

S6 Revision: Relationship between Kinetic Equations and Particle Motion

Let me briefly go back to basics. Suppose I have a collection of particles in phase space described by the phase-space coordinate \vec{q} - e.g., $\vec{q} = (\vec{r}, \vec{v})$.

Let us say that individual particle motion is described by $\left\{ \begin{array}{l} \text{but not necessarily, e.g.} \\ \vec{q} = (\vec{r}, \mu, w_{||}, \vartheta) \end{array} \right.$

$$\dot{\vec{q}} = \vec{V}(\vec{q}, t) - \text{generalised phase velocity, e.g. } \vec{V} = \left(\vec{v}, \frac{q_{\perp}}{m_{\perp}} (\vec{E} + \vec{v} \times \vec{B}) \right) \quad (62)$$

Then conservation of particles implies (cf. MHD Notes §12.1):

$$\frac{d}{dt} \int_{\Gamma} d\vec{q} P(\vec{q}, t) = - \int_{\partial\Gamma} (P\vec{V}) \cdot d\vec{S} = - \int_{\partial\Gamma} d\vec{q} \frac{\partial}{\partial \vec{q}} \cdot (P\vec{V}) \quad (63)$$

$\int_{\Gamma} d\vec{q} P(\vec{q}, t)$ is # of particles in phase volume Γ
 $P(\vec{q}, t)$ is phase-space density of particles
 $\int_{\partial\Gamma} (P\vec{V}) \cdot d\vec{S}$ is flux of particles in or out

Now make Γ infinitesimal and get

$$\boxed{\frac{\partial P}{\partial t} + \frac{\partial}{\partial \vec{q}} \cdot (\vec{V} P) = 0} \quad (64)$$

Vlasov equation follows for $\vec{q} = (\vec{r}, \vec{v})$, and $P = f$.

\uparrow More precisely Klimontovich equation if we use precise \vec{E} and \vec{B} . Vlasov eqn if we separate the macroscopic fields, shove the microscopic ones into collisions, then neglect collisions [see §1.6 and §1.8 of the KT Notes]

Now suppose we change phase-space variables

$$(\vec{r}, \vec{v}) \rightarrow \vec{q}$$

Eq. (64) holds, but $P \neq f$. Indeed, the derivation (63) required P to be normalized in such a way that

$\int d\vec{q} P = N$ the # of particles. Therefore

$$\int d\vec{r} d\vec{v} f = \int d\vec{q} \underbrace{\left| \det \frac{\partial(\vec{r}, \vec{v})}{\partial \vec{q}} \right|}_{\equiv J} f \quad \Rightarrow \quad P = Jf. \quad (65)$$

standard stuff.

Thus, if we stick with the definition of f in the "original" variables \vec{r}, \vec{v} , then

$$P(\vec{q}) = J(\vec{q}) \underbrace{\tilde{f}(\vec{r}(\vec{q}), \vec{v}(\vec{q}))}_{\text{colloquially, I often call this } f(\vec{q})} \quad (66)$$

and f satisfies

$$\boxed{\frac{\partial Jf}{\partial t} + \frac{\partial}{\partial \vec{q}} \cdot (\vec{v} Jf) = 0} \quad (67)$$

↑ collisions can be added
 $\left(\frac{\partial Jf}{\partial t}\right)_c$.

An example is

$$\vec{q} = (\vec{r}, \mu, w_{\parallel}, \vartheta) \leftarrow \text{the variables in which we ended up writing KMHD}$$

where

$$\mu = \frac{w_{\perp}^2}{2B} = \frac{|\vec{v} - \vec{u} \cdot (\mathbb{1} - \hat{b}\hat{b})|^2}{2B} \quad (68)$$

$$w_{\parallel} = (\vec{v} - \vec{u}) \cdot \hat{b}$$

and ϑ is defined by

$$\vec{v} = \vec{u} + \sqrt{2B\mu} \cos \vartheta \hat{x} + \sqrt{2B\mu} \sin \vartheta \hat{y} + w_{\parallel} \hat{z}$$

$\hat{z} \equiv \hat{b}$

Then

$$\det \frac{\partial(\vec{r}, \vec{v})}{\partial \vec{q}} = \det \frac{\partial \vec{v}}{\partial(\mu, w_{||}', \vartheta)} = \det \begin{pmatrix} \sqrt{\frac{B}{2\mu}} \cos \vartheta & \sqrt{\frac{B}{2\mu}} \sin \vartheta & 0 \\ 0 & 0 & 1 \\ -\sqrt{2B\mu} \sin \vartheta & \sqrt{2B\mu} \cos \vartheta & 0 \end{pmatrix}$$

$$= -B \quad (69)$$

Thus, the kinetic equation in these variables is

$$\frac{\partial Bf}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\dot{\vec{r}} Bf) + \frac{\partial}{\partial \mu} (\dot{\mu} Bf) + \frac{\partial}{\partial w_{||}'} (\dot{w}_{||}' Bf) + \frac{\partial}{\partial \vartheta} (\dot{\vartheta} Bf) = \left(\frac{\partial Bf}{\partial t} \right)_c \quad (70)$$

Note that (67) can, in fact, be rewritten as

$$\frac{\partial f}{\partial t} + \dot{\vec{r}} \cdot \nabla f + \dot{\mu} \frac{\partial f}{\partial \mu} + \dot{w}_{||}' \frac{\partial f}{\partial w_{||}'} + \dot{\vartheta} \frac{\partial f}{\partial \vartheta} = \left(\frac{\partial f}{\partial t} \right)_c \quad (71)$$

because and, more generally,

(67) is, in fact

$$\boxed{\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{q}} = \left(\frac{\partial f}{\partial t} \right)_c} \quad (72)$$

because

$$\frac{\partial J}{\partial t} + \frac{\partial}{\partial \vec{q}} \cdot (\vec{v} J) = 0 \quad (73)$$

Proof This simply expresses conservation of phase volume, viz., let us calculate

$$\frac{d}{dt} \int_{\delta \Gamma} d\vec{r} d\vec{v} = \frac{d}{dt} \delta x \delta y \delta z \delta v_x \delta v_y \delta v_z =$$

$$= \left(\int_{\delta \Gamma} d\vec{r} d\vec{v} \right) \left(\frac{\delta \dot{x}}{\delta x} + \frac{\delta \dot{y}}{\delta y} + \frac{\delta \dot{z}}{\delta z} + \frac{\delta \dot{v}_x}{\delta v_x} + \frac{\delta \dot{v}_y}{\delta v_y} + \frac{\delta \dot{v}_z}{\delta v_z} \right)$$

infinitesimal phase vol.

$$= \left(\int_{\delta\Gamma} d\vec{r} d\vec{v} \right) \left(\frac{\partial}{\partial \vec{r}} \cdot \vec{r} + \frac{\partial}{\partial \vec{v}} \cdot \vec{v} \right) = 0$$

(this is in fact generically true for any Hamiltonian system) (74)

But this also means

$$0 = \frac{d}{dt} \int_{\delta\Gamma} d\vec{r} d\vec{v} = \frac{d}{dt} \int_{\delta\Gamma} d\vec{q} J(\vec{q}, t) =$$

$$= \frac{d}{dt} J(\vec{q}, t) \delta q_1 \dots \delta q_6 = \left(\int_{\delta\Gamma} J d\vec{q} \right) \left[\sum_i \frac{\delta q_i}{\delta q_i} + \frac{1}{J} \frac{dJ}{dt} \right]$$

Thus,

$$\frac{1}{J} \left(\frac{\partial J}{\partial t} + \vec{v} \cdot \frac{\partial J}{\partial \vec{q}} \right) + \frac{\partial}{\partial \vec{q}} \cdot \vec{v} = 0,$$

$$\begin{matrix} \frac{\partial}{\partial \vec{q}} \cdot \vec{v} & \frac{\partial J}{\partial t} + \frac{d\vec{q}}{dt} \cdot \frac{\partial J}{\partial \vec{q}} \\ \parallel & \parallel \\ \vec{v} & \vec{v} \end{matrix}$$

which is the same as (73), q.e.d.

