3. Inter-species collisions & relaxation processes.

I have so far derived a CI for collisions between particles of the same species, electrons (but it would work exactly the same way for ions), and found that they would push these particles towards a local Maxwellian equilibrium.

That would suggest that in an actual plasma, collisions would push electrons and (several species of) ions towards states with different mean velocities and temperatures. But clearly, there will also be collisions between different species that would push them towards some kind of equilibrium with each other — this is interesting and we would like to be able to describe these properties, and the rates at which various relaxation processes would occur.

Well, the rates we can perhaps work out on the twiddle level first.

3.1 Simple estimates

For the ion-ion collisions, it is just (71) with e→i:

$$\boxed{\nu_{ii} \sim \frac{Z^4 e^4 n_i \Lambda_{ii}}{m_i^{1/2} T_i^{3/2}}} \quad , \qquad (72)$$

n_e/Z because of quasineutrality

So
$$\frac{\nu_{ii}}{\nu_{ee}} \sim Z^3 \sqrt{\frac{m_e}{m_i}} \left(\frac{T_e}{T_i}\right)^{3/2} \frac{\Lambda_{ii}}{\Lambda_{ee}} \ll 1 \qquad (73)$$

[generally

Electron-ion collisions:

As usual, the mfp of an electron before colliding with an ion is

$$\lambda_{ei} \sim \frac{1}{n_i \sigma_{ei}} \sim \frac{1}{n_i}\left(\frac{T_e}{Ze^2}\right)^2 \sim \frac{T_e^2}{e^4 n_e Z}$$

All velocities are just v_{the}, not relative to ions, because $v_{the} \gg v_{thi}$

how thick on the grounds ions are, to collide with $n_i = n_e/Z$

cross-section for a (putative) head-on collision $\sigma_{ei} \sim \pi d_{ei}^2$

$$\frac{m_e v_{the}^2}{2} \sim \frac{Ze^2}{d_{ei}}$$

electron kinetic energy

ion Coulomb potential on electron

So,

$$\boxed{\nu_{ei} \sim \frac{v_{the}}{\lambda_{ei}}\Lambda_{ei} \sim \frac{Z e^4 n_e \Lambda_{ei}}{m_e^{1/2} T_e^{3/2}}}, \quad (74)$$

to correct for the fact that there are many glancing collisions within the Debye sphere rather than one head-on collision

Whence

$$\frac{\nu_{ei}}{\nu_{ee}} \sim Z\frac{\Lambda_{ei}}{\Lambda_{ee}} \quad (75)$$

The ratio can be significant for heavy ions.

Note that since light electrons bounce off heavy ions, this is a bit like like bouncing off a wall, so the electron energy barely changes, while momentum clearly does, by order unity.

Ion-electron collisions:

Ions' lived experience of collisions with electrons is very different: they are heavy, so they are basically being bombarded, with each kick changing their velocity by only a small amount.

Physically, this should look like a Brownian motion of a heavy particle through a sea of light ones. The easiest way to estimate their collision frequency and get the right result is to posit that ν_{ie} is just their friction rate with the electrons:

this will turn out to be correct

mean flow velocities

$$\left(\frac{\partial \vec{u}_i}{\partial t}\right)_c = -\nu_{ie}\left(\vec{u}_i - \vec{u}_e\right) \qquad (76)$$

By the same token,

$$\left(\frac{\partial \vec{u}_e}{\partial t}\right)_c = -\nu_{ei}\left(\vec{u}_e - \vec{u}_i\right) \qquad (77)$$

But overall, momentum must be conserved by collisions:

$$m_i n_i \left(\frac{\partial \vec{u}_i}{\partial t}\right)_c + m_e n_e \left(\frac{\partial \vec{u}_e}{\partial t}\right)_c = 0, \qquad (78)$$

whence

$$\boxed{\nu_{ie} = \frac{m_e}{m_i}\overset{Z}{\frac{n_e}{n_i}}\,\nu_{ei} \sim \frac{Z^2 m_e^{1/2} e^4 n_e \Lambda_{ei}}{m_i T_e^{3/2}}} \qquad (79)$$

Thus, to summarize,

$$\nu_{ee} : \nu_{ei} : \nu_{ii} : \underset{\substack{\uparrow \\ \text{slowest}}}{\nu_{ie}} \sim 1 : Z\frac{\Lambda_{ei}}{\Lambda_{ee}} : Z^3\sqrt{\frac{m_e}{m_i}}\left(\frac{T_e}{T_i}\right)^{3/2}\frac{\Lambda_{ii}}{\Lambda_{ee}} : Z^2\frac{m_e}{m_i} \qquad (80)$$

Intuitively, we expect that

- electrons become locally Maxwellian at the rate ν_{ee}
- mean flows relax to each other at the rate ν_{ei} [see (77)] $\Leftrightarrow$ electrons become isotropic in the ion frame at this rate
- ions become locally Maxwellian at the rate ν_{ii}
- energies equalise at the rate ν_{ie} (because ei collisions do not change electron energy)

We are going to show all this systematically in what follows.

3.2 Inter-species CI

... but to do that, we need to derive the CI for interspecies collisions.

Actually I could (and you might think should) have done this above, by keeping track of species sums and indices.

But I was lazy and did not want to obscure the structure of the derivation by subscript clutter. So it is now your job to go through my derivation and restore species dependence - a good exercize, actually.

The result is: the LB CI (58) becomes

$$\boxed{\frac{Df_\alpha}{Dt} = \sum_{\alpha''} \frac{16\pi^3 q_\alpha^2 q_{\alpha''}^2}{V m_\alpha} \frac{\partial}{\partial \vec{v}} \cdot \int d\vec{v}'' \sum_{\vec{k}} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta(\vec{k}\cdot(\vec{v}-\vec{v}''))}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} \cdot \left[\frac{f_{\alpha''}(\vec{v}'')}{m_\alpha}\frac{\partial f_\alpha}{\partial \vec{v}} - \frac{f_\alpha(\vec{v})}{m_{\alpha''}}\frac{\partial f_{\alpha''}}{\partial \vec{v}''}\right] \equiv \sum_{\alpha''}\left(\frac{\partial f_\alpha}{\partial t}\right)_{LB\,\alpha''}} \tag{81}$$

an the Landau approximation (69)

Coulomb log.

$$\boxed{\frac{Df_\alpha}{Dt} = \sum_{\alpha''} \frac{2\pi q_\alpha^2 q_{\alpha''}^2 \Lambda}{m_\alpha} \frac{\partial}{\partial \vec{v}} \cdot \int \frac{d\vec{v}''}{w}\left(\mathbb{1} - \frac{\vec{w}\vec{w}}{w^2}\right) \cdot \left[\frac{f_{\alpha''}(\vec{v}'')}{m_\alpha}\frac{\partial f_\alpha}{\partial \vec{v}} - \frac{f_\alpha(\vec{v})}{m_{\alpha''}}\frac{\partial f_{\alpha''}}{\partial \vec{v}''}\right] \equiv \sum_{\alpha''}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\alpha''}} \tag{82}$$

Obviously, one can, in the same manner as I have done above, prove that

- particles are conserved species by species
- momentum and energy are conserved when summed over species
- H-theorem is satisfied for

$$S = -\sum_\alpha \int d\vec{v}\, f_\alpha \ln f_\alpha \tag{83}$$

and the equilibrium is a Maxwellian with mean flows and temperatures the same across all species.

