5. Braginskii MHD.

In this last part of the course, I will expand the tentative transport calculations with started with resistivity etc. and show you how to calculate the friction force (so resistivity), the heat fluxes, and the momentum fluxes in a plasma embedded in a magnetic field — effectively, we will be arranging a marriage between KMHD and collisions.

All this is based on seminal paper by Braginskii (1965), which proved that one could become eternally famous by being the first to do a standard but very difficult calculation properly. This is called "classical transport". In fusion science, this was followed by decades of "neoclassical transport" (one featuring trapped-particle physics in curved fields) — you will find an intro to that in F. Parra's notes (and while Braginskii is their patron saint, Felix is the reigning emperor of the neoclassical calculators — a joint emperor with Per Helander perhaps, whose book is the bible of the field). Sadly, I will not have time to cover neoclassical transport.

5.1 Collisional contributions to electron distribution function

Let us recall how electrons were described in our derivation of KMHD.

From $$\frac{\partial f_e}{\partial t} + \vec v\cdot\nabla f_e - \frac{e}{m_e}\left(\vec E + \frac{\vec v\times\vec B}{c}\right)\cdot\frac{\partial f_e}{\partial\vec v} = \left(\frac{\partial f_e}{\partial t}\right)_c \quad (239)$$

we go to $\vec v = \vec w + \vec u_e$ and so

$$\underbrace{\frac{df_e}{dt_e}}_{\frac{\partial}{\partial t}+\vec u_e\cdot\nabla} + \vec w\cdot\nabla f_e + \Bigg(\underbrace{-\frac{e}{m_e}\frac{\vec w\times\vec B}{c}}_{\boxed{-\Omega_e\left(\frac{\partial f_e}{\partial\vartheta}\right)_{w_\perp,w_\parallel}}} - \vec w\cdot\nabla\vec u_e + \underbrace{\vec a_e}_{-\frac{e}{m_e}\left(\vec E+\frac{\vec u_e\times\vec B}{c}\right)-\frac{d\vec u_e}{dt_e}}\Bigg)\cdot\frac{\partial f_e}{\partial\vec w} = \left(\frac{\partial f_e}{\partial t}\right)_c \quad (240)$$

$\Omega_e<0$!!!

forces in a frame moving with $\vec u_e$

This distribution function now is required to satisfy $\int d\vec w\,\vec w f_e = 0$ (241), whereas $\vec u_e$ is found from the momentum equation

$$-m_e n_e\vec a_e = -\nabla\cdot\hat P_e + \vec R_{ei} \quad (242)$$

$\uparrow$ friction force

For electrons, because m_e is small, we usually think of the momentum equation as the equation for the Electric field — generalized Ohm's law:

$$\vec E = \underbrace{-\frac{\vec u_e\times\vec B}{c}}_{-\frac{\vec u_i\times\vec B}{c}+\frac{\vec j\times\vec B}{cen_e}} - \frac{m_e}{e}\frac{d\vec u_e}{dt} - \frac{\nabla\cdot\hat P_e}{en_e} + \frac{\vec R_{ei}}{en_e} \quad (243)$$

$-\frac{\vec u_i\times\vec B}{c}$: $\uparrow$ ⊥ e. field in fluid frame

$\frac{\vec j\times\vec B}{cen_e}$: $\uparrow$ "Hall term"

$\frac{m_e}{e}\frac{d\vec u_e}{dt}$: electron inertia

$\frac{\nabla\cdot\hat P_e}{en_e}$: "thermoelectric term"

$\frac{\vec R_{ei}}{en_e}$: friction force ("Ohmic")

$$\boxed{\vec R_{ei} = \int d\vec w\, m_e\vec w\left(\frac{\partial f_e}{\partial t}\right)_{ei}}$$

When deriving ideal MHD, or KMHD, one argues all these terms (except $\vec{u}_i$) to be small in the Larmor-radius expansion — it is those terms that people interested in non-MHD reconnection focus on (some of them — Hall, lowest-order-in-$k\rho_e$ parts of $\nabla\cdot\hat{P}_e$ — do not in fact break flux conservation, but can modify the structure and dynamics of reconnecting sheets; others can step in do resistivity's job — electron inertia, FLR terms in $\hat{P}_e$. The simplest, I think, introduction this kind of physics is the appendix of the paper by Zocco & AAS-2011).

The grounds for neglecting $\vec{R}_{ei}$ in (243) is that $\frac{\nu_{ei}}{\Omega_e} \ll 1$, but we already know, from reconnection experience, that ideal MHD will always find a way to access resistivity. Let us now allow it to do this formally by ordering

$$\boxed{\frac{\nu_{ei}}{\Omega_e} \sim 1}$$

We will, however, continue assuming $\frac{\omega}{\Omega_e} \ll 1$, i.e.,

$$\boxed{\omega \sim k u_e \sim k v_{the} \ll \Omega_e \sim \nu_{ei}} \quad (244)$$

↑ in fact $\ll 1$ because we will assume $u_e \sim u_i \sim v_{thi} \ll v_{the}$ in the $\sqrt{m_e/m_i}$ expansion

~~[illegible]~~

~~[illegible]~~

With this ordering, the collision term is as large as the Ω_e term:

$$\Omega_e\left(\frac{\partial f_e}{\partial\vartheta}\right)_{w_\perp,w_\parallel} + \left(\frac{\partial f_e}{\partial t}\right)_c = \cancel{\frac{df_e}{dt_e}} + \vec{w}\cdot\nabla f_e + (\vec{a}_e - \cancel{\vec{w}\cdot\nabla\vec{u}_e})\cdot\frac{\partial f_e}{\partial\vec{w}} \qquad (245)$$

$$\vec{a}_e = \frac{\nabla\cdot\hat{P}_e}{m_e n_e} - \frac{\vec{R}_{ei}}{m_e n_e} \quad \text{from (242)}$$

$\frac{\omega}{\Omega_e} \sim \frac{k u_e}{\Omega_e} \sim k\rho_e\sqrt{\frac{m_e}{m_i}}$ assuming $u_e\sim v_{thi}$

$\frac{k v_{the}}{\Omega_e} \sim k\rho_e$

$\frac{k p_e}{m_e n_e v_{the}\Omega_e} \sim \frac{k \cancel{n_e} \cancel{m_e} v_{the}^{\cancel{2}}}{\cancel{m_e}\cancel{n_e}\cancel{v_{the}}\Omega_e} \sim k\rho_e$

$\frac{\nu_{ei} u_e}{\Omega_e v_{the}} \sim \frac{\nu_{ei}}{\Omega_e}\sqrt{\frac{m_e}{m_i}}$

$\sim \frac{k u_e}{\Omega_e} \sim k\rho_e\sqrt{\frac{m_e}{m_i}}$

We have several small parameters, so let us keep careful track:

$k\rho_e$, $\sqrt{\frac{m_e}{m_i}}$, $\frac{\nu_{ei}}{\Omega_e} \sim 1$ for now but we will have ~~an~~ subsidiary expansion in it before long!

Formally, let us order $\boxed{k\rho_e \sim \sqrt{\frac{m_e}{m_i}}}$ (246) and banish forever the 2nd-order terms — e. inertia $\frac{df_e}{dt_e}$ and the "Coriolis force" $\vec{w}\cdot\nabla\vec{u}_e$

$$\underbrace{\Omega_e\left(\frac{\partial f_e}{\partial\vartheta}\right)_{w_\perp,w_\parallel}}_{\text{0-th order}} + \left(\frac{\partial f_e}{\partial t}\right)_c = \underbrace{\vec{w}\cdot\nabla f_e + \vec{a}_e\cdot\frac{\partial f_e}{\partial\vec{w}}}_{\text{1st order}} \qquad (247)$$

= 0 to lowest order.

