Kinetic theory of Stellar systems: Problem set

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1) Eddington inversion
Any distribution $f(\mathcal{E})$ that is a function only of the energy of the particles $\mathcal{E} = \frac{1}{2}m|\boldsymbol{v}|^2 + m\phi(\boldsymbol{x})$ of the mean potential $\phi(\boldsymbol{x})$ is a quasi-steady-state for a stellar system. For a given potential, finding an $f(\mathcal{E})$ which generates it requires an 'Eddington inversion'.

a) Consider a monotonic spherical potential $\phi(r) < 0$ with $\phi(r) \to 0$ as $r \to \infty$, self-consistently generated by the distribution $f(\mathcal{E})$, which consists of only bound orbits. Argue that the mass density $\rho(r)$ can be written purely as a function of potential. Thus by formally writing the mass density as a function of $f(\mathcal{E})$ and differentiating with respect to potential, show that

$$\frac{\mathrm{d}\rho}{\mathrm{d}\phi} = -4\pi \int_{m\phi}^{0} \mathrm{d}\mathcal{E}\, \frac{f(\mathcal{E})}{\sqrt{2(\frac{\mathcal{E}}{m} - \phi)}}$$

This is an Abel integral equation. Its solution is given by

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}\mathcal{E}} \int_{\mathcal{E}}^{0} \mathrm{d}\phi\, \frac{\frac{\mathrm{d}\rho}{\mathrm{d}\phi}}{\sqrt{\phi - \frac{\mathcal{E}}{m}}}$$

(*optional: show this*)

This formula for the distribution function in terms of the potential is known as the *'Eddington inversion'*.

b) The Plummer Sphere is a spherical distribution of matter with potential and density given by

$$\phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}.$$

Find the distribution of matter $\rho(r)$ that generates this potential. Using an Eddington inversion of otherwise, find the distribution function f that generates this matter distribution.

2) Isochrone Potential and Bertrand's theorem
We consider the dynamics of a test particle within a central potential, $\phi(r)$. Such a potential is generically integrable, and is characterised by the two actions J_r and L, respectively the radial and azimuthal actions, with associated orbital frequencies Ω_r, Ω_ϕ. Show that

$$\left(\frac{\partial J_r}{\partial E}\right)_L = \frac{1}{\Omega_r}; \quad \left(\frac{\partial J_r}{\partial L}\right)_E = -\frac{\Omega_\phi}{\Omega_r} \tag{0.1}$$

with E and L, respectively the energy and angular momentum of the orbit.

In Henon (1959), Michel Hénon introduced the 'isochrone potential' as the central potential

$$\phi(r) = -\frac{GM_{\mathrm{tot}}}{b + \sqrt{b^2 + r^2}} \tag{0.2}$$

with b the system's scale length. He showed in particular that this potential is the most generic potential for which $\Omega_r(E, L) = \Omega_r(E)$, i.e., for which the radial frequency is only a function of the orbit energy. For the isochrone potential, one finds

$$\Omega_r = \frac{(-2E)^{3/2}}{GM_{\mathrm{tot}}} \quad \frac{\Omega_\phi}{\Omega_r} = \frac{1}{2}\left(1 + \frac{L}{\sqrt{L^2 + 4GM_{\mathrm{tot}}b}}\right) \tag{0.3}$$

Relying on the fact that partial derivative commute, or otherwise, use the isochrone potential to prove Bertrand's theorem: *The only central potentials for which all bound orbits are closed are the Keplerian and Harmonic Potentials.*

3) Linear response in homogeneous stellar systems.
The dimensionless response Matrix of a stellar system is given by

$$M_{pq}(\omega) = (2\pi)^3 \sum_{\boldsymbol{k}} \int \mathrm{d}\boldsymbol{J}\, \frac{\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{J}}}{\omega - \boldsymbol{k}\cdot\boldsymbol{\Omega}(\boldsymbol{J})} [\phi_{\boldsymbol{k}}^{(p)}(\boldsymbol{J})]^* \phi_{\boldsymbol{k}}^{(q)}(\boldsymbol{J}) \tag{0.4}$$

