

§11 Electrostatic Drift Kinetics: ITB Instability

So far, I have focused on regimes in which the dynamics of \vec{B} were of natural interest — this led to the $\beta \gg 1$ limit (astrophysics except, e.g., solar corona) and a focus on pressure anisotropies and associated instabilities.

I am now going to change tack and explore the opposite regime — one in which \vec{B} does not change at all. Intuitively, this will describe $\beta \ll 1$, but I will postpone ~~thinking~~ a careful consideration of exactly ~~when~~ how to get this electrostatic limit and instead first explore what kind of wild life exists in that part of the woods. If it turns out to be interesting, we will dwell further on the question of ~~where exactly in parameter space~~ where exactly in parameter space all that is.

~~For this we need to consider~~

So, let us assume $\vec{B} = B_0 \hat{z} = \text{const}$ and $\delta \vec{B} = 0$ exactly.

This implies

$$c \nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} = 0$$

$$\boxed{\vec{E} = -\nabla \phi} \quad (207)$$

and the DK equation is — from (149) (note that all higher order corrections are 0 because \vec{B} does not vary)

$$\frac{\partial f}{\partial t} + \vec{v}_E \cdot \nabla_{\perp} f + v_{\parallel} \nabla_{\parallel} f + \frac{q}{m} E_{\parallel} \frac{\partial f}{\partial v_{\parallel}} = 0, \quad (208)$$

11.1 Electrostatic Limit

where $\vec{V}_E = c \frac{\vec{E} \times \vec{B}}{B^2} = + \frac{c}{B_0} \hat{z} \times \nabla_{\perp} \phi$ (209)

So

$$\frac{\partial f}{\partial t} + \frac{c}{B_0} \underbrace{\{\phi, f\}}_{\text{"}} + v_{\parallel} \nabla_{\parallel} f - \frac{q}{m} (\nabla_{\parallel} \phi) \frac{\partial f}{\partial v_{\parallel}} = 0 \quad (210)$$

$$\hat{z} \cdot (\nabla_{\perp} \phi \times \nabla_{\perp} f) = \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial x}$$

Consider again a static equilibrium with $\phi = 0$.

Then $v_{\parallel} \nabla_{\parallel} f_0 = 0$ (211)

so we can pick any equilibrium distribution that does not have a parallel variation.

This time, I want to study a distribution that has some profile in space, but is Maxwellian,

i.e., $f_0 = \frac{n_0(x)}{\pi^{3/2} v_{th}^3(x)} e^{-\frac{v_{\perp}^2 + v_{\parallel}^2}{v_{th}^2(x)}}$, $v_{th}(x) = \sqrt{\frac{2T(x)}{m}}$ (212)

This is about as simple as it gets, but we shall see that it is nevertheless interesting.

Let us now perturb eq. (210) around the eq-m (212):

$$\frac{\partial \delta f}{\partial t} + \frac{c}{B_0} \underbrace{\{\phi, f_0\}}_{\text{"}} + v_{\parallel} \nabla_{\parallel} \delta f - \frac{q}{m} (\nabla_{\parallel} \phi) \frac{\partial f_0}{\partial v_{\parallel}} = 0$$

~~scribble~~

$$- \frac{c}{B_0} \frac{\partial \phi}{\partial y} \frac{\partial f_0}{\partial x} - \frac{2v_{\parallel}}{v_{th}^2} f_0$$

\swarrow \searrow
 $v_{\parallel} \nabla_{\parallel} \frac{q\phi}{T} f_0$ because $\frac{mv_{th}^2}{2} = T$

$$\frac{\rho v_{th}}{2} = \left(\frac{cmv_{th}^2}{2qB_0} \frac{\partial}{\partial y} \frac{q\phi}{T} \right) \text{scribble}$$

$$\frac{\partial \delta f}{\partial t} + v_{\parallel} \nabla_{\parallel} \delta f = -v_{\parallel} \nabla_{\parallel} \frac{q\phi}{T} f_0 + \frac{\rho v_{th}}{2} \frac{\partial}{\partial y} \frac{q\phi}{T} \frac{\partial f_0}{\partial x} \quad (213)$$

This equation immediately teaches us something very useful when $\alpha = e$ (electrons). 11.2 Electron Response

Electrons are much faster than ions:

$$\frac{v_{the}}{v_{thi}} = \sqrt{\frac{T_e m_i}{T_i m_e}} \gg 1 \text{ assum } T_e \sim T_i \quad (214)$$

If we look for solutions with $\omega \ll k_{\parallel} v_{the}$, we can neglect all terms in (213) except those $\propto v_{\parallel}$.