5.1.1 0-th order

The 0-th order solution is now not simply a gyrotropic function but a Maxwellian:

$$\left(\frac{\partial f_e}{\partial t}\right)_c = \underset{\substack{\uparrow\\ \text{enforces}\\ \text{isotropy in } \vec{u}_i \text{ frame}}}{\nu_D\,\mathcal{L}[f_e]} + \underset{\substack{\uparrow\\ \text{enforces Maxwellianity}\\ \text{in } \vec{u}_e \text{ frame.}}}{\left[\frac{\partial f_e}{\partial t}\right]_e} \qquad (248)$$

Since we are working to lowest order in $\sqrt{\frac{m_e}{m_i}}$, the difference between the $\vec{u}_i$ and $\vec{u}_e$ frames does not matter at this order (but will matter at the next one!)

Formally, you might ask for a proof that the Maxwellian is the only solution. That's easy: $\int d\vec{w}\ (247)\cdot(1+\ln f_0)$ gives (to lowest order)

$$\underbrace{\Omega_e \int \underset{\overset{\parallel}{dw_\perp w_\perp dw_\parallel d\vartheta}}{d\vec{w}} \left(\frac{\partial}{\partial\vartheta} f_0 \ln f_0\right)_{w_\perp, w_\parallel}}_{\overset{\parallel}{0}} + \int d\vec{w}\, \ln f_0 \underset{\uparrow}{\left(\frac{\partial f_0}{\partial t}\right)_e} = 0 \quad (249)$$

vanishes iff f_0 is a local Maxwellian (as in the proof of the H theorem)

So, we have

$$f_e = f_0 + \delta f\,, \quad f_0 = \frac{n_e}{(2\pi T_e/m_e)^{3/2}}\, e^{-\frac{m_e w^2}{2T_e}} \qquad (250)$$

5.1.2 1st order

$$\Omega_e \left(\frac{\partial \delta f}{\partial\vartheta}\right)_{w_\perp, w_\parallel} + \nu_D \mathcal{L}[f_0 + \delta f] + \left(\frac{\partial \delta f}{\partial t}\right)_{c,e} = \text{frame difference}$$

0 to lowest order, but not to 1st because of the frame difference

$$f_0 = \frac{n_e}{(\pi v_{the}^2)^{3/2}}\, e^{-\frac{(\vec{w}'+\vec{u}_i-\vec{u}_e)^2}{v_{the}^2}} \approx \underset{\substack{\uparrow\\ \text{annihilated}\\ \text{by } \mathcal{L}}}{f_0(w')} + (\vec{u}_i-\vec{u}_e)\cdot \underbrace{\frac{\partial f_0}{\partial \vec{w}}}_{-\frac{2\vec{w}\cdot(\vec{u}_i-\vec{u}_e)}{v_{the}^2} f}$$

$$\vec{w}' = \vec{v} - \vec{u}_i$$

$$\mathcal{L}\left[-\frac{2\vec{w}\cdot(\vec{u}_i-\vec{u}_e)}{v_{the}^2} f_0\right] = \frac{2\vec{w}\cdot(\vec{u}_i-\vec{u}_e)}{v_{the}^2} f_0 \text{ because}$$

$$\mathcal{L}[\vec{w}] = -\vec{w}$$

$$= \vec{w}\cdot\nabla f_0 + \underset{\overset{\parallel}{\frac{\nabla p_e - \vec{R}_{ei}}{m_e n_e}}}{\vec{a}_e}\cdot\frac{\partial f_0}{\partial \vec{w}} = \vec{w}\cdot\left[\frac{\nabla n_e}{n_e} - \frac{3}{2}\frac{\nabla T_e}{T_e} + \frac{w^2}{v_{the}^2}\frac{\nabla T_e}{T_e}\right] f_0$$

to lowest order, pressure of the f_0 is isotropic

$$- \frac{\nabla p_e - \vec{R}_{ei}}{\cancel{m_e} n_e}\cdot\frac{\cancel{m_e}}{T_e}\vec{w} f_0$$

$$= \vec{w}\cdot\left[\cancel{\frac{\nabla p_e}{p_e}} + \left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right)\frac{\nabla T_e}{T_e} - \cancel{\frac{\nabla p_e}{p_e}} + \frac{\vec{R}_{ei}}{p_e}\right] f_0 \qquad (251)$$

$\vec{R}_{ei}$ itself depends on δf : let's work this out.

By definition,

$$\vec{R}_{ei} = \int d\vec{w}'\, m_e \underset{\overset{\uparrow}{\vec{v}-\vec{u}_i}}{\vec{w}'} \left(\frac{\partial f_e}{\partial t}\right)_{c,i} = \int d\vec{w}'\, m_e \vec{w}'\, \nu_D(w')\, \mathcal{L}[f_e]$$

$$= -m_e \int d\vec{w}'\, \nu_D(w')\, \vec{w}' f_e \quad \text{because } \mathcal{L}[\vec{w}'] = -\vec{w}'$$

↑ but this function is a function of $\vec{w} = \vec{v} - \vec{u}_e = \vec{w}' + \vec{u}_i - \vec{u}_e$

$$= -m_e \int d\vec{w}'\, \nu_D(w')\, \vec{w}'\left[\underset{\overset{\parallel}{\cancel{f_0(w')} + \delta f(\vec{w}')}}{f_e(\vec{w}')} + (\vec{u}_i - \vec{u}_e)\cdot\frac{\partial f_e}{\partial \vec{w}'}\right]$$

vanishes under integration

same order

$$= -m_e \int d\vec{w}'\, \nu_D(w')\, \vec{w}'\, \delta f(\vec{w}')$$

$$+ \underbrace{\frac{2 m_e}{v_{the}^2}\int d\vec{w}'\, \nu_D(w')\, \vec{w}'\vec{w}'\cdot(\vec{u}_i - \vec{u}_e)}_{\overset{\parallel}{-m_e n_e \nu_{ei}(\vec{u}_e - \vec{u}_i)}} \qquad (252)$$

friction force for a Maxwellian previously calculated!

OK, let us assemble everything:

$$\Omega_e\left(\frac{\partial\delta f}{\partial\vartheta}\right)_{w_\perp,w_\parallel} + \frac{2\vec{w}\cdot(\vec{u}_i-\vec{u}_e)}{v_{the}^2}\nu_D(w)f_0 + \nu_D(w)\mathcal{L}[\delta f] + \left(\frac{\partial\delta f}{\partial t}\right)_{c,e} = \vec{w}\cdot\left[\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right) - \frac{m_e}{T_e}\nu_{ei}(\vec{u}_e-\vec{u}_i) - \frac{m_e}{p_e}\int d\vec{w}'\,\nu_D(w')\,\vec{w}'\,\delta f(\vec{w}')\right]f_0$$

$$\Rightarrow\ \Omega_e\frac{\partial\delta f}{\partial\vartheta} + \underbrace{\nu_D(w)\mathcal{L}[\delta f] + \frac{2f_0}{n_e v_{the}^2}\vec{w}\cdot\int d\vec{w}'\,\nu_D(w')\,\vec{w}'\,\delta f(\vec{w}')} + \left(\frac{\partial\delta f}{\partial t}\right)_{c,e} = \vec{w}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right) - \frac{2}{v_{the}^2}\left[\nu_{ei}-\nu_D(w)\right](\vec{u}_e-\vec{u}_i)\right\} \qquad (253)$$

This "modified" Lorentz operator is a rather marvellous thing: it scatters particles but makes sure to conserve their momentum, so that δf does not develop a mean $\vec{w}$ — which it is not allowed to do by definition of $\vec{w}$! Indeed:

$$\int d\vec{w}\,\vec{w}\left[\nu_D(w)\mathcal{L}[\delta f] + \frac{2f_0}{n_e v_{the}^2}\vec{w}\cdot\int d\vec{w}'\,\nu_D(w')\,\vec{w}'\,\delta f(\vec{w}')\right]$$

$$= \cancel{-\int d\vec{w}\,\nu_D(w)\,\vec{w}\,\delta f} + \underbrace{\left(\int d\vec{w}\,\frac{2f_0}{n_e v_{the}^2}\vec{w}\vec{w}\right)}_{\frac{2}{3}\frac{1}{n_e v_{the}^2}\int d\vec{w}\,w^2 f_0 = 1}\cdot\cancel{\int d\vec{w}'\,\nu_D(w')\,\vec{w}'\,\delta f(\vec{w}')}$$

$= 0$, q.e.d.