Another way of proving this, purely ~~calculus~~ based on vector differentiation, is found in Ap.A of Parra Notes-2

OK, so I can always write the kinetic equation, in phase variables \vec{q} , as

$$\left(\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} = \left(\frac{\partial f}{\partial t} \right)_c \right) \quad (74)$$

particle motion.

Thus, instead of changing variables in the Vlasov eqn, I can just work out $\dot{\vec{q}}$ for any new opportune variables and then sub. this into (74).

This turns out to be a useful insight...

§7. Particle Motion in a Strong Magnetic Field.

7.1 Constant Fields

Let us start with a 1st-year undergraduate problem:
a charged particle in crossed, constant \vec{E} and \vec{B} fields:

$$\frac{d\vec{v}}{dt} = \frac{q}{m} \left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right) \quad (75)$$

This splits into

$$\frac{dv_{\parallel}}{dt} = \frac{q}{m} E_{\parallel} \Rightarrow v_{\parallel}(t) = v_{\parallel 0} + \frac{q}{m} E_{\parallel} t \quad (76)$$

and

$$\frac{d\vec{v}_{\perp}}{dt} = \frac{q}{m} \left(\vec{E}_{\perp} + \frac{\vec{v}_{\perp} \times \vec{B}}{c} \right) \quad (77)$$

↑ inhomogeneous
↑ homogeneous

→ particular integral is $\vec{v}_{\perp} = \vec{v}_{\perp E}$ where

$$\vec{E}_{\perp} + \frac{\vec{v}_{\perp E} \times \vec{B}}{c} = 0 \Rightarrow \vec{v}_{\perp E} = c \frac{\vec{E} \times \vec{B}}{B^2} \quad \text{E} \times \text{B drift} \quad (78)$$

So full solution is $\vec{v}_{\perp} = \vec{w}_{\perp} + \vec{v}_{\perp E}$, where the hom. soln satisfies

$$\frac{d\vec{w}_{\perp}}{dt} = \left(\frac{qB}{mc} \right) \vec{w}_{\perp} \times \hat{z} \quad \text{NB: } \frac{dw_{\perp}^2}{dt} = 0 \quad (79)$$

$\omega = \frac{qB}{mc}$
 $w_{\perp} = \text{const}$

Let $\vec{w}_{\perp} = w_{\perp} \cos \vartheta \hat{x} + w_{\perp} \sin \vartheta \hat{y}$

$$\frac{d\vec{w}_{\perp}}{dt} = \left[\hat{x} w_{\perp} (-\sin \vartheta) + \hat{y} w_{\perp} \cos \vartheta \right] \dot{\vartheta} = \Omega \left[\hat{x} w_{\perp} \sin \vartheta - \hat{y} w_{\perp} \cos \vartheta \right] \quad (80)$$

So $\boxed{\dot{\vartheta} = -\Omega} \Rightarrow \vartheta = -\Omega t + \vartheta_0$

Assemble:

$$\vec{v}(t) = v_{\parallel}(t) \hat{b} + \vec{v}_{\perp E} + w_{\perp} \left[\hat{x} \cos \vartheta + \hat{y} \sin \vartheta \right] \quad (81)$$

Get $\vec{r}(t)$ by integrating the above;

$$\vec{r}(t) = \vec{r}_0 + (v_{||0}t + \frac{qE_{||}}{2m}t^2)\hat{b} + c \frac{\vec{E}_{\perp} \times \vec{B}}{B^2} t + \dots$$

$$\left\{ -\frac{w_{\perp}}{\Omega} [\hat{x} \sin \vartheta - \hat{y} \cos \vartheta] \right\} \quad (82)$$

" Larmor radius

$$\rightarrow -\frac{\vec{w}_{\perp} \times \hat{b}}{\Omega}$$

The idea now is that if \vec{E} and \vec{B} are functions of t and \vec{r} , particle motion will look similar to (82) as long the characteristic time and length scales of the fields' variation are long compared to Ω^{-1} and ρ :

$$\rho_* \equiv \frac{\rho}{l} \ll 1 \quad \text{and} \quad \frac{\omega}{\Omega} \ll 1 \quad (83)$$

If $\omega \sim \frac{v_{th}}{l}$, then $\rho_* = \frac{\rho}{l} \sim \frac{\rho \omega}{v_{th}} \sim \frac{\omega}{\Omega}$

Another key thing to order is the size of the electric field \therefore from (76):

$$\omega v_{th} \sim \frac{q}{m} E_{||} \Rightarrow \cancel{E_{||}} E_{||} \sim \frac{m \omega v_{th}}{q} \sim \frac{m v_{th}^2}{q l} \quad (84)$$

and from (78),

$$E_{\perp} \sim \frac{v_{E \perp} B}{c} \sim \frac{q B}{m c} \frac{m v_{E \perp}}{q} \sim \frac{\Omega}{\omega} \frac{m \omega v_{th}}{q} \frac{v_{E \perp}}{v_{th}}$$

$$\sim \frac{1}{\rho_*} Ma E_{||} \gg E_{||} \quad \text{as long as} \quad Ma \gg \rho_* \quad (85)$$

this is called "high-flow ordering"

$Ma \sim \rho_*$ - "low-flow ordering"

The motion is begging to be formally recast in different variables:

$$\vec{v} = v_{\parallel} \hat{b} + \vec{v}_E + w_{\perp} (\hat{x} \cos \vartheta + \hat{y} \sin \vartheta) \quad (86)$$

~~scribbled out text~~

where v_{\parallel} , \vec{u} , w_{\perp} are expected to turn out to be slow ($\sim \omega, l$) functions of time and space, whereas \hat{e}_{\perp} will rotate fast. Formally,

$$(\vec{r}, \vec{v}) \rightarrow (\vec{r}, w_{\perp}, v_{\parallel}, \vartheta) \equiv \vec{q}(\vec{r}, \vec{v}, t) \quad (87)$$

and, in order to have a kinetic eqn in terms of \vec{q} , we need $\dot{\vec{q}}$ and plug it into (74).