On the rhs, this is fine if

$$\frac{\frac{\rho_e v_{the}}{2} \frac{\partial}{\partial y} \frac{e\phi}{T_e} \frac{\partial f_{0e}}{\partial x}}{v_{\parallel} \nabla_{\parallel} \frac{e\phi}{T} f_{0e}} \sim \frac{k_y \rho_e v_{the} / 2 L_e}{k_{\parallel} v_{the}} \sim \frac{k_y \rho_e}{k_{\parallel} L_e} \ll 1 \quad (215)$$

↖ grad. length of e eqn variation

This leaves us with

$$v_{\parallel} \nabla_{\parallel} \delta f_e \approx v_{\parallel} \nabla_{\parallel} \frac{e\phi}{T_e} f_{0e}$$

$$\delta f_e \approx \frac{e\phi}{T_e} f_{0e} \quad (216)$$

This is just "Boltzmann response". Integrating,

$$\boxed{\frac{\delta n_e}{n_{0e}} = \frac{e\phi}{T_e}} \quad (217)$$

Since, by quasineutrality, in an e-i plasma, where $q_i = Ze$, $n_{0e} = Zn_{0i}$, $\delta n_e = Z\delta n_i$, we have, from (217),

$$\frac{e\phi}{T_e} = \frac{\delta n_i}{n_{0i}} \quad (218)$$

Thus, if we can solve the ion kinetic equation, ^(in terms of ϕ) then we can calculate δn_i , which will then immediately give us ϕ via (218).

So let us consider now

$$\frac{\partial \delta f_i}{\partial t} + v_{\parallel} \nabla_{\parallel} \delta f_i = -v_{\parallel} \nabla_{\parallel} \frac{Z e \phi}{T_i} f_{0i} + \frac{p_i v_{thi}}{2} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i} \frac{\partial f_{0i}}{\partial x}$$

Of course I can solve this in the usual way, (219) but let me first examine the "fluid dynamics" of this system by taking moments of (219):

$\int d^3 \vec{w}$ (219): [NB: the distinction between (μ, v_{\parallel}) and $(w_{\perp}, v_{\parallel})$ variables does not matter because $B = B_0 = \text{const}$]

11.3 Fluid ITG

$$\frac{\partial \delta n_i}{\partial t} + \nabla_{\parallel} n_{0i} u_{\parallel i} = \frac{p_i v_{thi}}{2} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i} \frac{d n_{0i}}{dx}$$

$$\boxed{\frac{\partial}{\partial t} \frac{\delta n_i}{n_{0i}} + \nabla_{\parallel} u_{\parallel i} = - \frac{p_i v_{thi}}{2 L_n} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i}}, \quad - \frac{d \ln n_{0i}}{dx} = \frac{1}{L_n}$$

(220)

$\int d^3 \vec{w} v_{\parallel}$ (219):

$$\frac{\partial}{\partial t} n_{0i} u_{\parallel i} + \nabla_{\parallel} \frac{\delta p_{\parallel i}}{m_i} = - \nabla_{\parallel} \frac{Z e \phi}{T_i} \left(\frac{p_{0i}}{m_i} \right) = \frac{n_{0i} T_{0i}}{m_i} \frac{Z T_e}{T_i} \frac{\delta n_i}{n_{0i}} \text{ via (218)}$$

(221)

$\int d^3 \vec{w} m_i v_{\parallel}^2$ (219):

$$\frac{\partial}{\partial t} \delta p_{\parallel i} + \nabla_{\parallel} \left(\int d^3 \vec{w} m_i v_{\parallel}^3 \delta f_i \right) = \frac{p_i v_{thi}}{2} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i} \frac{\partial p_{0i}}{\partial x}$$