OK, now, to solve (253), we separate the gyrotropic and gyroangle-dependent parts:

$$\delta f = \langle\delta f\rangle_\vartheta + \widetilde{\delta f} \qquad (254)$$

5.1.3 Gyrotropic part of δf

Average eq. (253) over the gyroangle:

Since
$$\langle \mathcal{L}[\delta f]\rangle_\vartheta = \frac{1}{2}\left\langle \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\delta f}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2\delta f}{\partial\vartheta^2}\right\rangle_\vartheta$$
$$= \frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\langle\delta f\rangle}{\partial\xi}, \qquad (255)$$

we have

$$\frac{1}{2}\nu_D(w)\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\langle\delta f\rangle}{\partial\xi} + \frac{2f_0(w)w}{n_e v_{the}^2}\,\xi \underbrace{\int d\vec{w}'\,\nu_D(w')\,w'\xi'\langle\delta f\rangle}_{= 2\pi\int_0^\infty dw'\,\underbrace{w'^3\nu_D(w')}_{\text{const, so take out of integral}}\int_{-1}^{1} d\xi'\,\xi'\langle\delta f\rangle}$$

because $\langle\vec{w}\rangle = w_\parallel\hat{b} = \xi w\hat{b}$

$$+\left(\frac{\partial\langle\delta f\rangle}{\partial t}\right)_{c,e} =$$

linearised e-e CO

$$= \xi w f_0(w)\,\hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right) - \frac{2[\nu_{ei}-\nu_D(w)]}{v_{the}^2}(\vec{u}_e-\vec{u}_i)\right\} \qquad (256)$$

$$\boxed{\nu_D(w)\left[\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\langle\delta f\rangle}{\partial\xi} + \frac{4\pi w^4 f_0(w)}{n_e v_{the}^2}\,\xi\int_0^\infty dw'\int_{-1}^{1}d\xi'\,\xi'\langle\delta f\rangle\right]}$$

I am going to do the lazy thing again and refuse to deal with the e-e collision operator (plutonium plasma). If you want to learn (as you should do) how to do things properly I yet again address you to the lecture notes by F. Parra and M. Kunz (who do it by two different methods)

W/o the e-e collisions, (256) becomes

$$\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\langle\delta f\rangle}{\partial\xi} + \frac{4\pi w^4 f_0(w)}{n_e v_{the}^2}\,\xi\int_0^\infty dw'\int_{-1}^{1} d\xi'\,\xi'\langle\delta f\rangle$$

$$= \xi\,\frac{w f_0(w)}{\nu_D(w)}\,\hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right) - \frac{2\left[\nu_{ei}-\nu_D(w)\right]}{v_{the}^2}(\vec{u}_e-\vec{u}_i)\right\}$$

$$\equiv \xi\, C(w) \qquad (257)$$

Let us look for a solution in the form (obviously)

$$\langle\delta f\rangle = \xi\, h(w) \qquad (258)$$

$$-h(w) + \frac{8\pi w^4 f_0(w)}{3 n_e v_{the}^2}\int_0^\infty dw'\, h(w') = C(w) \qquad (259)$$

Note that both sides of this equation integrate to 0:

$$\frac{8\pi}{3 n_e v_{the}^2}\int_0^\infty dw\, w^4 f_0(w) = \frac{8}{3\sqrt{\pi}}\underbrace{\int_0^\infty dx\, x^4 e^{-x^2}}_{\frac{1}{2}\cdot\frac{3}{2}\frac{\sqrt{\pi}}{2}} = 1 \qquad (260)$$

$$\int_0^\infty dw\, C(w) = \int_0^\infty dw\,\frac{w f_0(w)}{\nu_D(w)}\,\hat{b}\cdot\left\{\underset{①}{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right)} - \underset{②}{\frac{2\left[\nu_{ei}-\nu_D(w)\right]}{v_{the}^2}(\vec{u}_e-\vec{u}_i)}\right\}$$

$$① \propto \int_0^\infty dw\, w^4 f_0(w)\left(\frac{w^2}{v_{the}^2}-\frac{5}{2}\right) \propto \int_0^\infty dx\, e^{-x^2}x^4\left(x^2-\frac{5}{2}\right) = 0 \qquad (261)$$

$$② \propto \int_0^\infty dw\, w f_0(w)\left[\frac{\nu_{ei}}{\nu_D(w)}-1\right] \propto \int_0^\infty dx\, x e^{-x^2}\left(\frac{4}{3\sqrt{\pi}}x^3-1\right) = \frac{1}{2}-\frac{1}{2} = 0$$

Recall that

$$\nu_D(w) = \frac{4\pi Z e^4 n_e \Lambda_{ei}}{m_e^2 w^3}$$

$$\nu_{ei} = \frac{4\sqrt{2\pi}}{3}\frac{Z e^4 n_e \Lambda_{ei}}{m_e^{1/2} T_e^{3/2}}$$

$$\Rightarrow \frac{\nu_D}{\nu_{ei}} = \frac{4\pi}{m_e^2 w^3}\frac{m_e^{1/2}T_e^{3/2}}{4\sqrt{2\pi}}\cdot 3 = 3\sqrt{\frac{\pi}{2}}\left(\frac{v_{the}}{\sqrt{2}w}\right)^3 = \frac{3\sqrt{\pi}}{4}\left(\frac{v_{the}}{w}\right)^3 \qquad (262)$$

This means that

$$h(w) = -C(w) \qquad (263)$$

is perfectly good particular solution. But there is also a homogeneous solution:

$$h_{hom}(w) = \frac{8\pi w^4 f_0(w)}{3 n_e v_{the}^2}\,\alpha, \qquad (264)$$

where the constant α is determined from the requirement that $\int d\vec{w}\, w_\parallel \langle \delta f \rangle = 0$. Let us implement this:

$$\int d\vec{w}\, w_\parallel\, \xi\, h_{hom}(w) = \alpha \int_0^\infty dw\, w^3\, \frac{4\pi}{3}\, \frac{8\pi w^4 f_0(w)}{3 n_e v_{the}^2} =$$

$$= \frac{32\pi^2}{9}\alpha \int_0^\infty dw\, w^7 \frac{e^{-w^2/v_{the}^2}}{\pi^{3/2} v_{the}^5} = \frac{32\sqrt{\pi}}{9}\alpha v_{the}^3 \underbrace{\int_0^\infty dx\, x^7 e^{-x^2}}_{= \frac{1}{2}\int_0^\infty dy\, y^3 e^{y} = 3}$$

$$= \frac{32\sqrt{\pi}}{3}\alpha\, v_{the}^3 \qquad (264a)$$

$$\int d\vec{w}\, w_\parallel\, \xi\, C(w) = \frac{4\pi}{3}\int_0^\infty dw\, w^3 \frac{w f_0(w)}{\nu_D(w)}\, \hat{b}\cdot\{\dots\}$$

