

## §9. Back to KMHD: Mirror Instability & Barres Damping

### 9.1. Formal Calculation

Now that I have the kinetic equation for KMHD, and have explained at length how such kinetic equations are derived, I want to complete the job of doing the linear theory. Recall that in a simple, homogeneous, static ~~equilibrium~~ equilibrium considered in §4, I derived, for the perturbations:

$$\begin{cases} -\omega \delta \vec{B} = k_{||} \vec{u}_{\perp} \\ -\omega \frac{\delta B}{B_0} = -\vec{k}_{\perp} \cdot \vec{u}_{\perp} \end{cases} \quad \leftarrow \text{eq. (43), from induction eqn}$$

and

$$-\omega p_0 \vec{u} = -\vec{k}_{\perp} \left( \delta p_{\perp} + \frac{B_0 \delta B}{4\pi} \right) - \hat{z} k_{||} \left[ \delta p'_{||} + (p_{||0} - p'_{||0}) \frac{\delta B}{B_0} \right] \\ + \delta \vec{B} k_{||} \left( p_{||0} - p'_{||0} + \frac{B_0^2}{4\pi} \right) \quad \text{eq. (44)}$$

Now we want  
to extract  
modes with  
 $\delta B \neq 0$

{ the firehose instability, which were  
extracted by taking  $\perp$  part of (44)  
and then  $\vec{k}_{\perp} \times$  of it.  
These modes had  $\delta \vec{B} = -\frac{k_{||}}{\omega} \vec{u}_{\perp}$  and  
everything else = 0

In view of (43),  $\frac{\delta B}{B_0} = \frac{\vec{k}_{\perp} \cdot \vec{u}_{\perp}}{\omega}$ , so let us  
multiply eq. (44) by  $\vec{k}_{\perp}$ : (15b)

$$-\omega p_0 \underbrace{\vec{k}_{\perp} \cdot \vec{u}_{\perp}}_{\omega \frac{\delta B}{B_0}} = -k_{\perp}^2 \left( \delta p_{\perp} + \frac{B_0 \delta B}{4\pi} \right) + \underbrace{\vec{k}_{\perp} \cdot \delta \vec{B}}_{||} k_{||} \left( p_{||0} - p'_{||0} + \frac{B_0^2}{4\pi} \right) \\ - \frac{k_{||}}{\omega} \vec{k}_{\perp} \cdot \vec{u}_{\perp} = -k_{||} \frac{\delta B}{B_0}$$

$$\omega^2 \frac{\delta B}{B_0} = k_{\perp}^2 \left( \frac{\delta p_{\perp}}{p_0} + v_A^2 \frac{\delta B}{B_0} \right) + k_{\parallel}^2 \left( \frac{\delta p'_{\parallel 0} - p'_{\parallel 0}}{p_0} + v_A^2 \right) \frac{\delta B}{B_0} \quad (157)$$

} } }  
 inertia pressure tension

If we could calculate  $\delta p_{\perp}$ , we would be done.  
 But we can't, because we have the kinetic eqn,  
 which we are about to linearise.

Note that I have not done anything about the  
 $\parallel$  part of eq. (44), which is

$$-\omega p_0 u_{\parallel} = k_{\parallel} [\delta p'_{\parallel} + (p_{\parallel 0} - p'_{\parallel 0}) \frac{\delta B}{B_0}], \quad (158)$$

but actually I can obviate that by taking my  
 $\parallel$  kinetic variable to be  $v_{\parallel}$ . Then (158) will  
 automatically be contained in the linearisation  
 of the kinetic equation and I won't need to  
 worry about calculating  $\delta p'_{\parallel}$ .

So, the kinetic equation I want to use is (108):

$$\frac{\partial f}{\partial t} + \vec{V}_E \cdot \nabla f + v_{\parallel} \nabla_{\parallel} f + \left( \frac{q E_{\parallel}}{m_{\parallel}} \vec{b} \cdot \frac{D \vec{V}_E}{D t} - \mu \nabla_{\parallel} B \right) \frac{\partial f}{\partial v_{\parallel}} = 0$$

↑ ↑  
 same as  $\vec{U}_{\perp}$  —————

~~Derive from first principle~~

~~Derive from first principle~~

~~Derive from first principle~~

~~Derive from first principle~~

If, in equilibrium,  $\vec{u}_\perp = 0$ ,  $E_\parallel = 0$ ,  $\vec{B}_0 = B_0 \hat{z}$ , this equation becomes simply

$$\frac{\partial f_0}{\partial t} + v_{\parallel} \frac{\partial f_0}{\partial z} = 0$$

and we are free to choose any equilibrium that is not a function of  $t$  or  $z$ . Let's have

$$f_0 = f_0(\mu, v_{\parallel}) \quad . \quad (159)$$

Then

$$f = f_0(\mu, v_{\parallel}) + \delta f(t, \vec{r}, \mu, v_{\parallel}).$$

We will only need the  $\perp$  pressure:

$$\begin{aligned}
 p_{\perp} &= \cancel{\int d^3 \vec{w} \frac{m w_{\perp}^2}{2} f} = 2\pi \underbrace{\int dw_{\perp} w_{\perp}}_{\parallel} \underbrace{\int dv_{\parallel} \frac{m w_{\perp}^2}{2} f} = \\
 &= 2\pi m B^2 \underbrace{\int d\mu \mu}_{\parallel} \underbrace{\int dv_{\parallel} f}_{\parallel} = \underbrace{\frac{d w_{\perp}}{2}}_{\parallel} = \underbrace{m \mu B}_{\parallel} \\
 &\quad B_0 f_0 B \quad f_0 + \delta f \\
 &= 2\pi m B_0^2 \left(1 + 2 \frac{\delta B}{B_0}\right) \iint d\mu \mu dv_{\parallel} (f_0 + \delta f) \\
 &= 2\pi m B_0^2 \underbrace{\left\{ d\mu \mu \int dv_{\parallel} \left[ f_0 + 2 \frac{\delta B}{B_0} f_0 + \delta f \right] \right\}}_{\parallel} = \\
 &= p_{\perp 0} + 2 \frac{\delta B}{B_0} p_{\perp 0} + 2\pi m B_0^2 \underbrace{\int d\mu \mu \int dv_{\parallel} \delta f}_{\parallel} \quad . \quad (160)
 \end{aligned}$$

