular to $\hat{\mathbf{b}}$), but the vectors themselves are only defined up to a rotation within this plane, as shown in figure 1. Obviously, this arbitrary rotation of the basis will not affect the motion of the particle, as we will see to lowest order in $\rho_* \ll 1$.

Since we want to determine the functions $v_{\|}(t)$, $w_{\perp}(t)$ and $\varphi(t)$, it will be convenient to write them as functions of $\mathbf{v}$ and $\mathbf{r}$, quantities for which we know the time evolution equations (see (3.6) and (3.7)). We can invert relation (3.13) to find

$$v_{\|} = \mathbf{v} \cdot \hat{\mathbf{b}}(\mathbf{r}, t), \tag{3.15}$$

$$w_{\perp} = \sqrt{|\mathbf{v} - \mathbf{v}_E|^2 - [\mathbf{v} \cdot \hat{\mathbf{b}}(\mathbf{r}, t)]^2}, \tag{3.16}$$

$$\varphi = -\arctan\left(\frac{(\mathbf{v} - \mathbf{v}_E(\mathbf{r}, t)) \cdot \hat{\mathbf{e}}_2(\mathbf{r}, t)}{(\mathbf{v} - \mathbf{v}_E(\mathbf{r}, t)) \cdot \hat{\mathbf{e}}_1(\mathbf{r}, t)}\right). \tag{3.17}$$

Note that the end result is that we have changed from the phase space coordinates

$$\mathbf{X} = (X_1, X_2, X_3, X_4, X_5, X_6) = (\mathbf{r}, \mathbf{v}) \tag{3.18}$$

to the new phase space coordinates

$$\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) = (\mathbf{r}, v_{\|}, w_{\perp}, \varphi). \tag{3.19}$$

3.2. *Evolution equations for* $\{\mathbf{r}, v_{\|}, w_{\perp}, \varphi\}$

We proceed to calculate the evolution equations

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \dot{\mathbf{Q}}(\mathbf{Q}, t). \tag{3.20}$$

Note the distinction between

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}, \frac{\mathrm{d}w_\perp}{\mathrm{d}t}, \frac{\mathrm{d}\varphi}{\mathrm{d}t} \right), \tag{3.21}$$

and

$$\dot{\mathbf{Q}} = (\dot{\mathbf{r}}, \dot{v}_\parallel, \dot{w}_\perp, \dot{\varphi}) : \tag{3.22}$$

the set of differential equations (3.20) can be written as $\mathrm{d}\mathbf{Q}/\mathrm{d}t = \mathbf{f}(\mathbf{Q})$, where $\mathrm{d}\mathbf{Q}/\mathrm{d}t$ is the time derivative of the vector $\mathbf{Q}$, and $\dot{\mathbf{Q}}$ is the function $\mathbf{f}(\mathbf{Q})$.

To find the time derivative of any of the new phase space coordinates, we use the transformations $Q_i(\mathbf{r}, \mathbf{v}, t)$ for $i = 1, 2, 3, 4, 5, 6$, given by (3.15), (3.16) and (3.17). Differentiating with respect to time and using the chain rule, we find

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \frac{\partial Q_i}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \nabla Q_i + \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \nabla_v Q_i. \tag{3.23}$$

Using (3.6) and (3.7), this equation becomes

$$\dot{Q}_i = \underbrace{\frac{\partial Q_i}{\partial t}}_{\sim Q_i v_t/L} + \underbrace{\mathbf{v} \cdot \nabla Q_i}_{\sim Q_i v_t/L} + \underbrace{\frac{Ze}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v Q_i}_{\sim Q_i \Omega \sim Q_i \rho_*^{-1} v_t/L}. \tag{3.24}$$

This expression gives $\dot{Q}_i$ as a function of $\mathbf{X} = (\mathbf{r}, \mathbf{v})$. With the expression $\mathbf{v}(v_\parallel, w_\perp, \varphi)$ in (3.13), we get $\dot{Q}_i$ as a function of $\mathbf{Q} = (\mathbf{r}, v_\parallel, w_\perp, \varphi)$.

Applying (3.24) to $\{\mathbf{r}, v_\parallel, w_\perp, \varphi\}$, and after several vector manipulations (see Appendix A), we obtain

$$\dot{\mathbf{r}} = v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + w_\perp \hat{\mathbf{e}}_\perp = O(v_t), \tag{3.25}$$

$$\begin{aligned} \dot{v}_\parallel = {} & w_\perp \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \cdot \hat{\mathbf{e}}_\perp + w_\perp^2 \hat{\mathbf{e}}_\perp \cdot \nabla \hat{\mathbf{b}} \cdot \hat{\mathbf{e}}_\perp + w_\perp \hat{\mathbf{e}}_\perp \cdot \nabla \hat{\mathbf{b}} \cdot \mathbf{v}_E \\ & + \frac{Ze}{m} \mathbf{E} \cdot \left[\hat{\mathbf{b}} + \frac{1}{\Omega} \hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right] = O\left(\frac{v_t^2}{L} \right), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \dot{w}_\perp = {} & -v_\parallel \left[\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right] \cdot \hat{\mathbf{e}}_\perp - \left[\frac{\partial \mathbf{v}_E}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E \right] \cdot \hat{\mathbf{e}}_\perp \\ & - w_\perp \hat{\mathbf{e}}_\perp \cdot (v_\parallel \nabla \hat{\mathbf{b}} + \nabla \mathbf{v}_E) \cdot \hat{\mathbf{e}}_\perp = O\left(\frac{v_t^2}{L} \right), \end{aligned} \tag{3.27}$$

$$\begin{aligned} \dot{\varphi} = {} & \Omega + \left[\frac{\partial \hat{\mathbf{e}}_1}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{e}}_1 \right] \cdot \hat{\mathbf{e}}_2 + w_\perp \hat{\mathbf{e}}_\perp \cdot \nabla \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 \\ & + \frac{v_\parallel}{w_\perp} \left[\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right] \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{e}}_\perp) \\ & + \frac{1}{w_\perp} \left[\frac{\partial \mathbf{v}_E}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \mathbf{v}_E \right] \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{e}}_\perp) \\ & + \hat{\mathbf{e}}_\perp \cdot (v_\parallel \nabla \hat{\mathbf{b}} + \nabla \mathbf{v}_E) \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{e}}_\perp) = \Omega + O\left(\frac{v_t}{L} \right). \end{aligned} \tag{3.28}$$

The particular form of $\dot{\mathbf{r}}$, $\dot{v}_\parallel$, $\dot{w}_\perp$ and $\dot{\varphi}$ is not relevant to understand the expansion in ρ_*. There are two important points to emphasize:

- To lowest order, our choice of phase space coordinates has separated the slow and fast time scales. The coordinate φ is the only coordinate with a very large time derivative, of order Ω. Hence, it is the only coordinate that changes an amount of order unity in the fast time scale Ω^{-1}.