see (257)

$$= \frac{4\pi}{3}\underbrace{\frac{1}{w^3\nu_D(w)}}_{\text{const!}} \int_0^\infty dw\, w^7 f_0(w)\, \hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) - \right.$$

$$\left. - \frac{2\nu_{ei}}{v_{the}^2}\left[1 - \frac{3\sqrt{\pi}}{4}\left(\frac{v_{the}}{w}\right)^3\right](\vec{u}_e - \vec{u}_i)\right\}$$

$$= \frac{4\pi}{3}\frac{n_e v_{the}^5}{\pi^{3/2} w^3 \nu_D(w)} \int_0^\infty dx\, x^7 e^{-x^2}\, \hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(x^2 - \frac{5}{2}\right) - \right.$$

$$\left. - \frac{2\nu_{ei}}{v_{the}^2}\left(1 - \frac{3\sqrt{\pi}}{4}\frac{1}{x^3}\right)(\vec{u}_e - \vec{u}_i)\right\}$$

$$- \frac{4}{3\sqrt{\pi}}\frac{n_e v_{the}^5}{w^3\nu_D}\, \hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\frac{1}{2}\left(4! - \frac{5}{2}\cdot 3!\right) - \frac{2\nu_{ei}}{v_{the}^2}(\vec{u}_e - \vec{u}_i)\left(\frac{1}{2}\cdot 3! - \frac{3\sqrt{\pi}}{4}\frac{3\sqrt{\pi}}{8}\right)\right\}$$

$$= \frac{4}{\sqrt{\pi}}\frac{n_e v_{the}^5}{w^3\nu_D}\, \hat{b}\cdot\left\{\frac{3}{2}\frac{\nabla T_e}{T_e} - \frac{2\nu_{ei}}{v_{the}^2}(\vec{u}_e - \vec{u}_i)\left(1 - \frac{3\pi}{32}\right)\right\} \qquad (264b)$$

Requiring that this be cancelled by α gives

$$\alpha = \frac{3}{8\pi}\,\frac{n_e v_{the}^2}{w^3 \nu_D}\,\hat{b}\cdot\{\dots\} \Rightarrow h_{hom}(w) = \frac{w f_0(w)}{\nu_D(w)}\,\hat{b}\cdot\{\dots\}$$

Assembling terms,

$$\boxed{\langle\delta f\rangle = -\frac{w_\parallel f_0(w)}{\nu_D(w)}\,\hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-4\right) - \frac{2\nu_{ei}}{v_{the}^2}\left[\frac{3\pi}{32}-\frac{3\sqrt{\pi}}{4}\frac{v_{the}^3}{w^3}\right](\vec{u}_e-\vec{u}_i)\right\}} \quad (265)$$

5.1.4 Parallel friction force

Let us not delay at least some gratification and see how this result allows us to amend our previous calculation of the e-i friction force. Its definition is (252) and from $\langle\delta f\rangle$ we can now calculate its $\parallel$ component ($\langle\delta f\rangle$'s contribution to $\vec{R}_{\perp ei}$ vanish):

$$R_{\parallel ei} = -m_e n_e \nu_{ei}(u_{\parallel e}-u_{\parallel i}) - m_e \underbrace{\int d\vec{w}\,\nu_D(w)\,w_\parallel \langle\delta f\rangle}$$

$$\hookrightarrow -\int d\vec{w}\, w_\parallel^2 f_0(w)\,\hat{b}\cdot\{\dots\} = -\int_0^\infty dw\, w^4 f_0(w)\,\frac{4\pi}{3}\,\hat{b}\cdot\{\dots\}$$

$$= -\frac{4\pi}{3}\,\frac{n_e v_{the}^2}{\pi^{3/2}}\int_0^\infty dx\, x^4 e^{-x^2}\,\hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}(x^2-4) - \frac{2\nu_{ei}}{v_{the}^2}\left(\frac{3\pi}{32}-\frac{3\sqrt{\pi}}{4}\frac{1}{x^3}\right)(\vec{u}_e-\vec{u}_i)\right\}$$

$$= -\frac{\cancel{4\pi}}{\cancel{3\sqrt{\pi}}}\, n_e v_{the}^2\left[\frac{\nabla_\parallel T_e}{T_e}\,\frac{\cancel{\sqrt{\pi}}}{2}\,\frac{1}{\cancel{2}}\cdot\frac{\cancel{3}}{\cancel{2}}\left(\frac{5}{2}-4\right) - \frac{2\nu_{ei}}{v_{the}^2}\left(\frac{3\pi}{32}\cdot\frac{\cancel{\sqrt{\pi}}}{2}\cdot\frac{\cancel{3}}{\cancel{4}} - \frac{\cancel{3\sqrt{\pi}}}{\cancel{4}}\cdot\frac{1}{2}\right)(\vec{u}_{e\parallel}-\vec{u}_{i\parallel})\right]$$

$$= -\, n_e v_{the}^2\left[-\frac{3}{4}\,\frac{\nabla_\parallel T_e}{T_e} - \frac{\nu_{ei}}{v_{the}^2}\left(\frac{3\pi}{32}-1\right)(\vec{u}_{e\parallel}-\vec{u}_{i\parallel})\right]$$

$$= +\frac{n_e}{m_e}\,\frac{3}{2}\nabla_\parallel T_e + n_e \nu_{ei}\left(\frac{3\pi}{32}-1\right)(\vec{u}_{e\parallel}-\vec{u}_{i\parallel})$$

Thus,

$$\boxed{R_{\| ei} = -\frac{3\pi}{32} m_e n_e \nu_{ei} (u_{\| e} - u_{\| i}) - \frac{3}{2} n_e \nabla_{\|} T_e} \quad (266)$$

This is the familiar Spitzer-Härm result In the Z=1 correct calculation, the coeff. becomes 0.51

This bit is a new feature. In the Z=1 case, the coeff. is 0.71

The physics of the new term is this: electrons coming from low temperatures collide more often and so lose more momentum than those from high temperatures. The result is net friction between them.

This is called the "thermoelectric effect": put this back into (243) and discover that collisions lead to a contribution to parallel current — I will discuss this in § 5.2.

5.1.5 Non-gyrotropic part of δf

Let us now subtract (256) from (253):

$$\Omega_e \frac{\partial \delta\tilde{f}}{\partial \vartheta} + \nu_D(w) \mathcal{L}[\delta\tilde{f}] + \frac{2 f_0(w)}{n_e v_{the}^2} \vec{w}_\perp \cdot \int d\vec{w}' \nu_D(w') \vec{w}_\perp' \delta\tilde{f}(\vec{w}')$$

$$+ \left(\frac{\partial \delta\tilde{f}}{\partial t}\right)_{c,e} = \vec{w}_\perp \cdot \left\{ \frac{\nabla T_e}{T_e} \left(\frac{w^2}{v_{the}^2} - \frac{5}{2} \right) - \frac{2\nu_{ei}}{v_{the}^2} \left[1 - \frac{3\sqrt{\pi}}{4} \frac{v_{the}^3}{w^3} \right] (\vec{u}_e - \vec{u}_i) \right\} f_0(w)$$

(267)

I am going to solve this in the subsidiary expansion

$$\frac{\nu_{ei}}{\Omega_e} \ll 1 \quad \text{– but still} \gg \sqrt{\frac{m_e}{m_i}} \sim k\rho_e$$

$$\delta\tilde f = \delta\tilde f^{(0)} + \delta\tilde f^{(1)} \qquad (268)$$

This means that to lowest order, I can just drop all collisional terms:

$$\Omega_e \frac{\partial \delta\tilde f^{(0)}}{\partial\vartheta} = \vec w_\perp \cdot \frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) f_0(w) \qquad (269)$$

$$\hookrightarrow = \frac{\partial}{\partial\vartheta}\vec w_\perp \times \hat b$$ – check this by letting $\vec w_\perp = w_\perp(\hat x\cos\vartheta + \hat y\sin\vartheta)$

So, solution:

Note that $\int d\vec w\, \vec w\, \delta\tilde f^{(0)} = 0$, so all is well.

$$\boxed{\delta\tilde f^{(0)} = \frac{\vec w_\perp \times \hat b}{\Omega_e}\cdot\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) f_0(w)} \qquad (270)$$

This part of δf is entirely independent of collisions, it is an FLR effect!