Formally (no approximations yet!)

$$\begin{aligned} \dot{\vec{q}} &= \left(\frac{\partial \vec{q}}{\partial t} \right)_{\vec{r}, \vec{v}} + \dot{\vec{r}} \cdot \left(\frac{\partial \vec{q}}{\partial \vec{r}} \right)_{\vec{v}, t} + \dot{\vec{v}} \cdot \left(\frac{\partial \vec{q}}{\partial \vec{v}} \right)_{\vec{r}, t} = \\ &= \frac{\partial \vec{q}}{\partial t} + \vec{v} \cdot \nabla \vec{q} + \frac{q}{m} (\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \cdot \frac{\partial \vec{q}}{\partial \vec{v}} = \underbrace{\hspace{10em}}_{= E_{\parallel} \hat{b}} \\ &= \frac{\partial \vec{q}}{\partial t} + (w_{\parallel} \hat{b} + \vec{v}_E + \vec{w}_{\perp}) \cdot \nabla \vec{q} + \frac{q}{m} (\vec{E} + \frac{\vec{v}_E \times \vec{B}}{c} + \frac{B}{c} \hat{b} \times \hat{b}) \cdot \frac{\partial \vec{q}}{\partial \vec{v}} \\ &= \frac{D \vec{q}}{Dt} + \vec{w}_{\perp} \cdot \nabla \vec{q} + \left[\frac{q}{m} \underbrace{\hspace{10em}}_{E_{\parallel} \hat{b}} + \Omega \underbrace{\hspace{10em}}_{(\vec{w}_{\perp} \times \hat{b})} \right] \cdot \frac{\partial \vec{q}}{\partial \vec{v}} \end{aligned} \quad (88)$$

Most straightforwardly,

$$\dot{\vec{r}} = \vec{v} = w_{\parallel} \hat{b} + \vec{v}_E + \vec{w}_{\perp} \quad (89)$$

$$\langle \dot{\vec{r}} \rangle = w_{\parallel} \hat{b} + \vec{v}_E, \text{ anticipating the need to gyroaverage} \quad (90)$$

Slightly more work to calculate $\dot{\mathbf{v}}_{\parallel}$. Since

$$\mathbf{v}_{\parallel} = \hat{\mathbf{b}} \cdot \mathbf{v} \quad (91)$$

we have

$$\begin{aligned} \dot{\mathbf{v}}_{\parallel} &= \underbrace{(\hat{\mathbf{b}} \cdot \mathbf{v})}_{\parallel \hat{\mathbf{b}} + \hat{\mathbf{w}}_{\perp} + \hat{\mathbf{v}}_E} \cdot \frac{D\hat{\mathbf{b}}}{Dt} + \hat{\mathbf{w}}_{\perp} \cdot (\nabla \hat{\mathbf{b}}) \cdot \underbrace{(\hat{\mathbf{v}})}_{\parallel \hat{\mathbf{b}} + \hat{\mathbf{w}}_{\perp} + \hat{\mathbf{v}}_E} + \\ &+ \hat{\mathbf{b}} \cdot \left[\cancel{\dots} + \frac{q}{m} (\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c}) + \Omega \hat{\mathbf{w}}_{\perp} \times \hat{\mathbf{b}} \right] \\ &= \hat{\mathbf{w}}_{\perp} \cdot \frac{D\hat{\mathbf{b}}}{Dt} + \hat{\mathbf{w}}_{\perp} \hat{\mathbf{w}}_{\perp} : \nabla \hat{\mathbf{b}} + \cancel{\dots} + \frac{q}{m} E_{\parallel} \\ \langle \dot{\mathbf{v}}_{\parallel} \rangle &= \langle \hat{\mathbf{w}}_{\perp} \hat{\mathbf{w}}_{\perp} \rangle : \nabla \hat{\mathbf{b}} + \cancel{\dots} + \frac{q}{m} E_{\parallel} \quad (92) \\ &\quad \uparrow \text{all terms of order } \sim \frac{v_{th}^2}{L} \\ &= \frac{w_{\perp}^2}{2} (\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}}) : \nabla \hat{\mathbf{b}} = \frac{w_{\perp}^2}{2} \nabla \cdot \hat{\mathbf{b}} = -\frac{w_{\perp}^2}{2B} \nabla_{\parallel} B \end{aligned}$$

$$\rightarrow = -\mu \nabla_{\parallel} B - \hat{\mathbf{b}} \cdot \frac{D\hat{\mathbf{v}}_E}{Dt} + \frac{q}{m} E_{\parallel} \quad (93) \quad (94)$$

Since $\hat{\mathbf{w}}_{\perp} = (\mathbf{v} - \hat{\mathbf{v}}_E) \cdot (\mathbb{1} - \hat{\mathbf{b}}\hat{\mathbf{b}})$ and $w_{\perp} = |\hat{\mathbf{w}}_{\perp}| = \sqrt{\hat{\mathbf{w}}_{\perp} \cdot \hat{\mathbf{w}}_{\perp}}$

$$\dot{w}_{\perp} = \frac{1}{2w_{\perp}} 2\hat{\mathbf{w}}_{\perp} \cdot \dot{\hat{\mathbf{w}}}_{\perp} = \frac{\hat{\mathbf{w}}_{\perp} \cdot \dot{\hat{\mathbf{w}}}_{\perp}}{w_{\perp}} =$$

$$\cancel{\dots}$$

$$\begin{aligned} &= \frac{\hat{\mathbf{w}}_{\perp} \cdot \dot{\hat{\mathbf{v}}}}{w_{\perp}} \left[\hat{\mathbf{v}} - \frac{D\hat{\mathbf{v}}_E}{Dt} - \hat{\mathbf{w}}_{\perp} \cdot \nabla \hat{\mathbf{v}}_E - \hat{\mathbf{v}}_{\parallel} \hat{\mathbf{b}} - \hat{\mathbf{v}}_{\parallel} \left(\frac{D\hat{\mathbf{b}}}{Dt} + \hat{\mathbf{w}}_{\perp} \cdot \nabla \hat{\mathbf{b}} \right) \right] \\ &= \frac{\hat{\mathbf{w}}_{\perp} \cdot \hat{\mathbf{v}}}{w_{\perp}} \left[\frac{q}{m} (\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c}) + \Omega \hat{\mathbf{w}}_{\perp} \times \hat{\mathbf{b}} - \frac{D\hat{\mathbf{v}}_E}{Dt} - \hat{\mathbf{w}}_{\perp} \cdot \nabla \hat{\mathbf{v}}_E \right. \\ &\quad \left. - \hat{\mathbf{v}}_{\parallel} \left(\frac{D\hat{\mathbf{b}}}{Dt} + \hat{\mathbf{w}}_{\perp} \cdot \nabla \hat{\mathbf{b}} \right) \right] \end{aligned}$$

$$= - \frac{\vec{w}_\perp}{w_\perp} \cdot \left(\frac{D\vec{v}_E}{Dt} + v_{||} \frac{D\hat{b}}{Dt} \right) - \frac{1}{w_\perp} \vec{w}_\perp \vec{w}_\perp : (\nabla \vec{v}_E + v_{||} \nabla \hat{b}) \quad (95)$$