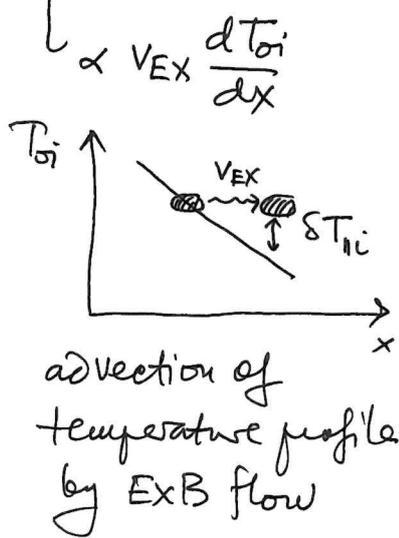
heat flux

$T_{0i} \frac{d n_{0i}}{dx} + n_{0i} \frac{d T_{0i}}{dx}$

$T_{0i} \left(\frac{\partial \delta n_i}{\partial t} + n_{0i} \nabla_{\parallel} u_{\parallel i} \right) - \frac{p_i v_{thi}}{2} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i} n_{0i} \frac{d T_{0i}}{dx} =$

$$\frac{\partial}{\partial t} \frac{\delta T_{ii}}{T_{oi}} + \frac{\nabla_{\parallel} \delta q_{ii}}{\rho_{oi}} = \nabla_{\parallel} u_{ii} - \frac{\rho_i v_{thi}}{2LT} \frac{\partial}{\partial y} \frac{Z e \phi}{T_i} \quad (222)$$

heat flux
 can be neglected
 if $\omega \gg k_{\parallel} v_{thi}$,
 and can then also
 neglect $\nabla_{\parallel} u_{ii} \sim k_{\parallel} v_{thi} \frac{u_{ii}}{v_{thi}}$
 compressional heating



So let us assume

$$\omega \gg k_{\parallel} v_{thi} \quad (223)$$

and worry later whether this is satisfied by the solution that we get. Note that this amounts to taking the fluid limit: δq_{ii} was the non-closed term in the equations, still requiring δf_i to calculate. Now it is gone and we have a closed system!

Physically, this is neglect of ion Landau damping because it would be $\propto e^{-\omega / (k_{\parallel} v_{thi})^2} \ll 1$, same principle (in fact exactly the same approximation) as I made when deriving the ion sound waves in the KT course (see L.Notes §3.8).

So, eq. (222) is now used (218)

$$\frac{\partial}{\partial t} \frac{\delta T_{ii}}{T_{oi}} \approx - \underbrace{\frac{\rho_i v_{thi}}{2LT}}_{\omega_{*T}} \frac{\partial}{\partial y} \frac{Z T_e}{T_i} \frac{\delta n_i}{n_{oi}} \quad (224)$$

$$\omega_{*T} = \frac{k_y \rho_i v_{thi}}{2LT} \frac{Z T_e}{T_i} \text{ "drift frequency"}$$

We shall see that (223) will be satisfied if

$$\omega_{*T} \gg \omega \quad (225)$$

and, therefore,

$$\frac{\delta T_{ii} / T_{oi}}{\delta n_i / n_{oi}} \sim \frac{\omega_{*T}}{\omega} \gg 1 \quad (226)$$

This comes rather handy in dealing with eq. (221),

where

$$\frac{\partial u_{ii}}{\partial t} = -\nabla_{||} \left(\underbrace{\frac{\delta p_{iii}}{m_i n_{oi}} + \frac{Z T_{oe}}{T_{oi}} \frac{\delta n_i}{n_{oi}}}_{\text{these two terms are small compared to first, by (226)}} \right) \approx -\frac{v_{thi}^2}{2} \nabla_{||} \frac{\delta T_{ii}}{T_{oi}} \quad (227)$$

$$\frac{\delta T_{ii}}{m_i} + \frac{T_{oi}}{m_i} \frac{\delta n_i}{n_{oi}}$$

these two terms are small compared to first, by (226)