To get something collisional, let us go to next order:

$$\Omega_e \frac{\partial \delta\tilde f^{(1)}}{\partial\vartheta} = -\nu_D(w)\,\mathcal{L}\left[\delta\tilde f^{(0)}\right] - \frac{2f_0(w)}{n_e v_{the}^2}\vec w_\perp\cdot\int d\vec w'\,\nu_D(w')\,\vec w_\perp'\,\delta\tilde f^{(0)}(\vec w')$$
$$- \left(\frac{\partial\delta\tilde f^{(0)}}{\partial t}\right)_{c,e} - \frac{2\nu_{ei}}{v_{the}^2}\left[1 - \frac{3\sqrt{\pi}}{4}\frac{v_{the}^3}{w^3}\right]\vec w_\perp\cdot(\vec u_e - \vec u_i)\, f_0(\vec w) \qquad (271)$$

[illegible]

It is not hard to see that $\mathcal{L}[\vec w_\perp] = -\vec w_\perp$, so

$$\mathcal{L}\left[\delta\tilde f^{(0)}\right] = -\delta\tilde f^{(0)} \qquad (272)$$

$$\int d\vec{w}\, \nu_D(w)\, \vec{w}_\perp\, \delta\tilde{f}^{(0)} = \int d\vec{w}\, \nu_D(w)\, \vec{w}_\perp \frac{\vec{w}_\perp \times \hat{b}}{\Omega_e} \cdot \frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{th}^2} - \frac{5}{2}\right) f_0(w)$$

$$= \left\{\left[\int d\vec{w}\, \nu_D(w)\, \underset{\uparrow}{\vec{w}_\perp \vec{w}_\perp}\left(\frac{w^2}{v_{th}^2} - \frac{5}{2}\right) f_0\right] \times \frac{\hat{b}}{\Omega_e}\right\} \cdot \frac{\nabla T_e}{T_e}$$

averaging this gives $\left(\mathbb{1} - \hat{b}\hat{b}\right)\frac{w^2}{2}(1-\xi^2)$

$$= \int_0^\infty dw\, w^2 \cancel{2}\pi \underbrace{\int_{-1}^{1} d\xi\, (1-\xi^2)}_{=\frac{4}{3}} \frac{1}{\cancel{2}} \nu_D(w)\, w^2 \left(\frac{w^2}{v_{th}^2} - \frac{5}{2}\right) f_0 \frac{\hat{b}}{\Omega_e} \times \frac{\nabla T_e}{T_e}$$

$$= \underbrace{\frac{\cancel{4}\pi}{\cancel{3}} \int_0^\infty dw\, w^4 \left(\frac{w^2}{v_{th}^2} - \frac{5}{2}\right) \underbrace{\nu_D(w)}_{=\frac{\cancel{3}\sqrt{\pi}}{\cancel{4}}\frac{v_{the}^3}{w^3}\nu_{ei}} f_0}_{=} \frac{\hat{b} \times \nabla T_e}{\Omega_e \underbrace{T_e}_{=\frac{m_e v_{the}^2}{2}}} =$$

$$\nu_{ei}\, \pi^{3/2} \cancel{v_{the}^3} \int_0^\infty dw\, w \left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) \frac{n_e}{\cancel{\pi^{3/2} v_{the}^3}} e^{-w^2/v_{the}^2}$$

$$= \frac{\hat{b} \times \nabla T_e}{\Omega_e} \frac{\nu_{ei} n_e}{m_e} \int_0^\infty dx \left(x - \frac{5}{2}\right) e^{-x} = -\frac{3}{2} \frac{n_e \nu_{ei}}{m_e} \frac{\hat{b} \times \nabla T_e}{\Omega_e} \qquad (273)$$

Thus,

$$\Omega_e \frac{\partial \delta\tilde{f}^{(1)}}{\partial \vartheta} = \nu_D(w)\, \delta\tilde{f}^{(0)} \text{[illegible]} - \left(\frac{\partial \delta\tilde{f}^{(0)}}{\partial t}\right)_{ee}$$

$$+ \frac{2 f_0}{v_{the}^2} \nu_{ei}\, \vec{w}_\perp \cdot \left[\frac{3}{2 m_e} \frac{\hat{b} \times \nabla T_e}{\Omega_e} - \left(1 - \frac{3\sqrt{\pi}}{4}\frac{v_{th}^3}{w^3}\right)(\vec{u}_e - \vec{u}_i)\right] \qquad (274)$$

I will yet again ignore e-e collisions and integrate using $\vec{w}_\perp = \frac{\partial}{\partial \vartheta} \vec{w}_\perp \times \hat{b}$, $\vec{w}_\perp \times \hat{b} = -\frac{\partial \vec{w}_\perp}{\partial \vartheta}$

$$\boxed{\begin{aligned} \delta\tilde{f}^{(1)} = &- \frac{\nu_D(w)}{\Omega_e^2} \vec{w}_\perp \cdot \frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) f_0 \\ &+ \frac{\nu_{ei}}{\Omega_e}\left(\vec{w}_\perp \times \hat{b}\right) \cdot \left[\frac{3}{2}\frac{\hat{b} \times \nabla T_e}{\Omega_e T_e} - \left(1 - \frac{3\sqrt{\pi}}{4}\frac{v_{the}^3}{w^3}\right)\frac{2}{v_{the}^2}(\vec{u}_e - \vec{u}_i)\right] f_0 \end{aligned}} \qquad (275)$$

Ex: Check that this has no mean flow!

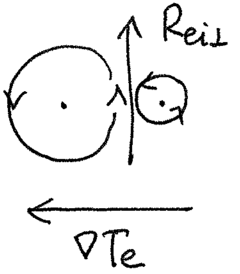
5.1.6 Perpendicular friction force

I have in fact already computed this. By definition,

$$\vec{R}_{ei\perp}^{(1)} = -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i)_\perp - m_e \underbrace{\int d\vec{w}\, \nu_D(w)\, \vec{w}_\perp\, \tilde{\delta f}^{(0)}(\vec{w})}_{= \; -\frac{3}{2} \frac{n_e \nu_{ei}}{m_e} \frac{\hat{b}\times\nabla T_e}{\Omega_e} \text{ from (273)}}$$

$$\boxed{\begin{aligned} \vec{R}_{ei\perp}^{(1)} &= -m_e n_e \nu_{ei} (\vec{u}_e - \vec{u}_i)_\perp \\ &\quad + \frac{3}{2} n_e \nu_{ei} \frac{\hat{b}\times\nabla T_e}{\underset{\substack{\uparrow \\ \text{NB } \Omega_e < 0}}{\Omega_e}} \end{aligned}} \quad (276)$$

The physics of the ∇T_e contribution is similar to the physics of the $\nabla_\parallel T_e$ one:



Gyrating particles on the colder side have smaller velocities, so collide more than the the ones on the hotter side. The result is net friction in the direction $\hat{b}\times\nabla T_e$.