In this exercise, we set out to show that for homogeneous systems this response matrix reduces to the dielectric function of plasma physics. *(Hint: if you are a plasma physicist and you are getting annoyed about factors of 2π and L throughout this calculation, start in $(\boldsymbol{x}, \boldsymbol{v})$ space from the beginning and then note that the $\boldsymbol{p}$ are vectors of integers, not $2\pi/L \times$ integers. You should be able to reverse engineer all the results you want)*
(a) Assume that the system is placed within a periodic 3D box of size L, and that the mean potential vanishes, i.e., $\phi_0 = 0$, so that unperturbed trajectories are straight lines. The system's angle-action coordinates and the associated orbital

frequencies are given by

$$\boldsymbol{\theta} = \frac{2\pi}{L}\boldsymbol{x}, \quad \boldsymbol{J} = \frac{L}{2\pi}\boldsymbol{v}, \quad \boldsymbol{\Omega} = \frac{2\pi}{L}\boldsymbol{v}. \tag{0.5}$$

The system's instantaneous potential and DF are linked by

$$\phi(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{v}'\, f(\boldsymbol{x}',\boldsymbol{v}')U(\boldsymbol{x},\boldsymbol{x}'), \tag{0.6}$$

where $U(\boldsymbol{x},\boldsymbol{x}') = -\frac{Gm}{|\boldsymbol{x}-\boldsymbol{x}'|}$ is the gravitational pairwise interaction. Making use of the periodicity of the system and assuming $U(\boldsymbol{\theta},\boldsymbol{\theta}')$ is translationally invariant show that,

$$U(\boldsymbol{\theta},\boldsymbol{\theta}) = -\frac{Gm}{L\pi}\sum_{\boldsymbol{p}\neq\boldsymbol{0}}\frac{e^{i\boldsymbol{p}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')}}{|\boldsymbol{p}|^2} \tag{0.7}$$

Using this, show that the response matrix becomes

$$M_{\boldsymbol{pq}}(\omega) = \frac{GL^2m}{\pi}\delta_{\boldsymbol{pq}}\frac{1}{|\boldsymbol{p}|^2}\int \mathrm{d}\boldsymbol{v}\,\frac{\boldsymbol{p}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}}{\bar{\omega}-\boldsymbol{p}\cdot\boldsymbol{v}}, \tag{0.8}$$

where $\bar{\omega} = (2\pi)^{-1}L\omega$.

(b) Assume that the system's mean distribution function (DF) follows the Maxwellian distribution:

$$f_0(\boldsymbol{v}) = \frac{n_0}{(2\pi\sigma^2)^{3/2}}e^{-|\boldsymbol{v}|^2/(2\sigma^2)} \tag{0.9}$$

where n_0 is the system's mean density, and σ is the velocity dispersion. Show that, in this case, the expression for the system's response matrix reduces to

$$M_{\boldsymbol{pq}}(\omega) = \delta_{\boldsymbol{pq}}\left(\frac{L}{L_{\mathrm{J}}}\right)^2\frac{1}{|\boldsymbol{p}|^2}\left[1+\zeta Z(\zeta)\right], \tag{0.10}$$

where we have

$$\zeta = \frac{\bar{\omega}}{\sqrt{2}|\boldsymbol{p}|\sigma}, \quad Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{e^{-u^2}}{u-\zeta}, \quad L_{\mathrm{J}} = \sqrt{\frac{\pi\sigma^2}{Gmn_0}}. \tag{0.11}$$

The length L_{J} is refered to as the 'Jean's length'. What is the main difference between this expression for a self-gravitating system and the analogous one for an electrostatic plasma? What happens for a system with $L > L_{\mathrm{J}}$? (Hint: write down the dispersion relation and then look for purely growing solutions $\omega = i\gamma$ for real γ, noting that Z is purely imaginary for a purely imaginary ζ).
(c) *(optional)* A globular cluster has a mass density of $n_0 m \sim 10^4 M_\odot\,\mathrm{pc}^{-3}$, a velocity dispersion of $\sim 6\mathrm{km\,s^{-1}}$. Compute the Jean's length for such a globular cluster and comment. Why would a similar calculation be less appropriate for the Milky Way?