• However, the time scale separation is not completely successful. The gyrophase φ appears in sines and cosines inside the function $\hat{\mathbf{e}}_\perp(\mathbf{r}, \varphi, t)$. This is a crucial observation because it means that even though the gyrophase φ is the only coordinate that changes by order unity in the fast time scale, the other coordinates, $\mathbf{r}$, $v_\|$ and $w_\perp$, have small oscillations with frequency Ω due to the sines and cosines of φ appearing in (3.25), (3.26) and (3.27). It is desirable to eliminate those oscillations. We do so to lowest order in ρ_* in the next subsection.

3.3. *Integrating the motion to lowest order in* ρ_*

We need to integrate equations (3.20),

$$Q_i(t) = \int^t \dot{Q}_i(\mathbf{r}(t'), v_\|(t'), w_\perp(t'), \varphi(t'), t')\,\mathrm{d}t'. \tag{3.29}$$

To calculate this integral, we first split it into time intervals Δt of the order of several gyroperiods, $\Delta t \sim \Omega^{-1}$,

$$Q_i(t) = \sum_j \int_{t_j}^{t_j+\Delta t} \dot{Q}_i(\mathbf{r}(t'), v_\|(t'), w_\perp(t'), \varphi(t'), t')\,\mathrm{d}t', \tag{3.30}$$

where $t_j + \Delta t = t_{j+1}$. In each short time interval Δt, the changes of $\mathbf{r}$, $v_\|$ and $w_\perp$ are small because the order of magnitude estimates in (3.25), (3.26) and (3.27) imply that

$$\int_{t_j}^{t_j+\Delta t} \dot{\mathbf{r}}(t')\,\mathrm{d}t' \sim \rho^* L \ll L, \tag{3.31}$$

$$\int_{t_j}^{t_j+\Delta t} \dot{v}_\|(t')\,\mathrm{d}t' \sim \rho^* v_t \ll v_t, \tag{3.32}$$

$$\int_{t_j}^{t_j+\Delta t} \dot{w}_\perp(t')\,\mathrm{d}t' \sim \rho^* v_t \ll v_t. \tag{3.33}$$

Using equation (3.28) and the fact that the position of the particle barely changes, we find that the change of the gyrophase in a time interval Δt is

$$\int_{t_j}^{t_j+\Delta t} \dot{\varphi}(t')\,\mathrm{d}t' \simeq \int_{t_j}^{t_j+\Delta t} \Omega(\mathbf{r}(t'), t')\,\mathrm{d}t' \simeq \Omega(\mathbf{r}(t_j), t_j)\Delta t. \tag{3.34}$$

Thus, the changes of $\mathbf{r}$, $v_\|$ and $w_\perp$ in a time interval Δt can be written as

$$\begin{aligned}&\int_{t_j}^{t_j+\Delta t} \dot{Q}_i(\mathbf{r}(t'), v_\|(t'), w_\perp(t'), \varphi(t'), t')\,\mathrm{d}t' \\ &\quad\simeq \int_{t_j}^{t_j+\Delta t} \dot{Q}_i(\mathbf{r}(t_j), v_\|(t_j), w_\perp(t_j), \varphi(t_j) + \Omega(\mathbf{r}(t_j), t_j)(t' - t_j), t_j)\,\mathrm{d}t'.\end{aligned} \tag{3.35}$$

To calculate the integral (3.35), we define the gyroaverage of a function f,

$$\langle f \rangle_\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{r}, v_\|, w_\perp, \varphi, t)\,\mathrm{d}\varphi, \tag{3.36}$$

as the average over a period of φ holding $\mathbf{r}$, $v_\|$, $w_\perp$ and t fixed. With this average, we split the function $\dot{Q}_i$ into a gyrophase independent piece, $\langle \dot{Q}_i \rangle_\varphi$, and a gyrophase dependent piece,

$$\widetilde{\dot{Q}}_i = \dot{Q}_i - \langle \dot{Q}_i \rangle_\varphi. \tag{3.37}$$

With the separation in $\langle \dot{Q}_i \rangle_\varphi$ and $\widetilde{\dot{Q}}_i$, equation (3.35) becomes

$$\begin{aligned}\int_{t_j}^{t_j+\Delta t} & \dot{Q}_i(\mathbf{r}(t'), v_\parallel(t'), w_\perp(t'), \varphi(t'), t')\,\mathrm{d}t' \\ & \simeq \langle \dot{Q}_i \rangle_\varphi(\mathbf{r}(t_j), v_\parallel(t_j), w_\perp(t_j), t_j)\Delta t \\ & \quad + \frac{1}{\Omega(\mathbf{r}(t_j), t_j)} \int_{\varphi(t_j)}^{\varphi(t_j+\Delta t)} \widetilde{\dot{Q}}_i(\mathbf{r}(t_j), v_\parallel(t_j), w_\perp(t_j), \varphi', t_j)\,\mathrm{d}\varphi'. \end{aligned} \tag{3.38}$$

where we have use the change of variable $\varphi' = \varphi(t_j) + \Omega(\mathbf{r}(t_j), t_j)(t' - t_j)$ and the approximation $\varphi(t_j + \Delta t) \simeq \varphi(t_j) + \Omega(\mathbf{r}(t_j), t_j)\Delta t$ in the second integral. Substituting (3.38) into equation (3.30), we find

$$\begin{aligned} Q_i(t) \simeq & \sum_j \langle \dot{Q}_i \rangle_\varphi(\mathbf{r}(t_j), v_\parallel(t_j), w_\perp(t_j), t_j)\Delta t \\ & + \sum_j \frac{1}{\Omega(\mathbf{r}(t_j), t_j)} \int_{\varphi(t_j)}^{\varphi(t_j+\Delta t)} \widetilde{\dot{Q}}_i(\mathbf{r}(t_j), v_\parallel(t_j), w_\perp(t_j), \varphi', t_j)\,\mathrm{d}\varphi'. \end{aligned} \tag{3.39}$$