Exercise 1. Work carefully through the multispecies calculation, repeating and, where necessary, generalising the steps in my derivation, culminating in a set of clean notes to yourself that lead to the results outlined on p.46.

As we saw in §3.1, the rates of collisions between species are not the same, so relaxation will happen at different rates. This gives us something worth studying.

---

§3.3 Electron-Ion Collisions

Let us first interest ourselves in the e-i collisions, which, as we saw in (80) are the fastest of the lot (by a factor of Z than e-e — so only faster for heavy ions and by $\sqrt{\frac{m_i}{m_e}}$ than i-i — so, to lowest order, electrons are done relaxing before the ions even notice!)

Let us see what they do, assuming formally

$$Z \sim 1, \quad \frac{T_i}{T_e} \sim 1, \quad \sqrt{\frac{m_e}{m_i}} \ll 1 \Rightarrow \frac{v_{thi}}{v_{the}} \sim \sqrt{\frac{m_e}{m_i}} \ll 1. \tag{84}$$

and a single ion species.

We have

$$\left(\frac{\partial f_e}{\partial t}\right)_i = \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_e}\frac{\partial}{\partial \vec{v}}\cdot \int d\vec{v}''\, \overbrace{\frac{1}{w}\left(\mathbb{1}-\frac{\vec{w}\vec{w}}{w^2}\right)}^{\equiv \hat{U}(\vec{w})}\cdot \left[\underbrace{\frac{f_i(\vec{v}'')}{m_e}\frac{\partial f_e}{\partial \vec{v}}}_{①} - \underbrace{\frac{f_e(\vec{v})}{m_i}\frac{\partial f_i}{\partial \vec{v}''}}_{②}\right] \qquad (85)$$

$$\frac{②}{①} \sim \frac{f_e}{m_i}\frac{f_i}{v_{thi}}\frac{m_e v_{the}}{f_i f_e} \sim \sqrt{\frac{m_e}{m_i}} \ll 1$$

In fact, ② can be neglected to 2 orders, i.e., up to $O(\frac{m_e}{m_i}$

Indeed, consider

$$\hat{U}(\underset{\text{els}}{\vec{v}}-\underset{\text{ions}}{\vec{v}''}) \approx \hat{U}(\vec{v}) - v''_k \frac{\partial U_{ij}(\vec{v})}{\partial v_k} + O\left(\frac{m_e}{m_i}\right) \qquad (86)$$

$$\frac{v_{thi}}{v_{the}} \sim \sqrt{\frac{m_e}{m_i}}$$

and

$$\int d\vec{v}''\, \hat{U}(\vec{w})\cdot \frac{f_e(\vec{v})}{m_i}\frac{\partial f_i}{\partial \vec{v}''} =$$

$$\frac{f_e(\vec{v})}{m_i}\left[\hat{U}(\vec{v})\underbrace{\int d\vec{v}''\frac{\partial f_i}{\partial \vec{v}''}}_{0} - \int d\vec{v}''\, \vec{v}''\cdot\frac{\partial \hat{U}(\vec{v})}{\partial \vec{v}}\cdot\frac{\partial f_i}{\partial \vec{v}''} + \dots\right]$$

$$= \frac{f_e(\vec{v})}{m_i} n_i \frac{\partial}{\partial \vec{v}}\cdot\hat{U}(\vec{v}) \qquad (87)$$

— this has an extra factor of $\sqrt{\frac{m_e}{m_i}}$, so $\frac{②}{①} \sim \frac{m_e}{m_i}$

Great, we have

$$\left(\frac{\partial f_e}{\partial t}\right)_i \approx \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_e^2}\frac{\partial}{\partial \vec{v}}\cdot \underbrace{\left[\int d\vec{v}''\, \hat{U}(\vec{v}-\vec{v}'') f_i(\vec{v}'')\right]}_{\equiv \hat{D}_{ei}(\vec{v})}\cdot\frac{\partial f_e}{\partial \vec{v}} \qquad (88)$$

pure diffusion in velocity space (in line with the anticipatory intuition in § 3.1 that an electron is a drunken man stumbling from one immovable ion "lamppost" to another, in $\vec{v}$ space obviously).

To lowest order,

$$\hat{D}_{ei}(\vec{v}) \approx \hat{U}(\vec{v})\, n_i, \qquad (89)$$

but we can do better and keep the $O\left(\sqrt{\frac{m_e}{m_i}}\right)$ part because, as we saw on p. 48, the approximation (88) is valid to two orders. This is easy: using (86),

$$\hat{D}_{ei}(\vec{v}) \approx n_i \hat{U}(\vec{v}) - \int d\vec{v}''\, \vec{v}'' \cdot \frac{\partial \hat{U}}{\partial \vec{v}} f_i(\vec{v}'') =$$

$$= n_i \left[\hat{U}(\vec{v}) - \vec{u}_i \cdot \frac{\partial \hat{U}(\vec{v})}{\partial \vec{v}}\right] \approx n_i \hat{U}(\vec{v} - \vec{u}_i) \qquad (90)$$

But now we can change variables to measure electron velocities in the frame of the ions:

$$\vec{w} = \vec{v} - \vec{u}_i$$

whereupon (88) becomes

$$\boxed{\left(\frac{\partial f_e}{\partial t}\right)_i = \frac{2\pi Z e^4 n_e \Lambda_{ei}}{m_e^2} \frac{\partial}{\partial \vec{w}} \cdot \frac{1}{w}\left(\mathbb{1} - \frac{\vec{w}\vec{w}}{w^2}\right) \cdot \frac{\partial f_e}{\partial \vec{w}}} \qquad (91)$$

Because the diffusion coefficient here is $\propto$ a projection operator on the plane $\perp \vec{w}$, the operator cannot change the energy distribution of the electrons, it just diffuses them in angles. This operator is known as the <u>Lorentz pitch-angle-scattering operator</u> and has a simple form when recast in polar coordinates:

$$\boxed{\left(\frac{\partial f_e}{\partial t}\right)_i = \frac{4\pi Z e^4 n_e \lambda_{ei}}{m_e^2 w^3}\hat{\mathcal{L}}[f_e]}, \qquad (92)$$

where

$$\hat{\mathcal{L}}[f_e] = \frac{1}{2} w^3 \frac{\partial}{\partial \vec{w}}\cdot\frac{1}{w}\left(\mathbb{1}-\frac{\vec{w}\vec{w}}{w^2}\right)\cdot\frac{\partial f_e}{\partial \vec{w}}$$

in the plane $\perp \vec{w}$, so unaffected by $\hat{U}$
$\hat{U}\cdot\hat{e}_{\theta,\phi} = \frac{1}{w}\hat{e}_{\theta,\phi}$

$$\frac{\partial}{\partial \vec{w}} = \hat{e}_w\frac{\partial}{\partial w} + \hat{e}_\theta\frac{1}{w}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{w\sin\theta}\frac{\partial}{\partial\phi},$$

$\hat{e} = \frac{\vec{w}}{w}$ annihilated by $\hat{U}$

So

$$\hat{\mathcal{L}}[f_e] = \frac{1}{2}w^3\frac{\partial}{\partial\vec{w}}\cdot\frac{1}{w^2}\left(\hat{e}_\theta\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)f_e$$

$$= \frac{1}{2}\cancel{w^3}\frac{1}{\cancel{w}\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\frac{1}{\cancel{w^2}}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}\frac{1}{\cancel{w}\sin\theta}\frac{\partial}{\partial\phi}\right]f_e$$

$$= \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]f_e \qquad (93)$$

$\xi = \cos\theta$

$$= \frac{1}{2}\left(\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right)f_e$$

$$\vec{w} = w\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}$$

The effect of this operator is to isotropise electrons (in the rest frame of the ions) — again, as expected.