4) Using Rostoker's principle to derive the Balescu–Lenard equation.
*It is easy to get lost in the mathematics required to derive the Balescu–Lenard equation. This question works through a different, and slightly less rigorous, derivation of the Balescu–Lenard equation based on an idea known as 'Rostoker's principle'. The idea is to posit that the complicated dressing (by the dielectric response) in a plasma or stellar system can still be thought of as a collection of uncorrelated interactions, provided we replace the 'bare' Coulombic/Newtonian interaction $\psi(\boldsymbol{r},\boldsymbol{r}')$ of each two-body encounter with its 'dressed' counterpart Ψ^d. This is to say that the objects that are deflecting off of each other are not single stars but rather correlated clouds of stars that can be treated as particles. Note we do not need to deal explicitly with the fluctuations here, so we drop the '0' subscript on mean-field quantities. In completing this question, you made find it useful to read [Hamilton (2021) MNRAS **501**, 3371]*

a) (*interaction of stars via a dressed potential*) Consider a 'test' star with coordinates $(\boldsymbol{\theta},\boldsymbol{J})$ and a 'field' star with coordinates $(\boldsymbol{\theta}',\boldsymbol{J}')$. Rostoker's principle says that we can forget about all the other stars, and treat the interaction of these two stars as if they were an isolated system with specific two-body Hamiltonian (units of (velocity)^2):

$$h = H(\boldsymbol{J}) + H(\boldsymbol{J}') + U^d(\boldsymbol{\theta},\boldsymbol{J},\boldsymbol{\theta}',\boldsymbol{J}'),$$

where $H(\boldsymbol{J})$ is the mean-field Hamiltonian. Here $U^d(\boldsymbol{\theta},\boldsymbol{J},\boldsymbol{\theta}',\boldsymbol{J}')$ is the dressed specific potential energy between a particle at phase-space location $(\boldsymbol{\theta},\boldsymbol{J})$ and a particle at $(\boldsymbol{\theta},\boldsymbol{J})$. (It consists of the usual Newtonian attraction plus collective effects; if we ignore these then $U^d \to -Gm/|\boldsymbol{r}-\boldsymbol{r}'|$). Let us expand the dressed susceptibility coefficients ψ^d as a Fourier series in the angle variables:

$$U^d(\boldsymbol{\theta},\boldsymbol{J},\boldsymbol{\theta}',\boldsymbol{J}') = m\sum_{\boldsymbol{kk}'}e^{i(\boldsymbol{k}\cdot\boldsymbol{\theta}-\boldsymbol{k}'\cdot\boldsymbol{\theta}')}\psi^d_{\boldsymbol{kk}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}'\cdot\boldsymbol{\Omega}')$$

Here $\psi^d_{\boldsymbol{k},\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}'\cdot\boldsymbol{\Omega}')$ is the dressed potential interaction we derived in the lectures, and we have put in the correct frequency dependence ($\omega = \boldsymbol{k}'\cdot\boldsymbol{\Omega}'$) basically because we know what the answer has to look like, but also because it is physically clear that these are the only frequencies available to the system (i.e., only the 'bath' stars can effect the dressed potential).

