COLLISIONAL PLASMA PHYSICS

[Statistical Physics of Plasmas]

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1. Introduction

- This course was first designed by Felix Parra in 2015 as part of our coordinated sequence of courses on plasma physics — meant to complement "Kinetic Theory", "Advanced Fluid Dynamics" (MHD), and "Collisionless Plasma Physics" (KMHD) but to be also comprehensible on its own.

- It concerns an extremely fundamental challenge in theoretical physics: how to describe interactions between many (classical) particles exerting forces on each other ("collisions") and hence how to predict various relaxation processes whereby the system pushes itself towards thermal equilibrium. For neutral particles, interacting via short-range forces, this was covered by Paul Dellar in his part of the Kinetic Theory course. In plasmas, particles are charged and so interact via long-range, Coulombic forces. As a result of that, head-on collisions in which particles get within their naively estimated cross section of each other are rare.

You might recall from the KT course (if you have taken it) that this was estimated (in §1.4 of my Lecture Notes) by balancing their Coulomb potential energy and their kinetic energy:

$$\frac{e^2}{d} \sim \frac{m v_{th}^2}{2} \sim T \qquad (1)$$

distance of closest approach

The resulting distance is, however, very small compared to the typical distance between particles $\Delta r \sim n^{-1/3}$:

$$\frac{d}{\Delta r} \sim \frac{e^2 n^{1/3}}{T} \ll 1 \qquad (2)$$

$$\sim N_D^{-2/3}, \quad N_D = n\lambda_D^3 \text{ plasma parameter}$$

in a weakly coupled plasma, by definition.

This means that the majority of collisions are "glancing" encounters, each deflecting the participating particles only a little.

Thus, collisions lead to diffusion of particles in the velocity space (accumulation of small kicks to their momentum), and mathematically are expected to be described by a Fokker-Planck operator (§1.8).

~~[illegible]~~

~~[illegible]~~

- If you have followed Robbie Ewart's KT lectures on galactic dynamics, this situation is familiar to you from the theory of ~~[illegible]~~ interaction between mutually gravitating masses - and indeed there are near precise parallels (but also important differences) between the theories of collision operators in galactic dynamics and in plasma physics (in fact the former largely followed the latter in this area ~~[illegible]~~ - you will see this if follow the lecture notes by Chris Hamilton and J-B Fouvry, or indeed read their papers). The key difference is, of course that gravity is not shielded whereas electricity is: in a plasma, the Coulombic interactions are only long-range within the Debye sphere.

That does not invalidate what I have told you about collisions:

$$\frac{d}{\lambda_D} \sim \frac{e^2}{T}\left(\frac{ne^2}{T}\right)^{1/2} \sim \frac{1}{n\lambda_D^3} = \frac{1}{N_D} \ll 1 \quad (3)$$

So there is plenty of space inside the Debye sphere for Coulombic glancing collisions to occur. Indeed if we estimate the total potential energy of all the particles in a Debye sphere, we get (almost trivially), via (3),

$$N_D \cdot \frac{e^2}{\lambda_D} \sim \frac{e^2}{d} \sim T. \qquad (4)$$

Thus, collisions are perfectly capable of changing the overall state of the system — if, as I showed in §2.1, over times that are long compared to collective particle motions that can occur on Debye scales and above:

$$\frac{\nu_e}{\omega_{pe}} \sim \frac{v_{the}}{\lambda_{mfp}} \frac{1}{\omega_{pe}} \sim \frac{\lambda_D}{\underset{\nwarrow \frac{1}{nd^2}}{\lambda_{mfp}}} \sim \frac{1}{N_D} \ll 1 \qquad (5)$$

- In my KT lectures, I focussed on the faster, collisionless dynamics, which culminated in the emergence, via QLT, of stable, non-thermal (in general) distributions like one with a plateau resulting from an initially unstable bump-on-tail state.

These are non-universal states dictated by initial conditions and the requirement to saturate specific instabilities (the saturation is, of course, not always quasilinear, as I explained in §2.4, but the general point stands).

There is the question of what happens on longer times — when collisions kick in. That is where the theory of collisions comes in.

[In fact, it comes in much earlier because collisionless dynamics causes the distribution function to develop highly filamentary structure in phase space — see §5.5.]

So let us tackle collisions.

I shall not do it here in the way that it is done in the majority of textbooks, viz., either by considering binary particle interactions and building the collision operator out of Boltzmann's integral coupled with Rutherford's cross-section for a Coulomb collision or by generalising the BBGKY scheme to plasmas.

You could call these, respectively, the "Landau way" and the "Bogoliubov–Klimontovich way".

You will find an excellent exposition of the former approach in Felix Parra's original lecture notes broadly follow Klimontovich 1967 but

for this course and of the latter one in Matt Kunz's lecture notes for the Princeton graduate course on "Irreversible Processes in Plasmas" (which also ~~discuss~~ contain two other derivations and tons of insight). I will adopt instead the shortest path to plasma collision integrals that I know and that casts them explicitly ~~~~ as consequences of physical assumptions and mathematical deductions about sub-Debye-scale ~~electric~~ fluctuations in a plasma — it is a hybrid of the ~~~~ schemes I have learned from Klimontovich's book (1967) and a paper by Kadomtsev & Pogutse (1970), although I have not done the detailed historical research to pin ~~down~~ down ~~~~ who did it this way first (or whether someone did).

In this approach, the theory will be very explicitly quasilinear, although that approximation will not be enough and an assumption amounting to Boltzmann's Stoßzahlansatz will be required to break correlations and make the system irreversible.

It is fundamentally quite similar (and, I think, ultimately equivalent, bar some

mathematical niceties) to the BBGKY-based derivation, but is much shorter.
I will try to sketch this equivalence a posteriori, but might not have time to cover it in the actual lectures.

- The main outcome of this first part of the course will be Landau's collision operator and, consequently, the conclusion that a plasma will be pushed by collisions to a local Maxwellian equilibrium with some mean density, velocity, and temperature — it will furthermore be the unavoidable consequence of the H-theorem that the mean velocity and temperature must be the same for all species of particles.
How fast the relaxation to the local Maxwellian and to a single $\vec{u}$ and T occurs and in what order for which species will prompt consideration of collisions between two species with vastly different masses — electrons and ions — and of the simplifications and consequences associated with the existence of the small parameter m_e/m_i.
- We shall then consider how collisions erase local gradients — the theory of transport.

While the derivation of viscosity and thermal conductivity in a plasma is the application of the general Chapman–Enskog scheme, which is not conceptually different from the way in which it is done in neutral gases, at least one example of it, iconic in plasma physics, is worth working through — it is known as the Spitzer–Härm problem and involves the calculation of plasma's resistivity.

- This is the diffusion coefficient of the magnetic field with which you are familiar if ~~you~~ have followed Michael Barnes' (or my) MHD lectures. This will give me a natural way to segue to the part of this course that deals with Resistive MHD, which was largely omitted in those MHD lectures. This is (primarily) the theory of magnetic reconnection, starting with the derivation of a resistive instability called the tearing mode and proceeding thence to nonlinear reconnection regimes known as Sweet–Parker and fast reconnection. These processes are ubiquitous and ~~[illegible]~~ indispensable ~~[illegible]~~ in most plasma dynamics when magnetic fields are present

– and they are also fundamentally and conceptually interesting as they tell you how topological constraints imposed by flux freezing in ideal MHD are relaxed.

- Finally, I will return to viscosity and thermal conductivity – but in a magnetised plasma where the Larmor frequency is large compared to the collision rate – the limit that was studied (mostly) collisionlessly in the lectures on KMHD, drift kinetics, etc. Collisionless transport – heat fluxes and momentum fluxes (stresses) – are anisotropic in such a plasma with respect to the local direction of the magnetic field (you already know this from all the discussion of pressure anisotropies in such a plasma – except now we will assume that β is low enough for such configurations not to be catastrophically unstable).

  The collisional (fluid) limit with such anisotropic fluxes is known as Braginskii MHD. In its lowest-order form (in ρ_*, the Larmor radius), it is very easily derivable from KMHD + collision operator but that only gives parallel transport (along $\vec{B}$). To get perpendicular transport, one has to go to two extra orders — that is a bit of a mess of a derivation, but I will try to sketch the most optimal path to the answer with as little pain as possible.

- Felix Parra (and Sarah Newton, who taught the course after 2019) also covered an introduction to collisional transport in tokamaks — the so-called "Neoclassical Theory". I may or may not have time to do this, but in any case, you have his lecture notes and also a superb book on the subject by Helander & Sigmar.

You will also find guidance to other textbooks in M. Kunz's lectures

- You should not view my notes — handwritten or typed, as time permits — as your sole source of instruction. At the very least, you should supplement them with the lecture notes by Felix Parra (Oxford 2019) and Matt Kunz (Princeton 2021). These are members of my tribe, so I quote them first, but of course there is a ton of other textbook-level literature, which you should have at least glancing interactions with (I may give you some pointers later on).