Now linearize (108) and calculate  $\delta f$ :

$$-i\omega \delta f + ik_{\parallel} v_{\parallel} \delta f + \left(\frac{q}{m} E_{\parallel} - \mu ik_{\parallel} \delta B\right) \frac{\partial f_0}{\partial v_{\parallel}} = 0$$

Hence

$$\delta f = \frac{1}{i(\omega - k_{\parallel} v_{\parallel})} \left( \frac{q}{m} E_{\parallel} - ik_{\parallel} \mu \delta B \right) \frac{\partial f_0}{\partial v_{\parallel}} \quad (162)$$

we shall need this

Remember that  $E_{\parallel}$  is computed from the quasineutrality constraint (61) :

$$\sum_{\alpha} q_{\alpha} n_{\alpha} = 0 \quad \Rightarrow \quad \sum_{\alpha} q_{\alpha} \delta n_{\alpha} = 0 \quad (163)$$

So we need to know density as well : similarly to (160), we have

$$\begin{aligned} n &= 2\pi B \int d\mu \int dv_{\parallel} f = 2\pi B_0 \left( 1 + \frac{\delta B}{B_0} \right) \int d\mu \int dv_{\parallel} (f_0 + \delta f) \\ &= n_0 + \underbrace{\frac{\delta B}{B_0} n_0}_{\delta n} + 2\pi B_0 \int d\mu \int dv_{\parallel} \delta f \end{aligned}$$

Sum over species :

$$\sum_{\alpha} q_{\alpha} 2\pi B_0 \int d\mu \int dv_{\parallel} \delta f_{\alpha} = 0 \quad (164)$$

$\uparrow$   
(162)

$$\frac{1}{i} E_{\parallel} \sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} 2\pi B_0 \int d\mu \int dv_{\parallel} \frac{1}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}} =$$

$$= ik_{\parallel} \delta B \sum_{\alpha} q_{\alpha} 2\pi B_0 \int d\mu \int dv_{\parallel} \frac{1}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}}$$

$$E_{\parallel} = ik_{\parallel} \frac{\delta B}{B_0} \frac{\sum_{\alpha} q_{\alpha} 2\pi B_0^2 \int d\mu \int dv_{\parallel} \frac{1}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}}}{\sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} 2\pi B_0 \int d\mu \int dv_{\parallel} \frac{1}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}}} \quad (165)$$

$\propto (\omega, k_{\parallel})$

-50 -

Thus, from (162),

$$\delta f = \frac{k_{\parallel}}{\omega - k_{\parallel} v_{\parallel}} \frac{\delta B}{B_0} \left[ \frac{q}{m} \chi(\omega, k_{\parallel}) - \mu B_0 \right] \frac{\partial f_0}{\partial v_{\parallel}} \quad (166)$$

~~From~~ From (160),

$$\delta p_{\perp} = k_{\parallel} \frac{\delta B}{B_0} 2\pi B_0^2 \int d\mu \int dv_{\parallel} \frac{\frac{q}{m} \chi - \mu B_0}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}} + 2 \frac{\delta B}{B_0} p_{\perp 0} \quad (167)$$

Quite a lot of generality is possible in these calculations, but I want to show you a simple case of a "bi-Maxwellian" equilibrium with no species drifts:

$$f_0 = \frac{n_0}{\pi^{3/2} V_{th\perp}^2 V_{th\parallel}^2} e^{-\frac{2\mu B_0}{V_{th\perp}^2} - \frac{v_{\parallel}^2}{V_{th\parallel}^2}} \quad (168)$$

Then

$$2\pi B_0 \int d\mu f_0 = \frac{n_0}{\sqrt{\pi} V_{th\parallel}} e^{-\frac{v_{\parallel}^2}{V_{th\parallel}^2}} \quad (169)$$

$$2\pi B_0^2 \int d\mu \mu f_0 = \frac{V_{th\perp}^2}{2} \frac{n_0}{\sqrt{\pi} V_{th\parallel}} e^{-\frac{v_{\parallel}^2}{V_{th\parallel}^2}} \quad (170)$$

$$2\pi B_0^3 \int d\mu \mu^2 f_0 = \frac{V_{th\perp}^4}{2} \underbrace{\frac{n_0}{\sqrt{\pi} V_{th\parallel}}} \underbrace{e^{-\frac{v_{\parallel}^2}{V_{th\parallel}^2}}}_{\text{''''}} \quad (171)$$

Then

$F_M(v_{\parallel})$  1D Maxwellian

$$\delta p_{\perp} = 2 p_{\perp 0} \frac{\delta B}{B_0} + k_{\parallel} \frac{\delta B}{B_0} \cancel{\frac{m v_{th\perp}^2}{2}} \int dv_{\parallel} \frac{\frac{q}{m} \chi - V_{th\perp}^2}{\omega - k_{\parallel} v_{\parallel}} \frac{\partial F_M(v_{\parallel})}{\partial v_{\parallel}} - \frac{2 v_{\parallel}}{V_{th\parallel}^2} F_M(v_{\parallel}) \quad //$$

$$= 2P_{\perp 0} \frac{\delta B}{B_0} - \frac{\delta B}{B_0} \frac{V_{th\perp}^2}{V_{th\parallel}^2} \left( q\chi - m V_{th\perp}^2 \right) \underbrace{\int dV_{\parallel} \frac{k_{\parallel} V_{\parallel}}{\omega - k_{\parallel} V_{\parallel}} F_M(V_{\parallel})}_{\parallel} =$$

