

Collisionless Plasma Physics - Part II

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§1. Intro.

In the introduction to kinetic theory, my starting point was the Vlasov ^(-Lauday) equation (1.30) (§1.6 of my KT notes)

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \cdot \frac{\partial \vec{v}}{\partial \vec{r}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c \quad (1)$$

coupled to Maxwell's equations for \vec{E} and \vec{B} in terms of charge densities and currents. I then quickly set $\vec{B}=0$ and spent all of my time on investigating a collisionless plasma in electrostatic approximation, described by the Vlasov-Poisson system.

On a completely separate track (§12 and onwards in my notes), I looked at the fluid dynamics of a conducting medium threaded by magnetic field — while I often called this medium "plasma", I refused to engage with its microscopic structure and instead derived equations for its density, velocity and pressure from basic requirements imposed by conservation laws.

Indeed kinetic descriptions are often harder to wrap one's mind around than fluid ones and so we like imagining plasmas as fluids, characterised by density, velocity, and ~~pressure~~ also pressure, temperature or some generalisations thereof. ~~Based on classical~~

Indeed, even in the very kinetic lectures on such things as longitudinal plasma waves (§3), I occasionally appealed to fluid-dynamical description: e.g.,

"Langmuir hydrodynamics" (Ex. 3.1), "hydrodynamics of sound waves" (Ex. 3.6), etc.

~~Because it's easier~~ Such descriptions reduce the dimensionality of the phase space from 6 to 3 and thus make things "simpler", expressing them in terms of "intuitive" quantities - but why are these quantities intuitive? Well, they are, of course, densities of conserved ~~not~~ quantities (# of particles, momentum, energy), but our predilection to limiting ourselves to them unless we absolutely must (like in the case of Landau damping) is rooted in the fact that gases that we are used to (e.g. our atmosphere) are very collisional. In application to eq. (1), this would mean that the collision term — the rhs of the eqn — were dominant, i.e.,

$$\frac{\nu_2}{\nu} \gg \omega_m$$

↑
typical coll. freq. typical frequency of motions,
of species 2 waves, etc.

Then, to lowest order in $\frac{\nu_2}{\nu}$ expansion,

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = 0 \quad \Rightarrow \quad f_\alpha = \frac{n_\alpha}{(2\pi V_{th\alpha})^{3/2}} e^{-\frac{|\vec{v} - \vec{U}_\alpha|^2}{V_{th\alpha}^2}} \quad (2)$$

So the distribution function of species α is a local Maxwellian with, for each species, some density n_α , flow velocity \vec{U}_α , and temperature $T_\alpha = n_\alpha V_{th\alpha}^2 / 2$ (it is convenient to let different species have different flow velocities and temperatures, because these often equalize at a later rate

than the distribution converges to a Maxwellian).

See ongoing course on Collisional P.P.

Therefore I will be interested in a regime where collisions

The dirty secret of MHD description is that in fact, to get closed equations, I needed collisions to be dominant — spot where that happened in §12! (I will point this out in what follows.)

Here I will be interested in a physical regime where collisions are not so dominant:

$$\gamma_d \sim \omega \text{ or even } \gamma_d \ll \omega \quad (3)$$

but I will make a concerted effort to preserve as much of the fluid approach as possible.

Indeed we will see that, in the presence of a strong magnetic field ("strong" in a sense shortly to be quantified), one can preserve quite a lot of fluid approach — thus making an explicit and aesthetically pleasing link between kinetics and MHD.

§2. From kinetics to fluid dynamics. 2.1 Peculiar Kinetics

I would like to make a preliminary technical step that will facilitate formulating plasma kinetics explicitly as a generalization of the fluid theory.