Since $\Delta t \sim \Omega^{-1} \ll t \sim L/v_t$, the first term in the right side of (3.39) can be approximated by an integral in time. Recalling that $\varphi(t_j + \Delta t) = \varphi(t_{j+1})$ and ignoring the slow variation in time of $\mathbf{r}$, $v_\parallel$ and $w_\perp$, we can sum all the integrals in the second term in the right side of (3.39) to obtain one integral in φ from the initial value of φ to the final value of φ. Thus, equation (3.39) can be written as

$$Q_i(t) \simeq \int^t \langle \dot{Q}_i \rangle_\varphi(\mathbf{r}(t'), v_\parallel(t'), w_\perp(t'), t')\,\mathrm{d}t' + \widetilde{Q}_i(t), \tag{3.40}$$

where

$$\widetilde{Q}_i(t) \simeq \frac{1}{\Omega(\mathbf{r}(t), t)} \int^{\varphi(t)} \widetilde{\dot{Q}}_i(\mathbf{r}(t), v_\parallel(t), w_\perp(t), \varphi', t)\,\mathrm{d}\varphi'. \tag{3.41}$$

As an example of $\widetilde{Q}_i$, we calculate the small correction to the position $\tilde{\mathbf{r}}$. Equation (3.41) for $\mathbf{r}$ gives

$$\begin{aligned} \tilde{\mathbf{r}} & \simeq \frac{1}{\Omega} \int^\varphi \tilde{\dot{\mathbf{r}}}(\varphi')\,\mathrm{d}\varphi' = \frac{1}{\Omega} \int^\varphi w_\perp \hat{\mathbf{e}}_\perp(\varphi')\,\mathrm{d}\varphi' = \frac{1}{\Omega} \int^\varphi w_\perp(\cos\varphi'\,\hat{\mathbf{e}}_1 - \sin\varphi'\,\hat{\mathbf{e}}_2)\,\mathrm{d}\varphi' \\ & = \frac{w_\perp}{\Omega}(\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2) = \frac{w_\perp}{\Omega}\hat{\mathbf{b}} \times \hat{\mathbf{e}}_\perp. \end{aligned} \tag{3.42}$$

The integral $\tilde{\mathbf{r}}$ represents the Larmor gyration. To see this, compare this term to the term proportional to ρ in (2.12).

The validity of approximation (3.40) can be checked by differentiating it to determine whether $\mathrm{d}Q_i/\mathrm{d}t$ is equal to $\dot{Q}_i$. By differentiating (3.40), we find $\mathrm{d}Q_i/\mathrm{d}t = \langle \dot{Q}_i \rangle_\varphi + \mathrm{d}\widetilde{Q}_i/\mathrm{d}t$. To calculate the derivative of $\widetilde{Q}_i$, we use

$$\frac{\mathrm{d}\widetilde{Q}_i}{\mathrm{d}t} = \frac{\partial \widetilde{Q}_i}{\partial t} + \dot{\mathbf{r}} \cdot \nabla \widetilde{Q}_i + \dot{v}_\parallel \frac{\partial \widetilde{Q}_i}{\partial v_\parallel} + \dot{w}_\perp \frac{\partial \widetilde{Q}_i}{\partial w_\perp} + \dot{\varphi} \frac{\partial \widetilde{Q}_i}{\partial \varphi} = \Omega \frac{\partial \widetilde{Q}_i}{\partial \varphi} + O\left(\frac{v_t}{L}\widetilde{Q}_i\right). \tag{3.43}$$

where we have used the order of magnitude estimates in equations (3.25), (3.26), (3.27) and (3.28). Hence, $\mathrm{d}\widetilde{Q}_i/\mathrm{d}t \simeq \widetilde{\dot{Q}}_i$, and as a result $\mathrm{d}Q_i/\mathrm{d}t \simeq \langle \dot{Q}_i \rangle_\varphi + \widetilde{\dot{Q}}_i = \dot{Q}_i$ as expected.

From the order of magnitude estimates in equations (3.25), (3.26) and (3.27), we find

$$\dot{\widetilde{Q}}_i = O\left(\frac{v_t}{L} Q_i\right), \tag{3.44}$$

and hence

$$\widetilde{Q}_i \sim \rho_* Q_i \ll Q_i. \tag{3.45}$$

For example, $\tilde{\mathbf{r}}$ in (3.42) is of order $\rho_* L$. The piece $\widetilde{Q}_i$ is small in size because the highly oscillatory time derivative $\dot{\widetilde{Q}}_i$ averages to zero, giving only a small contribution to $Q_i(t)$. Using the estimate in (3.45), equation (3.40) becomes to lowest order

$$Q_i(t) = \int^t \langle \dot{Q}_i \rangle_\varphi (\mathbf{r}(t'), v_\|(t'), w_\perp(t'), t')\, \mathrm{d}t' + O(\rho_* Q_i). \tag{3.46}$$

Ignoring the corrections small in ρ_*, it is tempting to write equation (3.46) as

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} \simeq \langle \dot{Q}_i \rangle_\varphi. \tag{3.47}$$

This expression is correct in the sense that once integrated in time, it gives the value of Q_i up to a correction small in ρ_*. However, it is obvious that (3.47) is different to the lowest order expression $\mathrm{d}Q_i/\mathrm{d}t = \dot{Q}_i$. The reason for this difference is that the small correction $\widetilde{Q}_i$, of order ρ_* (see (3.41)), has a large time derivative that competes with the slow time derivative of the largest piece of Q_i. For example, the oscillation in position given in (3.42) has instantaneous velocities of order v_t, but its average velocity is almost zero, contributing a very small piece to the total displacement $\mathbf{r}$. We will use (3.47) with the understanding that it is only valid when integrated in time.

Using (3.25), (3.26) and (3.27), $\nabla \cdot \mathbf{B} = 0$ and Faraday's induction law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ (see Appendix B), equation (3.47) gives

$$\boxed{\begin{aligned} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &\simeq \langle \dot{\mathbf{r}} \rangle_\varphi = v_\| \hat{\mathbf{b}} + \mathbf{v}_E, && (3.48)\\ \frac{\mathrm{d}v_\|}{\mathrm{d}t} &\simeq \langle \dot{v}_\| \rangle_\varphi = \frac{Ze}{m}\left(\hat{\mathbf{b}} + \frac{1}{\Omega}\hat{\mathbf{b}} \times \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t}\right) \cdot \mathbf{E} - \frac{w_\perp^2}{2B}\hat{\mathbf{b}} \cdot \nabla B, && (3.49)\\ \frac{\mathrm{d}w_\perp}{\mathrm{d}t} &\simeq \langle \dot{w}_\perp \rangle_\varphi = \frac{w_\perp}{2B}\frac{\mathrm{d}B}{\mathrm{d}t}, && (3.50) \end{aligned}}$$

where

$$\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t} = \frac{\partial \hat{\mathbf{b}}}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \nabla \hat{\mathbf{b}} \simeq \frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \tag{3.51}$$

and

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\partial B}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \nabla B \simeq \frac{\partial B}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla B. \tag{3.52}$$

We do not need to consider $\mathrm{d}\varphi/\mathrm{d}t$ because φ does not appear in equations (3.48), (3.49) and (3.50).