2. From Fluctuations to Collision Operators

2.1 Klimontovich Description of a Plasma

Back in §1.5 of my KT Lectures, my starting point for the microscopic, exact description of a plasma was a random phase-space density called the Klimontovich distribution function:

$$F_\alpha(\vec{r},\vec{v},t) = \sum_{i=1}^{N_\alpha} \delta^3(\vec{r}-\vec{r}_i^{(\alpha)}(t))\,\delta^3(\vec{v}-\vec{v}_i^{(\alpha)}(t)) \quad (1)$$

exact particle trajectory

Back then, I quickly averaged over sub-Debye scales to obtain the smooth phase-space density that was the repository of all information in the collisionless description. Now I will take the object (1) seriously and consider how it might evolve with time.

To simplify notation, I will often use the phase-space variable $\vec{Q} = (\vec{r},\vec{v})$ and also drop species indices everywhere where it does not cause ambiguity. It will sometimes be your job, by way of exercise, to restore them and thus generalise my calculations.

So: $$F(\vec{Q},t) = \sum_i \delta^{(6)}(\vec{Q} - \vec{Q}_i(t)) \qquad (2)$$

Let us calculate the time derivative

$$\frac{\partial F}{\partial t} = -\sum_i \frac{d\vec{Q}_i}{dt} \cdot \frac{\partial}{\partial \vec{Q}} \delta(\vec{Q} - \vec{Q}_i)$$

↑ these do not depend on $\vec{Q}$, only on time

$$= -\frac{\partial}{\partial \vec{Q}} \cdot \sum_i \frac{d\vec{Q}_i}{dt} \delta(\vec{Q} - \vec{Q}_i)$$

↑ these satisfy particle eqns of motion:

$$\frac{d\vec{Q}_i}{dt} = \vec{V}(\vec{Q}_i) \qquad (3)$$

generalized phase-space velocity

exact microscopic fields

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i \qquad (4)$$

$$\frac{d\vec{v}_i}{dt} = \frac{q}{m}\left(\vec{E}^{\,\mathrm{micro}}(\vec{r}_i,t) + \frac{\vec{v}_i \times \vec{B}^{\,\mathrm{micro}}(\vec{r}_i,t)}{c}\right) \qquad (5)$$

NB: $\vec{V}$ is only a function of particle's position in phase space; information about other particles enters via the fields.

This gives us

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \vec{Q}} \cdot \sum_i \underbrace{\vec{V}(\vec{Q}_i)}_{=\,\vec{V}(\vec{Q})} \delta(\vec{Q} - \vec{Q}_i)$$

$\vec{V}(\vec{Q})$ because multiplies δ function

$$= -\frac{\partial}{\partial \vec{Q}} \cdot \vec{V}(\vec{Q}) \underbrace{\sum_i \delta(\vec{Q}-\vec{Q}_i)}_{= F}$$

So,
$$\boxed{\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \vec{Q}} \cdot \vec{V}(\vec{Q})\, F} \qquad (6)$$

This is Liouville's equation — just a statement of conservation of probability: phase-space density F evolves by a continuity equation in 6D phase space.

Furthermore, the flow $\vec{V}(\vec{Q})$ is incompressible:

$$\frac{\partial}{\partial \vec{Q}} \cdot \vec{V}(\vec{Q}) = 0, \qquad (7)$$

which can be checked directly and immediately from (4)-(5) — this is the property of the Lorentz force, reflecting the fact that particles' motion is Hamiltonian. So (6) becomes

$$\frac{\partial F}{\partial t} + \vec{V}(\vec{Q}) \cdot \frac{\partial F}{\partial \vec{Q}} = 0 \qquad (8)$$

or, returning to $(\vec{r}, \vec{v})$ notation,

$$\boxed{\frac{\partial F}{\partial t} + \vec{v} \cdot \nabla F + \frac{q}{m}\left(\vec{E}^{\,micro} + \frac{\vec{v} \times \vec{B}^{\,micro}}{c}\right) \cdot \frac{\partial F}{\partial \vec{v}} = 0} \qquad (9)$$

— Klimontovich equation, exact.

This is just like Vlasov's equation but ~~these~~ this time there is no collision ~~operator~~ integral here because the microscopic field are explicitly present and so the kinetic equation already includes Coulombic interactions.

Equ (9) is closed by Maxwell's equations for microscopic fields:

$$\nabla \cdot \vec{E}^{(micro)} = 4\pi \sum_\alpha q_\alpha \int d\vec{v}\, F_\alpha(\vec{r},\vec{v},t) \tag{10}$$

$$\nabla \cdot \vec{B}^{(micro)} = 0 \tag{11}$$

$$\nabla \times \vec{E}^{(micro)} + \frac{1}{c}\frac{\partial \vec{B}^{(micro)}}{\partial t} = 0 \tag{12}$$

$$\nabla \times \vec{B}^{(micro)} - \frac{1}{c}\frac{\partial \vec{E}^{(micro)}}{\partial t} = \frac{4\pi}{c}\sum_\alpha q_\alpha \int d\vec{v}\,\vec{v}\, F_\alpha(\vec{r},\vec{v},t) \tag{13}$$

2.2 Splitting off Collisions

Now let us separate the microscopic fields into mean (macroscopic) and fluctuating parts in the same way as I did in §1.4:

$$F_\alpha = \underbrace{\langle F_\alpha \rangle}_{\equiv f_\alpha} + \delta F_\alpha \tag{14}$$

Note that I am about to diverge slightly from the way this theory is presented in ~~...~~ the textbooks that derive the collision integrals via the BBGKY scheme: my average is the coarse-graining average over sub-Debye scales, their average is the average over many realizations of the initial conditions of the particles trajectories (an ensemble average). The underlying assumption is that averaging over an ensemble will have the effect of smoothing the distribution function in the same way as the coarse graining does. This is not obviously guaranteed if you are a mathematician but it is OK if you are physicist (as long as you stay aware).

Averaging the Klimontovich equation (9) gives the Vlasov-Landau equation:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\nabla f_\alpha + \frac{q}{m}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial f_\alpha}{\partial \vec{v}} =$$

$$= -\frac{q_\alpha}{m_\alpha}\left\langle\left(\delta\vec{E} + \frac{\vec{v}\times\delta\vec{B}}{c}\right)\cdot\frac{\partial\delta F_\alpha}{\partial\vec{v}}\right\rangle \equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_c \qquad (15)$$

where $\vec{E}$ and $\vec{B}$ are mean fields, which satisfy averaged Maxwell's equations (with $F_\alpha \to f_\alpha$) (since Maxwell's equations are linear, averaging them is a painless ~~[illegible]~~ procedure) whereas $\delta\vec{E}$ and $\delta\vec{B}$ satisfy the microscopic Maxwell's equations with charge and current densities expressed in terms of δF_α.

The latter implies that the rhs of (15) is quadratic in δF_α. In order to close this equation, we will need an expression for the correlation function $\langle \delta F_\alpha \delta F_{\alpha'} \rangle$ in terms of f_α and $f_{\alpha'}$ — that is the challenge of deriving the collision integral.

It cannot have escaped your notice that what I have just done is actually the exact same thing that I did in §2 in splitting $f = f_0 + \delta f$ and separating the evolution of the equilibrium and fluctuations — the only difference is the nature of the average: this time it is the sub-Debye scale coarse-graining (rather than a larger-scale space and time average in §2.2) so the "equilibrium" (or, to be more correct, mean) distribution is f and the fluctuations are the fields responsible for inter-particles interactions. The theory that we are going to construct for this new situation is actually going to be quite close in spirit to QLT — but with some distinct underlying assumptions.

2.3 General ~~[illegible]~~ Form of Collision Integrals

The general form is of course just one in (15) but what I want to do here is set us up for a calculation of CI's by making some simplifying assumptions first.

First, as long as particle motion is non-relativistic ($v \ll c$), we can ignore $\delta \vec{B}$ in our treatment of fluctuations associated with particle interactions, i.e., we can ~~treat~~ work in the electrostatic approximation, just like we did in the KT lectures, but this time it is ~~not~~ an arbitrary simplifying imposition but an actual reasonable approximation. If you are wondering why assuming $v \ll c$ is not <u>always</u> enough to justify neglect of magnetic fields, that is because at larger ($\gtrsim \lambda_D$) scales, the electric fields are screened and so are generally much smaller than our present $\delta \vec{E}$ associated with unscreened Coulombic interactions.

You can, if you have courage and stamina, undertake generalising the upcoming derivations by retaining $\delta \vec{B}$ and allowing $v \sim c$.

[Klimontovich 1967 does treat this case.]

This will set you on the arduous but worthwhile journey to relativistic CIs – I hope you will be well equipped for it by the end of these Lectures.