Change variables in (1) as follows:

$$(t, \vec{r}, \vec{v}) \rightarrow (t, \vec{r}, \vec{w}), \quad \vec{w} = \vec{v} - \vec{u}_a(t, \vec{r}) \quad (4)$$

peculiar
velocity

Thus, we explicitly separate the flow of species α ("ordered motion") from the particle's "disordered motion". Here

$$\vec{u}_\alpha(t, \vec{r}) = \frac{1}{n_\alpha} \int d^3 \vec{v} \vec{v} f_\alpha(t, \vec{r}, \vec{v}) - \text{exact!} \quad (5)$$

Then

$$\begin{aligned} \left(\frac{\partial}{\partial t} \right)_w &= \left(\frac{\partial}{\partial t} \right)_w + \left(\frac{\partial \vec{w}}{\partial t} \right)_w \cdot \left(\frac{\partial}{\partial \vec{w}} \right)_{\vec{v}} = \\ &= \left(\frac{\partial}{\partial t} \right)_w - \frac{\partial \vec{u}_\alpha}{\partial t} \cdot \frac{\partial}{\partial \vec{w}} \\ \left(\nabla \right)_v &= \left(\nabla \right)_w + \left(\nabla \vec{w} \right)_v \cdot \frac{\partial}{\partial \vec{w}} = \left(\nabla \right)_w - (\nabla \vec{u}_\alpha) \cdot \frac{\partial}{\partial \vec{w}} \\ \frac{\partial}{\partial \vec{v}} &= \frac{\partial}{\partial \vec{w}} \end{aligned} \quad \left. \right\} \quad (6)$$

This gives us, from eq.(1),

$$\begin{aligned} \frac{\partial f_\alpha}{\partial t} - \frac{\partial \vec{u}_\alpha}{\partial t} \cdot \frac{\partial f_\alpha}{\partial \vec{w}} + (\vec{u}_\alpha + \vec{w}) \cdot \nabla f_\alpha - (\vec{u}_\alpha + \vec{w}) \cdot (\nabla \vec{u}_\alpha) \cdot \frac{\partial f_\alpha}{\partial \vec{w}} \\ + \frac{q_\alpha}{m_\alpha} \left[\vec{E} + \frac{(\vec{u}_\alpha + \vec{w}) \times \vec{B}}{c} \right] \cdot \frac{\partial f_\alpha}{\partial \vec{w}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_c \end{aligned} \quad (7)$$

Let us introduce compact notation

$$\frac{d}{dt_\alpha} = \frac{\partial}{\partial t} + \vec{u}_\alpha \cdot \nabla \quad \begin{matrix} \text{(convective derivative)} \\ \text{wrt species } \alpha \end{matrix}$$

and group terms:

$$\begin{aligned} \frac{df_\alpha}{dt_\alpha} + \vec{w} \cdot \nabla f_\alpha + \left[\frac{q_\alpha}{m_\alpha} \vec{w} \times \vec{B} - \vec{w} \cdot \nabla \vec{u}_\alpha + \left(\frac{q_\alpha}{m_\alpha} \left(\vec{E} + \frac{\vec{u}_\alpha \times \vec{B}}{c} \right) - \frac{d\vec{u}_\alpha}{dt_\alpha} \right) \right] \frac{\partial f_\alpha}{\partial \vec{w}} \\ = \left(\frac{\partial f_\alpha}{\partial t} \right)_c \end{aligned} \quad (8)$$

\vec{a}_α
acceleration independent
of \vec{w}

The definition of \vec{u}_2 in (5) now becomes a constraint on the new distribution function $f_2(t, \vec{r}, \vec{w})$:

$$\int d^3 \vec{w} \vec{w} f_2 = 0 \quad (9) \quad \text{i.e. } \int d^3 \vec{w} m_2 \vec{w}$$

~~Integrability condition for the distribution function~~

Taking the first moment of eq. (8) and using (9) gives us an evolution eqn for \vec{u}_2 — this is exactly equivalent to taking the \vec{v} moment of eq. (1):

$$\nabla \cdot \underbrace{\int d^3 \vec{w} m_2 \vec{w} \vec{w} f_2}_{\substack{\text{pressure} \\ \hat{P}_2}} + \int d^3 \vec{w} m_2 \vec{w} [\dots] \cdot \frac{\partial f_2}{\partial \vec{w}} = \int d^3 \vec{w} m_2 \vec{w} \left(\frac{\partial f_2}{\partial \vec{w}} \right)_c \quad (10)$$

by parts,
then use (9)

\hat{P}_2 \vec{R}_2
pressure tensor friction force

$$\nabla \cdot \hat{P}_2 - m_2 n_2 \vec{a}_2 = \vec{R}_2$$

\uparrow sub from p. 4

$$\boxed{m_2 n_2 \frac{d \vec{u}_2}{dt_2} = -\nabla \cdot \hat{P}_2 + q_2 n_2 \left(\vec{E} + \frac{\vec{u}_2 \times \vec{B}}{c} \right) + \vec{R}_2} \quad (11)$$

Momentum eqn for species 2

So, our new kinetic equation is (8), and it depends on fields \vec{E} , \vec{B} and \vec{u}_2 , which are calculated in terms of f_2 and each other via Maxwell's equations and eq. (11).