To see this formally, we can prove a kind of H-theorem for this operator: clearly, if we say that

$$f_e = \underset{\text{isotropic}}{f_0(w)} + \delta f(\vec{w}), \qquad (94)$$

then f_0 will not be changed by (93) and we have

$$\left(\frac{\partial\,\delta f}{\partial t}\right)_c = \nu_D(w)\,\hat{\mathcal{L}}[\delta f] \qquad (95)$$

The change of the norm of δf due to e-i collisions is

$$\frac{d}{dt}\int d\vec{w}\,\delta f^2 = 2\int d\vec{w}\,\delta f\,\hat{\mathcal{L}}[\delta f] =$$

$$= \int \underbrace{d\vec{w}}_{\hookrightarrow\, dw\, w^2\, d\xi\, d\phi}\, \nu_D(w)\,\delta f\left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2}{\partial\phi^2}\right]\delta f$$

$$= -\int dw\, w^2 \nu_D(w) \iint d\xi\, d\phi \left[(1-\xi^2)\left(\frac{\partial\delta f}{\partial\xi}\right)^2 + \frac{1}{1-\xi^2}\left(\frac{\partial\delta f}{\partial\phi}\right)^2\right]$$

$$= 0 \text{ iff } \frac{\partial\delta f}{\partial\xi} = 0 \text{ and } \frac{\partial\delta f}{\partial\phi} = 0 \qquad (96)$$

Note that you can obtain the same result by asking at which point the electron entropy $S_e = -\int d\vec{w}\, f_e \ln f_e$ will stop being increased by the operator.

Thus, e-i collisions will isotropize f_e at the rate

$$\nu_D(w) = \frac{4\pi Z e^4 n_e \Lambda_{ei}}{m_e^2 w^3} \sim \frac{Z e^4 n_e \Lambda_{ei}}{m_e^2 v_{the}^3} \qquad (97)$$

$$\sim \frac{Z e^4 n_e \Lambda_{ei}}{m_e^{1/2} T_e^{3/2}} \sim \nu_{ei} \text{ according to (74).}$$

The e-e collisions will help at almost the same rate, by pushing f_e towards a Maxwellian with some mean flow $\vec{u}_e$ — but that is of course only isotropic in the frame moving with $\vec{u}_i$ if $\vec{u}_e = \vec{u}_i$. This means that our next order of business will be to show that e-i collisions push the e and i mean-flow velocities towards each other.

Exercise 2. Work out what happens if the plasma contains many ion species – all heavier than electrons, but with different densities and different masses.

3.4 Electron-ion friction force

There are no w derivatives in (93), so energy is conserved by this operator (there is, of course, e-i energy exchange, but it comes in at ~~[illegible]~~ $O(m_e/m_i)$, with the restoration of ② in (85) – ion drag on electrons; we will compute it later, when we study i-e collisions)

This is not so with momentum – obviously, when electrons bounce off ions, they deposit momentum! Let us then calculate the friction force on electrons due to e-i collisions (which is the only collisional friction on them because e-e collisions conserve momentum, obviously):

$$\vec{R}_{ei} = \int d\vec{v}\, m_e \vec{v} \left(\frac{\partial f_e}{\partial t}\right)_i = \int d\vec{w}\, m_e (\vec{w} + \cancel{\vec{u}_i}) \left(\frac{\partial f_e}{\partial t}\right)_i$$

0 because $\int d\vec{w} \left(\frac{\partial f_e}{\partial t}\right)_i = 0$

$$= m_e \int d\vec{w}\, \nu_D(w) \vec{w}\, \hat{\mathcal{L}}[\delta f] \quad \leftarrow \text{use (91)}$$

$$= m_e \int d\vec{w}\, \overbrace{\nu_D(w) \vec{w}}^{(w^3)} \frac{1}{2} \frac{\partial}{\partial \vec{w}} \cdot \frac{1}{w} \left(\mathbb{1} - \frac{\vec{w}\vec{w}}{w^2}\right) \cdot \frac{\partial f_e}{\partial \vec{w}}$$

~~[illegible]~~

by parts →

$$= -\frac{m_e}{2} \int d\vec{w}\, \nu_D(w)\, w^2 \left(\mathbb{1} - \frac{\vec{w}\vec{w}}{w^2}\right) \cdot \frac{\partial f_e}{\partial \vec{w}}$$

← NB: projector annihilates all other terms by parts again

$$= \frac{m_e}{2}\int d\vec{w}\, f_e \frac{\partial}{\partial w_j} \underbrace{\nu_D(w)\, w^2}_{\shortparallel} \left(\delta_{ij} - \frac{w_i w_j}{w^2}\right) =$$

$$\boxed{\underbrace{\nu_D(w)\, w^3}_{\text{independent of } w} \cdot \frac{1}{w}}$$

$$= \frac{m_e}{2}\int d\vec{w}\, f_e\, \nu_D(w)\, w^3 \frac{\partial}{\partial w_j} \frac{1}{w}\left(\delta_{ij} - \frac{w_i w_j}{w^2}\right)$$

$$= \frac{m_e}{2}\int d\vec{w}\, f_e\, \nu_D(w)\, w^3 \left[-\frac{w_i}{w^3} - \frac{(\cancel{3}+1)\, w_i}{w^3} + w_i w_j \frac{\cancel{3} w_j}{w^5}\right]$$

$$= - m_e \int d\vec{w}\, f_e\, \nu_D(w)\, \vec{w} \qquad (98)$$

↑ Suppose this is isotropic in the frame of mean electron flow (e.g. a local Maxwellian), i.e

$$f_e = f_e(|\vec{v} - \vec{u}_e|) = f_e(|\vec{w} + \vec{u}_i - \vec{u}_e|) \qquad (99)$$

Then

$$\vec{R}_{ei} = - m_e \int d\vec{w}\, \nu_D(w)\, \vec{w} \cdot \left[\underbrace{f_e(\vec{w})}_{0} + (\vec{u}_i - \vec{u}_e)\cdot \frac{\vec{w}}{w}\frac{\partial f_e}{\partial w}\right]$$

$$= - m_e (\vec{u}_i - \vec{u}_e) \cdot \int d\vec{w}\, \nu_D(w) \frac{\vec{w}\vec{w}}{w} \frac{\partial f_e}{\partial w}$$

$$= m_e(\vec{u}_e - \vec{u}_i)\cdot \int_0^\infty dw\, \underbrace{w^3 \nu_D(w)}_{\text{independent of } w} \frac{\partial f_e}{\partial w} \underbrace{\overset{\text{angles}}{\int d\Omega}\, \frac{\vec{w}\vec{w}}{w^2}}_{\overset{\shortparallel}{\frac{1}{3}\mathbb{1}\cdot 4\pi}}$$

$$= -\frac{(4\pi)^2 Z e^4 n_e \Lambda_{ei}}{3 m_e} (\vec{u}_e - \vec{u}_i)\, f_e(0)$$

$$\hookrightarrow \frac{n_e}{(\pi v_{the}^2)^{3/2}} = \frac{n_e m_e^{3/2}}{(2\pi T_e)^{3/2}}$$

if Maxwellian

$$\boxed{\vec{R}_{ei} = -m_e n_e \underbrace{\frac{4\sqrt{2\pi}}{3}\, \frac{Z e^4 n_e \Lambda_{ei}}{m_e^{1/2} T_e^{3/2}}}_{\substack{\equiv \\ \nu_{ei}}} (\vec{u}_e - \vec{u}_i)} \qquad (100)$$

ν_{ei} official definition (=, rather than ~)

So, again unsurprisingly (lack of surprises is the name of the game!), the friction force pushes electron mean flow towards ion mean flow, ~~at~~ the ν_{ei} rate.