Note also that

$$P_{\perp 0} = n_0 \frac{m V_{th\perp}^2}{2} \quad \leftarrow \text{see (170)}$$

and

$$P_{\parallel 0} = 2\pi B_0 \int d\mu \int dV_{\parallel} m V_{\parallel}^2 f_0$$

$$= n_0 \frac{m V_{th\parallel}^2}{2} \quad \leftarrow \text{see (16g)}$$

$$\left. \begin{aligned} & -n_0 + \omega \int dV_{\parallel} \frac{F_M(V_{\parallel})}{\omega - k_{\parallel} V_{\parallel}} \\ & = -n_0 - \frac{\omega}{k_{\parallel}} \int dV_{\parallel} \frac{F_M(V_{\parallel})}{V_{\parallel} - \frac{\omega}{k_{\parallel}}} \end{aligned} \right\}$$

$$= -n_0 [1 + \zeta Z(\zeta)]$$

$$\text{where } \zeta = \frac{\omega}{|k_{\parallel}| V_{th\parallel}} \quad \text{and}$$

$$Z(\zeta) = \int dx \frac{e^{-x}}{x - \zeta}$$

plasma dispersion function

$$= 2P_{\perp 0} \frac{\delta B}{B_0} + \frac{\delta B}{B_0} \frac{P_{\perp 0}}{P_{\parallel 0}} (q\chi n_0 - 2P_{\perp 0}) [1 + \zeta Z(\zeta)] \quad (172)$$

By a similar calculation,

$$\begin{aligned} \chi &= \frac{-\sum_{\alpha} q_{\alpha} \cancel{\frac{V_{th\perp\alpha}^2}{2}} \cancel{\frac{q}{2V_{th\parallel\alpha}^2}} \int dV_{\parallel} \frac{V_{\parallel}}{\omega - k_{\parallel} V_{\parallel}} F_{M\alpha}(V_{\parallel})}{-\sum_{\alpha} \frac{q_{\alpha}^2}{m_{\alpha}} \frac{2}{V_{th\parallel\alpha}^2} \int dV_{\parallel} \frac{V_{\parallel}}{\omega - k_{\parallel} V_{\parallel}} F_{M\alpha}(V_{\parallel})} = \\ &= \frac{\sum_{\alpha} n_{\alpha} q_{\alpha} \frac{P_{\perp 0\alpha}}{P_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})]}{\sum_{\alpha} \frac{q_{\alpha}^2 n_{\alpha}^2}{P_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})]} \quad (173) \end{aligned}$$

Finally, note that (172) is for a single species, while for our dispersion relation (157), we need total  $\delta p_{\perp}$ , so

$$\begin{aligned} \delta p_{\perp} &= 2p_{\perp 0} \frac{\delta B}{B} + \frac{\delta B}{B} \left\{ \chi \sum_{\alpha} n_{0\alpha} q_{\alpha} \frac{p_{\perp 0\alpha}}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right. \\ &\quad \left. - 2 \sum_{\alpha} \frac{p_{\perp 0\alpha}^2}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right\} = \\ &= \frac{\left( \sum_{\alpha} n_{0\alpha} q_{\alpha} \frac{p_{\perp 0\alpha}}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right)^2}{\sum_{\alpha} \frac{q_{\alpha}^2 n_{0\alpha}^2}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})]} \end{aligned} \quad (174)$$

Let me normalize this to  $p_0 v_A^2 = \frac{B_0^2}{4\pi}$  and call  $\beta_{\perp\alpha} = \frac{p_{\perp 0\alpha}^2}{B_0^2}$

$$\begin{aligned} \frac{\delta p_{\perp}}{p_0 v_A^2} &= \frac{\delta B}{B} \left\{ \sum_{\alpha} \beta_{\perp\alpha} - \sum_{\alpha} \beta_{\perp\alpha} \frac{p_{\perp 0\alpha}}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] + \right. \\ &\quad \left. + \frac{\left( \sum_{\alpha} n_{0\alpha} q_{\alpha} \frac{p_{\perp 0\alpha}}{p_{\parallel 0\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right)^2}{2 \sum_{\alpha} \frac{q_{\alpha}^2 n_{0\alpha}^2}{\beta_{\parallel\alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})]} \right\} \end{aligned} \quad (175)$$

Whence (157) becomes

$$\boxed{\omega^2 = k_{\perp}^2 v_A^2 [1 + \{ \dots \}] + k_{\parallel}^2 v_A^2 \left[ 1 + \frac{1}{2} \cancel{\sum_{\alpha} (\beta_{\perp\alpha} - \beta_{\parallel\alpha})} \right]} \quad (176)$$

We can study this dispersion relation ad nauseam, but the thing to do is play with a simple limit.

Here is the winning ordering:

$$\frac{1}{\beta} \sim \left| 1 - \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right| \sim 3 \ll 1 \quad (177)$$

You will see it works once we have ~~done~~ worked through what it implies.

Under this ordering  $Z(\zeta_\alpha) \approx i\sqrt{\pi}$ , so

$$\{ \dots \} \approx \sum_\alpha \beta_{\perp\alpha} \left( 1 - \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right) - \sum_\alpha \beta_{\perp\alpha} \left( \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right) \zeta_\alpha i\sqrt{\pi} \approx 1$$

~~$\sum_\alpha \beta_{\perp\alpha} \left( \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right) \zeta_\alpha i\sqrt{\pi}$~~

$$+ \frac{\mathcal{O}\left(\frac{1}{\beta^2}\right)}{\mathcal{O}\left(\frac{1}{\beta}\right)} \leftarrow \text{this term negligible} \quad [\text{this is } \circledast \text{ in (175)}]$$

↑ since  $V_{the} \gg V_{thi}$ ,  
the electron term  
can be neglected  
here

$$\approx \sum_\alpha \beta_{\perp\alpha} \left( 1 - \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right) - \beta_{\perp i} \zeta_i i\sqrt{\pi} \quad (178)$$

Also, in (176),

$$\frac{\omega^2}{k_i^2 V_A^2} \sim \zeta_i^2 \beta_i \sim \frac{1}{\beta_i} \ll 1, \text{ so lhs can be neglected.}$$