I am on the brink of being able to work with MHD variables. Recall that what we cared about in §12 was the flow of mass:

Let us see ~~whether~~ how these quantities evolve.

Taking the $\int d^3\vec{w}$ moment of eq. (8), we get

$$\frac{d \ln \alpha}{dt \alpha} - \int d^3 \vec{W} \vec{W} \cdot (\nabla \vec{U}_2) \cdot \frac{\partial f_\alpha}{\partial \vec{W}} = 0, \quad (13)$$

by parts

Where all other terms vanish by (9).

$$\frac{d n_\alpha}{dt} + (\nabla \cdot \vec{u}_\alpha) n_\alpha = 0 \quad (14)$$

Equivalently, and more conventionally,

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + \nabla \cdot (\mathbf{n}_\alpha \vec{u}_\alpha) = 0, \quad (15) \quad \text{via } \sum_\alpha m_\alpha$$

the continuity equation for species s , whence

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}, \quad (16)$$

The mass continuity equation, same as in MHD.

Now sum eq. (ii) over spec

Now sum eq. (11) over species : $\sum_{\alpha} \vec{m}_{\alpha} n_{\alpha} \frac{d\vec{u}_{\alpha}}{dt_{\alpha}} = - \nabla \cdot \underbrace{\sum_{\alpha} \vec{p}_{\alpha}}_{\stackrel{\wedge}{\sim} \text{ "}} + \underbrace{\left(\sum_{\alpha} q_{\alpha} n_{\alpha} \right)}_{\stackrel{\wedge}{\sim} \text{ "}} \vec{E} + \frac{\left(\sum_{\alpha} q_{\alpha} n_{\alpha} \vec{u}_{\alpha} \right) \times \vec{B}}{c}$

\hat{P}
total
pressure
tensor

\circ (friction between species
(conserves overall momentum))

Now

$$\begin{aligned}
 \sum_{\alpha} m_{\alpha} n_{\alpha} \frac{d\vec{u}_{\alpha}}{dt_{\alpha}} &= \sum_{\alpha} \frac{d}{dt_{\alpha}} m_{\alpha} n_{\alpha} \vec{u}_{\alpha} - \sum_{\alpha} m_{\alpha} \vec{u}_{\alpha} \frac{dn_{\alpha}}{dt_{\alpha}} = \\
 &= \frac{\partial}{\partial t} \rho \vec{u} + \sum_{\alpha} \left[\vec{u}_{\alpha} \cdot \nabla m_{\alpha} n_{\alpha} \vec{u}_{\alpha} + m_{\alpha} n_{\alpha} \vec{u}_{\alpha} (\nabla \cdot \vec{u}_{\alpha}) \right] \\
 &= \frac{\partial}{\partial t} \rho \vec{u} + \nabla \cdot \sum_{\alpha} m_{\alpha} n_{\alpha} \vec{u}_{\alpha} \vec{u}_{\alpha} \quad \begin{matrix} \text{drift of species} \\ \text{wrt mass} \\ \text{flow} \end{matrix} \\
 &= \frac{\partial}{\partial t} \rho \vec{u} + \nabla \cdot \left[\rho \vec{u} \vec{u} + \sum_{\alpha} m_{\alpha} n_{\alpha} (\vec{u}'_{\alpha} \vec{u} + \vec{u} \vec{u}'_{\alpha} + \vec{u}'_{\alpha} \vec{u}'_{\alpha}) \right] \\
 &= \cancel{\rho \frac{\partial \vec{u}}{\partial t} + \vec{u} \cancel{\frac{\partial^2}{\partial t^2}} + \rho \vec{u} \cdot \nabla \vec{u} + \vec{u} \cancel{\nabla \cdot (\rho \vec{u})}} \quad \begin{matrix} \text{because } \sum m_{\alpha} n_{\alpha} \vec{u}'_{\alpha} = 0 \\ \text{by eq. (16)} \end{matrix} \\
 &\quad + \nabla \cdot \sum_{\alpha} m_{\alpha} n_{\alpha} \vec{u}'_{\alpha} \vec{u}'_{\alpha} \\
 &= \cancel{\rho \frac{d\vec{u}}{dt}} + \nabla \cdot \sum_{\alpha} m_{\alpha} n_{\alpha} \vec{u}'_{\alpha} \vec{u}'_{\alpha} \quad (18)
 \end{aligned}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