(NB: Exercise 2 can continue here.)

3.5 Resistivity

The above calculation can give us immediate insight into a meaningful physical problem: what is the resistivity of a (homogeneous) plasma?

By definition, $\vec{E} = \underset{\uparrow\text{ resistivity}}{\eta} \vec{j}$, (101)

where $\vec{j} = Zen_i\vec{u}_i - en_e\vec{u}_e = en_e(\vec{u}_i - \vec{u}_e)$ (102)

$= \frac{e \cancel{n_e}}{m_e \cancel{n_e} \nu_{ei}} R_{ei}$ according to (100).

If we now demand a steady state — so a force balance between electic force on electrons and friction, we get

$$-en_e\vec{E}+\vec{R}_{ei}=0 \Rightarrow \vec{R}_{ei}=en_e\vec{E} \qquad (103)$$

and so

$$\boxed{\vec{E}=\underbrace{\frac{m_e\nu_{ei}}{e^2 n_e}}_{\shortparallel}\vec{j}} \qquad (104)$$

η easy enough.

There is, however a nuance: (100) for R_{ei} was based on (99), which, really, followed from nowhere, so, technically, $\vec{j}\underset{(98)}{=}en_e(\vec{u}_i-\vec{u}_e)=-e\int d\vec{w}\,\vec{w}f_e$

[illegible] (105)

and f_e here still has to be calculated from the electron kinetic equation: in steady state and ignoring spatial gradients of anything,

$$-\frac{e}{m_e}\vec{E}\cdot\frac{\partial f_e}{\partial\vec{v}}=\left(\frac{\partial f_e}{\partial t}\right)_e+\left(\frac{\partial f_e}{\partial t}\right)_i \qquad (106)$$

Solving (106) [illegible] for f_e and then using (105) to infer η is called the Spitzer–Härm problem. This takes into account the fact that, under the action of $\vec{E}$, electrons are not quite as isotropic and Maxwellian as assumed in §3.4 — they are to lowest order (in small $\vec{E}$), and so it is a perturbative calculation, which leads to an order-unity correction of η (which depends on Z and involves ν_{ee} as well as ν_{ei} — ~~it~~ the resulting prefactor varies roughly between 0.5 and 0.3). The calculation is long — you will find two different versions of it in Kunz's and Parra's notes (and the original in the Spitzer & Härm paper). Here I will do a baby version of it ignoring e-e collisions and keeping only the Lorentz operator for e-i collisions — this is quantitatively correct for $Z \gg 1$ (when $\nu_{ei} \gg \nu_{ee}$) — plutonium plasma!

$$-\frac{e}{m_e}\vec{E}\cdot\frac{\partial f_e}{\partial \vec{w}} \simeq \nu_D(w)\hat{\mathcal{L}}[f_e] \qquad (107)$$

Let $f_e = \underbrace{f_0(w)}_{\text{isotropic}} + \delta f(\vec{w})$ and assume $\vec{E}$ small,

viz. $\frac{e}{m_e}E\frac{1}{v_{the}} \ll \nu_{ei} \sim \frac{Ze^4 n_e \Lambda_{ei}}{m_e^{1/2}T_e^{3/2}}$

$$\text{or } E \ll \frac{Ze^3 n_e \Lambda_{ei}}{T_e} \sim \frac{e\Lambda_{ei}Z}{\lambda_{De}^2} \sim E_D \qquad (108)$$

"Dreicer field"

Then

$$-\frac{e}{m_e}\vec{E}\cdot\frac{\partial f_0}{\partial \vec{w}} \simeq \nu_D(w)\hat{\mathcal{L}}[\delta f] \qquad (109)$$

(we shall return later to what happens if $E \gtrsim E_D$)

$$-\frac{e}{m_e}\vec{E}\cdot\frac{\vec{w}}{w}\frac{\partial f_0}{\partial w} = \frac{1}{2}\nu_D(w)\,w^3\,\frac{\partial}{\partial\vec{w}}\cdot\frac{1}{w}\left(\mathbb{1}-\frac{\vec{w}\vec{w}}{w^2}\right)\cdot\frac{\partial\delta f}{\partial\vec{w}} \qquad (110)$$

This is easily integrated. You can work out for yourselves how to show that this solution is the only sensible solution:

$$\delta f = C(w)\,\vec{w}\cdot E \qquad (111)$$

Then,

$$\text{rhs of } (110) = \frac{1}{2}\nu_D(w)\,w^3\,\frac{\partial}{\partial\vec{w}}\cdot\frac{1}{w}\left(\mathbb{1}-\frac{\vec{w}\vec{w}}{w^2}\right)\cdot\Big[\vec{E}C(w) + \underbrace{\vec{w}\cdot\vec{E}\,\frac{\cancel{\vec{w}}}{w}\frac{\partial C}{\partial w}}_{\text{annihilated by projector}}\Big]$$

$$= \frac{1}{2}\nu_D(w)\,w^3\,\vec{E}\cdot\Big[\frac{\partial}{\partial\vec{w}}\cdot\underbrace{\frac{C(w)}{w}}_{\text{derivative of this annihilated by projector}}\left(\mathbb{1}-\frac{\vec{w}\vec{w}}{w^2}\right)\Big]$$

$$= -\frac{1}{2}\nu_D(w)\,\cancel{w^2}\,C(w)\,\vec{E}\cdot\left[3\frac{\vec{w}}{\cancel{w^2}}+\frac{\vec{w}}{\cancel{w^2}}-2\vec{w}\frac{1}{\cancel{w^2}}\right] =$$

$$= -\nu_D(w)\,C(w)\,\cancel{\vec{E}\cdot\vec{w}} = \text{lhs of } (110) = -\cancel{\vec{E}\cdot\vec{w}}\,\frac{e}{m_e}\frac{1}{w}\frac{\partial f_0}{\partial w}$$

$$\hookrightarrow C(w) = \frac{e}{m_e\nu_D(w)}\frac{1}{w}\frac{\partial f_0}{\partial w}$$

$$\delta f = \frac{e}{m_e\nu_D(w)}\vec{E}\cdot\frac{\vec{w}}{w}\frac{\partial f_0}{\partial w} \qquad (112)$$