So, our equations are

$$\frac{Df_\alpha}{Dt} = -\frac{q_\alpha}{m_\alpha}\left\langle \delta\vec{E}\cdot\frac{\partial \delta F_\alpha}{\partial \vec{v}}\right\rangle$$

$$= \frac{\partial}{\partial\vec{v}}\cdot\left[\frac{q}{m}(\nabla\varphi)\,\delta F_\alpha\right], \qquad (16)$$

where the "full time derivative" on the lhs contains all of the Vlasov evolution operator and $\delta\vec{E} = -\nabla\varphi$ — different φ than in my KT lectures, now satisfying Poisson's equation with the microscopic charge density:

$$-\nabla^2\varphi = 4\pi\sum_\alpha q_\alpha\int d\vec{v}\,\delta F_\alpha \qquad (17)$$

I like working in Fourier space, where the above equation is easily solved without recourse to tedious Green's functions:

$$\varphi_k = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d\vec{v}\, \delta F_{k\alpha} \tag{18}$$

Since my average was a coarse-graining average, I will also assume that only $\vec{k}$'s corresponding to the coarse-graining scale and below it matter, so all variation of f_α and fields above that scale (spatial variation of the mean fields) can be ignored, i.e., viewed as represented by the $\vec{k}=0$ mode. Effectively, this means that I am treating my coarse-graining average exactly as did the average that allowed me to split $f = f_0 + \delta f$ in §2 of my KT Notes. And so, exactly like I did there, I can now write

$$\frac{Df_\alpha}{Dt} = \frac{\partial}{\partial \vec{v}} \cdot \left(\frac{q_\alpha}{m_\alpha} \sum_k i\vec{k} \langle \varphi_k^* \delta F_{\alpha k} \rangle \right) \tag{19}$$

$$= \frac{\partial}{\partial \vec{v}} \cdot \sum_{\alpha'} \frac{4\pi q_\alpha q_{\alpha'}}{m_\alpha} \sum_k \frac{i\vec{k}}{k^2} \int d\vec{v}' \langle \delta F_{\alpha k}(\vec{v})\, \delta F^*_{\alpha' k}(\vec{v}') \rangle$$

You see that the rhs – the prototypical CI – is expressed in terms of the correlation function

$$C_{k\alpha\alpha'}(\vec{v},\vec{v}') = \langle \delta F_{k\alpha}(\vec{v})\, \delta F^*_{k\alpha'}(\vec{v}') \rangle \qquad (20)$$

This establishes explicitly the principle already alluded to ~~before~~ in §2.2 of these Notes: the evolution of f is determined by the 2nd-order correlator of δF, which, if we want a closed equation, it is now our task to express in terms of f. All derivations of CI's are schemes for doing this, usually requiring some "closure" assumptions at some stage. This is of course just a reiteration of the message of §2.3 of my KT Lectures: to know about ~~[illegible]~~ the mean (there, f_0), we need to understand the fluctuations around it (there, δf).

Let me make another simplifying step, purely for the purpose of reducing algebraic clutter: I want to derive the collision integral accounting only for electron-electron interactions. This amounts to picking out only the term with $\alpha=\alpha'=e$ (and $q_\alpha=-e$) in (19). This is by no means all there is — not even for electrons, but ~~[illegible]~~ it will make algebra a bit more compact and excuse us from having to have species indices and sums everywhere. Effectively, this means replacing (18) with

$$\varphi_k = -\frac{4\pi e}{k^2}\int d\vec{v}\,\delta F_k \qquad (21)$$

and (19) with

$$\frac{Df}{Dt} = -\frac{\partial}{\partial\vec{v}}\cdot\left(\frac{e}{m}\sum_k i\vec{k}\langle\varphi_k^*\delta F_k\rangle\right)$$

$$= -\frac{\partial}{\partial\vec{v}}\cdot\left(\frac{e}{m}\sum_{\vec{k}}\vec{k}\,\mathrm{Im}\langle\varphi_k^*\delta F_k\rangle\right) \qquad (22)$$

(independent of the e-only simplification)

I have made an additional step: the imaginary part was distilled out of the $\vec{k}$ sum by splitting the latter in two equal parts and replacing $\vec{k}\to-\vec{k}$ in one of them (same trick as in the QLT calculation in §6 of the KT notes).

Equivalently, via (21),

$$\boxed{\frac{Df}{Dt} = \frac{\partial}{\partial\vec{v}}\cdot\left[\frac{\omega_{pe}^2}{\underbrace{n_e}_{\text{mean electron density}}}\sum_k\frac{\vec{k}}{k^2}\int d\vec{v}'\,\mathrm{Im}\,C_k(\vec{v},\vec{v}')\right]} \qquad (23)$$

I stress that the contribution to (19) from electron-ion collisions is <u>not</u> negligible compared to e-e ones. Later on, I shall ask you to generalise the upcoming calculation to multiple species – keeping track of α^*'s and α's will be a useful way to go through it again carefully, without getting mesmerised into copying my formulae. Collisions between different particles will prove interesting and consequential (especially ones between e's and i's!) and we will study their effects at length.

Now, however, I want to focus on the general expression (23) and see what can be done about calculating the correlation function $C_k(\vec{v},\vec{v}')$.

Let me observe that since

$$C_k^*(\vec{v},\vec{v}') = \langle \delta F_k^*(\vec{v})\, \delta F_k(\vec{v}') \rangle = C_k(\vec{v}',\vec{v}), \qquad (24)$$

$\mathrm{Im}\, C_k(\vec{v},\vec{v}') \neq 0$ only if $C_k(\vec{v},\vec{v}')$ is <u>not</u> symmetric wrt swapping $\vec{v} \leftrightarrow \vec{v}'$.

2.4 Fluctuations and Two-particle correlations

Let us take a stab at calculating the correlation function in (23):

$$C_k(\vec{v},\vec{v}\,') = \iint \frac{d\vec{r}\,d\vec{r}\,'}{V^2}\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}\,')} \underbrace{\langle \delta F(\vec{r},\vec{v})\,\delta F(\vec{r}\,',\vec{v}\,')\rangle}_{\equiv\, C(\vec{Q},\vec{Q}\,')} \qquad (25)$$

where

$$C(\vec{Q},\vec{Q}\,') = \langle F(\vec{Q})F(\vec{Q}\,')\rangle - f(\vec{Q})f(\vec{Q}\,')$$

$$= \underbrace{\langle \sum_i \delta(\vec{Q}-\vec{Q}_i)\,\delta(\vec{Q}\,'-\vec{Q}_i)\rangle}_{=\,\langle \sum_i \delta(\vec{Q}-\vec{Q}_i)\,\delta(\vec{Q}\,'-\vec{Q})\rangle \,=\, \delta(\vec{Q}\,'-\vec{Q})f(\vec{Q})}$$

$$+ \underbrace{\langle \sum_{i\neq j}\sum \delta(\vec{Q}-\vec{Q}_i)\,\delta(\vec{Q}\,'-\vec{Q}_j)\rangle}_{=\, f_2(\vec{Q},\vec{Q}\,')}$$

$f_2(\vec{Q},\vec{Q}\,')$ 2-particle distribution function

$$- f(\vec{Q})f(\vec{Q}\,')$$

$$= \delta(\vec{Q}\,'-\vec{Q})f(\vec{Q}) + \underbrace{f_2(\vec{Q},\vec{Q}\,') - f(\vec{Q})f(\vec{Q}\,')}_{=\, g_2(\vec{Q},\vec{Q}\,')\ \text{cumulant}} \qquad (26)$$

If particles are completely independent of each other,

$$f_2(\vec{Q},\vec{Q}\,') = \frac{N-1}{N} f(\vec{Q})f(\vec{Q}\,'), \qquad (27)$$

(assuming particles are indistinguishable)

So $C(\vec{Q},\vec{Q}') = \delta(\vec{Q}'-\vec{Q}) f(\vec{Q})$ (28)

and (25) becomes

$$C_k(\vec{v},\vec{v}') = \underbrace{\frac{1}{V^2}\int d\vec{r}\, f(\vec{r},\vec{v})}_{\frac{f}{V}} \delta(\vec{v}-\vec{v}') \qquad (29)$$

$\frac{f}{V}$ because f does not change over the small box we are in

Exercise. Calculate the spectrum of electric fluctuations arising from this — thermal particle noise.

But this is symmetric in $\vec{v} \leftrightarrow \vec{v}'$ (and at any rate manifestly real), so the rhs of (23) $= 0$.

This is not terribly surprising: assuming the particles are uncorrelated means they have no interaction — the collision integral duly vanishes.

Thus, information about collisions will be contained in the correlations between pairs of particles (binary collisions), $g_2(\vec{Q},\vec{Q}') \neq 0$.

Eq. (23) with only g_2 left in the rhs is the first equation of the BBGKY hierarchy.

In the traditional approach, one continues by writing an evolution equation for g_2, which, naturally turns out to depend on g_3, etc.

This, however, can be truncated because each next $(n+1)$st particle correlator turns out to be smaller than n-th by N_D^{-1} — we shall show this a posteriori, once we derive our CI. This amounts to treating the nonlinearity in the Klimontovich equation perturbatively, i.e., calculating δF to linear order only – the quasilinear approximation! That is exactly how I will proceed: solve for δF linearly in terms of f, put the result into eq. (23) and thus get a closed equation for f – this is just the same approach as QLT (§6 of my KT Lectures).