↑ all terms of order $\frac{v_{th}^2}{c}$

$$\langle \dot{w}_\perp \rangle = - \frac{w_\perp}{2} (\mathbb{1} - \hat{b}\hat{b}) : (\nabla \vec{v}_E + v_{||} \nabla \hat{b}) =$$

$$= - \frac{w_\perp}{2} (\nabla \cdot \vec{v}_E - \hat{b}\hat{b} : \nabla \vec{v}_E + v_{||} \nabla \cdot \hat{b})$$

→ Note that $\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} = \nabla \times (\vec{v}_E \times \vec{B}) - c \nabla \times E_{||} \hat{b} =$

$$= - \frac{\vec{v}_E \times \vec{B}}{c} + E_{||} \hat{b} \text{ identity!}$$

$$= -\vec{v}_E \cdot \nabla \vec{B} + \vec{B} \cdot \nabla \vec{v}_E - \vec{B} \nabla \cdot \vec{v}_E - c \nabla \times E_{||} \hat{b}$$

$$\frac{d\vec{B}}{dt} = \vec{B} \cdot \nabla \vec{v}_E - (\nabla \cdot \vec{v}_E) \vec{B} - c \nabla \times E_{||} \hat{b}$$

$$\frac{1}{B} \frac{dB}{dt} = \hat{b}\hat{b} : \nabla \vec{v}_E - \nabla \cdot \vec{v}_E - \frac{c}{B} \hat{b} \cdot (\nabla \times E_{||} \hat{b})$$

$$= \hat{b} \cdot (\nabla \times \hat{b})$$

Therefore,

$$\langle \dot{w}_\perp \rangle = \frac{w_\perp}{2} \left(\frac{1}{B} \frac{dB}{dt} + \frac{cE_{||}}{B} \hat{b} \cdot (\nabla \times \hat{b}) + v_{||} \frac{\nabla_{||} B}{B} \right) =$$

$$= \frac{\mu}{w_\perp} \left(\frac{DB}{Dt} + cE_{||} \hat{b} \cdot (\nabla \times \hat{b}) \right) \quad (96)$$

For $\mu = \frac{w_\perp^2}{2B}$, we have

$$\dot{\mu} = \frac{w_\perp \dot{w}_\perp}{B} - \frac{w_\perp^2}{2B^2} \left(\frac{DB}{Dt} + \vec{w}_\perp \cdot \nabla B \right) = \frac{w_\perp \dot{w}_\perp}{B} - \mu \left(\frac{D}{Dt} + \vec{w}_\perp \cdot \nabla \right) \ln B \quad (97)$$

$$\langle \dot{\mu} \rangle = \mu \left(\frac{D \ln B}{Dt} + \frac{cE_{||}}{B} \hat{b} \cdot (\nabla \times \hat{b}) \right) - \mu \frac{D \ln B}{Dt}$$

$$= \mu \frac{cE_{||}}{B} \hat{b} \cdot (\nabla \times \hat{b}) \quad (98)$$

$$(84) \rightarrow \mu \frac{cmv_{th}^2}{2 \cdot 0} \frac{1}{l} \sim \mu \frac{v_{th}}{l} \frac{\rho}{l} \text{ 1st order}$$

So μ is considered to lowest order

Finally we need $\dot{\vartheta}$.

Recall, from p. 30, that

$$\dot{\vec{w}}_{\perp} = E_{\parallel} \hat{b} + \Omega \vec{w}_{\perp} \times \hat{b} - \frac{D\vec{v}_E}{Dt} - \vec{w}_{\perp} \cdot \nabla \vec{v}_E - v_{\parallel} \hat{b} - v_{\parallel} \left(\frac{D\hat{b}}{Dt} + \vec{w}_{\perp} \cdot \nabla \hat{b} \right) \quad (99)$$

$$\vec{w}_{\perp} = w_{\perp} (\hat{x} \cos \vartheta + \hat{y} \sin \vartheta)$$

$$w_{\perp} (\hat{x} \sin \vartheta - \hat{y} \cos \vartheta)$$

↓

$$\dot{\vartheta} = -\Omega + \text{small terms of order } \frac{v_{th}}{l} \quad (100)$$

↑
we shall see that we need not calculate them explicitly

Thus, particles gyrate quickly and then change their v_{\parallel} and w_{\perp} slowly while drifting with velocity \vec{v}_E .
Magnetic moment μ is conserved.

§ 8 Derivation of Drift Kinetics from Particle Motion

8.1 High Flow

The reason this is all useful is that we can now have a systematic procedure for derivation of kinetic equations (which in this context are usually referred to as "drift kinetic").

Indeed, from (74), the kinetic equation (exact) is

$$\underbrace{\frac{\partial f}{\partial t} + \vec{r} \cdot \nabla f + \dot{\mu} \frac{\partial f}{\partial \mu} + \dot{v}_{\parallel} \frac{\partial f}{\partial v_{\parallel}}}_{\text{call this } \hat{L} f} + \dot{v}_{\perp} \frac{\partial f}{\partial v_{\perp}} = 0 \quad (101)$$

-Ω + ...

Then
$$\Omega \frac{\partial f}{\partial v_{\perp}} = \hat{L} f + (\dot{v}_{\perp} + \Omega) \frac{\partial f}{\partial v_{\perp}} \quad (102)$$

Let $f = \langle f \rangle + \delta f$. Then

↑ gyroaveraged
distribution

$$\Omega \frac{\partial \delta f}{\partial v_{\perp}} = \hat{L} (\langle f \rangle + \delta f) + (\dot{v}_{\perp} + \Omega) \frac{\partial \delta f}{\partial v_{\perp}} \quad (103)$$

In the ρ_* expansion, the lhs \gg rhs, so, to lowest order,

$$\Omega \frac{\partial \delta f_0}{\partial v_{\perp}} = 0 \Rightarrow \delta f_0 = 0 \quad (104)$$

~~Therefore~~ Thus, δf is in fact 1st-order in ρ_* .

~~Therefore~~

~~Therefore~~

~~Therefore~~

~~Therefore~~

Thus, $f = \langle f \rangle + \delta f_1 + \delta f_2 + \dots$ in powers of ρ_* .

Then, eq. (103) to lowest order is

$$\Omega \frac{\partial \delta f_1}{\partial t} = \hat{L} \langle f \rangle \quad (105)$$

We can extract an equation for $\langle f \rangle = f(\vec{r}, \mu, w_{||}, t)$ by averaging (105) over \mathcal{D} : ↑
our previous notation

$$\langle \hat{L} \rangle \langle f \rangle = 0 \quad (106)$$

$$\text{or } \frac{\partial f}{\partial t} + \langle \dot{\vec{r}} \rangle \cdot \nabla f + \langle \dot{\mu} \rangle \frac{\partial f}{\partial \mu} + \langle \dot{w}_{||} \rangle \frac{\partial f}{\partial w_{||}} = 0 \quad (107)$$

$$\left(\vec{v}_E + v_{||} \hat{b} \right) \quad \text{eq. (89)}$$

$$\left(0(\rho_*) \right) \quad \text{eq. (98)}$$

$$\left(\frac{q}{m} E_{||} - \hat{b} \cdot \frac{D \vec{v}_E}{Dt} - \mu \nabla_{||} B \right) \quad \text{eq. (93)}$$

This is exactly eq. (58) ~~but with $\vec{u} \equiv \vec{v}_E$~~ .