Substitute this into (105):

$$J = -e\int d\vec{w}\,\vec{w}\,\frac{e}{m_e\nu_D(w)}\vec{E}\cdot\frac{\vec{w}}{w}\frac{\partial f_0}{\partial w}$$

$$= -\frac{e^2\vec{E}}{m_e}\cdot\int d\vec{w}\,\underbrace{\frac{\vec{w}\vec{w}}{w^2}}_{\text{average over angles } \frac{1}{3}\mathbb{1}\cdot 4\pi}\frac{w}{\nu_D(w)}\frac{\partial f_0}{\partial w}$$

$$= -\frac{\cancel{4\pi} e^2}{3m_e}\vec{E}\int_0^\infty dw\,\frac{w^3}{\nu_D(w)}\frac{\partial f_0}{\partial w} =$$

$$\nu_D(w) = \frac{\cancel{4\pi} Z e^4 n_e \Lambda_{ei}}{m_e^2 w^3} = \frac{3\sqrt{\pi}}{4}\nu_{ei}\frac{v_{the}^3}{w^3}$$

as defined in (100)

$$= -\frac{16\sqrt{\pi}e^2}{9m_e\nu_{ei}}\vec{E}\int_0^\infty dw\,\frac{w^6}{v_{the}^3}\frac{\partial f_0}{\partial w} =$$

by parts

$$= +\frac{32}{3}\frac{\sqrt{\pi}e^2}{m_e\nu_{ei}}\vec{E}\int_0^\infty dw\,\frac{w^5}{v_{the}^3}f_0$$

if Maxwellian, $\frac{n_e e^{-w^2/v_{the}^2}}{v_{the}^3\pi^{3/2}}$

$$= \frac{32}{3\pi}\frac{e^2 n_e}{m_e\nu_{ei}}\vec{E}\underbrace{\int_0^\infty dx\,x^5 e^{-x^2}}_{=\frac{1}{2}\int_0^\infty dy\,y^2 e^{-y} = 1} \qquad (113)$$

Thus,

$$\boxed{\vec{E} = \underbrace{\frac{3\pi}{32}\frac{m_e\nu_{ei}}{e^2 n_e}}_{=\eta}\vec{j}} \qquad (114) \quad \text{cf. } (104)$$

η, so an order-unity correction, as promised. The prefactor < 1 because the effect of the electric field is to accelerate electrons and make the current flow a bit more vigorously (larger conductivity)

The effect of e-e collisions at finite Z is to decelerate them again, increasing resistivity a bit – that's what Spitzer & Härm solved at the price of great suffering but greater virtue.

NB: This is the first example of transport theory – collisional responce to imposed fields (or gradients). More on that in due course!

3.6 Runaway Electrons

We can now milk the friction-force calculation for another interesting nugget of physics.

Recall (98):

$$\vec{R}_{ei} = -m_e \int d\vec{w}\, \nu_D(w)\, \vec{w}\, f_e$$

On p.53, I went on to calculate $\vec{R}_{ei}$ by assuming that f_e is an isotropic (and, later, Maxwellian) distribution with a mean flow $\vec{u}_e \sim \vec{u}_i \ll v_{the}$ (so an expansion could be done in $\vec{u}_e - \vec{u}_i$).

Now let us consider what might happen if $\vec{u}_e \sim v_{the} \gg \vec{u}_i$ (i.e., electrons are a kind of beam).

$$f_e = f_e(|\underbrace{\vec{v}-\vec{u}_e}_{\equiv\, \vec{w}'}|), \quad \vec{w} = \vec{v} - \vec{u}_i = \vec{w}' + \vec{u}_e - \underset{\uparrow\ \text{can be neglected everywhere}}{\vec{u}_i} \qquad (115)$$

Then

$$\vec{R}_{ei} = -m_e \int d\vec{w}'\, \nu_D(|\vec{w}'+\vec{u}_e|)\,(\vec{w}'+\vec{u}_e)\, f_e(w')$$

$$= -\frac{4\pi Z e^4 n_e \lambda_{ei}}{m_e} \underbrace{\int d\vec{w}'\, \frac{\vec{w}'+\vec{u}_e}{|\vec{w}'+\vec{u}_e|^3}\, f_e(w')}$$

$$= \int_0^\infty dw' w'^2 \int_{-1}^{1} d\xi \int_0^{2\pi} d\phi\, \frac{\overset{\text{only compont} \parallel \vec{u}_e \text{ survives}}{\vec{w}' + \vec{u}_e}}{(w'^2+u_e^2+2u_e w'\xi)^{3/2}}\, f_e(w')$$

↑ angle θ is between $\vec{w}'$ and $\vec{u}_e$

$$= 2\pi \frac{\vec{u}_e}{u_e} \int_0^\infty dw' w'^2 f_e(w') \int_{-1}^{1} d\xi\, \frac{w'\xi + u_e}{(w'^2+u_e^2+2u_e w'\xi)^{3/2}}$$

$$\overset{\uparrow}{=}\; 2\pi \vec{u}_e \int_0^\infty dx\, x^2 f_e(x u_e) \underbrace{\int_{-1}^{1} d\xi \frac{x\xi+1}{(x^2+1+2x\xi)^{3/2}}}$$

$$x = \frac{w'}{u_e}$$

$$= \frac{1}{2}\int_{-1}^{1} d\xi \frac{2x\xi+1+x^2+1-x^2}{(x^2+1+2x\xi)^{3/2}} =$$

$$= \frac{1}{2}\int_{-1}^{1} d\xi \left[\frac{1}{\sqrt{x^2+1+2x\xi}} + \frac{1-x^2}{(x^2+1+2x\xi)^{3/2}}\right]$$

$$\overset{\nearrow}{=} \frac{1}{4x}\int_{(x-1)^2}^{(x+1)^2} dy \left(\frac{1}{\sqrt{y}} + \frac{1-x^2}{y^{3/2}}\right) = \frac{1}{2x}\left(\sqrt{y} - \frac{1-x^2}{\sqrt{y}}\right)\Bigg|_{(x-1)^2}^{(x+1)^2}$$

$$y = 2\xi+1+x^2$$

$$= \frac{1}{2x}\Big[\underbrace{x+1-|x-1| - (1-x^2)\left(\frac{1}{x+1} - \frac{1}{|x-1|}\right)}\Big]$$

$$\parallel$$

$$\cancel{x}+1-\cancel{x}+1-(1-x^2)\frac{\cancel{x}-1-\cancel{x}-1}{x^2-1} = 0 \text{ if } x \geqslant 1$$

$$x+\cancel{1}-\cancel{1}+x-(\cancel{1-x^2})\frac{\cancel{1}-x-\cancel{1}-x}{\cancel{1-x^2}} = 4x \text{ if } x<1$$

$$= 2\cdot H(1-x) \qquad (116)$$

So then

$$\vec{R}_{ei} = -\frac{16\pi^2 Z e^4 n_e \Lambda_{ei}}{m_e}\vec{u}_e \underbrace{\int_0^1 dx\, x^2 f_e(xu_e)}_{\parallel \atop \frac{1}{u_e^3}\int_0^{u_e} dw'\, w'^2 f_e(w')} \qquad (117)$$

Let $f_e(w')$ be a Maxwellian. Then

$$\int_0^{u_e} dw'\, w'^2 f_e(w') = \int_0^{u_e} dw'\, w'^2 \frac{n_e\, e^{-w'^2/v_{the}^2}}{(\pi v_{the}^2)^{3/2}} =$$

$$= n_e \int_0^{u_e/v_{the}} \frac{dx\,x^2}{\pi^{3/2}}\, e^{-x^2} = \frac{n_e}{\pi^{3/2}} \left(-\frac{\partial}{\partial\lambda}\right) \int_0^{u_e/v_{the}} dx\, e^{-\lambda x^2} \Big|_{\lambda=1}$$

$$= \frac{n_e}{\pi^{3/2}} \left(-\frac{\partial}{\partial\lambda} \frac{1}{\sqrt{\lambda}} \int_0^{\sqrt{\lambda}\, u_e/v_{the}} dy\, e^{-y^2}\right)_{\lambda=1} =$$

$$= \frac{n_e}{2\pi} \left[-\frac{\partial}{\partial\lambda} \frac{1}{\sqrt{\lambda}} \mathrm{erf}\left(\frac{\sqrt{\lambda}\, u_e}{v_{the}}\right)\right]_{\lambda=1} =$$