I will still need an assumption though: linear theory, which, as before, will be done by Laplace transform, will require an initial condition, and I will assume that that has uncorrelated particles, satisfying (28). This is exactly Boltzmann's Stoßzahlansatz: the system "starts" with uncorrelated particles before the collision, then is advanced a short time (linearly) as the interaction occurs and particles become correlated, and then proceeds to the next "time step", which again starts with uncorrelated particles.

i.e., the initial condition is always the thermal particle noise

compared to time over which f changes significantly – but long compared to streaming time, $kvt \gg 1$.

[In the BBGKY/Klimontovich approach, one istead solves for g_2 neglecting g_3 and assuming that $g_2 = 0$ initially. – not, technically speaking, exactly equivalent, but basically is.]

2.5 Quasilinear Collision Integrals

2.5.1 Laplace-transform slu for fluctuations

So, let us do to Klimontovich equation what we did to Vlasov equation when working out linear theory: for electrons, $\delta F = F - f$ satisfies

$$\frac{\partial \delta F}{\partial t} + \vec{v}\cdot\nabla\delta F + \frac{e}{m}(\nabla\varphi)\cdot\frac{\partial f}{\partial \vec{v}} + \text{neglected nonlinear terms} = 0 \qquad (30)$$

NB: microscopic field!

including
$$\frac{e}{m}(\nabla\varphi)\cdot\frac{\partial \delta F}{\partial \vec{v}}$$
$$-\frac{e}{m}\left\langle(\nabla\varphi)\cdot\frac{\partial \delta F}{\partial \vec{v}}\right\rangle$$
$$-\frac{e}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial \delta F}{\partial \vec{v}}$$

Fourier:

$$\frac{\partial \delta F_k}{\partial t} + i\vec{k}\cdot\vec{v}\,\delta F_k = -i\frac{e}{m}\varphi_k\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}} \qquad (31)$$

Laplace:

$$\delta\hat{F}_k(p) = \int_0^\infty dt\, e^{-pt}\,\delta F_k(t) \qquad (32)$$

$$-g_k + p\,\delta\hat{F}_k + i\vec{k}\cdot\vec{v}\,\delta\hat{F}_k = -i\frac{e}{m}\hat{\varphi}_k\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}} \qquad (33)$$

← time dependence assumed slow compared to δF ("Bogoliubov hypothesis")

NB! see p. 28a

$$\delta\hat{F}_k(p) = -i\frac{e}{m}\frac{\hat{\varphi}_k(p)}{p+i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}} + \underbrace{\frac{g_k}{p+i\vec{k}\cdot\vec{v}}}_{\equiv\,\hat{h}_k(p)} \qquad (34)$$

Whence, via Poisson equ,

$$\hat{\varphi}_k(p) = -\frac{4\pi e}{k^2}\int d\vec{v}\,\frac{\hat{h}_k(p)}{\epsilon(p,\vec{k})}, \qquad (35)$$

"dressed" distribution

where $\epsilon(p,\vec{k}) = 1 - i\frac{\omega_{pe}^2}{n_e k^2}\int d\vec{v}\,\frac{1}{p+i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_0}{\partial\vec{v}}$ (36)

Inverse Laplace:

$$\delta F_k(t) = \int_{-i\infty+\sigma}^{+i\infty+\sigma}\frac{dp}{2\pi i}\,e^{pt}\,\delta\hat{F}_k(p)$$

$p=-i\omega+\sigma \longrightarrow$

$$= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{(-i\omega+\sigma)t}\,\delta F_{k\omega}, \qquad (37)$$

where $\delta F_{k\omega} = \delta\hat{F}_k(-i\omega+\sigma) =$

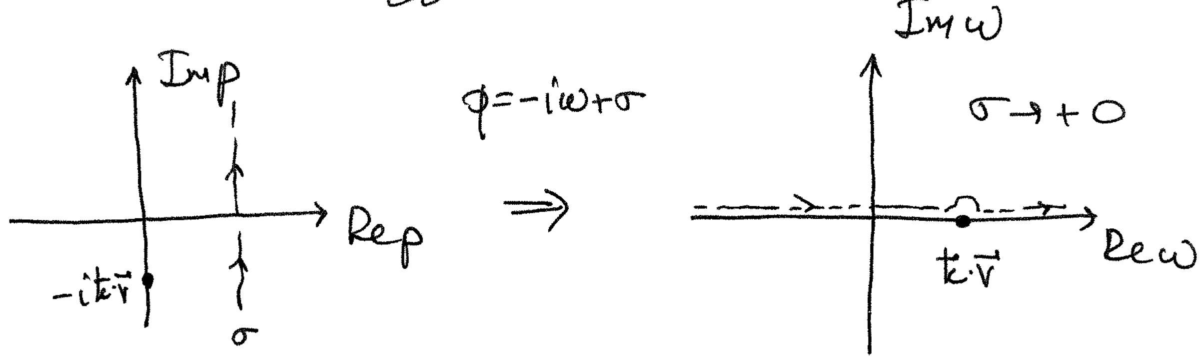
$$= \frac{e}{m}\,\frac{\varphi_{k\omega}}{\omega-\vec{k}\cdot\vec{v}+i\sigma}\,\vec{k}\cdot\frac{\partial f}{\partial\vec{v}} + h_{k\omega} \qquad (38)$$

$$h_{k\omega} = \hat{h}_k(-i\omega+\sigma) = \frac{i g_k(\vec{v})}{\omega-\vec{k}\cdot\vec{v}+i\sigma} \qquad (39)$$

$$\varphi_{k\omega} = \hat{\varphi}_k(-i\omega+\sigma) = -\frac{4\pi e}{k^2}\int d\vec{v}\,\frac{h_{k\omega}}{\epsilon_{k\omega}} \qquad (40)$$

$$\epsilon_{k\omega} = \epsilon(-i\omega+\sigma,\vec{k}) = 1 + \frac{\omega_{pe}^2}{n_e k^2}\int d\vec{v}\,\frac{1}{\omega-\vec{k}\cdot\vec{v}+i\sigma}\,\vec{k}\cdot\frac{\partial f}{\partial\vec{v}} \qquad (41)$$

I have rebranded the Laplace-transformed functions to look almost like Fourier-transformed ones. ~~because~~ The reason for this rearrangement is that I am going to assume that $\hat{\varphi}_k(p)$ and, therefore, $\delta\hat{F}_k(p)$ have no poles in $\mathrm{Re}\,p > 0$, i.e., f is <u>stable</u>, so I may let $\sigma \to +0$ and integrate along the real line.



I will keep σ in the denominator to remind me how to circumvent the ballistic pole $\omega=\vec{k}\cdot\vec{v}$.

The assumption that f is stable can be justified as follows. We are interested in collisional relaxation, which will happen generally slower than (collisionless) instabilities might flare up. So let them flare up, saturate, and change f in such a way as to shut themselves down (e.g., quasilinearly, as the bump-on-tail instability did in §6 of the KT Lectures). After that, we have an f (which to lowest order is $\approx f_0$) that is stable or, at worst, marginally stable, i.e., $\epsilon_{k\omega}$ might have zeros on the real-ω line and certainly at $\mathrm{Im}\,\omega=\mathrm{Re}\,p<0$, but not at $\mathrm{Re}\,p>0$.

Note. On p. 28, I treated f as effectively constant on the timescales of the linear physics that evolved δF. When applied to the relaxation times of the two-point correlation function of δF, this is known as Bogoliubov hypothesis — correlations relax faster than the mean. With your knowledge of collisionless theory in mind, you might wonder how this can be true if f contains, e.g., Langmuir waves with frequencies $\sim \omega_{pe}$ — while some of the δF dynamics is on the same timescales!

In the original Soviet literature, they seem to be aware of this problem and explicitly separate the two asymptotic limit of plasma physics:

- "approximation of self-consistent field" — collisionless Vlasov equation
- "homogeneous approximation", where they assume f to have no spatial variation and derive the CI quasilinearly, essentially in the same way I do here.

If these in the end are to be combined into one equation, the only proper way wd seem to be to assume that, for δF,

$$kv \gg \omega_{pe}, \text{ i.e., } k\lambda_{De} \gg 1 \qquad (41a)$$

(Landau approximation) — i.e., we are explicitly ~~as~~ coarse-graining from F to f

on Sub-Debye scale. That wd mean that in (38), we ought to keep only $h_{k\omega}$ (i.e., the only evolution of $\delta F_{k\omega}$ from the initial condition g_k that we wd include is ballistic) and, accordingly, $\epsilon_{k\omega} \approx 1$ everywhere. That will be the difference between the Balescu-Lenard & Landau CIs, and the above discussion suggests that Landau's is the only consistent limit.

I will nevertheless proceed with the BL derivation to give you the idea of the standard results and will return to this discussion afterwards, when you know what I am talking about!