Finally, let us bring in (220)

and Fourier transform (in t, z and y but not x):

$$+(\omega - \omega_{*n}) \frac{\delta n_i}{n_{oi}} = k_{||} u_{ii} \leftarrow (220)$$

$$\left(\frac{k_y^2 v_{thi}^2 Z T_{oe}}{2 L_n T_{oi}} \right)$$

$$\omega u_{ii} = \frac{k_{||}^2 v_{thi}^2}{2} \frac{\delta T_{ii}}{T_{oi}} \leftarrow (227)$$

$$\omega \frac{\delta T_{ii}}{T_{oi}} = \omega_{*T} \frac{\delta n_i}{n_{oi}} \leftarrow (224)$$

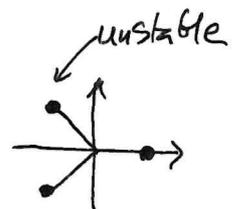
$$\boxed{\omega^2 (\omega - \omega_{*n}) = \frac{k_{||}^2 v_{thi}^2}{2} \omega_{*T}}$$

Dispersion relation (228)

Simplest case: $\omega_{*n} \ll \omega$. Then

$$\omega^3 = \frac{k_{||}^2 v_{thi}^2}{2} \omega_{*T}$$

$$\omega = \left(\frac{k_{||}^2 v_{thi}^2}{2} |\omega_{*T}| \right)^{1/3} \left(-\frac{1}{2} \text{sgn} k_y + \frac{\sqrt{3}}{2} i \right) \quad (229)$$



This is called the ion-temperature-gradient (ITG) instability, which is the main culprit behind turbulent transport in tokamaks.

Note that

$$\frac{\omega}{k_{\parallel} v_{thi}} \sim \left(\frac{\omega_{*T}}{k_{\parallel} v_{thi}} \right)^{1/3} \gg 1 \text{ as per (223)}$$

$$\frac{\omega}{\omega_{*T}} \sim \left(\frac{k_{\parallel} v_{thi}}{\omega_{*T}} \right)^{2/3} \ll 1 \text{ as per (225)}$$

assump $\omega_{*T} \gg k_{\parallel} v_{thi}$.

Instability feedback loop:

Eq. (224): $v_{Ex} \frac{dT_{oi}}{dx}$ creates δT_{ii}

Eq. (227): $-\nabla_{\parallel} \delta T_{ii}$ is \parallel pressure gradient, which creates u_{\parallel}

Eq. (220): \parallel compression $\nabla_{\parallel} u_{\parallel}$ creates δn_i

Eq. (218) & (217): $\delta n_i \rightarrow \delta n_e$ via quasineutrality
 $\rightarrow \phi$ via electron response

Voilà.

BUT: At fixed k_y , $\gamma \propto k_{\parallel}^{2/3}$, so fastest growing modes will break the fluid limit (223)

This means that in fact we must do kinetic theory after all. Let us remind to (219) and do this properly.

11.4 Kinetic ITG instability

From (219),

$$\begin{aligned}
 \delta f_i &= \frac{1}{k_{\parallel} v_{\parallel} - \omega} \left\{ -i k_{\parallel} v_{\parallel} \frac{Z e \phi}{T_i} f_{oi} + \right. \\
 &\quad \left. + i \frac{k_y \rho_i v_{thi}}{2} \frac{Z e \phi}{T_i} \left[\frac{d \ln n_{oi}}{dx} + \left(\frac{\omega_{\perp}^2 + v_{\parallel}^2}{v_{thi}^2} - \frac{3}{2} \right) \frac{d \ln T_{oi}}{dx} \right] f_{oi} \right\} \\
 &\quad \quad \quad \frac{Z T_e}{T_i} \frac{e \phi}{T_e} \quad \quad \quad - \frac{1}{L_n} \quad \quad \quad - \frac{1}{L_T} \\
 &= \frac{1}{k_{\parallel} v_{\parallel} - \omega} \frac{e \phi}{T_e} f_{oi} \left[- \frac{Z T_e}{T_i} k_{\parallel} v_{\parallel} - \omega_{*n} - \omega_{*T} \left(\frac{\omega_{\perp}^2 + v_{\parallel}^2}{v_{thi}^2} - \frac{3}{2} \right) \right] \quad (2)
 \end{aligned}$$