We get

$$0 = 1 + \sum_\alpha \beta_{\perp\alpha} \left( 1 - \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} \right) - \beta_{\perp i} \zeta_i i\sqrt{\pi} + \frac{k_{\parallel i}^2}{k_i^2} \left[ 1 + \frac{1}{2} \sum_\alpha \beta_{\perp\alpha} \left( \frac{P_{\parallel\alpha}}{P_{\perp\alpha}} - 1 \right) \right]$$

$$-\beta_i i\sqrt{\pi} \frac{i\omega}{|k_{\parallel i}| V_{thi}} \approx \sum_\alpha \beta_{\perp\alpha} \left( \frac{P_{\perp\alpha}}{P_{\parallel\alpha}} - 1 \right) - 1$$

$\omega / |k_{\parallel i}| V_{thi} / i$

$\uparrow$   
distinction between  $\perp$   
 $\omega \parallel$  is small here

$$- \frac{k_{\parallel i}^2}{k_i^2} \left[ 1 + \frac{1}{2} \sum_\alpha \beta_{\perp\alpha} \left( 1 - \frac{P_{\parallel\alpha}}{P_{\perp\alpha}} \right) \right] \quad (179)$$

Clearly rhs is real, so  $i\omega$  is real  $\Rightarrow \omega = i\gamma$   
and  $-i\omega = \gamma$  :

$$\equiv \Delta$$

$$\gamma \approx \frac{|k_{\parallel}| v_{thi}}{\sqrt{\pi} \beta_i} \left\{ \sum_{\perp} \beta_{\perp \alpha} \left( \frac{P_{\perp \alpha}}{P_{\parallel \alpha}} - 1 \right) - \frac{1}{\frac{k_{\parallel}^2}{k_{\perp}^2} \left[ 1 + \frac{1}{2} \sum_{\perp} \beta_{\perp \alpha} \left( 1 - \frac{P_{\parallel \alpha}}{P_{\perp \alpha}} \right) \right]} \right\} > 0 \quad (180)$$

This, if positive,  
will produce  
instability

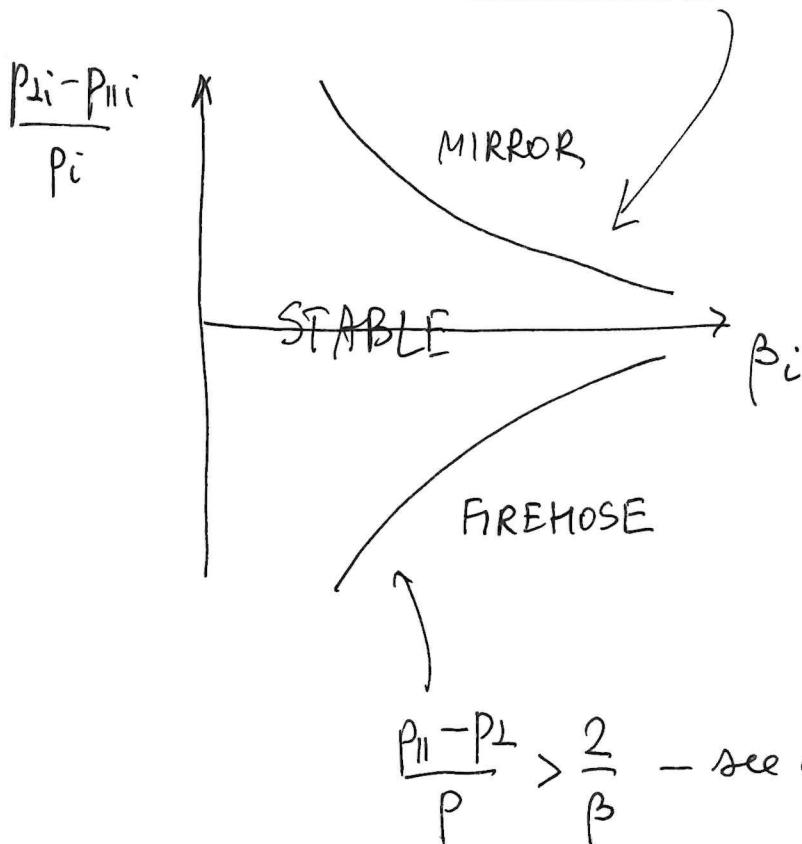
↑ this is stabilising  
but small if

condition for  
instability

$k_{\parallel} \ll k_{\perp}$  (so instability is highly oblique)

For simplicity, if electrons have no anisotropy,  
the stability boundary is

$$\boxed{\frac{P_{\perp i} - P_{\parallel i}}{P_i} > \frac{1}{\beta_i}} \quad (181)$$



This actually  
checks out  
rather well  
in the solar  
wind  
(observationally).

$$\frac{P_{\parallel} - P_{\perp}}{P} > \frac{2}{\beta} - \text{see eq. (47)}$$

What is the physics of this instability?

Looking back at (15b), we see that the "unstable" part of the dispersion relation (180) comes from the perpendicular pressure balance:

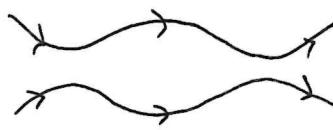
$$\underbrace{\delta p_{\perp} + \frac{B_0 \delta B}{4\pi}}_{\text{SS}} \approx 0 \quad (182)$$

$$\frac{B_0 \delta B}{4\pi} \left[ 1 - \sum_{\alpha} \beta_{1\alpha} \left( \frac{p_{1\alpha}}{p_{\parallel\alpha}} - 1 \right) + \sqrt{\pi} \beta_i \frac{\gamma}{|k_{\parallel}| V_{thi}} \right] = 0$$

↑                      ↑                      ↑  
 magnetic          non-resonant          resonant particle  
 pressure          particle pressure      pressure

We see that the effect of  $p_{1\alpha} > p_{\parallel\alpha}$  is to increase the number of  $w_{\perp} > w_{\parallel}$  particles in magnetic troughs ( $\delta B < 0$ ).