By the Gauss law,

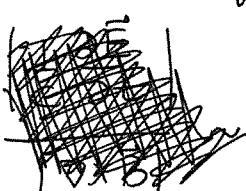
$$\sigma = \sum_{\alpha} q_{\alpha} n_{\alpha} = \frac{\nabla \cdot \vec{E}}{4\pi} \quad (19)$$

By Ampere - Maxwell law,

$$\vec{j} = \frac{c}{4\pi} \left(\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right) \quad (20)$$

The \vec{E} terms will be negligible but we'll need a good estimate for \vec{E} to prove this.

~~we have~~ So we have



$$\boxed{\rho \frac{d\vec{u}}{dt} = - \nabla \cdot (\hat{P} + \sum_{\alpha} m_{\alpha} n_{\alpha} \vec{u}'_{\alpha} \vec{u}'_{\alpha}) + \sigma \vec{E} + \frac{\vec{J} \times \vec{B}}{c}} \quad (21)$$

To get an estimate for \vec{E} , let us return to eq. (11) and rearrange it so:

$$\vec{E} = -\frac{\vec{U}_d \times \vec{B}}{c} + \underbrace{\frac{\nabla \cdot \hat{P}_d}{q_d n_d}}_{\sim 0} - \frac{\vec{R}_d}{q_d n_d} + \underbrace{\frac{m_d}{q_d} \frac{d\vec{U}_d}{dt_d}}_{\sim 0} \quad (22)$$

all of these are small, as follows:

$$\frac{\left| \frac{m_d}{q_d} \frac{d\vec{U}_d}{dt_d} \right|}{\left| \frac{\vec{U}_d \times \vec{B}}{c} \right|} \sim \frac{cm_d \omega U_d}{q_d n_d B} \sim \frac{\omega}{S_d} \ll 1 \quad \begin{array}{l} \text{assume Larmor motion} \\ \text{faster than all} \\ \text{relevant frequencies} \end{array}$$

↑
Larmor frequency

$$(23)$$

$$\frac{\left| \frac{\vec{R}_d}{q_d n_d} \right|}{\left| \frac{\vec{U}_d \times \vec{B}}{c} \right|} \sim \frac{cm_d n_d v_{dd}^2 U_d}{q_d n_d Y_d B} \sim \frac{v_{dd}^2}{S_d} \ll 1 \quad \begin{array}{l} \text{assume Larmor motion} \\ \text{faster than all} \\ \text{collisions} \end{array}$$

$$(24)$$

$$\vec{R}_d = - \sum_{\alpha'} m_d n_d v_{dd} (\vec{U}_{\alpha} - \vec{U}_{\alpha'}) \quad \text{drag force}$$

$$\frac{\left| \frac{\nabla \cdot \hat{P}_d}{q_d n_d} \right|}{\left| \frac{\vec{U}_d \times \vec{B}}{c} \right|} \sim \frac{ck \cancel{m_d n_d v_{thd}^2}}{q_d n_d U_d B} \sim \frac{k v_{thd}^2}{U_d S_d} \sim \frac{k v_{thi}^2}{U_i S_i} \quad \begin{array}{l} \text{assume } T_i \sim T_e \\ \text{if we assume } Ma \sim 1 \end{array}$$

$$(25)$$

$$\sim k p_i \frac{1}{Ma} \ll 1 \quad \text{if we assume } Ma \sim 1 \text{ and } k p_i \ll 1$$

AB: What if I want $Ma \ll 1$?