$$\frac{\partial f}{\partial t} + \vec{v}_E \cdot \nabla f + v_{||} \nabla_{||} f + \left(\frac{q}{m} E_{||} - \hat{b} \cdot \frac{D \vec{v}_E}{Dt} - \mu \nabla_{||} B \right) \frac{\partial f}{\partial w_{||}} = 0 \quad (108)$$

The only \perp motion of particles is here. The existence of this term is predicated on the ordering

$$Ma \sim \frac{v_E}{v_{th}} \sim 1 \quad (\text{"high flow"})$$

It is possible, however, and, indeed, common to ask what happens if

↑
eq. (108) in this framework is called the "high-flow drift-kinetic equation"

$$Ma \ll 1, \text{ namely, } Ma \sim \rho_*. \quad (109)$$

Since this is a bit of a hairy calculation, it's good to be convenient to rearrange things a bit. Recall that eq. (101) is in fact equivalent to eq. (70):

$$\frac{\partial Bf}{\partial t} + \nabla \cdot (\dot{\mathbf{r}} Bf) + \frac{\partial}{\partial \mu} (\dot{\mu} Bf) + \frac{\partial}{\partial \mathbf{v}_{||}} (\dot{\mathbf{v}}_{||} Bf) + \frac{\partial}{\partial \vartheta} (\dot{\vartheta} Bf) = 0 \quad (113)$$

I shall now call $\vec{q} = (\mathbf{r}, \mu, \mathbf{v}_{||})$, excluding ϑ . Then

$$\frac{\partial Bf}{\partial t} + \frac{\partial}{\partial q_i} \dot{q}_i Bf + \frac{\partial}{\partial \vartheta} [(-\Omega + \dots) Bf] = 0 \quad (114)$$

$f = \langle f \rangle + \delta f$ as before. Plug in and average:

$$\left\langle \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial q_i} \dot{q}_i \right) B (\langle f \rangle + \delta f) \right\rangle = 0$$

\uparrow
 $\langle \dot{q}_i \rangle + \delta \dot{q}_i$

$$\left(\frac{\partial B \langle f \rangle}{\partial t} + \frac{\partial}{\partial t} B \langle \delta f \rangle + \frac{\partial}{\partial q_i} \langle \dot{q}_i \rangle B \langle f \rangle + \frac{\partial}{\partial q_i} \langle \dot{q}_i \rangle B \langle \delta f \rangle \right)$$

$$+ \frac{\partial}{\partial q_i} \langle \delta \dot{q}_i \rangle B \langle f \rangle + \frac{\partial}{\partial q_i} \langle \delta \dot{q}_i \delta f \rangle B = 0$$

$$\rightarrow B \left(\frac{\partial \langle f \rangle}{\partial t} + \langle \dot{q}_i \rangle \frac{\partial \langle f \rangle}{\partial q_i} \right) + \langle f \rangle \left(\frac{\partial B}{\partial t} + \frac{\partial}{\partial q_i} \langle \dot{q}_i \rangle B \right)$$

our old drift-kinetic equation (106)

○ - just gyroaveraged version of (73)
Can check directly.

So the new, "low-flow" equation will have the form

$$\left(\frac{\partial}{\partial t} + \langle \dot{q}_i \rangle \frac{\partial}{\partial q_i} \right) \langle f \rangle + \underbrace{\frac{1}{B} \frac{\partial}{\partial q_i} B \langle \delta \dot{q}_i \delta f \rangle}_{\text{new term that needs calculation}} = 0 \quad (115)$$

One gets δf from the unaveraged version of (114), written to lowest order:

$$\Omega \frac{\partial \delta f}{\partial \tau} = \underbrace{\left(\frac{\partial}{\partial t} + \dot{q}_i \frac{\partial}{\partial q_i} \right)}_{\hat{L}} \langle \delta f \rangle \quad (116) \quad \text{This is exactly the same as eq. (105)}$$

$$= \underbrace{\left(\frac{\partial}{\partial t} + \langle \dot{q}_i \rangle \frac{\partial}{\partial q_i} \right)}_{\langle \hat{L} \rangle} \langle f \rangle + \delta \dot{q}_i \frac{\partial \langle f \rangle}{\partial q_i}$$

$\langle \hat{L} \rangle \langle f \rangle$ we have agreed that this is ρ^{*2} because to lowest order $\langle \hat{L} \rangle \langle f \rangle = 0$

Thus,

$$\Omega \frac{\partial \delta f}{\partial \tau} = \delta \dot{q}_i \frac{\partial \langle f \rangle}{\partial q_i} \quad (117)$$

What are these? They are

$$\delta \dot{\vec{r}} = \vec{w}_\perp \quad \text{from (89)} \quad (118)$$

$$\delta \dot{w}_\parallel = \vec{w}_\perp \cdot \left[\frac{D \hat{b}}{Dt} - (\nabla \cdot \vec{v}_E) \hat{b} \right] + \left[\vec{w}_\perp \vec{w}_\perp - \frac{w_\perp^2}{2} (\mathbb{1} - \hat{b} \hat{b}) \right] : \nabla \hat{b} \quad (119)$$

from (92) & (93)

$$\delta \dot{\mu} = \frac{1}{B} \vec{w}_\perp \cdot \left[\text{[scribble]} - \frac{D \vec{v}_E}{Dt} - v_\parallel \frac{D \hat{b}}{Dt} - \mu \nabla_\parallel B \right] - \frac{1}{B} \left[\vec{w}_\perp \vec{w}_\perp - \frac{w_\perp^2}{2} (\mathbb{1} - \hat{b} \hat{b}) \right] : (\nabla \vec{v}_E + v_\parallel \nabla \hat{b}) \quad (120)$$

from (97) & (96)

This is the same as the equation for the perturbation of the parallel velocity in the low-frequency limit.

These can be readily integrated wrt ϑ if one notices that

$$\begin{aligned} \vec{w}_\perp &= w_\perp (\hat{x} \cos \vartheta + \hat{y} \sin \vartheta) = w_\perp \frac{\partial}{\partial \vartheta} (\hat{x} \sin \vartheta - \hat{y} \cos \vartheta) = \\ &= \frac{\partial}{\partial \vartheta} \vec{w}_\perp \times \hat{b} \end{aligned} \quad (121)$$

and the less obvious fact (Ex. check explicitly):

$$\vec{w}_\perp \vec{w}_\perp - \frac{w_\perp^2}{2} (\mathbb{1} - \hat{b} \hat{b}) = \frac{\partial}{\partial \vartheta} \frac{1}{4} [\vec{w}_\perp (\vec{w}_\perp \times \hat{b}) + (\vec{w}_\perp \times \hat{b}) \vec{w}_\perp] \quad (122)$$