$\frac{\delta n_i}{n_{oi}}$ by (218), so take $\frac{1}{n_{oi}} \int d^3 \vec{w}$ (230):

$$1 = - \int d v_{\parallel} \frac{1}{k_{\parallel} v_{\parallel} - \omega} \left[\frac{Z T_e}{T_i} k_{\parallel} v_{\parallel} + \omega_{*n} + \omega_{*T} \left(\frac{v_{\parallel}^2}{v_{thi}^2} - \frac{1}{2} \right) \right] \frac{e^{-\frac{v_{\parallel}^2}{v_{thi}^2}}}{\sqrt{\pi} v_{thi}}$$

cancel with denominator and integrate

$$\begin{aligned}
 1 + \frac{Z T_e}{T_i} &= - \int d x \frac{1}{x - \zeta_i} \left[\frac{Z T_e}{T_i} \zeta_i + \zeta_{*n} + \zeta_{*T} \left(x^2 - \frac{1}{2} \right) \right] \frac{e^{-x^2}}{\sqrt{\pi}} \\
 &\quad \quad \quad \frac{v_{\parallel}}{v_{thi}} \quad \quad \quad \frac{\omega}{|k_{\parallel} v_{thi}|} \\
 &\quad \quad \quad \frac{1}{\sqrt{\pi}} \int d x \frac{x^2 e^{-x^2}}{x - \zeta_i} = \frac{1}{\sqrt{\pi}} \int d x \frac{(x^2 - \zeta_i^2 + \zeta_i^2) e^{-x^2}}{x - \zeta_i} \\
 &= \zeta_i^2 Z(\zeta_i) + \frac{1}{\sqrt{\pi}} \int d x \frac{(x - \zeta_i)(x + \zeta_i) e^{-x^2}}{x - \zeta_i} = \\
 &= \zeta_i^2 Z(\zeta_i) + \zeta_i
 \end{aligned}$$

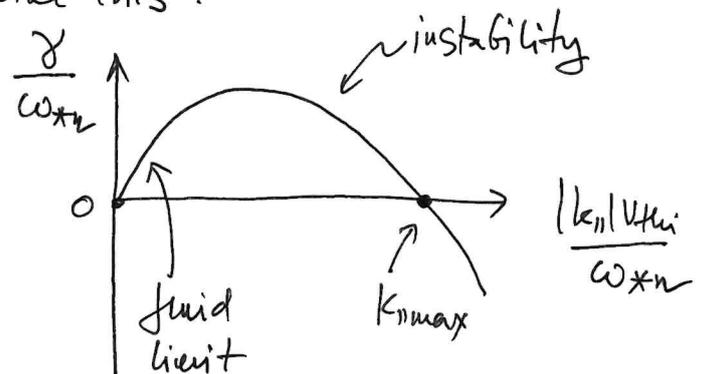
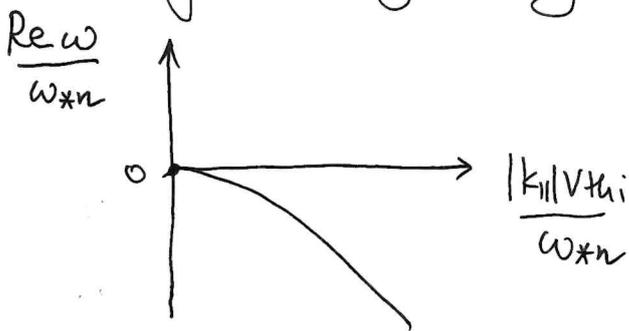
$$= - \left[\frac{Z T_e}{T_i} \zeta_i + \zeta_{*n} + \zeta_{*T} \left(\zeta_i^2 - \frac{1}{2} \right) \right] Z(\zeta_i) - \zeta_i \zeta_{*T}$$

This gives us the kinetic ITG dispersion relation:

$$1 + \frac{Z T_e}{T_i} + \zeta_{*T} \zeta_i + \left[\zeta_{*n} + \zeta_{*T} \left(\zeta_i^2 - \frac{1}{2} \right) + \frac{Z T_e}{T_i} \zeta_i \right] Z(\zeta_i) = 0 \quad (231)$$

Exercise. Expand in $\zeta_i \gg 1$ and recover (228), the fluid limit.