4. Summary: guiding center equations

The set of differential equations (3.48), (3.49) and (3.50) determines the average position of the particle, its average parallel velocity and its average perpendicular velocity. These equations ignore the small oscillations that happen at time scales of order Ω^{-1}.

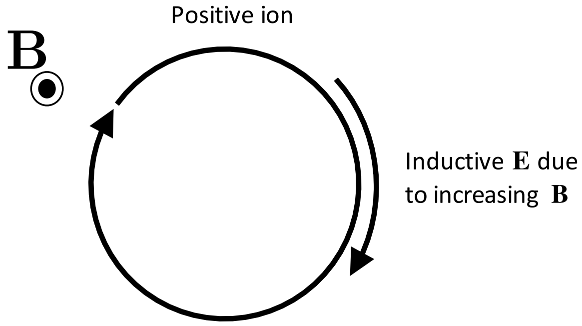

FIGURE 3. Inductive electric field due to $\mathrm{d}B/\mathrm{d}t > 0$ that accelerates the gyration of a positive ion and increases $w_\perp$. In this sketch, $\mathbf{B}$ is pointing to the the reader. Electrons are accelerated by $\mathrm{d}B/\mathrm{d}t > 0$ as well.

The position $\mathbf{r}$ is in fact the guiding center position and not the particle position. Keeping the correction (3.42), the position of the particle can be written as

$$\mathbf{r}(t) \simeq \mathbf{R}(t) + \frac{w_\perp(t)}{\Omega(\mathbf{r}(t), t)} \hat{\mathbf{b}}(\mathbf{r}(t), t) \times \hat{\mathbf{e}}_\perp(\mathbf{r}(t), \varphi(t), t), \tag{4.1}$$

with

$$\mathbf{R}(t) = \int^t [v_{||}(t') \hat{\mathbf{b}}(\mathbf{r}(t'), t') + \mathbf{v}_E(\mathbf{r}(t'), t')] \mathrm{d}t'. \tag{4.2}$$

We see that equation (3.48) gives the motion of the center $\mathbf{R}$ around which the particle gyrates. Similarly, equations (3.49) and (3.50) for $v_{||}$ and $w_\perp$ neglect small oscillations of order ρ_* with a characteristic frequency of order Ω.

The gyrophase $\varphi(t)$ does not enter in the guiding center motion. Once $\mathbf{r}(t)$, $v_{||}(t)$ and $w_\perp(t)$ are known, we can calculate the gyrophase using

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega(\mathbf{r}(t), t) + O(v_t/L). \tag{4.3}$$

To find φ accurately, we need the terms of $\dot{\varphi}$ of order v_t/L given in (3.28). These corrections of order v_t/L give a change in the gyrophase of order unity in a time of order L/v_t.

Equations (3.48), (3.49) and (3.50) give the position and the velocity of the guiding center accurately for times of order L/v_t. These equations are missing terms small by a factor of $\rho_* \ll 1$. The oscillatory rates $\widetilde{\dot{\mathbf{r}}}$, $\widetilde{\dot{v}}_{||}$ and $\widetilde{\dot{w}}_\perp$ only average to zero to lowest order. They contribute to the particle motion a non-zero correction of order ρ_*. These missing terms will lead to order unity corrections after times of order $\rho_*^{-1} L/v_t$. These corrections can be obtained from very tedious calculations. To minimize the number of such calculations in this course, we will talk about these terms in the next lecture, when we will need them to describe the statistics of magnetized plasmas.

There are other forms of the guiding center equations that prove useful. In particular, it is common to replace $w_\perp$ by another convenient variable. According to (3.50), the velocity of the gyration changes in time when B changes. The reason for this is the inductive electric field created by the time varying magnetic field (see figure 3). The effect of this inductive electric field is to keep constant the magnetic moment

$$\mu = \frac{w_\perp^2}{2B(\mathbf{r}, t)}. \tag{4.4}$$

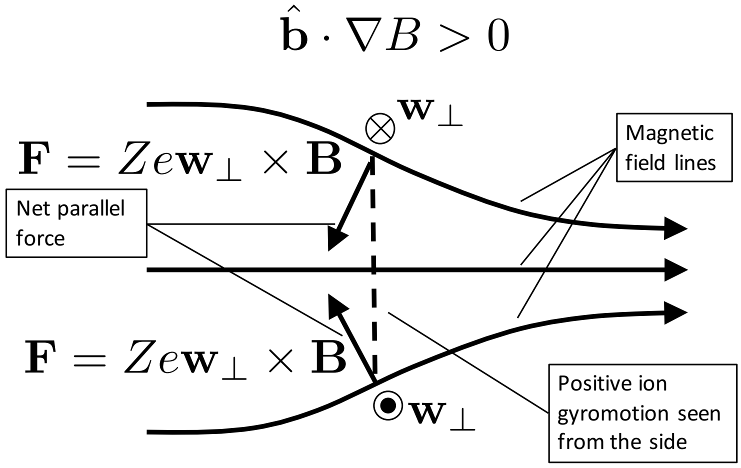

FIGURE 4. Net parallel force due to the magnetic force $Ze\mathbf{w}_\perp \times \mathbf{B}$ in a converging magnetic field.

Using (3.50), we find that its time derivative is

$$\boxed{\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{w_\perp}{B}\frac{\mathrm{d}w_\perp}{\mathrm{d}t} - \frac{w_\perp^2}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}t} \simeq 0.} \tag{4.5}$$

The magnetic moment μ is then approximately constant in time. Physically, the magnetic moment is proportional to the magnetic flux through the gyromotion. The area of the circle enclosed by the gyromotion is $A_g = \pi\rho^2 = \pi w_\perp^2/\Omega^2 = (\pi m^2/Z^2e^2)(w_\perp^2/B^2)$. Thus, the total magnetic flux through the area enclosed by the gyromotion is $A_gB = (2\pi m^2/Z^2e^2)\mu$. Note that the magnetic moment is not a constant of the motion, but an <u>adiabatic invariant</u> (see, for example, Landau & Lifshitz 1976, for a discussion of adiabatic invariants).

Using the magnetic moment, the equations for the guiding center motion become

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \simeq v_\| \hat{\mathbf{b}} + \mathbf{v}_E, \tag{4.6}$$

$$\frac{\mathrm{d}v_\|}{\mathrm{d}t} \simeq \frac{Ze}{m}\left(\hat{\mathbf{b}} + \frac{1}{\Omega}\hat{\mathbf{b}} \times \frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t}\right) \cdot \mathbf{E} - \mu\hat{\mathbf{b}} \cdot \nabla B. \tag{4.7}$$

We finish by giving some examples of guiding center motion.