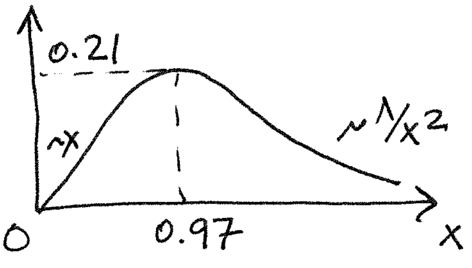
$$= \frac{n_e}{2\pi} \left[\frac{1}{2} \mathrm{erf}\left(\frac{u_e}{v_{the}}\right) - \frac{2}{\sqrt{\pi}} e^{-u_e^2/v_{the}^2} \frac{u_e}{v_{the}} \frac{1}{2}\right] \qquad (118)$$

Thus,

$$\vec{R}_{ei} = -\frac{4\pi Z e^4 n_e^2 \Lambda_{ei}}{m_e} \underbrace{\frac{\vec{u}_e}{u_e^3} \left[\mathrm{erf}\left(\frac{u_e}{v_{the}}\right) - \frac{2}{\sqrt{\pi}} \frac{u_e}{v_{the}} e^{-u_e^2/v_{the}^2}\right]}_{\begin{gathered}\parallel \\ \frac{2}{v_{the}^2} \frac{\vec{u}_e}{u_e} G\left(\frac{u_e}{v_{the}}\right)\end{gathered}} \qquad (119)$$

$$= -\frac{4\pi Z e^4 n_e^2 \Lambda_{ei}}{T_e} \frac{\vec{u}_e}{u_e} G\left(\frac{u_e}{v_{the}}\right)$$

$$G(x) = \frac{1}{2x^2}\left[\mathrm{erf}(x) - \frac{2x}{\sqrt{\pi}} e^{-x^2}\right]$$



This is called Chadrasekher's function.

Consider now the electron momentum equation for such a situation (assume everything is spatially homogeneous):

$$\int m_e \vec{v} \left| \frac{\partial f_e}{\partial t} + \underbrace{\vec{v}\cdot\cancel{\nabla} f_e}_{\text{ignore}} - \frac{e}{m_e}\vec{E}\cdot\frac{\partial f_e}{\partial \vec{v}} = \left(\frac{\partial f_e}{\partial t}\right)_i + \underbrace{\left(\frac{\partial f_e}{\partial t}\right)_e}_{\text{conserves momentum}} \right. \quad (120)$$

$$m_e n_e \frac{\partial \vec{u}_e}{\partial t} + e n_e \vec{E} = \vec{R}_{ei}$$

$$= -\underbrace{\frac{4\pi Z e^4 n_e^2 \Lambda_{ei}}{T_e}}_{\equiv\ e n_e E_D} \frac{\vec{u}_e}{u_e} G\left(\frac{u_e}{v_{the}}\right) \quad (121)$$

$e n_e E_D \leftarrow$ Dreicer field

$E_D \sim \frac{\nu_{ei} v_{the} m_e}{e}$ cf. (108) (p. 56)

Let us say the reason $\vec{u}_e$ is there is that $\vec{E}$ has accelerated electrons, i.e., $\vec{u}_e$ is in the $-\vec{E}$ direction.

So along that direction,

$$\frac{\partial u_e}{\partial t} = \frac{e}{m_e} E - \frac{e}{m_e} E_D G\left(\frac{u_e}{v_{the}}\right) \quad (122)$$

Steady state is only possible if

$$E < E_D \cdot \underbrace{\max G}_{\simeq\ 0.21} \quad (123)$$

Otherwise, whatever u_e, the rhs of (122) is > 0 and u_e will keep getting accelerated.

This is called (thermal) runaway electrons.

Physically, this happens because the ei collision frequency $\nu_D \propto \frac{1}{v^3}$, so the faster the electrons go the less they collide ~~[illegible]~~ and so the less friction with ions they experience — an unstable situation.

In fact, even for small E, sufficiently high-energy electrons can run away via the same physics:

$$\nu_D(v)\, v \lesssim \frac{e}{m_e} E$$

$$\sim \nu_{ei} \frac{v_{the}^3}{v^3} v \quad \Rightarrow \quad \frac{v_{the}^2}{v^2} \lesssim \frac{e}{m_e} \frac{E}{v_{the}\nu_{ei}} = \frac{E}{E_D}$$

$$\text{or} \quad \frac{v}{v_{the}} \gtrsim \sqrt{\frac{E_D}{E}} \qquad (124)$$

This is very bad for tokamaks: disruptions can create situations where electric field along field lines accelerates electrons to relativistic speeds and they can punch a hole through the hull. So studying these is a thing in fusion physics! (And one of the few cases where you can study relativistic particles in fusion context.)

3.7 Ion-Electron Collisions

Let us go back to (82) and study how a heavy species collides with a light one:

$$\left(\frac{\partial f_i}{\partial t}\right)_e = \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_i}\frac{\partial}{\partial \vec{v}}\cdot\int d\vec{v}''\,\hat{U}(\vec{w})\cdot\Big[\underbrace{\frac{f_e(\vec{v}'')}{m_i}\frac{\partial f_i}{\partial \vec{v}}}_{①} - \underbrace{\frac{f_i(\vec{v})}{m_e}\frac{\partial f_e}{\partial \vec{v}''}}_{②}\Big] \qquad (125)$$

$$= \frac{\partial}{\partial \vec{v}}\cdot\Big[\underbrace{\hat{D}(\vec{v})\cdot\frac{\partial f_i}{\partial \vec{v}}}_{①} - \underbrace{\vec{A}(\vec{v})f_i}_{②}\Big] \quad \text{FP form}$$

3.7.1 Diffusion

Since $\frac{①}{②} \sim \sqrt{\frac{m_e}{m_i}}$ à priori, the diffusion term requires less precicision in its calculation: recall that, to lowest order in the same expansion, $f_e(\vec{v}'') = f_e(v'')$ just isotropic — the fact that it is isotropic in the ion frame comes in at the next order (§3.3).

So let us just use that in ① and see what it becomes:

$$\hat{D}(\vec{v}) = \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_i^2}\int d\vec{v}''\,\hat{U}(\vec{w})\,f_e(v'') \qquad (126)$$

$\uparrow$ $\vec{v}-\vec{v}'' \approx -\vec{v}''$ because $\vec{v}$ is $\sim v_{thi}$ while $\vec{v}'' \sim v_{the}$

so all $\vec{v}$ dependence is lost to lowest order and

$$\int d\vec{v}''\,\hat{U}(\vec{v}'')\,f_e(v'') = \int d\vec{v}''\frac{f_e(v'')}{v''}\Big(\mathbb{1} - \underbrace{\frac{\vec{v}''\vec{v}''}{v''^2}}_{\hookrightarrow \frac{1}{3}\mathbb{1} \text{ after angle averaging}}\Big)$$

$$= \frac{8\pi}{3}\mathbb{1}\underbrace{\int_0^\infty dv''\,v''\,f_e(v'')}_{= \text{ if } f_e \text{ is Maxwellian}} \qquad (127)$$

$$\frac{1}{2}\int_0^\infty dv''^2\,\frac{n_e}{(v_{the}^2\pi)^{3/2}}e^{-v''^2/v_{the}^2} = \frac{n_e}{2\pi^{3/2}v_{the}}$$

Thus,

one of these = n_e/n_i

see (79)

$$\hat{D}(\vec{v}) = \frac{\cancel{2}\pi Z^2 e^4 \Lambda_{ei}}{m_i^2} \cdot \frac{8\pi}{3} \frac{n_e m_e^{1/2}}{\cancel{2}\pi^{3/2} T_e^{1/2}\sqrt{2}}$$

$$= \underbrace{\frac{4\sqrt{2\pi}}{3}\frac{Z e^4 \Lambda_{ei} n_e}{m_e^{1/2} T_e^{3/2}}}_{\substack{\parallel \\ \nu_{ei} \text{ as officially defined in (100) (p.54)}}} \frac{n_e m_e}{n_i m_i}\frac{T_e}{m_i} = \overset{\nu_{ie}}{\overset{\equiv}{\boxed{\frac{n_e m_e}{\cancel{n}_i m_i}\nu_{ei}}}}\frac{T_e}{m_i} \qquad (128)$$

It is not hard to make sense physically of the result that ions, under bombardment by electrons, diffuse at the rate (128).