Using all this, I can rewrite (117) as

$$\begin{aligned} \Omega \frac{\partial \delta f}{\partial \vartheta} &= - \Omega \frac{\partial \delta q_i}{\partial \vartheta} \frac{\partial \langle f \rangle}{\partial q_i} \\ \hookrightarrow \delta f &= - \delta q_i \frac{\partial \langle f \rangle}{\partial q_i} \end{aligned} \quad (123)$$

where δq_i 's are defined by $\delta q_i = - \Omega \frac{\partial \delta q_i}{\partial \vartheta}$, so

$$\delta \vec{r} = - \frac{\vec{w}_\perp \times \hat{b}}{\Omega} \quad (124)$$

$$\delta W_\parallel = - \left(\frac{\vec{w}_\perp \times \hat{b}}{\Omega} \right) \cdot \left[\frac{D \hat{b}}{Dt} - (\nabla \vec{v}_E) \cdot \hat{b} \right] + \frac{1}{4} \left[\vec{w}_\perp (\vec{w}_\perp \times \hat{b}) + (\vec{w}_\perp \times \hat{b}) \vec{w}_\perp \right] : \nabla \hat{b} \quad (125)$$

$$\begin{aligned} \delta \mu &= - \frac{\vec{w}_\perp \times \hat{b}}{B \Omega} \cdot \left[\text{[scribble]} - \frac{D \vec{v}_E}{Dt} - v_\parallel \frac{D \hat{b}}{Dt} - \mu \nabla_\parallel B \right] \\ &+ \frac{1}{4 B \Omega} \left[\vec{w}_\perp (\vec{w}_\perp \times \hat{b}) + (\vec{w}_\perp \times \hat{b}) \vec{w}_\perp \right] : (\nabla \vec{u} + v_\parallel \nabla \hat{b}) \end{aligned} \quad (126)$$

With these in hand, we know (123) and so can calculate the 2nd term in (115):

$$\frac{1}{B} \frac{\partial}{\partial q_i} B \langle \delta q_i \delta f \rangle = - \frac{1}{B} \frac{\partial}{\partial q_i} B \langle \delta q_i \delta q_j \rangle \frac{\partial \langle f \rangle}{\partial q_j} =$$

$$= - \langle \delta q_i \delta q_j \rangle \frac{\partial^2 \langle f \rangle}{\partial q_i \partial q_j} - \left(\frac{1}{B} \frac{\partial}{\partial q_i} B \langle \delta q_i \delta q_j \rangle \right) \frac{\partial \langle f \rangle}{\partial q_j} \quad (128)$$

$$\hookrightarrow \text{disappears because } \langle \delta q_i q_j \rangle = - \Omega \langle \frac{\partial \delta q_i}{\partial \vartheta} \delta q_j \rangle = \Omega \langle \delta q_i \frac{\partial \delta q_j}{\partial \vartheta} \rangle = - \langle \delta q_i \delta q_j \rangle \quad (129)$$

Thus, the low-flow drift-kinetic eqn is, from (115) and (128),

$$\left(\frac{\partial}{\partial t} + \dot{q}_i^{\text{eff}} \frac{\partial}{\partial q_i} \right) \langle f \rangle = 0 \quad (130)$$

where $\dot{q}_i^{\text{eff}} = \langle \dot{q}_i \rangle + \frac{1}{B} \frac{\partial}{\partial q_j} B \langle \delta q_j \delta \dot{q}_i \rangle$ (131)

Unpack this now:

$$\dot{\vec{r}}^{\text{eff}} = \vec{v}_E + v_{||} \hat{b} + \frac{1}{B} \nabla \cdot B \langle \delta \vec{r} \delta \dot{\vec{r}} \rangle + \frac{\partial}{\partial \mu} \langle \delta \mu \delta \dot{\vec{r}} \rangle + \frac{\partial}{\partial v_{||}} \langle \delta v_{||} \delta \dot{\vec{r}} \rangle \quad (132)$$

$$\dot{\mu}^{\text{eff}} = \underbrace{\langle \dot{\mu} \rangle^+}_{\frac{1}{B} \nabla \cdot B \langle \delta \vec{r} \delta \dot{\mu} \rangle} + \frac{\partial}{\partial \mu} \langle \delta \mu \delta \dot{\mu} \rangle + \frac{\partial}{\partial v_{||}} \langle \delta v_{||} \delta \dot{\mu} \rangle - \langle \delta \mu \delta \dot{v}_{||} \rangle \quad (133)$$

$$\dot{v}_{||}^{\text{eff}} = \underbrace{\langle \dot{v}_{||} \rangle^+}_{\frac{1}{B} \nabla \cdot B \langle \delta \vec{r} \delta \dot{v}_{||} \rangle} + \frac{\partial}{\partial \mu} \langle \delta \mu \delta \dot{v}_{||} \rangle + \frac{\partial}{\partial v_{||}} \langle \delta v_{||} \delta \dot{v}_{||} \rangle - \langle \delta v_{||} \delta \dot{\vec{r}} \rangle \quad (134)$$

So there are in fact only 4 averages to work out.

Note also that any averages involving an odd number of factors of \vec{w}_\perp vanish, as does anything that is in the form $\langle \text{scalar} \frac{\partial}{\partial \nu} \text{scalar} \rangle$, viz. the terms where $\text{scalar} = [\vec{w}_\perp (\vec{w}_\perp \times \hat{b}) + (\vec{w}_\perp \times \hat{b}) \vec{w}_\perp] : \nabla \hat{b}$ in (119), (120), (125) and (126). Furthermore, because $v_E \ll v_{th}$ in the low-flow ordering, all the terms ~~are~~ involving \vec{v}_E can be neglected. ~~Under these conditions~~

So, with all these simplifications on board,

$$\begin{aligned}
 (124) \quad \langle \delta \vec{F} \delta \vec{r} \rangle_{ij} &= - \left\langle \frac{\vec{w}_\perp \times \hat{b}}{\Omega} \vec{w}_\perp \right\rangle_{ij} = \cancel{\frac{w_\perp^2}{2\Omega}} - \epsilon_{ijmn} \underbrace{\langle w_m w_n \rangle}_{\frac{w_\perp^2}{2} (\delta_{mj} - b_m b_j)} b_n \frac{1}{\Omega} = \\
 &= - \frac{w_\perp^2}{2\Omega} \epsilon_{ijn} b_n
 \end{aligned} \tag{135}$$

So $\frac{1}{B} \nabla \cdot B \langle \delta \vec{F} \delta \vec{r} \rangle = \frac{1}{B} \nabla \times (B \frac{w_\perp^2}{2\Omega} \hat{b})$ ~~...~~

← this is $\propto \mu$ and $\nabla \times$ is taken at constant μ !

$$= \frac{\mu}{\Omega} \nabla \times (\hat{b} B) = \frac{\mu B}{\Omega} \nabla \times \hat{b} - \frac{\mu}{\Omega} \hat{b} \times \nabla B \tag{136}$$

$$\begin{aligned}
 (126) \quad \langle \delta \mu \delta \vec{r} \rangle &= - \left\langle \frac{\vec{w}_\perp \times \hat{b}}{\Omega} \vec{w}_\perp \right\rangle_{ij} \frac{1}{B} (-v_\parallel^2 \nabla_\parallel \hat{b} - \mu \nabla B)_i = \\
 &= - \frac{w_\perp^2}{2\Omega B} (v_\parallel^2 \nabla_\parallel \hat{b} + \mu \nabla B) \times \hat{b} = \\
 &= \frac{\mu}{\Omega} \hat{b} \times (v_\parallel^2 \hat{b} \cdot \nabla \hat{b} + \mu \nabla B) \tag{137}
 \end{aligned}$$

this term cancels with (136)

$$\frac{\partial \langle \delta \mu \delta \vec{r} \rangle}{\partial \mu} = \frac{1}{\Omega} \hat{b} \times (v_\parallel^2 \hat{b} \cdot \nabla \hat{b} + 2\mu \nabla B) \tag{138}$$