Treat the two-body interaction as a perturbation. To zeroth order in this perturbation, the test and field stars just follow

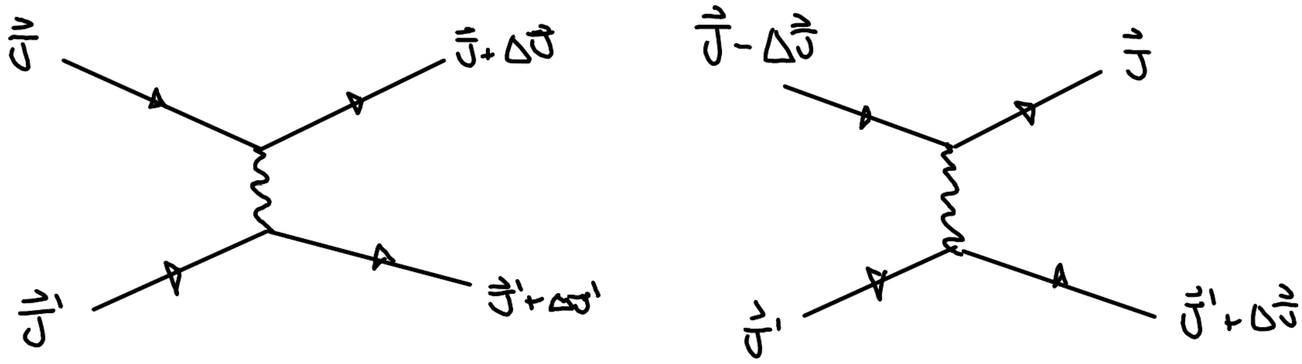

Figure 1: Two possibilities for 'scattering' in action coordinates. A test star's action changes by $\Delta \boldsymbol{J}$ during an 'encounter' with a field star whose action changes from $\boldsymbol{J}'$ to $\boldsymbol{J}' + \Delta\boldsymbol{J}'$. The DF $f(\boldsymbol{J})$ will be decremented if the test star is kicked out of $\boldsymbol{J}$ (left hand process), but will be incremented if the test star is kicked into state $\boldsymbol{J}$ from $\boldsymbol{J} - \Delta\boldsymbol{J}$ (right hand process). To write down the master equation we may also consider the inverse processes (by reversing the directions of the arrows).

their mean-field trajectories $\boldsymbol{\theta} = \boldsymbol{\theta}(t=0) + \boldsymbol{\Omega}t$ and $\boldsymbol{\theta}' = \boldsymbol{\theta}(t=0) + \boldsymbol{\Omega}'t$ indefinitely. To first order, the result of their interaction is to nudge each other to new actions $\boldsymbol{J} + \delta\boldsymbol{J}$ and $\boldsymbol{J}' + \delta\boldsymbol{J}'$ respectively. Show that

$$\delta\boldsymbol{J}(\boldsymbol{\theta}_0, \boldsymbol{J}, \boldsymbol{\theta}_0', \boldsymbol{J}', \tau) = -m \sum_{\boldsymbol{k}\boldsymbol{k}'} i\boldsymbol{k}\psi^d_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}') e^{i(\boldsymbol{k}\cdot\boldsymbol{\theta}_0 - \boldsymbol{k}'\cdot\boldsymbol{\theta}_0')} \frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')\tau} - 1}{i(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}')}.$$

The result for $\delta\boldsymbol{J}'$ is the same as for $\delta\boldsymbol{J}$ except we replace the first factor $i\boldsymbol{k} \to -i\boldsymbol{k}'$.

b) *The master equation* Following Rostoker, we consider the relaxation of our entire system to consist of nothing more than an uncorrelated set of dressed two-body encounters. Then it is easy to write down a master equation for the DF $f(\boldsymbol{J})$. To do so, we account for (a) test stars being nudged out of the state $\boldsymbol{J}$ and into some new state $\boldsymbol{J} + \Delta\boldsymbol{J}$ as illustrated in Figure 1a, and (b) test stars being nudged into the state $\boldsymbol{J}$ from $\boldsymbol{J} - \Delta\boldsymbol{J}$, as in Figure 1b. Processes (a) and (b) are both deterministic and time reversible, so we must also account for their inverses, which are the same diagrams but with the arrows pointing in the opposite direction.