Each collision conserves momentum, so the ion's velocity changes by

$$\Delta\vec{v}_i = -\frac{m_e}{m_i}\Delta\vec{v}_e \sim \frac{m_e}{m_i} v_{the} \sim \sqrt{\frac{m_e}{m_i}\frac{T_e}{T_i}}\, v_{thi} \ll v_{thi} \qquad (129)$$

↑ electron is deflected strongly

This is a small kick, and such kicks will accumulate as a random walk. The time between kicks is

$$\Delta t \sim \frac{1}{\underset{\text{time for electron to collide}}{\nu_{ei}}\ \underset{\text{\# of electrons per ion}}{Z}} \qquad (130)$$

So the diffusion coefficient is

$$D \sim \frac{\Delta v_i^2}{\Delta t} \sim \frac{m_e}{m_i}\frac{T_e}{\cancel{T_i}}\underset{\frac{\cancel{T_i}}{m_i}}{\underset{\simeq}{v_{thi}^2}}\,\nu_{ei}\underset{\frac{n_e}{n_i}}{\underset{\parallel}{Z}} \sim \frac{m_e n_e}{m_i n_i}\nu_{ei}\frac{T_e}{m_i} \qquad (131)$$

q.e.d.

3.7.2 Drag

Now let us look at the drag term ② in (125):

$$\vec{A}(\vec{v}) = \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_i m_e} \int d\vec{v}'' \, \hat{U}(\vec{v}-\vec{v}'') \cdot \frac{\partial f_e}{\partial \vec{v}''} \qquad (132)$$

Let us manipulate a little:

$$\hat{U}(\vec{v}-\vec{v}'') = \hat{U}(\vec{v}''-\vec{v}) = \hat{U}(\overbrace{\vec{v}''-\vec{u}_i}^{\equiv \vec{w}} - (\vec{v}-\vec{u}_i))$$

↑ because I know how to deal with fe in the ion frame

$$\approx \hat{U}(\underset{\substack{\uparrow \\ \text{new } \vec{w}!}}{\vec{w}}) - (\vec{v}-\vec{u}_i) \cdot \frac{\partial}{\partial \vec{w}} \hat{U}(\vec{w}) \qquad (133)$$

Then the integral in (132) has two terms:

$$\int d\vec{v}'' \, \hat{U}(\vec{w}) \cdot \frac{\partial f_e}{\partial \vec{v}''} \underset{\substack{\uparrow \\ \text{by parts}}}{=} -\int d\vec{v}'' f_e \underbrace{\frac{\partial}{\partial \vec{v}''} \cdot \hat{U}(\vec{w})}_{\parallel} =$$

$$\boxed{\frac{\partial}{\partial \vec{w}} \cdot \hat{U}(\vec{w}) = -\frac{2\vec{w}}{w^3} \quad \text{computed on p. 53}}$$

$$= 2 \int d\vec{v}'' \frac{\vec{w}}{w^3} f_e \qquad (134)$$

To lowest order, fe is isotropic, $\vec{w} \approx \vec{v}''$ and so this is zero, ~~but~~ so we need to keep the next order, but it is the exact same integral as in the e-i friction force, (98) (this is why I wanted $\vec{w} = \vec{v}'' - \vec{u}_i$)

$$\text{1st term of } \vec{A} = \frac{4\pi Z^2 e^4 \Lambda_{ei}}{m_i m_e} \int d\vec{v}'' \frac{\vec{w}}{w^3} f_e =$$

$$= \frac{m_e}{m_i n_i} \int d\vec{v}'' \underbrace{\frac{4\pi Z^2 e^4 n_e \Lambda_{ei}}{m_e^2 w^3}}_{\substack{\parallel \\ \nu_D(w) \text{ defined in (97)}}} \vec{w} f_e = -\frac{\vec{R}_{ei}}{m_i n_i} \qquad (135)$$

In contrast, the 2nd term of $\vec{A}$ is already small, so only the isotropic part of fe needs to be kept:

2nd term of $\vec{A} = -\frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_i m_e} \int d\vec{v}''(\vec{v}-\vec{u}_i)\cdot\left[\underbrace{\frac{\partial}{\partial\vec{w}}}\hat{U}(\vec{w})\right]\cdot\underbrace{\frac{\partial f_e}{\partial\vec{v}''}}_{\frac{\vec{v}''}{v''}\frac{\partial f_e}{\partial v''}} =$

$$\hookrightarrow \frac{\partial}{\partial w_k}\frac{1}{w}\left(\delta_{ij}-\frac{w_i w_j}{w^2}\right) =$$

$$= -\frac{w_k}{w^3}\delta_{ij} - \frac{w_j}{w^3}\delta_{ik} - \frac{w_i}{w^3}\delta_{jk} + 3\frac{w_i w_j w_k}{w^5}$$

Let $\vec{w} \approx \vec{v}''$ here and dot with v_j'';

get $-\left(\delta_{ik} - \frac{v_i'' v_k''}{v''^2}\right)\frac{1}{v''} = -\hat{U}(\vec{v}'')$

$$\approx \frac{2\pi Z^2 e^4 \Lambda_{ei}}{m_i m_e}(\vec{v}-\vec{u}_i)\cdot\int d\vec{v}''\underbrace{\hat{U}(\vec{v}'')}\frac{1}{v''}\frac{\partial f_e}{\partial v''}$$

$\hookrightarrow \frac{2}{3}\mathbb{1}\frac{1}{v''}$ after angle averaging

$$= \frac{4\pi}{3}\frac{Z^2 e^4 \Lambda_{ei}}{m_i m_e}(\vec{v}-\vec{u}_i)\underbrace{\int d\vec{v}''\frac{1}{v''^2}\frac{\partial f_e}{\partial v''}}_{\overset{\shortparallel}{4\pi\int_0^\infty dv''\frac{\partial f_e}{\partial v''}} = -4\pi f_e(0)}$$

$$= -\frac{16\pi^2}{3}\frac{Z^2 e^4 \Lambda_{ei}}{m_i m_e}(\vec{v}-\vec{u}_i)\underbrace{f_e(0)}_{\overset{\shortparallel}{\frac{n_e}{\pi^{3/2}v_{the}^3}}}$$

if Maxwellian

$$= -\frac{4\sqrt{2\pi}}{3}\frac{n_e m_e}{n_i m_i}\frac{Z e^4 n_e \Lambda_{ei}}{m_e^{1/2}T_e^{3/2}}(\vec{v}-\vec{u}_i)$$

= [redacted] $-\boxed{\frac{n_e m_e}{n_i m_i}\overset{=\nu_{ie}}{\nu_{ei}}}(\vec{v}-\vec{u}_i)$ (136)

You know you have got it right when the answer comes out so neat and simple.