Finally

$$(125) \quad \langle \delta w_\parallel \delta \vec{r} \rangle = - \left\langle \frac{\vec{w}_\perp \times \hat{b}}{\Omega} \vec{w}_\perp \right\rangle_{ij} (v_\parallel \nabla_\parallel \hat{b})_i = \frac{w_\perp^2}{2\Omega} v_\parallel (\nabla_\parallel \hat{b}) \times \hat{b} \tag{139}$$

$$\frac{\partial \langle \delta w_\parallel \delta \vec{r} \rangle}{\partial w_\parallel} = \frac{\mu B}{\Omega} (\hat{b} \cdot \nabla \hat{b}) \times \hat{b} \tag{140}$$

Assemble:

$$\vec{r}^{\circ} \text{eff} = \vec{v}_E + v_\parallel \hat{b} + \frac{1}{\Omega} \hat{b} \times \left(v_\parallel^2 \hat{b} \cdot \nabla \hat{b} + \mu \nabla B \right) + \frac{\mu B}{\Omega} \hat{b} \cdot (\nabla \times \hat{b}) \hat{b} \tag{141}$$

curvature drift
grad B drift
Banós drift (additional in \parallel direction)

The last average we need is, for (133) & (134),

$$\langle \delta \mu \delta \dot{v}_{\parallel} \rangle = - \left\langle \frac{\hat{w}_{\perp} \times \hat{b}}{\Omega} \hat{w}_{\perp} \right\rangle_{ij} \frac{1}{B} (-v_{\parallel}^2 \nabla_{\parallel} \hat{b} - \mu \nabla B)_i (v_{\parallel} \nabla_{\parallel} \hat{b})_j$$

\uparrow (126) \uparrow (119)

$$= \frac{\mu}{\Omega} \left[\hat{b} \times (v_{\parallel}^2 \hat{b} \cdot \nabla \hat{b} + \mu \nabla B) \right] \cdot (\hat{b} \cdot \nabla \hat{b}) w_{\parallel} = \frac{w_{\parallel} \mu^2}{\Omega} (\hat{b} \times \nabla B) \cdot \hat{z}$$

(142)

similar to

$$(137) \quad \frac{\partial}{\partial v_{\parallel}} \langle \delta \mu \delta \dot{v}_{\parallel} \rangle = \frac{\mu^2}{\Omega} (\hat{b} \times \nabla B) \cdot \hat{z}$$

(143)

For μ^{eff} in (133), we will also need

$$\frac{1}{B} \nabla \cdot B \langle \delta \mu \delta \dot{r} \rangle = \frac{1}{B} \nabla \cdot \left(B \frac{\mu}{\Omega} \hat{b} \times (v_{\parallel}^2 \hat{b} + \mu \nabla B) \right) =$$

\uparrow (137) \uparrow indep. of B, ∇ is at const μ

$$= \frac{\mu}{\Omega} \nabla \cdot \left[\hat{b} \times (v_{\parallel}^2 \hat{b} + \mu \nabla B) \right]$$

(144)

Assemble: $\langle \dot{\mu} \rangle$

$$\mu^{eff} = - \frac{\mu}{\Omega} \nabla \cdot \left[\hat{b} \times (v_{\parallel}^2 \hat{b} + \mu \nabla B) \right] - \frac{\mu^2}{\Omega} (\hat{b} \times \nabla B) \cdot \hat{z}$$

$$= - \frac{w_{\parallel}^2 \mu}{\Omega} \nabla \cdot (\hat{b} \times \hat{z}) - \frac{\mu^2}{\Omega} \left[\nabla \cdot (\hat{b} \times \nabla B) + (\hat{b} \times \nabla B) \cdot (\hat{b} \cdot \nabla \hat{b}) \right]$$

\uparrow $\langle \dot{\mu} \rangle$

$$= - \frac{w_{\parallel}^2 \mu}{\Omega} \nabla \cdot (\hat{b} \times \hat{z}) - \frac{\mu^2}{\Omega} \left[\hat{b} \hat{b} \cdot (\nabla \times \hat{b}) + \hat{b} \times (\hat{b} \cdot \nabla \hat{b}) \right] \cdot \nabla B$$

$$= - \frac{w_{\parallel}^2 \mu}{\Omega} \nabla \cdot (\hat{b} \times \hat{z}) - \frac{\mu^2}{\Omega} \hat{b} \cdot (\nabla \times \hat{b}) \hat{b} \cdot \nabla B$$

(145)

These are not very enlightening.

NB

$$v_B = \frac{\mu B}{\Omega} \hat{b} \cdot (\nabla \times \hat{b})$$

is Bañs drift

from (98)

For $\dot{v}_{||}^{\text{eff}}$ in (B4), we need

$$\frac{1}{B} \nabla \cdot B \langle \delta v_{||} \delta \vec{r}^2 \rangle = \frac{1}{B} \nabla \cdot B \frac{\mu B}{\Omega} v_{||} \vec{x} \times \hat{b} \quad (146)$$

↑
(139)

and $\frac{\partial}{\partial \mu} \langle \delta \mu \delta \dot{v}_{||} \rangle = \frac{2\mu v_{||}}{\Omega} (\hat{b} \times \nabla B) \cdot \vec{x} \quad (147)$

↑
(142)

Assemble:

$$\dot{v}_{||}^{\text{eff}} = \langle \dot{v}_{||} \rangle + \frac{\mu v_{||}}{\Omega} \left\{ \underbrace{-\frac{1}{B} \nabla \cdot (B^2 \vec{x} \times \hat{b}) + 2 (\hat{b} \times \nabla B) \cdot \vec{x}}_{\text{''}} \right\} =$$

$$\underbrace{-2 (\vec{x} \times \hat{b}) \cdot \nabla B + B \nabla \cdot (\hat{b} \times \vec{x})}_{\text{''}}$$

$$= \frac{q}{m} E_{||} - \mu \nabla_{||} B - \underbrace{\hat{b} \cdot \frac{D\vec{v}_E}{Dt}}_{\text{''}} + \frac{\mu v_{||} B}{\Omega} \nabla \cdot (\hat{b} \times \vec{x})$$

↑
from (93)

$$\begin{aligned} & -v_{||} (\nabla_{||} \vec{v}_E) \cdot \hat{b} \text{ to lowest order} \\ & = +v_{||} (\nabla_{||} \hat{b}) \cdot \vec{v}_E = +v_{||} \vec{x} \cdot \frac{c\vec{E} \times \hat{b}}{B} = \frac{\hat{b} \times \vec{x}}{B} \cdot c\vec{E} \\ & = +\frac{v_{||}}{B} (\hat{b} \times \vec{x}) \cdot c\vec{E} = +\frac{v_{||}}{\Omega} (\hat{b} \times \vec{x}) \cdot \frac{q}{m} \vec{E} \end{aligned}$$

$$= \frac{q}{m} \left(\hat{b} + \frac{v_{||}}{\Omega} \hat{b} \times \vec{x} \right) \cdot \vec{E} - \mu \nabla_{||} B + \frac{\mu v_{||} B}{\Omega} \nabla \cdot (\hat{b} \times \vec{x}) \quad (148)$$