In the unstable regime, the total non-resonant + magnetic pressure gets negative and the system, dynamically, now wants to make greater magnetic compressions and rarefactions:



- a destabilized slow wave,  $\delta p_{\perp}^{\text{nonres}}$   
 but, in the collisionless

plasma, it is aperiodic because it is ~~essentially~~ governed not by balance of pressure and inertia (like in MHD) but balance between magnetic + nonresonant pressure and the pressure of the resonant particles.

NB: When  $p_{\perp} = p_{\parallel}$ , it does not have

In order to see where the resonant piece of the pressure is coming from, it is quite useful to rewrite the kinetic equation using yet another set of ~~variables~~ variables:

$$(\vec{r}, \mu, v_{\parallel}) \rightarrow (\vec{r}, \mu, \varepsilon), \quad \varepsilon = \mu B + \frac{v_{\parallel}^2}{2} \quad (183)$$

$$\left( \frac{Df}{Dt} \right)_{\mu, \varepsilon} = \left( \frac{Df}{Dt} \right)_{\mu, \varepsilon} + \underbrace{\left( \frac{D\varepsilon}{Dt} \right)_{\mu, v_{\parallel}}}_{\text{particle energy}} \frac{\partial f}{\partial \varepsilon}$$

$$\frac{\partial f}{\partial v_{\parallel}} = v_{\parallel} \frac{\partial f}{\partial \varepsilon}$$

$$\mu \frac{DB}{Dt} = \mu \left( \frac{dB}{dt} + v_{\parallel} \nabla_{\parallel} B \right)$$

cancels with  
- $\mu \nabla_{\parallel} B$   
term in (108)

$$\frac{Df}{Dt} + v_{\parallel} \left( \frac{q}{m} E_{\parallel} - \vec{b} \cdot \frac{D\vec{V}_E}{Dt} - \mu \nabla_{\parallel} B \right) \frac{\partial f}{\partial \varepsilon}$$

$$+ \mu \left( \frac{dB}{dt} + v_{\parallel} \nabla_{\parallel} B \right) \frac{\partial f}{\partial \varepsilon} = 0$$

$$\frac{Df}{Dt} + \left[ v_{\parallel} \left( \frac{q}{m} E_{\parallel} - \vec{b} \cdot \frac{D\vec{V}_E}{Dt} \right) + \mu \frac{dB}{dt} \right] \frac{\partial f}{\partial \varepsilon} = 0 \quad (184)$$

When  $p_{\perp 2} = p_{\parallel 2}$ , eq. (182)

gives

$$\gamma = -\sqrt{\pi} \beta_i |k_{\parallel}| V_{thi} \quad (185)$$

Barnes damping

a.k.a.

"transit-time damping"  
or "magnetic pumping"

the resonant bit comes from here. This effect is usually called "betatron acceleration", referring to what happens when  $\delta B$  is damped ( $\omega_0$  pressure anisotropy) and particles receive this energy via "Landau" resonance where mirror force  $-\mu \nabla_{\parallel} B$  plays the role of electric field.

To reiterate:

Stable case: magnetic pressure opposes formation of  $\delta B$  perturbations — then  $\gamma < 0$  to compensate, so rotation acceleration moves energy from  $\delta B$  to particles,  $\delta B$  is Barnes-damped.

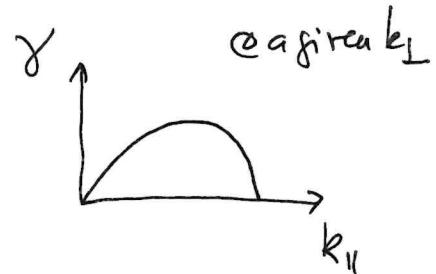
Unstable case: effective negative magnetic pressure favours  $\delta B$  perturbations — then  $\gamma > 0$  to compensate, so energy is moved from particles to  $\delta B$ ,  $\delta B$  is mirror-unstable.

Key takeaway: just like we did for  $p_{\parallel} > p_{\perp}$ , we have, for  $p_{\perp} > p_{\parallel}$  in high- $\beta$  plasma an instability with ~~a UV catastrophe at large  $k$~~ .

Indeed, (180) gives us

$$\gamma \approx \frac{|k_{\parallel}| v_{thi}}{\sqrt{\pi} \beta_i} \left\{ 1 - \frac{k_{\parallel}^2}{k_{\perp}^2} [\dots] \right\}$$

$$\left( \frac{\partial \gamma}{\partial k_{\parallel}} \right)_{k_{\perp}} \propto 1 - 3 \underbrace{\frac{k_{\parallel}^2}{k_{\perp}^2} [\dots]}_{\text{"at maximum } \frac{1}{3} \text{"}} = 0$$



Note that  
 $\gamma \ll |k_{\parallel}| v_{thi}$  as assumed in my derivation because

So

$$\gamma_{\max} = \frac{|k_{\parallel}| v_{thi}}{\sqrt{\pi} \beta_i} \frac{2}{3} \Lambda \quad (186)$$

where  $|k_{\parallel}| = k_{\perp} \sqrt{\frac{\Lambda}{3} [\dots]^{-1}}$  (187)

$\uparrow$  this can be arbitrarily large and larger it is the faster is the instability.

But this implies that the peak growth rate of this instability will be outside KMHD approximation (i.e. will involve FLR effects).

Further reading.

- More KMHD linear theory : Parra Notes sec IV  
(more systematic than my derivation)
- Mirror instability beyond KMHD approximation :  
Hellinger PoP 14, 082105 (2007)
- How mirror saturates :  
Rincon + MNRAS 447, L45 (2015)  
or Melville + MNRAS 459, 2701 (2016) for a qualitative picture (+ refs therein)  

Basically, as SB perturbations grow, they start trapping particles in regions of lower B - this effectively decreases the preponderance of  $w_{\perp} > w_{\parallel}$  particles in regions of higher SB and shuts down further growth
- Kunz + JPP 81, 325810501 (2015) - reduced KMHD (to KMHD what RMHD is to MHD), systematic description of linear pleyonis at  $k_{\parallel} \ll k_{\perp}$  as well as of free energy budgets.
- Kunz + JPP 86, 905860603 (2020) - how a sound wave manages to propagate in a collisionless high-beta plasma if it has finite amplitude - triggering fireholes and mirrors and thus creating its own "quasicollisional" environment.