↳ see §2.8 (pp 39-40)

2.5.2 Assembling the CI

Let us now calculate the collision integral (23). In fact trying to do this by calculating $C_k(\vec{v},\vec{v}')$ from (38) turns out to be a very algebraically painful exercise (tbh, I have not even tried), the more efficient way being to use (22):

$$\frac{Df}{Dt} = -\frac{\partial}{\partial \vec{v}}\cdot \frac{e}{m}\sum_{\vec{k}} \vec{k}\, \mathrm{Im}\langle \varphi_k^* \delta F_k\rangle$$

$$\frac{e}{m}\langle \varphi_k^* \delta F_k\rangle \underset{(37)}{=} \frac{e}{m}\iint \frac{d\omega\, d\omega'}{(2\pi)^2} e^{-i(\omega-\omega')}\langle \delta F_{k\omega}\varphi^*_{k\omega'}\rangle \qquad (42)$$

$$\frac{e}{m}\langle \delta F_{k\omega}\varphi^*_{k\omega'}\rangle \underset{(38)}{=} \underbrace{\frac{e^2}{m^2}\frac{\langle \varphi_{k\omega}\varphi^*_{k\omega'}\rangle}{\omega - \vec{k}\cdot\vec{v} + i\sigma}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}}}_{\overset{(40)}{=}} + \frac{e}{m}\langle h_{k\omega}\varphi^*_{k\omega'}\rangle \qquad (43)$$

"QL diffusion"

$$\frac{\omega_{pe}^4}{n_e^2 k^4 \epsilon_{k\omega}\epsilon^*_{k\omega'}}\iint d\vec{v}''\, d\vec{v}'\, \frac{C_{k\omega\omega'}(\vec{v}'',\vec{v}')}{\omega - \vec{k}\cdot\vec{v} + i\sigma}\,\vec{k}\cdot\frac{\partial f(\vec{v})}{\partial \vec{v}}$$

where $C_{k\omega\omega'}(\vec{v},\vec{v}') = \langle h_{k\omega}(\vec{v})\, h^*_{k\omega'}(\vec{v}')\rangle$

$$\frac{e}{m}\langle h_{k\omega}\varphi^*_{k\omega'}\rangle \overset{(40)}{=} -\frac{\omega_{pe}^2}{n_e k^2 \epsilon^*_{k\omega'}}\int d\vec{v}'\, \underbrace{\langle h_{k\omega}(\vec{v})\, h^*_{k\omega'}(\vec{v}')\rangle}_{=\ \frac{\epsilon_{k\omega}}{\epsilon_{k\omega}} C_{k\omega\omega'}(\vec{v},\vec{v}')} \qquad (44)$$

a kind of drag

useful trick $\longrightarrow \frac{\epsilon_{k\omega}}{\epsilon_{k\omega}} C_{k\omega\omega'}(\vec{v},\vec{v}')$

$$= -\frac{\omega_{pe}^2}{n_e k^2 \epsilon_{k\omega}\epsilon^*_{k\omega'}}\int d\vec{v}'\, C_{k\omega\omega'}(\vec{v},\vec{v}')\Big[\underbrace{1 + \frac{\omega_{pe}^2}{n_e k^2}\int d\vec{v}''\frac{1}{\omega - \vec{k}\cdot\vec{v}'' + i\sigma}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}''}}_{\epsilon_{k\omega} \text{ from } (41)}$$

Assemble:

1st term from (44)

$$\frac{Df}{Dt} = \frac{\partial}{\partial \vec{v}} \cdot \sum_k \vec{k}\, \mathrm{Im} \iint \frac{d\omega\, d\omega'}{(2\pi)^2}\, e^{-i(\omega-\omega')t} \left\{ \frac{\omega_{pe}^2}{n_e k^2} \int d\vec{v}'\, \frac{C_{k\omega\omega'}(\vec{v},\vec{v}')}{\epsilon_{k\omega}\epsilon^*_{k\omega'}} \right.$$

$$+ \frac{\omega_{pe}^4}{n^2 k^4} \frac{1}{\epsilon_{k\omega}\epsilon^*_{k\omega'}} \iint d\vec{v}''\, d\vec{v}' \left[\frac{C_{k\omega\omega'}(\vec{v},\vec{v}')}{\omega - \vec{k}\cdot\vec{v}'' + i\sigma}\, \vec{k}\cdot \frac{\partial f(\vec{v}'')}{\partial \vec{v}''} \right.$$

2nd term from (44)

(this vanishes, for reasons ~~...~~ to be explained in §2.6)

$$\left. - \frac{C_{k\omega\omega'}(\vec{v}'',\vec{v}')}{\omega - \vec{k}\cdot\vec{v} + i\sigma}\, \vec{k}\cdot \frac{\partial f(\vec{v})}{\partial \vec{v}} \right]$$

from (43)

$$= \frac{\partial}{\partial \vec{v}} \cdot \int d\vec{v}'' \left[\hat{D}(\vec{v},\vec{v}'')\cdot \frac{\partial f(\vec{v})}{\partial \vec{v}} - \hat{D}(\vec{v}'',\vec{v})\cdot \frac{\partial f(\vec{v}'')}{\partial \vec{v}''} \right] - \frac{\partial}{\partial \vec{v}}\cdot \vec{\Gamma} \equiv \left(\frac{\partial f}{\partial t}\right)_c \qquad (45)$$

where the kernel is:

$$\hat{D}(\vec{v}'',\vec{v}) = - \frac{\omega_{pe}^4}{n_e^2}\, \mathrm{Im} \sum_k \frac{\vec{k}\vec{k}}{k^4} \iint \frac{d\omega\, d\omega'}{(2\pi)^2} \frac{e^{-i(\omega-\omega')t}}{\epsilon_{k\omega}\epsilon^*_{k\omega'}} \times$$

$$\times \int d\vec{v}'\, \frac{C_{k\omega\omega'}(\vec{v},\vec{v}')}{\omega - \vec{k}\cdot\vec{v}'' + i\sigma} \qquad (46)$$

(Note that $\int d\vec{v}''\, \hat{D}(\vec{v},\vec{v}'')$ is basically our old QL diffusion matrix.)

and

$$\vec{\Gamma} = - \frac{\omega_{pe}^2}{n_e}\, \mathrm{Im} \sum_k \frac{\vec{k}}{k^2} \iint \frac{d\omega\, d\omega'}{(2\pi)^2} \frac{e^{-i(\omega-\omega')t}}{\epsilon_{k\omega}\epsilon^*_{k\omega'}} \int d\vec{v}'\, C_{k\omega\omega'}(\vec{v},\vec{v}') \qquad (47)$$

is an additional probability flux that will turn out to vanish.

For some of the future handling of (47), it is useful to observe that

$$\iint \frac{d\omega\, d\omega'}{(2\pi)^2} e^{-i(\omega-\omega')t} \frac{\overbrace{C_{k\omega\omega'}(\vec{v},\vec{v}')}^{=\langle h_{k\omega}(\vec{v})\, h^*_{k\omega'}(\vec{v}')\rangle}}{\epsilon_{k\omega}\epsilon^*_{k\omega'}} = \underbrace{\langle \tilde{h}_k(\vec{v})\, \tilde{h}_k^*(\vec{v}')\rangle}_{\equiv\, \tilde{C}_k(\vec{v},\vec{v}')}, \quad (48)$$

where $$\tilde{h}_k = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t} \frac{h_{k\omega}}{\epsilon_{k\omega}} \quad (49)$$

is the "dressed" version of the ~~piece~~ piece h_k of the perturbed distribution function — the piece that contains the linear response to the initial condition, as per (39).

This is a spatial Fourier transform of a real field:

$$\tilde{h}^*_{-k} = \frac{1}{2\pi}\int d\omega\, e^{i\omega t} \frac{h^*_{-k,\omega}}{\epsilon_{-k,\omega}} \underset{\omega\to-\omega}{=}$$

$$= \frac{1}{2\pi}\int d\omega\, e^{-i\omega t} \frac{h^*_{-k,-\omega}}{\epsilon_{-k,-\omega}}$$

Now, from (38):

$$h^*_{-k,-\omega} = \underbrace{\delta F^*_{-k,-\omega}}_{=\,\delta F_{k\omega} \text{ because } \delta F \text{ is real}} - \frac{e}{m} \frac{\overbrace{\varphi^*_{-k,-\omega}}^{=\,\varphi_{k\omega} \text{ because } \varphi \text{ is real}}}{-(\omega-\vec{k}\cdot\vec{v})-i0} (-\vec{k})\cdot\frac{\partial f}{\partial \vec{v}}$$

$$= \delta F_{k\omega} - \frac{e}{m}\frac{\varphi_{k\omega}}{\omega-\vec{k}\cdot\vec{v}+i0}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}} = h_{k\omega}$$

and from (41),

$$\epsilon^*_{-k,-\omega} = \epsilon_{k\omega} \text{ by the same manipulations.}$$

So, $\tilde{h}^*_{-k} = \tilde{h}_k$. (49a)

This means that

$$\tilde{C}_{-k}(\vec{v},\vec{v}') = \langle \tilde{h}_{-k}(\vec{v})\,\tilde{h}^*_{-k}(\vec{v}')\rangle = \langle \tilde{h}^*_k(\vec{v})\,\tilde{h}_k(\vec{v}')\rangle$$

$$= \tilde{C}^*_k(\vec{v},\vec{v}') = \tilde{C}_k(\vec{v}',\vec{v}) \quad \text{and} \quad (49b)$$

$$\vec{\Gamma} = -\frac{\omega_{pe}^2}{n_e}\sum_k \frac{\vec{k}}{k^2}\,\mathrm{Im}\,\tilde{C}_k(\vec{v},\vec{v}') = 0$$

if $\tilde{C}_k(\vec{v},\vec{v}') = \tilde{C}_k(\vec{v}',\vec{v})$ symmetric.