4.1. *Motion with small $\mathbf{E}_\perp$: low flow regime*

So far, we have considered that the perpendicular electric field $\mathbf{E}_\perp$ is as large as we have ordered it in (3.5), making the $\mathbf{E} \times \mathbf{B}$ drift of order the thermal speed v_t. This is known as the <u>high flow regime</u>. There are cases in which the perpendicular electric field is much smaller, of order

$$\mathbf{E}_\perp \sim \frac{mv_t^2}{ZeL}. \tag{4.8}$$

In this regime, the $\mathbf{E}\times\mathbf{B}$ drift is of order $\rho_* v_{ti}$. This is the <u>low flow regime</u> or <u>drift ordering</u>.

When $\mathbf{v}_E \sim \rho_* v_{ti}$, the guiding center equations (4.6) and (4.7) become to lowest order

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \simeq v_{\|}\hat{\mathbf{b}}, \tag{4.9}$$

$$\frac{\mathrm{d}v_{\|}}{\mathrm{d}t} \simeq \frac{Ze}{m}E_{\|} - \mu\hat{\mathbf{b}}\cdot\nabla B. \tag{4.10}$$

Thus, the particle only moves along the magnetic field line (note that it does not separate from it even if the magnetic field lines curve). The parallel velocity can be accelerated or decelerated by the parallel electric field and, unexpectedly, by the magnetic field. The magnetic field does not exert a force parallel to itself, but the direction of the magnetic field changes slightly with the position of the particle in its gyration around the guiding center, and the direction of the magnetic field at the particle position is different from the direction of the magnetic field at the guiding center. When $\hat{\mathbf{b}}\cdot\nabla B \neq 0$, the magnetic field lines have to converge or diverge due to $\nabla\cdot\mathbf{B} = 0$. This divergence or convergence causes the parallel acceleration $-\mu\hat{\mathbf{b}}\cdot\nabla B$ of the guiding center, as shown in figure 4. This parallel acceleration causes what is known as magnetic bottling.

To explain magnetic bottling, we consider a steady state plasma. In steady state, the electric field is well described by an electrostatic potential ϕ, $\mathbf{E} = -\nabla\phi$. With an electrostatic magnetic field, the total energy

$$\mathcal{E} = \frac{1}{2}mv_{\|}^2 + m\mu B(\mathbf{r}) + Ze\phi(\mathbf{r}) \tag{4.11}$$

is conserved. Indeed, using $\mathbf{E} = -\nabla\phi$, (4.9) and (4.10), we find

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = mv_{\|}\frac{\mathrm{d}v_{\|}}{\mathrm{d}t} + m\mu\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\nabla B + Ze\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\nabla\phi \simeq 0. \tag{4.12}$$

Equation (4.9) implies that the motion is along the magnetic field line. Then, limiting ourselves to a magnetic field line, and using the length along the line l as a coordinate, we obtain the equation of motion

$$\frac{\mathrm{d}l}{\mathrm{d}t} = v_{\|} = \pm\sqrt{\frac{2}{m}\left[\mathcal{E} - m\mu B(l) - Ze\phi(l)\right]}, \tag{4.13}$$

where we have used equation (4.11) to solve for $v_{\|}$ as a function of the constants $\mathcal{E}$ and μ. The particle motion is controlled by the effective potential $m\mu B(l) + Ze\phi(l)$. If $\mathcal{E}$ and μ are such that $\mathcal{E} > m\mu B(l) + Ze\phi(l)$ for all l, the particle moves uninterruptedly in the same direction it started. Conversely, if there is a location l_0 in which $\mathcal{E} = m\mu B(l_0) + Ze\phi(l_0)$, the particle will stop and bounce at l_0. In figure 5, we consider a situation in which $\phi(l) = 0$ and the magnetic field magnitude has a shape that leads to bounces for certain particles. Since $\phi(l) = 0$, the bounce points are determined by $B(l_0) = \mathcal{E}/m\mu$.

4.2. *Motion with large $\mathbf{E}_\perp$: high flow regime*

In the high flow regime, we have to use equations (4.6) and (4.7). Particles move off magnetic field lines due to the large $\mathbf{E}\times\mathbf{B}$ drifts. The parallel velocity is not only affected by the magnetic bottling force and the parallel electric field. There is an extra term,

$$\frac{Ze}{m\Omega}\left(\hat{\mathbf{b}}\times\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t}\right)\cdot\mathbf{E}. \tag{4.14}$$

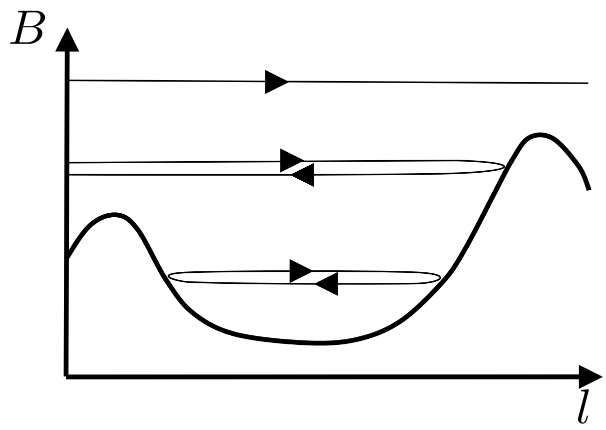

FIGURE 5. Sketch of B along a magnetic field line. Particles with $\mathcal{E}/m\mu$ smaller than the maximum in $B(l)$ will have a bounce point.

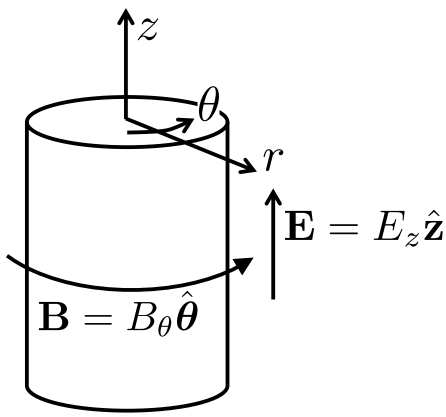

FIGURE 6. Cylindrical configuration with azimuthal magnetic field B_θ and axial electric field E_z.