## §10. Origin of Pressure Anisotropy

### 10.1 Qualitative Argument

So far I have taken the formal approach of picking a single, locally homogeneous equilibrium and linearising around it. The result is that if

$$\frac{|P_{\perp} - P_{\parallel}|}{P} \gtrsim \frac{1}{\beta} \quad (188)$$

any such "equilibrium" would be violently unstable to fast, microscale modes, which in fact take us outside the kMHD description. Their effect, once they saturate, will be, presumably to limit the dynamics to stable part of the parameter space — how that happens is a live research question, but the emerging answer is, crudely, that the plasma will, thanks to those instabilities, develop an effective collisionality that will pin pressure anisotropies locally to marginal levels.

Let me now inquire how ubiquitous the need for such non-kMHD adjustments of kMHD dynamics is likely to be.

At  $\beta \ll 1$ , there is no problem — we shall consider this limit in due course.

At  $\beta \gg 1$ , it is possible to ~~see~~ show that pressure anisotropies will be endemic by a very elementary

argument. We have already seen that, in KMF limit, particles conserve their magnetic moment:

$$\mu = \text{const}$$

But the mean of this, for particles of species  $\alpha$ , is

$$\langle \mu \rangle = \frac{1}{n_\alpha} \int d^3 \vec{w} \mu f_\alpha = \frac{P_{\perp\alpha}}{m_\alpha n_\alpha B} = \text{const} \quad (189)$$

~~$\frac{W_\perp^2}{2B}$~~

For the purposes of a qualitative discussion, let me pretend that  $n_\alpha = \text{const}$ . Then (189) means that, locally in a fluid element, every time your dynamics change  $B$ ,  $P_{\perp\alpha}$  must change proportionally:

$$\frac{1}{P_{\perp\alpha}} \frac{dP_{\perp\alpha}}{dt_\alpha} \sim \frac{1}{B} \frac{dB}{dt_\alpha} - \gamma_\alpha \frac{P_{\perp\alpha} - P_{\parallel\alpha}}{P_{\perp\alpha}} \quad (190)$$

$\uparrow$   
 $\mu$  conservation

If there are collisions,  
they will want to  
isotropize pressure

Thus, physically, we expect any dynamics that increase  $B$  to lead to  $P_{\perp} > P_{\parallel} \rightarrow$  mirror instability and any that decrease  $B$  to lead to  $P_{\perp} < P_{\parallel} \rightarrow$  firehose instability.

Such dynamics are ubiquitous  $\Rightarrow$  so will be the instabilities.

One simple way to see why instabilities were inevitable is to think of the kinematic dynamo problem. Take  $B$  that is sufficiently weak to exert no Lorentz force on the motions ( $\frac{B^2}{8\pi} \ll \rho u^2$ ) but sufficiently strong to lead to magnetized dynamics ( $S_{\alpha} \gg k u, \nu$ ). A generic 3D chaotic flow will amplify such a field multi-fold — in MHD. But in KMHD, every time  $B$  is doubled, so should  $p_{\perp}$  — the thermal energy. Clearly this is impossible, so  $\mu$  conservation must be broken — and it is, by microinstabilities.

Let us observe something interesting about the competition between  $\mu$  conservation and collisions in (190). Suppose  $\nu_{\alpha} \gg \omega$  (collisional limit, but still  $\nu_{\alpha} \ll \Omega_{\alpha}$ ). Then

$$\frac{P_{\perp\alpha} - P_{\parallel\alpha}}{P_{\perp\alpha}} \sim \frac{1}{\nu_{\alpha}} \frac{1}{B} \frac{\partial B}{\partial t_{\alpha}} = \frac{\hat{b}\hat{b} : \nabla \tilde{u}_{\alpha} - \nabla \cdot \tilde{u}_{\alpha}}{\nu_{\alpha}}$$

$$P_{\perp\alpha} - P_{\parallel\alpha} \sim \sum_{\alpha} \frac{P_{\perp\alpha}}{\nu_{\alpha}} (\hat{b}\hat{b} : \nabla \tilde{u}_{\alpha} - \nabla \cdot \tilde{u}_{\alpha}) \quad (191)$$

This goes into the momentum equation (39).

Thus, we get

$$\rho \frac{d\tilde{u}}{dt} \sim \nabla \cdot \hat{b}\hat{b} \frac{P_{\perp\alpha}}{\nu} (\hat{b}\hat{b} : \nabla \tilde{u} - \nabla \cdot \tilde{u}) \quad (192)$$

↑  
|| viscosity

Pressure anisotropy is || viscous stress!  
("Brayinskii MHD")

Let me now work some of this out more formally.

Namely, I want to derive evolution equations for  $p_{\perp}$  and  $p_{\parallel \perp}$  from the kinetic equation.

The most convenient form of it for the purpose of this calculation is ~~(53)~~ (53) :

$$\frac{Df_\alpha}{Dt_\alpha} + \frac{1}{B} \frac{DB}{Dt_\alpha} \frac{w_\perp}{2} \frac{\partial f_\alpha}{\partial w_\perp} + \left( \frac{q_\alpha}{m_\alpha} E_\parallel - \frac{D\vec{u}_\alpha \cdot \hat{b}}{Dt_\alpha} - \frac{w_\perp^2}{2B} \nabla_\parallel B \right) \frac{\partial f_\alpha}{\partial w_\parallel} = 0$$