~~The solution~~ The solution (which is easy to get numerically) of (231) generally looks like this:



There is a very easy trick to obtain stability boundary: on the stability boundary, $\gamma = 0$, so ζ_i is real. Therefore Im part of (231) only contains terms $\propto Z(\zeta_i)$:

$$\zeta_{*n} + \zeta_{*T} \left(\zeta_i^2 - \frac{1}{2} \right) + \frac{Z T_e}{T_i} \zeta_i = 0 \quad (232)$$

Whence also, from (231),

$$1 + \frac{Z T_e}{T_i} + \zeta_{*T} \zeta_i = 0 \quad (233)$$

frequency: $\zeta_i = - \left(1 + \frac{Z T_e}{T_i} \right) \zeta_{*T}^{-1} \quad (234)$

From (232) then

$$\zeta_{*n} - \frac{1}{2} \zeta_{*T} + \zeta_i \left(\zeta_i \zeta_{*T} + \frac{z T_e}{T_i} \right) = 0$$

(234) $\quad -1 - \frac{z T_e}{T_i}$

$$\zeta_{*T} \left(\zeta_{*n} - \frac{1}{2} \zeta_{*T} \right) + 1 + \frac{z T_e}{T_i} = 0$$

$$\frac{\omega_{*T} \left(\frac{1}{2} \omega_{*T} - \omega_{*n} \right)}{\left(|k_{\parallel}| v_{thi} \right)^2} = 1 + \frac{z T_e}{T_i}$$

This is $k_{\parallel \max}$ on the plot
↓

$$\frac{|k_{\parallel}| v_{thi}}{\omega_{*n}} = \sqrt{\frac{\frac{\omega_{*T}}{\omega_{*n}} \left(\frac{1}{2} \frac{\omega_{*T}}{\omega_{*n}} - 1 \right)}{1 + \frac{z T_e}{T_i}}} = \sqrt{\frac{\eta_i (\eta_i - 2)}{2 \left(1 + \frac{z T_e}{T_i} \right)}} \quad (235)$$

where $\eta_i = \frac{\omega_{*T}}{\omega_{*n}} = \frac{L_n}{L_T}$

Instability exists for $\eta_i > 2$, i.e., the critical temperature gradient is

$$\boxed{L_{Tc}^{-1} = 2 L_n^{-1}} \quad (236)$$

→ Note that $\gamma \propto \omega_{*n} \propto k_y$ - yet again we find unstable perturbations that grow faster as $k_y \rightarrow \infty$, so yet again drift kinetics is inadequate and we must access finite k_y effects to get to the peak of the instability.

This is obvious by dimensional analysis: $\gamma_{\max} = \omega_{*n} F(\eta_i)$

11.5 Justifying the electrostatic limit.

As promised, I want to come back to the question of how to justify $\delta\vec{B}=0$. As we have seen in the above calculation, a key ingredient of ITG dynamics was the perturbed cross-field flow

$$\vec{v}_E = \frac{c}{B_0} \hat{z} \times \nabla \phi \equiv \vec{u}_\perp$$

$$\vec{E} = -\nabla\phi + \text{neglected inductive part} - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

But in MHD, there would be ~~some mechanism~~ a mechanism for such a flow to drag with it field lines (frozen into it) - Alfvénic motion ($\delta\vec{B}_\perp/B_0 \sim \vec{u}_\perp/v_A$).

Let us estimate this effect.

$$\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} \Rightarrow \vec{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

So this equation by itself does not really tell us anything about ~~whether~~ whether $\partial\vec{B}/\partial t$ (equivalently, inductive electric field is negligible).

~~Let us recall where flux freezing came from: eq. (22):~~

$$\vec{E} = -\frac{\vec{u}_\alpha \times \vec{B}}{c} + \frac{\nabla \cdot \hat{P}_\alpha}{q_\alpha n_\alpha} - \frac{\vec{R}_\alpha}{q_\alpha n_\alpha} + \frac{m_\alpha}{q_\alpha} \frac{d\vec{u}_\alpha}{dt_\alpha}$$