This force came from the parallel component of the time derivative of the $\mathbf{E}\times\mathbf{B}$ drift,

$$\frac{Ze}{m\Omega}\left(\hat{\mathbf{b}}\times\frac{\mathrm{d}\hat{\mathbf{b}}}{\mathrm{d}t}\right)\cdot\mathbf{E} = -\frac{\mathrm{d}\mathbf{v}_E}{\mathrm{d}t}\cdot\hat{\mathbf{b}} \tag{4.15}$$

The acceleration that results from a change of direction of the $\mathbf{E}\times\mathbf{B}$ drift is a result of a combination of electromagnetic forces and a change in the momentum of the particle in the direction parallel to the magnetic field. This term represents the change in parallel momentum.

To show the importance of the term in (4.14), we study the simple cylindrical configuration in figure 6. The system is azimuthally symmetric. There is an azimuthal magnetic field $\mathbf{B} = B_\theta\hat{\boldsymbol{\theta}}$ and an axial electric field $\mathbf{E} = E_z\hat{\mathbf{z}}$. In the cylindrical coordinates $\{r,\theta,z\}$, equation (4.6) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{E_z}{B_\theta}, \tag{4.16}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v_\parallel}{r}, \tag{4.17}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 0, \tag{4.18}$$

that is, guiding centers rotate azimuthally due to their parallel velocity, and move radially inwards due to the $\mathbf{E} \times \mathbf{B}$ drift. Equation (4.7) gives

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{Ze}{m\Omega}\left[\hat{\mathbf{b}} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \nabla\hat{\mathbf{b}}\right)\right] \cdot \mathbf{E} = \frac{E_z v_\parallel}{B_\theta}\left[\hat{\boldsymbol{\theta}} \times (\hat{\boldsymbol{\theta}} \cdot \nabla\hat{\boldsymbol{\theta}})\right] \cdot \hat{\mathbf{z}}. \tag{4.19}$$

Since $\hat{\boldsymbol{\theta}} \cdot \nabla\hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}/r$, we finally obtain

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_z v_\parallel}{rB_\theta} = -\frac{v_\parallel}{r}\frac{\mathrm{d}r}{\mathrm{d}t}, \tag{4.20}$$

where we have used (4.16). Using equation (4.20), we can show that the azimuthal angular momentum $rv_\parallel$ is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}(rv_\parallel) = r\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + v_\parallel\frac{\mathrm{d}r}{\mathrm{d}t} = 0. \tag{4.21}$$

Thus, as particles move radially inwards due to the $\mathbf{E} \times \mathbf{B}$ drift, their parallel velocity accelerates because $v_\parallel \propto 1/r$.

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Appendix A. Derivation of $\dot{\mathbf{r}}$, $\dot{v}_\parallel$, $\dot{w}_\perp$ and $\dot{\varphi}$

In this appendix we show how to derive equations (3.25), (3.26), (3.27) and (3.28).

Applying the operator in (3.24) to $\mathbf{r}$, we obtain $\dot{\mathbf{r}} = \mathbf{v}$. Using (3.13), we rewrite this equation in the coordinates $\mathbf{Q} = (\mathbf{r}, v_\parallel, w_\perp, \varphi)$, obtaining (3.25).

To apply the operator in (3.24) to $v_\parallel$, we need $\partial v_\parallel/\partial t$, $\nabla v_\parallel$ and $\nabla_v v_\parallel$. Using the definition of $v_\parallel$ in (3.15), we obtain

$$\frac{\partial v_\parallel}{\partial t} = \frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot \mathbf{v}, \tag{A 1}$$

$$\nabla v_\parallel = \nabla \hat{\mathbf{b}} \cdot \mathbf{v}, \tag{A 2}$$

and

$$\nabla_v v_\parallel = \hat{\mathbf{b}}. \tag{A 3}$$

To obtain final expressions for $\partial v_\parallel/\partial t$ and $\nabla v_\parallel$, we use (3.13), finding

$$\frac{\partial v_\parallel}{\partial t} = \frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot (\mathbf{v}_E + w_\perp \hat{\mathbf{e}}_\perp) \tag{A 4}$$

and

$$\nabla v_\parallel = \nabla \hat{\mathbf{b}} \cdot (\mathbf{v}_E + w_\perp \hat{\mathbf{e}}_\perp). \tag{A 5}$$

Here, we have differentiated $\hat{\mathbf{b}} \cdot \hat{\mathbf{b}} = 1$ with respect to time and space to find $(\partial \hat{\mathbf{b}}/\partial t) \cdot \hat{\mathbf{b}} = 0 = \nabla \hat{\mathbf{b}} \cdot \hat{\mathbf{b}}$. With equations (A 3) , (A 4) and (A 5), we obtain (3.26).

To apply the operator in (3.24) to $w_\perp$, we need $\partial w_\perp/\partial t$, $\nabla w_\perp$ and $\nabla_v w_\perp$. Using the definition of $w_\perp$ in (3.16), we obtain

$$\begin{aligned} \frac{\partial w_\perp}{\partial t} &= -\frac{1}{\sqrt{|\mathbf{v} - \mathbf{v}_E|^2 - (\mathbf{v} \cdot \hat{\mathbf{b}})^2}} \left[(\mathbf{v} \cdot \hat{\mathbf{b}}) \frac{\partial \hat{\mathbf{b}}}{\partial t} \cdot \mathbf{v} + \frac{\partial \mathbf{v}_E}{\partial t} \cdot (\mathbf{v} - \mathbf{v}_E) \right] \\ &= -\left(v_\parallel \frac{\partial \hat{\mathbf{b}}}{\partial t} + \frac{\partial \mathbf{v}_E}{\partial t} \right) \cdot \hat{\mathbf{e}}_\perp - \frac{v_\parallel}{w_\perp} \frac{\partial}{\partial t} \cancelto{0}{(\hat{\mathbf{b}} \cdot \mathbf{v}_E)}, \end{aligned} \tag{A 6}$$

$$\nabla w_\perp = -\left(v_\parallel \nabla \hat{\mathbf{b}} + \nabla \mathbf{v}_E \right) \cdot \hat{\mathbf{e}}_\perp, \tag{A 7}$$

and

$$\nabla_v w_\perp = \frac{\mathbf{v} - \mathbf{v}_E - (\mathbf{v} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}}{\sqrt{|\mathbf{v} - \mathbf{v}_E|^2 - (\mathbf{v} \cdot \hat{\mathbf{b}})^2}} = \hat{\mathbf{e}}_\perp. \tag{A 8}$$

With equations (A 6), (A 7), (A 8) and (3.13), and using

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \hat{\mathbf{e}}_\perp = [\mathbf{E}_\perp + (w_\perp \hat{\mathbf{e}}_\perp + \mathbf{v}_E) \times \mathbf{B}] \cdot \hat{\mathbf{e}}_\perp = w_\perp (\hat{\mathbf{e}}_\perp \times \mathbf{B}) \cdot \hat{\mathbf{e}}_\perp = 0, \tag{A 9}$$

we obtain (3.27).