There is no great point in calculating the contribution from $\hat{\delta f}^{(1)}$, which are 2-nd order in ν_{ei}/Ω_e, $\vec{R}_{ei\perp}^{(2)}$, but be my guest and explore them!

It will be more interesting to use our δf to work out the electron heat flux.

P.S. Here is a slightly faster way of getting δf (due to Helander & Sigmar).

Write eq. (253), neglecting e-e collisions, in the following form:

$$\Omega_e \frac{\partial \delta f}{\partial \vartheta} + \nu_D(w)\mathcal{L}[\delta f] + \vec{w}\cdot\vec{U}[\delta f] = \vec{w}\cdot\vec{G}(w) \qquad (277)$$

short notation for corresponding terms

Seek solution in the form

$$\delta f = \vec{w}\cdot\vec{F}(w) \qquad (278)$$

Since $\frac{\partial \vec{w}}{\partial \vartheta} = -\vec{w}_\perp \times \hat{b}$ and $\mathcal{L}[\vec{w}] = -\vec{w}$, we get

$$-\Omega_e(\vec{w}_\perp\times\hat{b})\cdot\vec{F} - \nu_D(w)\vec{w}\cdot\vec{F} + \vec{w}\cdot\vec{U}[\vec{w}\cdot\vec{F}] = \vec{w}\cdot\vec{G} \qquad (279)$$

Since this must be satisfied $\forall \vec{w}$, we have

$$-\Omega_e \hat{b}\times\vec{F} - \nu_D\vec{F} + \vec{U}[\vec{w}\cdot\vec{F}] = \vec{G} \qquad (280)$$

Note that

$$\vec{U}[\vec{w}\cdot\vec{F}] = \frac{2f_0}{n_e v_{the}^2}\int d\vec{w}'\,\nu_D(w')\,\vec{w}'\vec{w}'\cdot\vec{F}(w')$$

$$= \frac{8\pi f_0}{3 n_e v_{the}^2}\int_0^\infty dw'\,w'^4\,\overset{\propto w'^{-3}}{\nu_D(w')}\,\vec{F}(w')$$

$$= \frac{8\pi f_0 w^3 \nu_D(w)}{3 n_e v_{the}^2}\int_0^\infty dw'\,w'\,\vec{F}(w') \equiv \nu_D\vec{V}[\vec{F}] \qquad (281)$$

$$-\Omega_e\hat{b}\times\vec{F} - \nu_D(\vec{F}-\vec{V}[\vec{F}]) = \vec{G} \qquad (282)$$

Taking the parallel component gives the gyrotropic part of the distribution:

$$F_\parallel - V_\parallel[F_\parallel] = -\frac{G_\parallel}{\nu_D} \qquad (283)$$

$$\boxed{F_\parallel = -\frac{G_\parallel}{\nu_D}} \text{ because } V_\parallel\left[\frac{G_\parallel}{\nu_D}\right] = 0 \qquad (284)$$

This has to be adjusted by adding the homogeneous solution $F_{\parallel hom} = \alpha(\ldots)$ and then calculating α to ensure $\int d\vec{w}\, w_\parallel \delta f = 0$ — same way I did it above.

The functional form of $G_\parallel$ adjusts accordingly

Now take $(282)_\perp$:

$$\Omega_e \hat{b}\times\vec{F}_\perp + \nu_D\left(\vec{F}_\perp - \vec{V}_\perp(\vec{F}_\perp)\right) = -\vec{G}_\perp \quad (285)$$

To 0th order in ν_D/Ω_e,

$$\hat{b}\times\Big|\ \Omega_e\hat{b}\times\vec{F}_\perp^{(0)} = -\vec{G}_\perp \quad \Rightarrow \quad \boxed{\vec{F}_\perp^{(0)} = \frac{\hat{b}\times\vec{G}_\perp}{\Omega_e}} \quad (286)$$

To 1st order,

$$\hat{b}\times\Big|\ \Omega_e\hat{b}\times\vec{F}_\perp^{(1)} = -\nu_D\left[\vec{F}_\perp^{(0)} - \vec{V}_\perp(\vec{F}_\perp^{(0)})\right] \quad (287)$$

$$\vec{F}_\perp^{(1)} = \frac{\nu_D}{\Omega_e}\left[\hat{b}\times\vec{F}_\perp^{(0)} - \vec{V}_\perp(\hat{b}\times\vec{F}_\perp^{(0)})\right]$$

$$\boxed{\vec{F}_\perp^{(1)} = -\frac{\nu_D}{\Omega_e}\left[\frac{\vec{G}_\perp}{\Omega_e} - \vec{V}_\perp\left(\frac{\vec{G}_\perp}{\Omega_e}\right)\right]} \quad (288)$$

adjusted with kom. solution.

Assembling (284), (286) and (288) via (278),

$$\delta f = -\left[\frac{w_\parallel \tilde{G}_\parallel}{\nu_D} + \left[\frac{\vec{w}_\perp\times\hat{b}}{\Omega_e} - \vec{w}_\perp\frac{\nu_D}{\Omega_e^2}\right]\cdot\vec{G}_\perp + \frac{\nu_D}{\Omega_e^2}\vec{w}_\perp\cdot\vec{V}_\perp(\vec{G}_\perp)\right., \quad (289)$$

which is the same solution as previously calculated.

This is a neater way to do it, perhaps.

Note that where $\vec{G}$ is multiplied ν_D/Ω_e^2, the $\vec{u}_e-\vec{u}_i$ term in it can be dropped.

Note that Helander's treatment of "Lorentz plasma" (his §4.2) is probably the shortest possible treatment of the material I am presenting here – shorter than mine because he does not go to $\vec{w}$ variables – but I wanted to preserve unity with KMHD treatment and some later convenience

5.2 Generalised Ohm's Law

To lowest order in our expansion,

$$\vec E + \frac{\vec u_e \times \vec B}{c} = -\frac{\nabla p_e}{e n_e} + \frac{\vec R_{ei}}{e n_e} \tag{290}$$

So resistivity in a magnetised plasma is anisotropic, but not by much. In any even, we can neglect $j_\perp \ll j_\parallel$ because we know from KMHD that $\vec u_{\perp e} \approx \vec u_{\perp i}$ to lowest order in $k\rho_i$ →

$$\left\{\begin{array}{l} -\overset{\eta_\parallel}{\boxed{\frac{3\pi}{32}\frac{m_e \nu_{ei}}{e^2 n_e}}}\,\overbrace{n_e(u_{\parallel e}-u_{\parallel i})}^{=-j_\parallel}\,\hat b \\ -\underset{\eta_\perp}{\boxed{\frac{m_e \nu_{ei}}{e^2 n_e}}}\,\underbrace{n_e(\vec u_{\perp e}-\vec u_{\perp i})}_{=-\vec j_\perp} \\ -\frac{3}{2}\frac{\nabla_\parallel T_e}{e}\hat b + \frac{3}{2}\frac{\nu_{ei}}{e\Omega_e}\hat b \times \nabla T_e \end{array}\right.$$

"Seebeck effect" "Nernst effect"

We can treat (290) as an equation for $\parallel$ current:

$$j_\parallel = \frac{1}{\eta_\parallel}E_\parallel + \frac{32}{3\pi}\frac{e}{m_e\nu_{ei}}\left(\nabla_\parallel p_e + \frac{3}{2}n_e\nabla_\parallel T_e\right) \tag{291}$$

Ohm

collisionless collisional (Seebeck)

thermoelectric effect (gradients drive currents)

We can also just recall that $\vec j = \frac{c}{4\pi}\nabla\times\vec B$ and treat (290) as the equation for the electric field to be put into the Faraday's Law

$$\frac{\partial \vec B}{\partial t} = -c\nabla\times\vec E = \nabla\times\left(\vec u_e\times\vec B + c\frac{\nabla \overset{n_eT_e}{p_e}}{en_e} - c\frac{\vec R_{ei}}{en_e}\right)$$

$$= \nabla\times(\vec u_e\times\vec B) - c\frac{\nabla n_e\times\nabla T_e}{en_e} - \nabla\times c\frac{\vec R_{ei}}{en_e} \tag{292}$$

Believed responsible for primordial magnetising seen in the lab for the first time by G. Gregori.

produces, rather than merely amplifies fields ⇝ "Biermann battery"

5.3 Electron ~~[illegible]~~ Heat Fluxes

5.3.1 Electron temperature eqn.

Let us now talk about flows of energy.