Multiply by  $\frac{m_\alpha w_\perp^2}{2}$  and integrate (dropping species indices) :

$$\begin{aligned} \frac{dP_\perp}{dt} + \hat{b} \cdot \nabla \underbrace{\int d^3 \vec{w} w_\parallel \frac{m w_\perp^2}{2} f}_{\substack{\parallel \\ q_\perp \text{ parallel flux} \\ \text{of } \perp \text{ heat}}} + \frac{1}{B} \frac{dB}{dt} \underbrace{\int d^3 \vec{w} \frac{m w_\perp^3}{4} \frac{\partial f}{\partial w_\perp}}_{\substack{\parallel \\ \text{by parts}}} + \\ + \frac{\nabla_\parallel B}{B} \underbrace{\int d^3 \vec{w} w_\parallel \frac{m w_\perp^3}{4} \frac{\partial f}{\partial w_\perp}}_{\substack{\parallel \\ \text{analogously}}} - \nabla \cdot \hat{b} - 2q_\perp \end{aligned}$$

$$\begin{aligned} & \int 2\pi dw_\perp dw_\parallel \frac{m w_\perp^4}{4} \frac{\partial f}{\partial w_\perp} \\ & \text{by parts} \\ & = - \int 2\pi dw_\perp dw_\parallel m w_\perp^3 f \\ & = - 2P_\perp \end{aligned}$$

$$+ \int d^3 \vec{w} \frac{m w_\perp^2}{2} \left( \frac{q}{m} E_\parallel - \frac{D\vec{u} \cdot \hat{b}}{Dt} - w_\parallel \hat{b} \hat{b} : \nabla \vec{u} - \frac{w_\perp^2}{2B} \nabla_\parallel B \right) \frac{\partial f}{\partial w_\parallel} = \text{colls.}$$

$$+ (\hat{b} \hat{b} : \nabla \vec{u}) P_\perp = (\hat{b} \hat{b} : \nabla \vec{u} - \nabla \cdot \vec{u}) P_\perp + \nabla \cdot \vec{u} P_\perp$$

$$= \frac{1}{B} \frac{dB}{dt} P_\perp = \frac{1}{n} \frac{dn}{dt} P_\perp$$

↑ cancels 2 in

$$\underbrace{\frac{dp_{\perp}}{dt} - \frac{p_{\perp}}{B} \frac{dB}{dt} - \frac{p_{\perp}}{n} \frac{dn}{dt}}_{P_{\perp} \frac{d}{dt} \ln \frac{p_{\perp}}{nB}} = -\underbrace{\nabla_{||} q_{\perp} - 2q_{\perp} \nabla \cdot \hat{b}}_{-\nabla \cdot q_{\perp} \hat{b} + q_{\perp} \nabla \cdot \hat{b} - 2q_{\perp} \nabla \cdot \hat{b}} + \text{colls} \quad (193)$$

$$\boxed{P_{\perp} \frac{d}{dt} \ln \frac{p_{\perp}}{nB} = -\nabla \cdot q_{\perp} \hat{b} - q_{\perp} \nabla \cdot \hat{b} + \text{colls}} \quad (194)$$

↑  
cf. (189)      heat fluxes, i.e.  
                        leakage of energy  
                        from/to fluid  
                        element

Now multiply (53) by  $m w_{||}^2$  and integrate:

$$\begin{aligned} \frac{dp_{||}}{dt} + \hat{b} \cdot \nabla \int d^3 \vec{w} w_{||} m w_{||}^2 f + \\ \underbrace{\int d^3 \vec{w} m w_{||}^2}_{\text{"parallel flux}} \underbrace{\frac{w_{\perp}}{2} \frac{\partial f}{\partial w_{\perp}}}_{\text{"of heat}} \\ + \frac{1}{B} \frac{dB}{dt} \int d^3 \vec{w} m w_{||}^2 \frac{w_{\perp}}{2} \frac{\partial f}{\partial w_{\perp}} \\ - \int d^3 \vec{w} m w_{||}^2 f = -p_{||} \end{aligned}$$

This can be derived, but it's also intuitive that collisions should push pressures together.

$$\begin{aligned} + \frac{\nabla_{||} B}{B} \int d^3 \vec{w} m w_{||}^3 \frac{w_{\perp}}{2} \frac{\partial f}{\partial w_{\perp}} + 2 \int d^3 \vec{w} \frac{m w_{\perp}^2}{2} w_{||} f \frac{\nabla_{||} B}{B} = \\ - \nabla \cdot \hat{b} - \int d^3 \vec{w} m w_{||}^3 f = -q_{||} \end{aligned}$$

$$+ \int d^3 \vec{w} m w_{||}^2 \left( \frac{q}{m} E_{||} - \frac{d \vec{u}}{dt} \cdot \hat{b} - w_{||} \hat{b} \cdot \nabla \vec{u} - \frac{w_{\perp}^2}{2} \frac{\nabla_{||} B}{B} \right) \frac{\partial f}{\partial w_{||}} = \text{colls}$$

because, after integration by parts,  $\int d^3 \vec{w} w_{||} f = 0$

$$3p_{||} (\hat{b} \cdot \nabla \vec{u}) = 3p_{||} \left( \frac{1}{B} \frac{dB}{dt} - \frac{1}{n} \frac{dn}{dt} \right)$$

$$\underbrace{\frac{dp_{||}}{dt} + 2 \frac{p_{||}}{B} \frac{dB}{dt} - 3 \frac{p_{||}}{n} \frac{dn}{dt}}_{P_{||} \frac{d}{dt} \ln \frac{P_{||} B^2}{n^3}} = \underbrace{-\nabla_{||} q_{||} - q_{||} \nabla \cdot \vec{b} + 2q_{\perp} \nabla \cdot \vec{b} + \text{colls}}_{-\nabla \cdot \vec{b} q_{||}} \quad (195)$$

$$P_{||} \frac{d}{dt} \ln \frac{P_{||} B^2}{n^3}$$

$$\boxed{P_{||} \frac{d}{dt} \ln \frac{P_{||} B^2}{n^3} = -\nabla \cdot \vec{b} q_{||} + 2q_{\perp} \nabla \cdot \vec{b} + \text{colls.}} \quad (196)$$

Eqs (194) & (196) are called

Chew-Goldberger-Low

equations (CGL), or,

sometimes, "double-adiabatic".