Let the *transition rate density* be $w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')$. This quantity is defined such that $w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')\mathrm{d}\Delta\boldsymbol{J}\mathrm{d}\Delta\boldsymbol{J}'\mathrm{d}\tau$ is the probability that a given test star with initial action $\boldsymbol{J}$ is scattered to the volume of phase space $\mathrm{d}\Delta\boldsymbol{J}$ around $\boldsymbol{J} + \Delta\boldsymbol{J}$, by a given field star with action $\boldsymbol{J}'$ that is itself scatted to the volume element $\mathrm{d}\Delta\boldsymbol{J}'$ around $\boldsymbol{J}' + \Delta\boldsymbol{J}'$, in a time interval $\mathrm{d}\tau$. Assuming the system's equilibrium state is invariant under time reversal, argue that f satisfies the master equation

$$\begin{aligned}\frac{\partial f(\boldsymbol{J})}{\partial t} &= (2\pi)^3 \int \mathrm{d}\boldsymbol{J}'\, \mathrm{d}\Delta\boldsymbol{J}\mathrm{d}\Delta\boldsymbol{J}' \\ &\times \frac{1}{2}\Big\{ w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')\left[-f(\boldsymbol{J})f(\boldsymbol{J}') + f(\boldsymbol{J}+\Delta\boldsymbol{J})f(\boldsymbol{J}'+\Delta\boldsymbol{J}')\right] \\ &+ w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}-\Delta\boldsymbol{J}, \boldsymbol{J}')\left[f(\boldsymbol{J}-\Delta\boldsymbol{J})f(\boldsymbol{J}') - f(\boldsymbol{J})f(\boldsymbol{J}'+\Delta\boldsymbol{J}')\right]\Big\}\end{aligned}$$

By expanding the integrand for weak interactions, i.e., for $\Delta\boldsymbol{J}, \Delta\boldsymbol{J}' \ll \boldsymbol{J}, \boldsymbol{J}'$, up to second order in small quantities, show that

$$\frac{\partial f(\boldsymbol{J})}{\partial t} = \frac{\partial}{\partial \boldsymbol{J}} \cdot \int \mathrm{d}\boldsymbol{J}' \left[A \cdot f(\boldsymbol{J}')\frac{\partial f}{\partial \boldsymbol{J}} + B \cdot f(\boldsymbol{J})\frac{\partial f}{\partial \boldsymbol{J}'}\right].$$

where A is a 3 matrix:

$$A(\boldsymbol{J}, \boldsymbol{J}') = \frac{(2\pi)^3}{2} \int \mathrm{d}\Delta\boldsymbol{J}\, \mathrm{d}\Delta\boldsymbol{J}'\, w(\Delta\boldsymbol{J}, \Delta\boldsymbol{J}'|\boldsymbol{J}, \boldsymbol{J}')\Delta\boldsymbol{J}^2 \equiv \frac{(2\pi)^3}{2}\frac{\langle\Delta\boldsymbol{J}^2\rangle_\tau}{\tau},$$

where $\langle\Delta\boldsymbol{J}^2\rangle_\tau$ is the expectiation value of $\Delta\boldsymbol{J}^2$ after a time interval τ for a given test star action $\boldsymbol{J}$ and field star action $\boldsymbol{J}'$. What is the analogous exprression for B?