↑
flux freezing

Take $\alpha=e$ and perturb the above equation

← note that in MHD all of that is dealt by subdominant terms, $E_{\parallel} \ll E_{\perp}$

$$-\nabla\phi - \frac{1}{c} \frac{\partial \delta A}{\partial t} = - \frac{\vec{u}_e \times \hat{z} B_0}{c} + \frac{\nabla \cdot \delta \hat{P}_e}{en_e} - \frac{m_e n_e v_{ei} (\vec{u}_e - \vec{u}_i)}{en_e} - \frac{m_e}{e} \frac{\partial \vec{u}_e}{\partial t} \quad (237)$$

Take \parallel part:

Assume a plasma consisting of e's and one ion species.

$$\nabla_{\parallel} = -\nabla_{\parallel} \phi - \frac{1}{c} \frac{\partial \delta A_{\parallel}}{\partial t} = - \frac{(\nabla \cdot \delta \hat{P}_e)_{\parallel}}{en_e} - \frac{m_e v_{ei}}{e} (u_{\parallel e} - u_{\parallel i}) - \frac{m_e}{e} \frac{\partial u_{\parallel e}}{\partial t} =$$

Note $j_{\parallel} = Z n_i u_{\parallel i} - e n_e u_{\parallel e} = e n_e (u_{\parallel i} - u_{\parallel e})$

$$= - \frac{(\nabla \cdot \delta \hat{P}_e)_{\parallel}}{en_e} + \frac{m_e v_{ei}}{e^2 n_e} j_{\parallel} - \frac{m_e}{e} \frac{\partial}{\partial t} \left(u_{\parallel i} - \frac{j_{\parallel}}{en_e} \right) = \frac{c}{4\pi} (\nabla \times \delta \vec{B})_{\parallel} = - \frac{c}{4\pi} \nabla^2 \delta A_{\parallel}$$

$$= - \frac{(\nabla \cdot \delta \hat{P}_e)_{\parallel}}{en_e} - \frac{c m_e v_{ei}}{4\pi e^2 n_e} \nabla^2 \delta A_{\parallel} - \frac{m_e}{e} \frac{\partial u_{\parallel i}}{\partial t} + \frac{\partial}{\partial t} \left(\frac{c m_e}{4\pi e^2 n_e} \nabla^2 \delta A_{\parallel} \right) =$$

$$= - \frac{(\nabla \cdot \delta \hat{P}_e)_{\parallel}}{en_e} - \frac{m_e}{e} \frac{\partial u_{\parallel i}}{\partial t} - \frac{1}{c} \left(\frac{\partial}{\partial t} + v_{ei} \right) d_e^2 \nabla^2 \delta A_{\parallel}$$

$\frac{c}{\omega_p e^2} = \frac{1}{c} d_e^2$ electron inertial scale

So we can work out what the magnetic perturbation is as follows:

usual MHD flux-freezing-breaking term

$$\frac{1}{c} \left[\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial t} + v_{ei} \right) d_e^2 \nabla^2 \right] \delta A_{\parallel} =$$

$$= -\nabla_{\parallel} \phi + \frac{(\nabla \cdot \delta \hat{P}_e)_{\parallel}}{en_e} + \frac{m_e}{e} \frac{\partial u_{\parallel i}}{\partial t} \quad (238)$$

↑ substitute (7b) perturbation

We can make $\delta A_{||}$ small by requiring

$$k^2 d_e^2 \gg 1 \quad \text{or} \quad v_e k^2 d_e^2 \gg \omega \sim \omega_*$$

Now note that

$$\frac{d_e^2}{\beta_i^2} = \frac{c^2 m_e Z^2 e^2 B_0^2}{4\pi e^2 n_e v_{thi}^2 m_i^2 c^2} = \frac{Z m_e B_0^2}{m_i 8\pi n_i T_i} = \frac{Z m_e}{m_i \beta_i}$$

$$k^2 d_e^2 = \underbrace{k^2}_{\uparrow} \underbrace{\beta_i^2}_{\uparrow} \frac{Z m_e}{m_i \beta_i} \gg 1 \quad \text{only possible if } \boxed{\beta_i \ll \frac{Z m_e}{m_i}} \quad (239)$$

↑ for the DK approximation to hold

Thus, formally, the electrostatic assumption is justified for very low β_i .

If you want a careful step-by-step analysis of how electrostatic limit works for temperature-gradient driven instabilities and also of what happens when magnetic perturbations are allowed, read

Adkins + arXiv: 2201.05670