So the low-flow DK eqn can now be written in its full glory:

$$\begin{aligned} \frac{\partial f}{\partial t} + (\vec{v}_E + \vec{v}_D) \cdot \nabla_{\perp} f + (v_{\parallel} + v_B) \nabla_{\parallel} f + \\ + \left[\frac{q}{m} (\hat{b} + \frac{v_{\parallel}}{\Omega} \hat{b} \times \vec{x}) \cdot \vec{E} - \mu \nabla_{\parallel} B + \frac{\mu v_{\parallel} B}{\Omega} \nabla \cdot (\hat{b} \times \vec{x}) \right] \frac{\partial f}{\partial v_{\parallel}} \\ + \left[\frac{v_B}{B} (q E_{\parallel} - \mu \nabla_{\parallel} B) - \frac{\mu v_{\parallel}^2}{\Omega} \nabla \cdot (\hat{b} \times \vec{x}) \right] \frac{\partial f}{\partial \mu} = 0 \end{aligned} \tag{149}$$

where $\vec{v}_D = \frac{1}{\Omega} \hat{b} \times (v_{\parallel}^2 \vec{x} + \mu \nabla B)$

$v_B = \frac{\mu B}{\Omega} \hat{b} \cdot (\nabla \times \hat{b})$

8.4 Adjusting variables

Curvature and ∇B drifts are natural, physical things, so it's good we've got them. ~~The~~ The other additional terms in (149) are quite ugly and not terribly intuitive. Such occurrences are often a sign that the variables that we used are not quite "right".

To see this let's observe that

$$\hat{b} \times \vec{x} = \nabla \times \hat{b} - \hat{b} \hat{b} \cdot (\nabla \times \hat{b})$$

So $\nabla \cdot (\hat{b} \times \vec{x}) = - \nabla \cdot [\hat{b} \hat{b} \cdot (\nabla \times \hat{b})]$ (150)

Thus,

$$\begin{aligned} \frac{\mu v_{\parallel} B}{\Omega} \nabla \cdot (\hat{b} \times \vec{x}) &= - \frac{\mu v_{\parallel} B}{\Omega} \nabla \cdot \left[\hat{b} \frac{\Omega}{\mu B} v_B \right] = - v_{\parallel} \nabla \cdot (v_B \hat{b}) \\ &= - v_{\parallel} v_B \nabla \cdot \hat{b} - v_{\parallel} \nabla_{\parallel} v_B \\ &= + \frac{v_{\parallel} v_B}{R} \nabla_{\parallel} B - v_{\parallel} \nabla_{\parallel} v_B \end{aligned} \tag{151}$$

$$\frac{\mu v_{||}^2}{\Omega} \nabla \cdot (\hat{b} \times \vec{x}) = - \frac{\mu v_{||}^2}{\Omega} \nabla \cdot \left(\hat{b} \frac{\Omega}{\mu B} v_B \right) = - \frac{v_{||}^2}{B} \nabla \cdot (v_B \hat{b})$$

$$= + \frac{v_{||}^2 v_B}{B^2} \nabla_{||} B - \frac{v_{||}^2}{B} \nabla_{||} v_B = - v_{||} \nabla_{||} \frac{v_B v_{||}}{B} \quad (152)$$

Let $\Delta\mu = - \frac{v_{||} v_B}{B}$. Then
our equation is

$$\begin{aligned} & v_{||} \nabla_{||} \frac{v_B v_{||}}{B} - v_{||}^2 v_B \nabla_{||} \frac{1}{B} \\ &= v_{||} \nabla_{||} \frac{v_B v_{||}}{B} + \frac{v_{||}^2 v_B}{B^2} \nabla_{||} B \end{aligned}$$

$$\frac{\partial f}{\partial t} + (\vec{v}_E + \vec{v}_D) \cdot \nabla_{\perp} f + (v_{||} + v_B) \nabla_{||} f +$$

$$+ \left[\frac{q}{m} \left(\hat{b} + \frac{v_{||}}{\Omega} \hat{b} \times \vec{x} \right) \cdot \vec{E} - (\mu + \Delta\mu) \nabla_{||} B - v_{||} \nabla_{||} v_B \right] \frac{\partial f}{\partial v_{||}}$$

$$= \left[\left(\frac{q}{m} E_{||} - \mu \nabla_{||} B \right) \frac{\partial \Delta\mu}{\partial v_{||}} + v_{||} \nabla_{||} \Delta\mu \right] \frac{\partial f}{\partial \mu} = 0 \quad (153)$$

Now the change of variables that is beginning to happen is

$$\begin{cases} \bar{\mu} = \mu + \Delta\mu = \mu - \frac{v_{||} v_B}{B} \\ \bar{v}_{||} = v_{||} + v_B \end{cases} \quad (154)$$

$$\frac{\partial f}{\partial t} + (\vec{v}_E + \vec{v}_D) \cdot \nabla_{\perp} f + \bar{v}_{||} \nabla_{||} f \equiv \frac{Df}{Dt}$$

$$+ \left[\frac{Dv_B}{Dt} + \frac{q}{m} \left(\hat{b} + \frac{v_{||}}{\Omega} \hat{b} \times \vec{x} \right) \cdot \vec{E} - \bar{\mu} \nabla_{||} B - v_{||} \nabla_{||} v_B \right] \frac{\partial f}{\partial v_{||}}$$

~~...~~ $\equiv v_{||} \nabla_{||} v_B +$
higher order

to lowest order from [...] $\frac{\partial f}{\partial v_{||}}$

$$- \left[\frac{D}{Dt} \Delta\mu - \left(\frac{q}{m} E_{||} - \mu \nabla_{||} B \right) \frac{\partial \Delta\mu}{\partial v_{||}} + \left(\frac{q}{m} E_{||} - \mu \nabla_{||} B \right) \frac{\partial \Delta\mu}{\partial v_{||}} + v_{||} \nabla_{||} \Delta\mu \right] \frac{\partial f}{\partial \mu}$$

$\equiv v_{||} \nabla_{||} \Delta\mu +$ higher order

So we have

$$\left[\frac{\partial f}{\partial t} + (\vec{v}_E + \vec{v}_D) \cdot \nabla_{\perp} f + \bar{v}_{\parallel} \nabla_{\parallel} f + \left[\frac{q}{m} \left(1 + \frac{v_{\parallel}}{\Omega} \hat{b} \times \hat{x} \right) \cdot \vec{E} - \mu \nabla_{\parallel} B \right] \frac{\partial f}{\partial v_{\parallel}} \right] = 0 \quad (155)$$

Thus, in adjusted variables, this is quite a nice equation.

I will return to the uses of low-flow drift kinetics, but now it is time to return to ~~the~~ the delights of KMHD \Leftrightarrow high-flow drift kinetics.