To apply the operator in (3.24) to φ, we need $\partial \varphi/\partial t$, $\nabla \varphi$ and $\nabla_v \varphi$. We start with the derivative of φ, defined in (3.17), with respect to time,

$$\begin{aligned} \frac{\partial \varphi}{\partial t} = -\frac{1}{[(\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_1]^2 + [(\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_2]^2} \Bigg[& (\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_1 \frac{\partial}{\partial t} [(\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_2] \\ & - (\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_2 \frac{\partial}{\partial t} [(\mathbf{v} - \mathbf{v}_E) \cdot \hat{\mathbf{e}}_1] \Bigg], \end{aligned} \tag{A 10}$$

where

$$\frac{\partial}{\partial t}[(\mathbf{v}-\mathbf{v}_E)\cdot\hat{\mathbf{e}}_i] = (\mathbf{v}-\mathbf{v}_E)\cdot\frac{\partial\hat{\mathbf{e}}_i}{\partial t} - \hat{\mathbf{e}}_i\cdot\frac{\partial\mathbf{v}_E}{\partial t} \tag{A 11}$$

for $i = 1, 2$. Using (3.13), expression (A 10) becomes

$$\begin{aligned}\frac{\partial\varphi}{\partial t} = {} & \left[-\cos^2\varphi\left(\frac{\partial\hat{\mathbf{e}}_2}{\partial t}\cdot\hat{\mathbf{e}}_1\right) + \sin^2\varphi\left(\frac{\partial\hat{\mathbf{e}}_1}{\partial t}\cdot\hat{\mathbf{e}}_2\right)\right] \\ & + \sin\varphi\cos\varphi\left(\frac{\partial\hat{\mathbf{e}}_2}{\partial t}\cdot\hat{\mathbf{e}}_2 - \frac{\partial\hat{\mathbf{e}}_1}{\partial t}\cdot\hat{\mathbf{e}}_1\right) \\ & - \frac{v_\|}{w_\perp}\left[\cos\varphi\left(\frac{\partial\hat{\mathbf{e}}_2}{\partial t}\cdot\hat{\mathbf{b}}\right) + \sin\varphi\left(\frac{\partial\hat{\mathbf{e}}_1}{\partial t}\cdot\hat{\mathbf{b}}\right)\right] \\ & + \frac{1}{w_\perp}(\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2)\cdot\frac{\partial\mathbf{v}_E}{\partial t}.\end{aligned} \tag{A 12}$$

Differentiating the expressions $\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_1 = 1 = \hat{\mathbf{e}}_2\cdot\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 = 0 = \hat{\mathbf{e}}_1\cdot\hat{\mathbf{b}} = \hat{\mathbf{e}}_2\cdot\hat{\mathbf{b}}$ with respect to time, we obtain $(\partial\hat{\mathbf{e}}_1/\partial t)\cdot\hat{\mathbf{e}}_1 = 0 = (\partial\hat{\mathbf{e}}_2/\partial t)\cdot\hat{\mathbf{e}}_2$, $(\partial\hat{\mathbf{e}}_2/\partial t)\cdot\hat{\mathbf{e}}_1 = -(\partial\hat{\mathbf{e}}_1/\partial t)\cdot\hat{\mathbf{e}}_2$, $(\partial\hat{\mathbf{e}}_1/\partial t)\cdot\hat{\mathbf{b}} = -(\partial\hat{\mathbf{b}}/\partial t)\cdot\hat{\mathbf{e}}_1$ and $(\partial\hat{\mathbf{e}}_2/\partial t)\cdot\hat{\mathbf{b}} = -(\partial\hat{\mathbf{b}}/\partial t)\cdot\hat{\mathbf{e}}_2$. Using these identities in (A 12), we obtain

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\hat{\mathbf{e}}_1}{\partial t}\cdot\hat{\mathbf{e}}_2 + \frac{1}{w_\perp}\left(v_\|\frac{\partial\hat{\mathbf{b}}}{\partial t} + \frac{\partial\mathbf{v}_E}{\partial t}\right)\cdot(\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2)\,. \tag{A 13}$$

Using the function $\hat{\mathbf{e}}_\perp$ in (3.14), we can rewrite this expression as

$$\frac{\partial\varphi}{\partial t} = \frac{\partial\hat{\mathbf{e}}_1}{\partial t}\cdot\hat{\mathbf{e}}_2 + \frac{1}{w_\perp}\left(v_\|\frac{\partial\hat{\mathbf{b}}}{\partial t} + \frac{\partial\mathbf{v}_E}{\partial t}\right)\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{e}}_\perp). \tag{A 14}$$

Treating $\nabla\varphi$ in the same way as $\partial\varphi/\partial t$, we obtain

$$\nabla\varphi = \nabla\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_2 + \frac{1}{w_\perp}(v_\|\nabla\hat{\mathbf{b}} + \nabla\mathbf{v}_E)\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{e}}_\perp). \tag{A 15}$$

Finally, the derivative with respect to $\mathbf{v}$ of φ gives

$$\begin{aligned}\nabla_v\varphi &= -\frac{1}{[(\mathbf{v}-\mathbf{v}_E)\cdot\hat{\mathbf{e}}_1]^2 + [(\mathbf{v}-\mathbf{v}_E)\cdot\hat{\mathbf{e}}_2]^2}\left[\big((\mathbf{v}-\mathbf{v}_E)\cdot\hat{\mathbf{e}}_1\big)\hat{\mathbf{e}}_2 - \big((\mathbf{v}-\mathbf{v}_E)\cdot\hat{\mathbf{e}}_2\big)\hat{\mathbf{e}}_1\right] \\ &= -\frac{1}{w_\perp}(\sin\varphi\,\hat{\mathbf{e}}_1 + \cos\varphi\,\hat{\mathbf{e}}_2) = \frac{1}{w_\perp}(\hat{\mathbf{e}}_\perp\times\hat{\mathbf{b}}).\end{aligned} \tag{A 16}$$

With equations (A 14), (A 15), (A 16) and (3.13), and using

$$(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot(\hat{\mathbf{e}}_\perp\times\hat{\mathbf{b}}) = [\mathbf{E}_\perp + (w_\perp\hat{\mathbf{e}}_\perp + \mathbf{v}_E)\times\mathbf{B}]\cdot(\hat{\mathbf{e}}_\perp\times\hat{\mathbf{b}}) = w_\perp B|\hat{\mathbf{e}}_\perp\times\hat{\mathbf{b}}|^2 = w_\perp B, \tag{A 17}$$

we obtain (3.28).