OK, assemble from (125), (128), (135) & (136):

$$\boxed{\left(\frac{\partial f_i}{\partial t}\right)_e \approx \nu_{ie}\frac{\partial}{\partial \vec{v}}\cdot\left[\underbrace{\frac{T_e}{m_i}\frac{\partial f_i}{\partial \vec{v}}}_{\text{ion diffusion discussed on p.65)}} + \underbrace{(\vec{v}-\vec{u}_i)f_i}_{\text{drag pushing ions towards their mean velocity}}\right] + \underbrace{\frac{\vec{R}_{ei}}{m_i n_i}\cdot\frac{\partial f_i}{\partial \vec{v}}}_{\text{ion-electron friction force pushing ion mean velocity to electron mean velocity}}} \qquad (137)$$

$$\vec{R}_{ie} = \int d\vec{v}\, \cancel{m_i}\vec{v}\, \frac{\vec{R}_{ei}}{\cancel{m_i} n_i}\cdot\frac{\partial f_i}{\partial \vec{v}}$$

$$= -\vec{R}_{ei}\frac{1}{n_i}\int d\vec{v} f_i = -\vec{R}_{ei} \quad \text{as ought be the case}$$

(the rest of the operator exerts no force)

3.8 Ion-Electron Collisional Energy Exchange

The main new effect that (137) allows us to calculate is the rate at which energy is exchanged between electrons and ions. This has two parts, frictional and diffusive.

3.8.1 Frictional Work

First,

$$Q_{ie,\,\text{frictional}} = \int d\vec{v}\, \frac{\cancel{m_i} v^2}{2}\, \frac{\vec{R}_{ei}}{\cancel{m_i} n_i}\cdot\frac{\partial f_i}{\partial \vec{v}} \overset{\text{by parts}}{=}$$

$$= -\vec{R}_{ei}\cdot\int\frac{d\vec{v}}{n_i}\vec{v} f_i = -\vec{R}_{ei}\cdot\vec{u}_i \qquad (138)$$

Energy loss by ions due friction force doing work.

3.8.2 Temperature Equilibration

Secondly,

$$Q_{ie,\,diff} = \nu_{ie}\int d\vec{v}\,\frac{m_i v^2}{2}\frac{\partial}{\partial\vec{v}}\cdot\left[\frac{T_e}{m_i}\frac{\partial f_i}{\partial\vec{v}} + (\vec{v}-\vec{u}_i)f_i\right]$$

$$= -\nu_{ie}\int d\vec{v}\left[T_e\,\vec{v}\cdot\frac{\partial f_i}{\partial\vec{v}} + m_i\vec{v}\cdot(\vec{v}-\vec{u}_i)f_i\right]$$

$$= 3\nu_{ie}n_iT_e - \nu_{ie}\int d\vec{v}\,m_i|\vec{v}-\vec{u}_i|^2 f_i$$

because $\int d\vec{v}\,(\vec{v}-\vec{u}_i)f_i = 0$ by definition of $\vec{u}_i$

$$= -3n_i\nu_{ie}(T_i - T_e) \qquad (139)$$

taking f_i to be Maxwellian (which it will be pushed to by i-i collisions, which are $\sqrt{\frac{m_i}{m_e}}$ faster than this process).

Thus, i-e collisions cause ion and electron temperatures to equalise, but slower than e-i flows equalize (ν_{ei}), electrons Maxwellianise (ν_{ee}), or ions Maxwellianise (ν_{ii}).

3.8.3 Ions as Brownian Particles

A useful further observation in this context is this. Let us use (100) for R_{ei} in (137):

$$\vec{R}_{ei} = -m_e n_e \nu_{ei}(\vec{u}_e - \vec{u}_i)$$

This gives us

$$\boxed{\left(\frac{\partial f_i}{\partial t}\right)_e = \nu_{ie}\frac{\partial}{\partial\vec{v}}\cdot\left[\frac{T_e}{m_i}\frac{\partial f_i}{\partial\vec{v}} + (\vec{v}-\vec{u}_e)f_i\right]} \qquad (140)$$

This reveals the ie collision operator for what it truly is: a diffusion operator on Brownian particles being pushed towards *electron* temperature and *electron* mean flow — i.e., being brought into equilibrium with an "ambient" Maxwellian species.

Given our heuristic calculation of ν_{ie} on p.44 (eq.(79)) as a rate of *momentum* transfer and of the diffusion coefficient on p.65 (eq.(131)), we in fact could have guessed (140) outright!

Exercise 3. Consider a population of Brownian particles satisfying the *Langevin equation*:

$$\dot{\vec{v}} + \gamma \vec{v} = \vec{\chi}(t), \qquad (141)$$

where γ is the friction coefficient and $\vec{\chi}(t)$ is a white-noise gaussian field satisfying

$$\langle \chi_i(t)\chi_j(t')\rangle = 2D\,\delta(t-t')\,\delta_{ij}. \qquad (142)$$

Work out the kinetic equation for these particles — the evolution equation for their distribution function f. You should discover that they have a collision operator similar to (140). What are γ and D in terms of ν_{ie}, T_e, m_i, etc.?

It is not hard to teach yourself (if you don't know already) how to derive such kinetic equations. Also prove the H-theorem for the operator (140) and so prove that it pushes the distribution function towards a Maxwellian.

Exercise 4. In the above calculations, I ended up, for convenience, using the Maxwellian as the distribution of the species that electrons or ions collided with. In fact, the collision operator for a given species of particles colliding with another, Maxwellian species can be worked out exactly - and has a useful structure: useful both because it captures the essential physics of collisions in a relatively compact way and because that is (part of) the operator one ends up using when one considers the *linearised* collision operators where to lowest order the distributions are Maxwellian.

So, assuming that species α collides with species α' and the latter is Maxwellian:

$$f_{\alpha'}(\vec{v}) = \frac{n_{\alpha'}}{(\pi v_{th\alpha'}^2)^{3/2}} e^{-\frac{|\vec{v}-\vec{u}_{\alpha'}|^2}{v_{th\alpha'}^2}}, \qquad (143)$$

Show that, in the rest frame of the latter species ($\vec{w} = \vec{v} - \vec{u}_{\alpha'}$),

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\alpha'} = \nu_{D\alpha\alpha'}(w)\,\hat{\mathcal{L}}[f_\alpha] \quad \leftarrow \text{pitch-angle scattering}$$

$$\leftarrow \text{energy diffusion \& drag}$$

$$+ \frac{1}{w^2}\frac{\partial}{\partial w}\, w^2 \nu_{S\alpha\alpha'}(w)\,\frac{m_\alpha}{m_\alpha + m_{\alpha'}}\left[\frac{T_{\alpha'}}{m_\alpha}\frac{\partial f_\alpha}{\partial w} + w f_\alpha\right] \tag{144}$$

and work out what $\nu_{D\alpha\alpha'}(w)$ and $\nu_{S\alpha\alpha'}(w)$ are (some of the integration tricks required were practiced in §3.6) — plot them, find asymptotics.

As I said above, this is part of the collision operator that you would use if you linearized around a Maxwellian (in this case α and α' can be the same species). Can you work out the other part?

Derive the linearized collision operator and prove that it is self-adjoint.

(All this is of course done in many books and notes — but I want you to produce a clean derivation for yourself, to use and to cherish.)