2.5.3 Doing the frequency integrals

Now let us calculate the diffusion matrix (46) by actually using the QL approximation (39) for $h_{k\omega}$ inside $C_{k\omega\omega'}(\vec{v},\vec{v}')$:

$$C_{k\omega\omega'} = \frac{\overbrace{\langle g_k(\vec{v})\,g_k^*(\vec{v}')\rangle}^{\equiv C_k(\vec{v},\vec{v}')}}{(\omega-\vec{k}\cdot\vec{v}+i\sigma)(\omega'-\vec{k}\cdot\vec{v}'-i\sigma)} \qquad (50)$$

Therefore, we can rewrite (46) as follows:

$$\hat{D}(\vec{v}'',\vec{v}) = -\frac{\omega_{pe}^4}{n_e^2}\,\mathrm{Im}\sum_k \frac{\vec{k}\vec{k}}{k^4}\int d\vec{v}'\,C_k(\vec{v},\vec{v}')\,I_k(\vec{v},\vec{v}',\vec{v}''), \qquad (51)$$

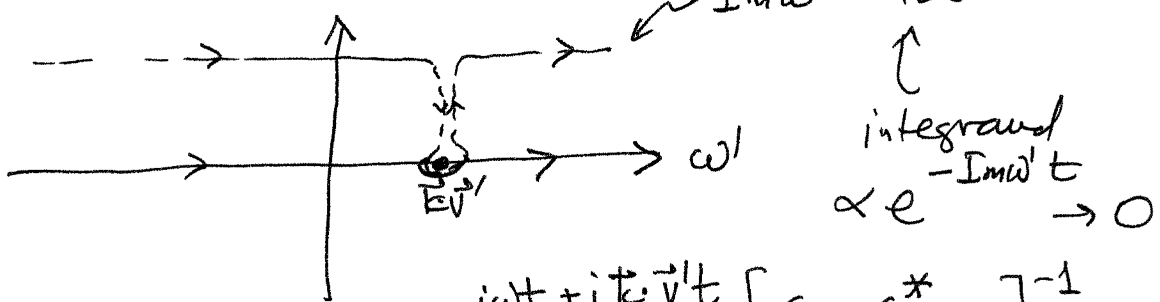
where

$$I_k(\vec{v},\vec{v}',\vec{v}'') = \iint \frac{d\omega\,d\omega'}{(2\pi)^3}\,\frac{e^{-i(\omega-\omega')t}}{\epsilon_{k\omega}\,\epsilon^*_{k\omega'}\,(\omega-\vec{k}\cdot\vec{v}+i\sigma)(\omega'-\vec{k}\cdot\vec{v}'-i\sigma)(\omega-\vec{k}\cdot\vec{v}''+i\sigma)}$$

(52)

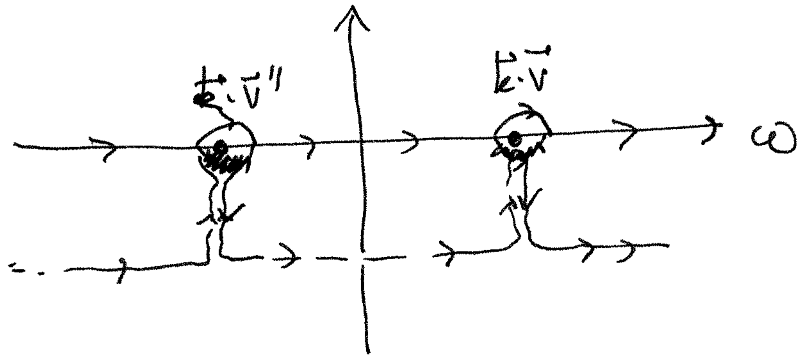
and this integral can be done by the usual trickery in complex plane. It is quite obviously a collection of poles.

Let me simplify life by assuming that $\epsilon^*_{k\omega'}$ has no ~~poles~~ zeroes either on the real line (real ω') or at $\mathrm{Im}\,\omega' > 0$ — i.e. that f is stable. Then we can do the ω' integral by shifting the integration contour so:



$$I_k(\vec v, \vec v', \vec v'') = \int \frac{d\omega}{2\pi}\, i\, \frac{e^{-i\omega t + i\vec k\cdot\vec v' t}\left[\epsilon_{k\omega}\epsilon^*_{\vec k, \vec k\cdot\vec v'}\right]^{-1}}{(\omega - \vec k\cdot\vec v + i\sigma)(\omega - \vec k\cdot\vec v'' + i\sigma)} \tag{53}$$

Now do the ω integral by shifting integration contour so:



(if $\epsilon_{k\omega}$ has any poles at $\mathrm{Im}\,\omega < 0$, these will give decaying contributions)

$$I_k(\vec v, \vec v', \vec v'') = \frac{e^{i\vec k\cdot\vec v' t}}{\epsilon^*_{\vec k, \vec k\cdot\vec v'}}\left[\frac{e^{-i\vec k\cdot\vec v t}}{\epsilon_{\vec k, \vec k\cdot\vec v}(\vec k\cdot\vec v - \vec k\cdot\vec v'')} + \frac{e^{-\vec k\cdot\vec v'' t}}{\epsilon_{\vec k, \vec k\cdot\vec v''}(\vec k\cdot\vec v'' - \vec k\cdot}\right.$$

$$= \frac{e^{-i\mathbf{k}\cdot(\vec{v}-\vec{v}')t}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}\,\epsilon^*_{\mathbf{k},\mathbf{k}\cdot\vec{v}'}}\;\underbrace{\frac{1}{\mathbf{k}\cdot(\vec{v}-\vec{v}'')}\left[1-\frac{e^{i\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}\,\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}''}}\right]}_{\parallel} \tag{54}$$

$$\underbrace{\frac{1}{\mathbf{k}\cdot(\vec{v}-\vec{v}'')}\left[1-\cos\mathbf{k}\cdot(\vec{v}-\vec{v}'')t\;\frac{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}''}}\right]}_{\text{finite as } \mathbf{k}\cdot(\vec{v}-\vec{v}'')\to 0} - i\,\underbrace{\frac{\sin\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}{\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}}_{\substack{it \\ \text{as } \mathbf{k}\cdot(\vec{v}-\vec{v}'')\to 0 \\ \text{so dominant} \\ \text{at } t\to\infty}}\;\frac{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}''}}$$

$$\approx -\frac{e^{i\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}-e^{-i\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}}{2\mathbf{k}\cdot(\vec{v}-\vec{v}'')t}\;\frac{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}''}}$$

$$= -\frac{i}{2}\underbrace{\int_{-t}^{t}dt'\,e^{i\mathbf{k}\cdot(\vec{v}-\vec{v}'')t'}}_{\to\, 2\pi\,\delta(\mathbf{k}\cdot(\vec{v}-\vec{v}''))}\;\frac{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}''}} \;\to\; -i\pi\,\delta(\mathbf{k}\cdot(\vec{v}-\vec{v}'')) \quad \text{as } t\to\infty$$

So,
$$I_k(\vec{v},\vec{v}',\vec{v}'') \to -i\pi\delta(\mathbf{k}\cdot(\vec{v}-\vec{v}''))\,\frac{e^{-i\mathbf{k}\cdot(\vec{v}-\vec{v}')t}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}\,\epsilon^*_{\mathbf{k},\mathbf{k}\cdot\vec{v}'}} \tag{54}$$

Putting this back into (51), we get

$$\boxed{\hat{D}(\vec{v}'',\vec{v}) = \frac{\pi\omega_{pe}^4}{n_e^2}\,\mathrm{Re}\sum_k \frac{\mathbf{k}\mathbf{k}}{k^4}\,\delta(\mathbf{k}\cdot(\vec{v}-\vec{v}''))\int d\vec{v}'\,\frac{C_k(\vec{v},\vec{v}')\,e^{-i\mathbf{k}\cdot(\vec{v}-\vec{v}')t}}{\epsilon_{\mathbf{k},\mathbf{k}\cdot\vec{v}}\,\epsilon^*_{\mathbf{k},\mathbf{k}\cdot\vec{v}'}}} \tag{55}$$

This is progress: everything is explicit and calculable with the only unknown the correlation function of the initial conditions $C_k(\vec{v},\vec{v}')$.