Double because, while (194)

confirms conservation of  $\mu$  (modulo heat fluxes),

(196) suggests that there is another adiabatic invariant in the system, ~~and~~ also modulo heat fluxes, whose conservation controls parallel pressure:

$$\frac{P_{||} B^2}{m n^3} = \text{const} \quad (197) - \text{cf. (189)}$$

$$\left( \frac{1}{n} \int d^3 \vec{W} W_{||}^2 \right) \frac{B^2}{n^2} \propto \langle J^2 \rangle \text{ by analogy with}$$

$J$  is called the "longitudinal invariant". It is

$$J = W_{||} l_{\alpha} \quad (198)$$

length of a fluid element of species  $\alpha$  along the magnetic-field line.

since collisions must not change total energy

$$\left[ \frac{d}{dt} \left( p_{\perp} + \frac{P_{||}}{2} \right) \right]_{\text{colls}} = 0$$

so the term here must be  
 $-2 \rightarrow (P_{||} - P_{\perp})$

ignoring  
 inters  
 colls.

here is why this is. We know from MHD (see my notes §B.7) that a separation vector between two fluid particles satisfies

$$\frac{d}{dt} \delta \vec{r} = \delta \vec{r} \cdot \nabla \vec{u} \quad (199)$$

which, if it is initially  $\parallel \vec{B}$  makes it  $\delta \vec{r} \propto \frac{\vec{B}}{n}$ , which also satisfies

because

$$\begin{aligned} \frac{d}{dt} \delta \vec{r} &= \vec{u}(\vec{r} + \delta \vec{r}) - \vec{u}(\vec{r}) \\ &\approx \delta \vec{r} \cdot \nabla \vec{u} \end{aligned}$$

$$\frac{d}{dt} \frac{\vec{B}}{n} = \frac{\vec{B}}{n} \cdot \nabla \vec{u} \quad (200)$$

if  $\vec{B}$  is frozen into the flow  $\vec{u}$ .

Therefore,  $l \propto \frac{B}{n} \Rightarrow J \propto w_{\parallel} \frac{B}{n} \Rightarrow \langle J^2 \rangle$  satisfies (197).

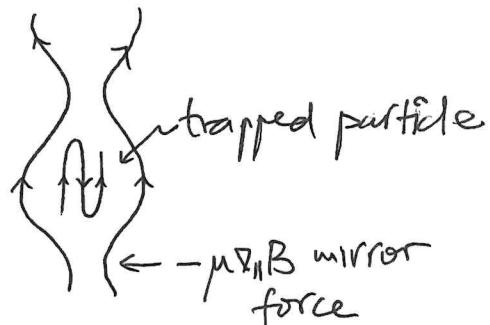
This type of adiabaticity is related to particles getting trapped (or just slowed down) in magnetic mirrors as they stream along field lines. Then

$$\oint dl w_{\parallel} = \text{const} \quad (201) \quad \text{"2nd adiabatic invariant"}$$

provided  $w_b \sim k_{\parallel} w_{\parallel} \gg \omega \sim k u$

$f_{\text{bounce}}$  frequency

(for passing particles, replace bounce average by field-line average)



Parenthetically, let me show why bouncing happens.

~~higher order~~ eps of particle motion to lowest order:

$$\dot{\vec{r}} = v_{\parallel} \hat{b}$$

assuming static fields  
and  $v_E = 0$ .

$$\dot{v}_{\parallel} = \frac{q}{m} E_{\parallel} - \mu \nabla_{\parallel} B$$

$$(202)$$

Energy of a particle:

$$E = \frac{mv_{||}^2}{2} + m\mu B(\vec{r}) + q\phi(\vec{r}) \quad (203)$$

potential,  $E_{||} = -\nabla_{||}\phi$

$$\dot{E} = \underbrace{mv_{||}\dot{v}_{||}}_{\cancel{\frac{m}{2}qV_{||}E_{||}}} + \underbrace{m\mu\vec{F}\cdot\nabla B}_{\cancel{m\mu V_{||}\nabla_{||}B}} + \underbrace{q\vec{r}\cdot\nabla\phi}_{\cancel{qV_{||}\nabla_{||}\phi}} = 0 \text{ conserved}$$

$$-\cancel{m\mu V_{||}\nabla_{||}B}$$

Then motion along the field:

$$\dot{l} = v_{||} = \pm \sqrt{\frac{2}{m} [2 - \mu B(l) - q\phi(l)]} \quad (204)$$

particle can move if  $[... < 0]$   
anywhere.

#### 10.4 Pressure Anisotropy

Finally calculate pressure anisotropy: (193)-(195)

$$\frac{d}{dt}(p_{\perp} - p_{||}) = (p_{\perp} + 2p_{||}) \frac{1}{B} \frac{dB}{dt} + (p_{\perp} - 3p_{||}) \frac{1}{n} \frac{dn}{dt} -$$

$$- \nabla \cdot [(q_{\perp} - q_{||}) \vec{B}] - 3q_{\perp} \nabla \cdot \vec{B} - 3\gamma(p_{\perp} - p_{||}) \quad (205)$$

this can  
be ignored in  
this limit  
because  
 $p_{\perp} - p_{||} \sim \frac{\omega}{\nu} \ll 1$

heat fluxes also  
cause pressure  
aniso

both changing  $B$   
and changing  $n$   
will produce  
pressure aniso.

Collisional limit:  $\nu \gg \omega \sim k u$ ,  $p_{\perp} - p_{||} \ll p_{\perp}, p_{||}$ .

Then

$$\frac{p_{\perp} - p_{||}}{p} \approx \frac{1}{\nu} \left\{ \underbrace{\frac{1}{B} \frac{dB}{dt} - \frac{2}{3} \frac{1}{n} \frac{dn}{dt}}_{\vec{B} \cdot \nabla u - \frac{1}{3} \nabla \cdot \vec{u}} - \frac{\nabla \cdot [(q_{\perp} - q_{||}) \vec{B}] + 3q_{\perp} \nabla \cdot \vec{B}}{3p} \right\} \quad (206)$$

Braginskii  
II stress  
(II viscosity)