c) *(The kinetic equation)* Now we put the results from (a) and (b) together. Stars are uniformly distributed in the angle variables, we can calculate the expectation value $\langle\Delta\boldsymbol{J}^2\rangle_\tau$ by averaging over initial phases $\boldsymbol{\theta}_0, \boldsymbol{\theta}_0'$. Thus we have

$$\langle\Delta\boldsymbol{J}^2\rangle_\tau = \int \frac{\mathrm{d}\boldsymbol{\theta}_0}{(2\pi)^3}\frac{\mathrm{d}\boldsymbol{\theta}_0'}{(2\pi)^3}\delta\boldsymbol{J}(\boldsymbol{\theta}_0, \boldsymbol{J}, \boldsymbol{\theta}_0', \boldsymbol{J}', \tau)^2$$

where $\delta\boldsymbol{J}$ is given in above. Plugging the above equation in your expression for A and taking the limit $\tau \to \infty$ show that

$$\begin{aligned}A(\boldsymbol{J}, \boldsymbol{J}') &= \frac{m^2}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{\theta}_0\, \mathrm{d}\boldsymbol{\theta}_0' \sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{\boldsymbol{q}\boldsymbol{q}'} \boldsymbol{k}\boldsymbol{q}\Psi^d_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{k}'\cdot\boldsymbol{\Omega}')\Psi^d_{\boldsymbol{q}\boldsymbol{q}'}(\boldsymbol{J}, \boldsymbol{J}', \boldsymbol{q}'\cdot\boldsymbol{\Omega}') \\ &\times e^{i(\boldsymbol{k}+\boldsymbol{q})\cdot\boldsymbol{\theta}_0}e^{-i(\boldsymbol{k}'+\boldsymbol{q}')\cdot\boldsymbol{\theta}_0'} \lim_{\tau\to\infty}\left[\frac{1}{\tau}\frac{e^{i(\boldsymbol{k}\cdot\boldsymbol{\Omega}-\boldsymbol{k}'\cdot\boldsymbol{\Omega}')\tau} - 1}{\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}'}\frac{e^{i(\boldsymbol{q}\cdot\boldsymbol{\Omega}-\boldsymbol{q}'\cdot\boldsymbol{\Omega}')\tau} - 1}{\boldsymbol{q}\cdot\boldsymbol{\Omega} - \boldsymbol{q}'\cdot\boldsymbol{\Omega}'}\right]\end{aligned}$$

Making use of the following two identities,

$$\Psi^d_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}') = [\Psi^d_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}')]^*, \quad \text{and} \quad \lim_{\tau\to\infty} \frac{\left(e^{ix\tau}-1\right)^2}{x^2\tau} = 2\pi\delta(x)$$

show that,

$$A(\boldsymbol{J},\boldsymbol{J}') = \pi(2\pi)^3 m^2 \sum_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{k}\boldsymbol{k}\delta\left(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}'\right)|\Psi^d_{\boldsymbol{k}\boldsymbol{k}'}|^2.$$

The result for B is identical to $A(\boldsymbol{J},\boldsymbol{J}')$ except we replace the factor $\boldsymbol{k}\boldsymbol{k}$ with $-\boldsymbol{k}\boldsymbol{k}'$. Putting the explicit formulae for $A(\boldsymbol{J},\boldsymbol{J}')$ and $B(\boldsymbol{J},\boldsymbol{J}')$ in to the kinetic equation, recover the Balescu–Lenard equation:

$$\frac{\partial f(\boldsymbol{J})}{\partial t} = \pi(2\pi)^3 m^2 \frac{\partial}{\partial \boldsymbol{J}} \cdot \sum_{\boldsymbol{k}\boldsymbol{k}'} \boldsymbol{k} \int \mathrm{d}\boldsymbol{J}'\, \delta\left(\boldsymbol{k}\cdot\boldsymbol{\Omega} - \boldsymbol{k}'\cdot\boldsymbol{\Omega}'\right)|\Psi^{\mathrm{d}}_{\boldsymbol{k}\boldsymbol{k}'}(\boldsymbol{J},\boldsymbol{J}',\boldsymbol{k}'\cdot\boldsymbol{\Omega}')|^2$$
$$\times \left(\boldsymbol{k}\cdot\frac{\partial}{\partial \boldsymbol{J}} - \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{J}'}\right) f(\boldsymbol{J}')f(\boldsymbol{J})$$

d) *(optional)* Where in the above calculation did the *Stosszahlansatz* enter in (at what part did we break correlations between particles)? How would the diagrams change if you wanted to make the stellar matter bosonic or fermionic (this is allegedly relevant for certain models of 'fuzzy dark matter')? Can you derive the collision operator that would result from them?