By the same method, we can calculate (49):

$$\tilde{h}_k = \int \frac{d\omega}{2\pi}\, \frac{e^{-i\omega t}}{\epsilon_{k\omega}}\, \frac{g_k(\vec{v})}{\omega - \vec{k}\cdot\vec{v} + i0} = -i\, \frac{e^{-i\vec{k}\cdot\vec{v}t}}{\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}}\, g_k(\vec{v}) \qquad (55a)$$

whence,

$$\tilde{C}_k(\vec{v},\vec{v}') = \frac{e^{-i\vec{k}\cdot(\vec{v}-\vec{v}')t}}{\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}\, \epsilon^*_{\vec{k},\vec{k}\cdot\vec{v}'}}\, C_k(\vec{v},\vec{v}') \qquad (55b)$$

and

$$\vec{\Gamma} = -\frac{\omega_{pe}^2}{n_e}\, \mathrm{Im} \sum_k \frac{\vec{k}}{k^2} \int d\vec{v}'\, \tilde{C}_k(\vec{v},\vec{v}') \qquad (55c)$$

2.5.4 Conservation laws

Let us see how our QL CI (45) with (55) & (55c) behaves vis-à-via the required conservation laws.

Particles: $\int d\vec{v} \left(\frac{\partial f}{\partial t}\right)_c = \int d\vec{v}\, \frac{\partial}{\partial \vec{v}}\cdot(\dots) = 0 \qquad (55d)$

regardless of anything — particle conservation was hard-wired.

Momentum: $\int d\vec{v}\, m\vec{v} \left(\frac{\partial f}{\partial t}\right)_c \underset{\substack{\uparrow \\ \text{by parts}}}{=} - \int d\vec{v}\, m\, (\dots)$

This has two pieces:

$$-\int d\vec{v}\, m \int d\vec{v}'' \left[\hat{D}(\vec{v},\vec{v}'')\cdot \frac{\partial f}{\partial \vec{v}} - \underset{\substack{\uparrow \\ \text{swap } \vec{v}\leftrightarrow\vec{v}''}}{\hat{D}(\vec{v}'',\vec{v})}\cdot \frac{\partial f}{\partial \vec{v}''}\right] = 0 \qquad (55e)$$

again w/o the need for the QL expression (55)

$$\int d\vec{v}\, m \vec{\Gamma} = -\frac{m\omega_{pe}^2}{n_e}\, \mathrm{Im} \sum_k \frac{\vec{k}}{k^2} \iint d\vec{v}\, d\vec{v}'\, \tilde{C}_k(\vec{v},\vec{v}') = 0 \qquad (55f)$$

only symmetric and, therefore, real part survives.

<u>Energy</u>:

$$\int d\vec{v}\, \frac{mv^2}{2} \left(\frac{\partial f}{\partial t}\right)_c = -\int d\vec{v}\, m\vec{v}\cdot(\ldots)$$

Two pieces again:

swap $\vec{v} \leftrightarrow \vec{v}''$ again

$$m \iint d\vec{v}\, d\vec{v}'' \left[\vec{v}\cdot\hat{D}(\vec{v},\vec{v}'')\cdot\frac{\partial f}{\partial \vec{v}} - \vec{v}\cdot\hat{D}(\vec{v}'',\vec{v})\cdot\frac{\partial f}{\partial \vec{v}''}\right] =$$

$$= m\iint d\vec{v}\, d\vec{v}''\, (\vec{v}-\vec{v}'')\cdot\hat{D}(\vec{v},\vec{v}'')\cdot\frac{\partial f}{\partial \vec{v}} = 0 \qquad (55g)$$

(55), because of $\delta(\vec{k}\cdot(\vec{v}-\vec{v}''))$ in it

So we do need the QL approximation to conserve energy, i.e., to prevent energy exchange between particles and fields.

Second piece:

$$m\int d\vec{v}\, \vec{v}\cdot\vec{\Gamma} = -\frac{m\omega_{pe}^2}{n_e}\, \mathrm{Im}\sum_k \frac{1}{k^2}\iint d\vec{v}\, d\vec{v}'\, \vec{k}\cdot\vec{v}\, \tilde{C}_k(\vec{v},\vec{v}')$$

$$= -\frac{m\omega_{pe}^2}{n_e}\, \mathrm{Im}\sum_k \frac{1}{k^2}\iint d\vec{v}\, d\vec{v}'\, \frac{1}{2}\Big[\vec{k}\cdot\vec{v}\, \tilde{C}_k(\vec{v},\vec{v}') - \vec{k}\cdot\vec{v}'\, \tilde{C}_{-k}(\vec{v}',\vec{v})\Big]$$

$$= -\frac{m\omega_{pe}^2}{2n_e}\, \mathrm{Im}\sum_k \frac{1}{k^2}\iint d\vec{v}\, d\vec{v}'\, \vec{k}\cdot(\vec{v}-\vec{v}')\, \tilde{C}_k(\vec{v},\vec{v}')$$

$\tilde{C}_{-k}(\vec{v}',\vec{v}) = \tilde{C}_k(\vec{v},\vec{v}')$ by (49b)

$= 0$ only if $\tilde{C}_k(\vec{v},\vec{v}')$ is symmetric and/or can pin $\vec{k}\cdot(\vec{v}-\vec{v}') = 0$, which even (55b) does not do! (55h)

2.6 Stoßzahlansatz & Lenard-Balescu CI

I now return to the calculation of two-particle correlations — the assumption that there are none led to (29):

$$\boxed{C_k(\vec{v},\vec{v}') = \frac{f}{V}\delta(\vec{v}-\vec{v}')} \qquad (29)$$

We can now assume that our linear theory for δF has captured all the relevant correlations and we can assume that (29) is satisfied not for the correlation function of δF_k but of its initial condition g_k.

This is exactly equivalent to Boltzmann's Stoßzahlansatz — no 2-particle correlations before the collision.

In the current approach, this is interpreted as assuming that, while linear response builds up correlations that then make up the CI, the nonlinear response, which acts on longer time scales (because amplitudes are small), destroys those correlations, so we can assume δF to be initially uncorrelated at the beginning of each "time step" that evolves f.

Implementing (29) in (55) leads to a very nice simplification:

$$\boxed{\hat{D}(\vec{v}'', \vec{v}) = \frac{\pi \omega_{pe}^4}{n_e^2 V} \, \mathrm{Re} \sum_{k} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta(\vec{k}\cdot(\vec{v}-\vec{v}''))}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} f(\vec{v})} \quad (56)$$

Note that explicit t dependence has disappeared, ~~[illegible]~~ and the same in (55c):

$$\vec{\Gamma} = -\frac{\omega_{pe}^2}{n_e V} \, \mathrm{Im} \sum_{k} \frac{\vec{k}}{k^2} \frac{f(\vec{v})}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} = 0, \quad (57)$$

as promised.

2.6.2 LB CI

Putting these back into (45), we get the Lenard–Balescu Collision Integral:

$$\boxed{\frac{Df}{Dt} = \frac{\pi \omega_{pe}^4}{n_e^2 V} \frac{\partial}{\partial \vec{v}} \cdot \int d\vec{v}'' \sum_{\vec{k}} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta(\vec{k}\cdot(\vec{v}-\vec{v}''))}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} \cdot \left[f(\vec{v}'') \frac{\partial f}{\partial \vec{v}} - f(\vec{v}) \frac{\partial f}{\partial \vec{v}''} \right] \equiv \left(\frac{\partial f}{\partial t}\right)_{LB}} \quad (58)$$

This ~~[illegible]~~ has the Fokker–Planck form [$\vec{\Gamma}$ having vanished in (45)]

$$\frac{Df}{Dt} = \frac{\partial}{\partial \vec{v}} \cdot \Big[\underbrace{\hat{D}(\vec{v}) \cdot \frac{\partial f}{\partial \vec{v}}}_{\text{diffusion}} - \underbrace{\vec{A}(\vec{v}) f}_{\text{drag}} \Big] \quad (59)$$

and all the right conservation laws, energy included [again, because $\vec{\Gamma}$ has vanished – see (55b)].