From $\int d\vec{w}\,\frac{m_e w^2}{2}$ (240), we get, in full generality

$$\underbrace{\frac{d}{dt_e}\frac{3}{2}p_e}_{\frac{3}{2}n_e\frac{dT_e}{dt_e}+\underbrace{\frac{3}{2}T_e\frac{dn_e}{dt_e}}_{\cancel{-\frac{3}{2}p_e\nabla\cdot\vec{u}_e}}} + \underbrace{\int d\vec{w}\,\frac{m_e w^2}{2}\,\vec{w}\cdot\nabla f_e}_{\nabla\cdot\vec{q}_e \text{ heat flux}} \qquad \boxed{-m_e\vec{a}_e\int d\vec{w}\,\vec{w}f_e = 0}$$

$$+\underbrace{\int d\vec{w}\,\frac{m_e w^2}{2}}\left(-\frac{e}{m_e}\cancel{\frac{\vec{w}\times\vec{B}}{c}} - \vec{w}\cdot\nabla\vec{u}_e + \cancel{\vec{a}_e}\right)\cdot\frac{\partial f_e}{\partial\vec{w}} =$$

vanishes under integrati by parts

$$= m_e\int d\vec{w}\,f_e\left(\vec{w}\vec{w}:\nabla\vec{u}_e + \frac{w^2}{2}\nabla\cdot\vec{u}_e\right) = \cancel{\frac{3}{2}p_e\nabla\cdot\vec{u}_e} + \hat{P}_e:\nabla\vec{u}_e$$

$p_e\mathbb{1} + \hat{\Pi}_e$

small (el. viscosity)

cancel

$$= \int d\vec{w}\,\frac{m_e w^2}{2}\left(\frac{\partial f_e}{\partial t}\right)_c = \int d\vec{v}\,\frac{m_e}{2}\left(v^2 - 2\vec{v}\cdot\vec{u}_e + \overset{0}{\cancel{u_e^2}}\right)\left(\frac{\partial f_e}{\partial t}\right)_c$$

$$= \underbrace{\int d\vec{v}\,\frac{m_e v^2}{2}\left(\frac{\partial f_e}{\partial t}\right)_c}_{\text{cons. of energy}} - \underbrace{m_e\vec{u}_e\cdot\int d\vec{v}\,\vec{v}\left(\frac{\partial f_e}{\partial t}\right)_c}_{\vec{u}_e\cdot\vec{R}_{ei}}$$

$$-\int d\vec{v}\,\frac{m_i v^2}{2}\left(\frac{\partial f_i}{\partial t}\right)_c = \vec{R}_{ei}\cdot\vec{u}_i + 3\nu_{ie}n_i(T_i - T_e)$$

$$= +\vec{R}_{ei}\cdot\underbrace{(\vec{u}_i - \vec{u}_e)}_{\frac{\vec{j}}{en_e}} - 3\,\frac{m_e}{m_i}n_e\nu_{ei}(T_e - T_i) \qquad (296)$$

this is very small in our expansion

Ohmic heating etc: $\eta_\parallel j_\parallel^2 + \ldots$

So, we have

$$\frac{3}{2}n_e\frac{dT_e}{dt_e}+\nabla\cdot\vec{q}_e+p_e\nabla\cdot\vec{u}_e+\hat{\Pi}_e:\nabla\vec{u}_e=\frac{\vec{R}_{ei}}{en_e}\cdot\vec{j}-3\frac{m_e}{m_i}n_e\nu_{ei}(T_e-T_i) \qquad (297)$$

compressional heating (adiabatic effect)

electron viscosity

electric heating

temperature relaxation

our focus here will be the e. heat flux

$$\boxed{\vec{q}_e=\int d\vec{w}\,\frac{m_e w^2}{2}\vec{w}\,f_e} \qquad (298)$$

To calculate it, all we need to do is substitute our δf (no heat flux in the Maxwellian f_0).

5.3.2 Parallel heat flux

From (265),

$$q_{\|e}=-\underbrace{\int d\vec{w}\,\frac{m_e w^2}{2}w_\|^2}_{=\,m_e 2\pi\frac{1}{2}\int_0^\infty dw\,w^6\cdot\frac{2}{3}}\frac{\overset{\frac{n_e}{\pi^{3/2}v_{the}^3}e^{-w^2/v_{the}^2}}{f_0(w)}}{\underset{\frac{3\sqrt{\pi}}{4}\nu_{ei}\frac{v_{the}^3}{w^3}}{\nu_D(w)}}\hat{b}\cdot\left\{\frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2}-4\right)-\frac{2\nu_{ei}}{v_{the}^2}\left(\frac{3\pi}{32}-\frac{3\sqrt{\pi}}{4}\frac{v_{the}^3}{w^3}\right)(\vec{u}_e-\vec{u}_i)\right\}$$

$$=-\frac{m_e n_e v_{the}^4}{\nu_{ei}}\frac{8}{9\pi}\int_0^\infty dx\,x^9e^{-x^2}\left[\frac{\nabla_\|T_e}{T_e}(x^2-4)-\frac{2\nu_{ei}}{v_{the}^2}\left(\frac{3\pi}{32}-\frac{3\sqrt{\pi}}{4x^3}\right)(u_e-u$$

$$=-\frac{n_e v_{the}^2}{\nu_{ei}}\frac{16}{9\pi}\nabla_\|T_e\frac{1}{2}\underbrace{\int_0^\infty dy\,y^4e^{-y}(y-4)}_{=4!}$$

$$+\,m_e n_e v_{the}^2\frac{16}{9\pi}(u_{\|e}-u_{\|i})\left[\frac{3\pi}{32}\cdot\frac{4!}{2}-\frac{3\sqrt{\pi}}{4}\overbrace{\int_0^\infty dx\,x^6e^{-x^2}}^{=\frac{\sqrt{\pi}}{2}\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}\right]$$

$$=-\frac{64}{3\pi}\frac{n_e v_{the}^2}{\nu_{ei}}\nabla_\|T_e+\frac{3}{4}m_e n_e v_{the}^2(u_{\|e}-u_{\|i}) \qquad (299)$$

So, $$\boxed{q_{\parallel e} = -\underbrace{\frac{128}{3\pi}\frac{n_e T_e}{m_e \nu_{ei}}}_{\parallel \text{ heat conductivity } \equiv \varkappa_\parallel} \nabla_\parallel T_e + \frac{3}{2} n_e T_e (u_{\parallel e} - u_{\parallel i})} \qquad (299)$$

$-\frac{3}{2}\frac{T_e}{e} j_\parallel$ current carries heat "Peltier effect"

Regular collisional diffusion $\sim n_e \frac{v_{the}^2}{\nu_{ei}} \sim n_e v_{the} \lambda_{mfp}$ → ∥ heat conductivity $\equiv \varkappa_\parallel$