5) Conservation laws and H-theorem.

a) The total mass, energy, and entropy of a stellar system are by

$$M = (2\pi)^d \int \mathrm{d}\boldsymbol{J}\, F(\boldsymbol{J})$$

$$E = (2\pi)^d \int \mathrm{d}\boldsymbol{J}\, H(\boldsymbol{J})F(\boldsymbol{J})$$

$$S = -(2\pi)^d \int \mathrm{d}\boldsymbol{J}\, F(\boldsymbol{J})\ln F(\boldsymbol{J})$$

show that the Balescu–Lenard equation conserves mass and energy. Show that the Balescu–Lenard equation increases the entropy of a system (i.e., $\frac{\partial S}{\partial t} \geqslant 0$).

b) (*Stellar equilibria*) Should it exist, we can define a Boltzmann distribution as $F_{\mathrm{B}}(J) = e^{-H(\boldsymbol{J})}$. Show that the Boltzmann distribution is not changed by the Balescu–Lenard equation. By writing

$$H = \frac{1}{2}m|\boldsymbol{v}|^2 + m\phi(\boldsymbol{x})$$

and assuming that the system enclosed a finite amount of mass (so that $\phi(r)\to 0$ as $r\to\infty$), argue that any self-gravitating solution to the Boltzmann distribution cannot exist. Therefore the Balescu–Lenard equation for self-gravitating systems describes the evaporation of stars from a system, rather than the relaxation of stars towards a thermal equilibrium.

c) (*optional—if you did the optional part of 3)*) Prove the conservation laws and H-theorem for the Balescu–Lenard collision operator for fermionic/bosonic matter (you will need to use a different function for the entropy)? What are the corresponding Stellar equilibria? Can they exist?

6) Kinetic Blockings. *(optional)*

We consider a generic long-range interacting Hamiltonian system in one spatial dimension (particles on a wire, HMF model, electrons in Magnetar-strength magnetic fields,... such systems do technically exist!). First, convince yourself that nothing in our derivation of the Balescu–Lenard equation required us to explicitly using the properties of the Poisson equation: it is a general derivation of particles with pair-wise interactions. Specialise to 1D systems (i.e., letting $\boldsymbol{k}=k$, $\boldsymbol{J}=J$, and $\boldsymbol{\Omega}(J)=\Omega(J)$), and consider an interaction potential, which is a function of only the difference in angles $\theta-\theta'$ between two particles

$$U(x[\theta,J],x'[\theta',J']) = U(\theta-\theta',J,J'). \tag{0.12}$$

Show that this makes the susceptibilities local in k, i.e., that

$$\psi_{k,k'}(J,J') \propto \delta_{kk'}, \tag{0.13}$$

therefore we have basis elements such that $\psi^{(p)}_k \propto \delta^{(p)}_k$. Show that the Balescu–Lenard collision operator reduces then reduces to the form

$$\frac{\partial f(J,t)}{\partial t} = 2\pi^2 m^2 \frac{\partial}{\partial J}\int \mathrm{d}J'\,|\Psi^{tot}(J,J')|^2 \delta_D(\Omega(J)-\Omega(J'))\left(\frac{\partial}{\partial J} - \frac{\partial}{\partial J'}\right) f(J,t)f(J',t)$$

give the form of the total dressed susceptibility $\Psi^{tot}|J,J'|^2$. What happens for a system with a monotonic frequency profile (i.e., $J\to\Omega(J)$ is one-to-one)? Intuitively, why can binary collisions not occur in such systems? How do such systems reach equilibrium in 1D?