2.7 H-theorem and Equilibrium

Finally, we can define entropy

$$S = -\int d\vec{v}\, f \ln f \qquad (60)$$

Its evolution is due only to the CI because the Vlasov part of the kinetic equation preserves it, or indeed any $\int d\vec{v}\, G(f)\ \forall G$,

$$\frac{dS}{dt} = -\int d\vec{v}\, \frac{\partial f}{\partial t} \ln f - \underbrace{\int d\vec{v}\, \frac{\partial f}{\partial t}}_{0}$$

$$\underset{\text{by parts}}{=} \int d\vec{v}\, \frac{1}{f} \frac{\partial f}{\partial \vec{v}} \cdot (\dots)$$

$$\overset{(58)}{=} \frac{\pi \omega_{pe}^4}{n_e^2 V} \sum_{k} \frac{1}{k^4} \iint d\vec{v}\, d\vec{v}''\, \frac{\delta(\vec{k}\cdot(\vec{v}-\vec{v}''))}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} \qquad (61)$$

$$\left[\frac{f(\vec{v}'')}{f(\vec{v})}\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{v}}\right)^2 - \vec{k}\cdot\frac{\partial f}{\partial \vec{v}}\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}''}\right] \geq 0 \text{ always}$$

$\downarrow$ symmetrize $\vec{v} \leftrightarrow \vec{v}''$

$$\boxed{\begin{aligned} &\frac{1}{2}\left[\frac{f(\vec{v}'')}{f(\vec{v})}\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{v}}\right)^2 - 2\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}}\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}''} + \frac{f(\vec{v})}{f(\vec{v}'')}\left(\vec{k}\cdot\frac{\partial f}{\partial \vec{v}''}\right)^2\right] \\ &= \frac{1}{2}\left[\sqrt{\frac{f(\vec{v}'')}{f(\vec{v})}}\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}} - \sqrt{\frac{f(\vec{v})}{f(\vec{v}'')}}\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}''}\right]^2 \end{aligned}}$$

Thus, entropy grows.

When will it stop growing? You need for that $[\ldots]^2 = 0$, or $\forall \vec{k}$,

$$\frac{1}{f(\vec{v})}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}} = \frac{1}{f(\vec{v}'')}\,\vec{k}\cdot\frac{\partial f}{\partial \vec{v}''} \qquad (62)$$

When $\vec{k}\cdot(\vec{v}-\vec{v}'')=0$ (enforced by the δ function),

or
$$\vec{k}\cdot\left[\frac{\partial \ln f}{\partial \vec{v}} - \frac{\partial \ln f}{\partial \vec{v}''}\right] = 0$$

So
$$\frac{\partial \ln f}{\partial \vec{v}} = \underbrace{\vec{a} + b}_{\text{const}}\vec{v} \Rightarrow \ln f = \overset{\text{normalisation}}{C} + \overset{\text{mean}}{\vec{a}\cdot\vec{v}} + \underset{\text{temperature}}{b\frac{v^2}{2}} \qquad (63)$$

This is just the statement that f is a Maxwellian with some mean:

$$\boxed{f = \frac{n_e}{(\pi v_{the}^2)^{3/2}}\, e^{-\frac{(\vec{v}-\vec{u})^2}{v_{the}^2}}} \qquad (64)$$

const

It is easy to check that this is indeed a fixed point of the LB integral (58) – and it is the only one because it has to be that to stop entropy from growing.

Thus, collisions will push the plasma towards thermal equilibrium.

At what rate though?

In order to work this out, we need to examine the $\vec{k}$ sum (integral) in (58).

2.8 Landau's Collision Integral

So, realistically, we will mostly be dealing with plasma states that are close to local Maxwellian (and are relaxing to it under the action of our CI), so estimates such as $v \sim v_{th}$ are meaningful.

To work out the contribution of different $\vec{k}$'s to the CI, let us consider the dielectric function (41)

$$\epsilon_{\vec{k},\vec{k}\cdot\vec{v}} = 1 + \frac{\omega_{pe}^2}{n_e k^2} \underbrace{\int d\vec{v}' \frac{1}{\vec{k}\cdot(\vec{v}-\vec{v}') + i0}\, \vec{k}\cdot\frac{\partial f}{\partial \vec{v}'}}_{\sim \frac{n_e}{v_{the}^2}}$$

$$\approx 1 \quad \text{if } k\lambda_{De} \gg 1 \tag{65}$$

$$\approx \frac{\text{const}}{k^2\lambda_{De}^2} \quad \text{if } k\lambda_{De} \ll 1 \tag{66}$$

Then

$$\frac{1}{V}\sum_{\vec{k}} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta\big(\vec{k}\cdot\overbrace{(\vec{v}-\vec{v}'')}^{\equiv \vec{W}}\big)}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} = \int \frac{d\vec{k}}{(2\pi)^3} \frac{\vec{k}\vec{k}}{k^4} \frac{\delta(\vec{k}\cdot\vec{W})}{|\epsilon_{\vec{k},\vec{k}\cdot\vec{v}}|^2} =$$

$$= \frac{1}{W}\int \frac{d\vec{k}_\perp}{(2\pi)^3} \frac{\vec{k}_\perp\vec{k}_\perp}{k_\perp^4} \underbrace{\frac{1}{|\epsilon_{\vec{k}_\perp,\vec{k}_\perp\cdot\vec{v}}|^2}}_{\simeq 1} \approx \frac{1}{8\pi^2 W}\left(\mathbb{1} - \frac{\vec{W}\vec{W}}{W^2}\right)\int \frac{dk_\perp}{k_\perp} \tag{67}$$

$\vec{k}_\perp = \vec{k}\cdot\left(\mathbb{1} - \frac{\vec{W}\vec{W}}{W^2}\right)$

$\simeq 1$ if we assume that our δF only contains $k\lambda_{De} \gg 1$ fluctuations

This diverges at both small and large scales, for clear physical reasons:

- at small scales, interactions are not weak (particles approach closely), $e\varphi \sim T \left(= \frac{m v_{th}^2}{2}\right)$, so clearly QL approximation is not justified
- at large scales, there are glancing interactions with ever larger number of particles via the long-range Coulomb potential, but we know this is in fact mitigated by Debye screening, which is neglected by the approximation $\epsilon \approx 1$. Indeed, if we renege on this approximation and istead use (66) at $k\lambda_{De} \ll 1$, this gets us an extra factor of $k_\perp^4$ in the integral, which comfortably heals the divergence. That is the reason why the LB CI is usually viewed as more general and inclusive of more physics that what we are about to get under (65). However, unless we wish to work in the approximation where f is located at very large scales only, both in space ($k\lambda_{De} \ll 1$) and in time ($\omega \ll \omega_{pe}$), it seems that using LB CI would be "double-counting" the dynamics that is already included in the Vlasov part of the kinetic equation.

this is how they do it in Galactic Dynamics

i.e., $f = f_0$ in the sense of my KT notes

[see discussion on pp. 28a-b.]

So, I think we ought to insist that the coarse-graining average that separated δF from f was over sub-Debye scales (i.e., $k \geq k_{min} \gg \lambda_{De}^{-1}$). Then we must bite the bullet of the divergent integral:

$$\Lambda \equiv \int \frac{dk_\perp}{k_\perp} = \ln \frac{k_{max}}{k_{min}} \underset{\substack{\uparrow \\ \text{non-rigorously}}}{=} \ln \frac{\lambda_{De}}{d} \sim N_D \qquad (68)$$

distance of closest approach ("Landau scale") $\frac{e}{d} \sim T$

This is called the Coulomb logarithm.

Obviously, in fact,

$$k_{max} \ll d^{-1} \text{ and } k_{min} \gg \lambda_{De}^{-1}$$

but since the divergence is logarithmic, this matters little (and the exact value of Λ can always be fit to experiment!) [Is this the context in which Landau quipped, dismissively, that "log is not even a function"?]

So, with these approximations, the LB CI (58) turns into the Landau (1936) CI:

$$\boxed{\frac{Df}{Dt} = \frac{\omega_{pe}^4 \Lambda}{8\pi n_e^2} \frac{\partial}{\partial \vec{v}} \cdot \int \frac{d\vec{v}''}{w} \left(\mathbb{1} - \frac{\vec{w}\vec{w}}{w^2} \right) \cdot \left[f(\vec{v}'') \frac{\partial f}{\partial \vec{v}} - f(\vec{v}) \frac{\partial f}{\partial \vec{v}''} \right]} \qquad (69)$$

This has all the same key physical properties as the LB CI (Fokker–Planck form, conservation laws, H-theorem, Maxwellian fixed point) and is the actual practical thing that people use for real calculations.

Landau first derived it as a specialisation of Boltzmann's CI to glancing collisions between Coulombically interacting particles with Rutherford collision cross-section (see Parra's notes).

Let us read off from (69) the collision frequency:

$$\nu_{ee} \sim \frac{\omega_{pe}^4}{n_e^2} \Lambda \, v_{the}^{-1+\cancel{3}-1-1} \frac{n_e}{\cancel{v_{the}^3}} \sim \frac{\omega_{pe}^4 \Lambda}{n_e v_{the}^3}$$

$$\sim \omega_{pe} \frac{\Lambda}{n_e \lambda_{De}^3} \sim \omega_{pe} \frac{\overset{\sim \ln N_D}{\Lambda}}{N_D}, \qquad (70)$$

which is exactly what we expected – (5) – apart from the appearance of Λ.

So, the theory seems to be sane.

In terms of primary parameters of the equilibrium:

$$\boxed{\nu_{ee} \sim \frac{e^4 n_e^{\cancel{2}} \Lambda_e}{\cancel{n_e} m_e^2 (T_e/m_e)^{3/2}} \sim \frac{e^4 n_e \Lambda_e}{m_e^{1/2} T_e^{3/2}}} \qquad (71)$$