At $Z=1$, the correct coefficients are 3.16 and 0.71

5.3.3 Diamagnetic heat flux

From (270),

$$q_{xe} = \int d\vec{w}\, \frac{m_e w^2}{2} \vec{w}_\perp \delta \tilde{f}^{(0)} = \int d\vec{w}\, \frac{m_e w^2}{2} \vec{w}_\perp \frac{\vec{w}_\perp \times \hat{b}}{\Omega_e} \cdot \frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) f$$

$$= \Bigg[\underbrace{\int d\vec{w}\, \frac{m_e w^2}{2} f_0(w)\, \vec{w}_\perp \vec{w}_\perp \left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right)}\Bigg] \cdot \frac{\hat{b} \times \nabla T_e}{\Omega_e T_e}$$

$$\| $$

$$\frac{m_e}{2}(\mathbb{1} - \hat{b}\hat{b}) \cancel{\frac{1}{2}} \cancel{2}\pi \underbrace{\int_{-1}^{1} d\xi\, (1 - \xi^2)}_{= \frac{4}{3}} \int_0^\infty dw\, w^6 \frac{n_e}{\pi^{3/2} v_{the}^3} e^{-w^2/v_{the}^2} \left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right)$$

$$= m_e n_e v_{the}^4 (\mathbb{1} - \hat{b}\hat{b}) \frac{2}{\cancel{3}\cancel{\sqrt{\pi}}} \underbrace{\int_0^\infty dx\, x^6 e^{-x^2}\left(x^2 - \frac{5}{2}\right)}_{= \frac{\cancel{\sqrt{\pi}}}{\cancel{2}} \frac{1}{2} \cdot \frac{\cancel{3}}{2} \cdot \frac{5}{2}\left(\frac{7}{2} - \frac{5}{2}\right)} = \frac{5}{8} m_e n_e v_{the}^4 (\mathbb{1} - \hat{b}\hat{b})$$

$$\boxed{q_{xe} = \frac{5}{2} \frac{n_e T_e}{m_e \Omega_e} \hat{b} \times \nabla T_e} \qquad (300)$$

NB: $\Omega_e < 0$!

This is sometimes called Righi–Leduc effect (or thermal Hall effect)

Physics of this is the same as of the ⊥ friction (276)

5.3.4 Perpendicular Heat Flux

From (275),

$$\vec{q}_{\perp e} = \int d\vec{w}\, \frac{m_e w^2}{2} \vec{w}_\perp \delta \tilde{f}^{(1)}$$

$$= +\int d\vec{w}\, \frac{m_e w^2}{2} \vec{w}_\perp \left\{ -\frac{\nu_D(w)}{\Omega_e^2} \vec{w}_\perp \cdot \frac{\nabla T_e}{T_e}\left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) f_0(w) \right.$$

$$\left. + \frac{\nu_{ei}}{\Omega_e} (\vec{w}_\perp \times \hat{b}) \cdot \left[\frac{3}{2} \frac{\hat{b} \times \nabla T_e}{\Omega_e T_e} - \left(1 - \frac{3\sqrt{\pi}}{4} \frac{v_{the}^3}{w^3}\right) \frac{2}{v_{the}^2} (\vec{u}_e - \vec{u}_i) \right] f_0 \right\}$$

$$= \left[\int d\vec{w}\, \frac{m_e w^2}{2} \vec{w}_\perp \vec{w}_\perp f_0(w) \right] \cdot \left\{ -\frac{\nu_D(w)}{\Omega_e^2} \left(\frac{w^2}{v_{the}^2} - \frac{5}{2}\right) \frac{\nabla T_e}{T_e} \right.$$

$$\left. + \frac{\nu_{ei}}{\Omega_e} \left[\frac{3}{2} \frac{\hat{b} \times (\hat{b} \times \nabla T_e)}{\Omega_e T_e} + \left(1 - \frac{3\sqrt{\pi}}{4} \frac{v_{the}^3}{w^3}\right) \frac{2}{v_{the}^2} \frac{\hat{b} \times \vec{j}}{e n_e} \right] \right\}$$

see p.127

$$\to m_e n_e v_{the}^4 (\mathbb{1} - \hat{b}\hat{b}) \frac{2}{3\sqrt{\pi}} \int_0^\infty dx\, x^6 e^{-x^2}$$

$$= m_e n_e v_{the}^4 \frac{\cancel{2}}{\cancel{3\sqrt{\pi}}} \left\{ -\underbrace{\int_0^\infty dx\, x^3 \left(x^2 - \frac{5}{2}\right) e^{-x^2} \frac{3\sqrt{\pi}}{4}}_{\to \frac{\cancel{3\sqrt{\pi}}}{4} \frac{1}{\cancel{2}} \left(-\frac{1}{2}\right)} \frac{\nu_{ei}}{\Omega_e^2} \frac{\nabla_\perp T_e}{T_e} \right.$$

$$\left. - \underbrace{\frac{3}{2} \int_0^\infty dx\, x^6 e^{-x^2}}_{\frac{\cancel{3}}{\cancel{2}} \frac{\sqrt{\pi}}{2} \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}} \frac{\nu_{ei}}{\Omega_e^2} \frac{\nabla_\perp T_e}{T_e} + \frac{2\nu_{ei}}{\Omega_e v_{the}^2 e n_e} (\hat{b} \times j) \cdot \underbrace{\int_0^\infty dx\, x^6 e^{-x^2} \left(1 - \frac{3\sqrt{\pi}}{4} \frac{1}{x^3}\right)}_{\frac{\sqrt{\pi}}{\cancel{2}} \frac{1}{2} \cdot \frac{\cancel{3}}{2} \cdot \frac{5}{2} - \frac{\cancel{3\sqrt{\pi}}}{4} \frac{1}{\cancel{2}}} \right\}$$

[illegible]

$$= m_e n_e v_{the}^4 \left\{ \frac{\nu_{ei}}{\Omega_e^2} \frac{\nabla_\perp T_e}{T_e} \overset{= -\frac{13}{16}}{\left(\frac{1}{\cancel{2}} - \frac{15}{16} \right)} + \frac{2\nu_{ei}}{\Omega_e v_{the}^2 e n_e} \hat{b} \times \vec{j} \overset{= \frac{3}{8}}{\left(\frac{5}{8} - \frac{1}{4} \right)} \right\}$$

So,

$$\boxed{\vec{q}_\perp = -\underbrace{\frac{13}{4}\frac{n_e T_e}{m_e}\frac{\nu_{ei}}{\Omega_e^2}}_{\substack{\perp\text{ heat conductivity}\\ \equiv \varkappa_\perp}}\nabla_\perp T_e + \underbrace{\frac{3}{2}T_e\frac{\nu_{ei}}{e\Omega_e}}_{\text{"Ettingshausen effect"}}\hat{b}\times\vec{j}} \qquad (301)$$

This is again diffusion but a different one than along the field.

$$\varkappa_\perp \sim n_e \nu_{ei} \rho_e^2$$

So the step of the ⊥ random walk is ρ_e, not λ_{mfp} — this is because electrons gyrate, so can't go very far across the field.

Key thing to carry from all this is that

$$\left.\begin{aligned}
\varkappa_\parallel &\sim n_e \frac{v_{the}^2}{\nu_{ei}}\\
\varkappa_\times &\sim \varkappa_\parallel \frac{\nu_{ei}}{\Omega_e}\\
\varkappa_\perp &\sim \varkappa_\parallel \frac{\nu_{ei}^2}{\Omega_e^2}
\end{aligned}\right\} \text{progressively smaller} \qquad (302)$$

collisionless effect

basically because it is harder to move across $\vec{B}$ than along it