Appendix B. Derivation of $\mathrm{d}\mathbf{r}/\mathrm{d}t$, $\mathrm{d}v_\|/\mathrm{d}t$ and $\mathrm{d}w_\perp/\mathrm{d}t$

In this appendix, we derive equations (3.48), (3.49) and (3.50).

From equation (3.25), it is obvious that

$$\langle\dot{\mathbf{r}}\rangle_\varphi = v_\|\hat{\mathbf{b}} + \mathbf{v}_E \tag{B 1}$$

because $\hat{\mathbf{e}}_\perp$, defined in (3.14), satisfies $\langle\hat{\mathbf{e}}_\perp\rangle_\varphi = 0$.

From equation (3.26), we find

$$\langle \dot{v}_\parallel \rangle_\varphi = w_\perp^2 \langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi : \nabla \hat{\mathbf{b}} + \frac{Ze}{m} \mathbf{E} \cdot \left[\hat{\mathbf{b}} + \frac{1}{\Omega} \hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right]. \tag{B 2}$$

In Einstein's index notation, the double contraction of any two matrices $\mathbf{M}$ and $\mathbf{N}$ is

$$\mathbf{M} : \mathbf{N} = M_{ij} N_{ji}. \tag{B 3}$$

We need the average $\langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi$. Using (3.14), we find

$$\hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp = \cos^2 \varphi \, \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 - \sin \varphi \cos \varphi (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1) + \sin^2 \varphi \, \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2. \tag{B 4}$$

In the orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}}\}$, this tensor becomes the matrix

$$\hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp = \begin{pmatrix} \cos^2 \varphi & -\sin \varphi \cos \varphi & 0 \\ -\sin \varphi \cos \varphi & \sin^2 \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B 5}$$

Taking the gyroaverage, we obtain

$$\langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi = \frac{1}{2} (\hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B 6}$$

Since the unit matrix $\mathbf{I}$ can be written as $\mathbf{I} = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{b}} \hat{\mathbf{b}}$, we finally get

$$\langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi = \frac{1}{2} (\mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}). \tag{B 7}$$

Using (B 7) and employing $\mathbf{I} : \nabla \hat{\mathbf{b}} = \nabla \cdot \hat{\mathbf{b}}$, equation (B 2) becomes

$$\langle \dot{v}_\parallel \rangle_\varphi = \frac{w_\perp^2}{2} \nabla \cdot \hat{\mathbf{b}} + \frac{Ze}{m} \mathbf{E} \cdot \left[\hat{\mathbf{b}} + \frac{1}{\Omega} \hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right]. \tag{B 8}$$

We can simplify this equation further by using $\nabla \cdot \mathbf{B} = 0$ to write

$$B \nabla \cdot \hat{\mathbf{b}} = -\hat{\mathbf{b}} \cdot \nabla B. \tag{B 9}$$

Using (B 9) in (B 8), we obtain

$$\langle \dot{v}_\parallel \rangle_\varphi = \frac{Ze}{m} \left[\hat{\mathbf{b}} + \frac{1}{\Omega} \hat{\mathbf{b}} \times \left(\frac{\partial \hat{\mathbf{b}}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla \hat{\mathbf{b}} \right) \right] \cdot \mathbf{E} - \frac{w_\perp^2}{2B} \hat{\mathbf{b}} \cdot \nabla B. \tag{B 10}$$

Finally, from (3.27), we obtain

$$\langle \dot{w}_\perp \rangle_\varphi = -v_\parallel w_\perp \langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi : \nabla \hat{\mathbf{b}} - w_\perp \langle \hat{\mathbf{e}}_\perp \hat{\mathbf{e}}_\perp \rangle_\varphi : \nabla \mathbf{v}_E. \tag{B 11}$$

Using (B 7), $\mathbf{I} : \nabla \hat{\mathbf{b}} = \nabla \cdot \hat{\mathbf{b}}$ and $\mathbf{I} : \nabla \mathbf{v}_E = \nabla \cdot \mathbf{v}_E$, equation (B 11) becomes

$$\langle \dot{w}_\perp \rangle_\varphi = -\frac{v_\parallel w_\perp}{2} \nabla \cdot \hat{\mathbf{b}} - \frac{w_\perp}{2} (\nabla \cdot \mathbf{v}_E - \hat{\mathbf{b}} \cdot \nabla \mathbf{v}_E \cdot \hat{\mathbf{b}}). \tag{B 12}$$

To simplify this equation, we use the fact that the orderings (3.4) and (3.5) imply

$$\mathbf{E} + \mathbf{v}_E \times \mathbf{B} = E_\parallel \hat{\mathbf{b}} \simeq 0. \tag{B 13}$$

Taking the curl of this equation, we obtain

$$\nabla \times \mathbf{E} + \nabla \times (\mathbf{v}_E \times \mathbf{B}) \simeq 0. \tag{B 14}$$

Using Faraday's induction law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, and

$$\nabla \times (\mathbf{v}_E \times \mathbf{B}) = \mathbf{B} \cdot \nabla \mathbf{v}_E - (\nabla \cdot \mathbf{v}_E)\mathbf{B} - \mathbf{v}_E \cdot \nabla \mathbf{B}, \tag{B 15}$$

equation (B 14) becomes

$$\mathbf{B} \cdot \nabla \mathbf{v}_E - (\nabla \cdot \mathbf{v}_E)\mathbf{B} \simeq \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}_E \cdot \nabla \mathbf{B}. \tag{B 16}$$

Projecting this equation on $\hat{\mathbf{b}}$, and using $(\partial \mathbf{B}/\partial t) \cdot \hat{\mathbf{b}} = \partial B/\partial t$ and $\nabla \mathbf{B} \cdot \hat{\mathbf{b}} = \nabla B$, we finally get

$$\hat{\mathbf{b}} \cdot \nabla \mathbf{v}_E \cdot \hat{\mathbf{b}} - \nabla \cdot \mathbf{v}_E \simeq \frac{1}{B} \left(\frac{\partial B}{\partial t} + \mathbf{v}_E \cdot \nabla B \right). \tag{B 17}$$

Using this expression and (B 9) in (B 12), we obtain

$$\langle \dot{w}_\perp \rangle_\varphi = \frac{w_\perp}{2B} \left[\frac{\partial B}{\partial t} + (v_\| \hat{\mathbf{b}} + \mathbf{v}_E) \cdot \nabla B \right]. \tag{B 18}$$

Equations (B 1), (B 10) and (B 18) give equations (3.48), (3.49) and (3.50).