Typically then $\hat{P}_d = \hat{P}_{od} + \delta \hat{P}_d$,

where $\delta \hat{P}_d \sim Ma \hat{P}_{od}$. Then

$$\nabla \cdot \hat{P}_d \sim \nabla \cdot \delta \hat{P}_d \sim Ma \hat{P}_{od}$$

and Ma cancels.

all scales small compared to Larmor radius

same as $\omega/S_d \ll 1$

if $k v_{thi} \sim k u_i \sim \omega$

-9-

Thus, to reiterate, assuming

$$v_{d\alpha} \sim \omega \sim k u_\alpha \sim k v_{th,i} \ll \Omega_\alpha \quad (26)$$

We get

$$\vec{E} = - \frac{\vec{u}_\alpha \times \vec{B}}{c} \quad (27)$$

to lowest order in this approximation.

This gives us two things:

- 1) $\vec{u}_{\perp\alpha} = c \frac{\vec{E} \times \vec{B}}{B^2} = \vec{u}_\perp$ independent of α , i.e., \perp to \vec{B} , all species just move at the $\vec{E} \times \vec{B}$ drift velocity.
- 2) We can now estimate the size of the \vec{E} terms in (21):

$$\left| \frac{\frac{1}{c} \frac{\partial \vec{E}}{\partial t}}{|\vec{J} \times \vec{B}|} \right| \sim \frac{\omega u_\alpha B}{c^2 B k} \sim \frac{\omega u_\alpha}{c^2 k} \sim \frac{u_\alpha^2}{c^2} \ll 1 \quad (28)$$

as long as everyting is non-relativistic

This is the same estimate as I have previously made to get rid of displacement current in MHD

$$\left| \frac{\sigma \vec{E}}{|\vec{J} \times \vec{B}|} \right| \sim \frac{ck E^2}{B \cdot ck B} \sim \frac{E^2}{B^2} \sim \frac{u_\alpha^2}{c^2} \ll 1 \quad \text{same again.} \quad (29)$$

Recall that in MHD, these estimates followed from Ohm's law — which is just (27), which we can

rewrite

$$\boxed{\vec{E} + \frac{\vec{u} \times \vec{B}}{c} = 0} \quad (30)$$

[NB: To get $\eta \vec{J}$, write (22) for $\alpha = e$ and keep $R_{\alpha e}$]

An immediate consequence of (28) is that all species drift w.r.t mass flow can only be parallel to \vec{B} :

$$\vec{q}'_2 = u'_{\parallel \alpha} \hat{b} \quad (32)$$

With all this additional information, the momentum eqn (21) becomes

$$\boxed{\rho \frac{d\vec{u}}{dt} = -\nabla \cdot (\hat{P} + \sum_{\alpha} m_{\alpha} n_{\alpha} u'^2_{\parallel \alpha} \hat{b} \hat{b} + \frac{B^2}{8\pi} \hat{I} - \frac{\vec{B} \vec{B}}{4\pi})} \quad (33)$$

\nearrow
Lorentz force
rewritten as

Maxwell Stress.

Another immediate

consequence of (28) is a closed equation ~~for~~ for \vec{B} in terms of \vec{u} : from Faraday's law,

$$\boxed{\frac{\partial \vec{B}}{\partial t} = -c \nabla \times \vec{E} = \nabla \times (\vec{u} \times \vec{B})}, \quad (34)$$

the familiar MHD induction equation.

So, (16), (33), and (34) look exactly like MHD equations with the exception of the pressure tensor and additional stress associated with ~~particles~~ species drifts.

In a collisional environment, there would be no drifts (because $\vec{R}_{\alpha} = 0$ would be the lowest-order approximation in (11)) and $\hat{P} = \rho \hat{I}$ would be isotropic, recovering MHD.

§3 Gyrotropic Plasmas

Let us go back to eq. (8) and observe the following.

$$\frac{q_2}{m_2} \frac{\vec{w} \times \vec{B}}{c} \cdot \frac{\partial f_2}{\partial \vec{w}} = -\frac{q_2 B}{m_2 c} \left(\frac{\partial f_2}{\partial \vec{w}} \right)_{w_\perp, w_\parallel} = -\nabla_{\vec{w}} \left(\frac{\partial f_2}{\partial \vec{w}} \right)_{w_\perp, w_\parallel} \quad (35)$$

change variables

$$\vec{w} = w_x \cos \theta \hat{x} + w_y \sin \theta \hat{y} + w_z \hat{z}$$

Where $\hat{x}, \hat{y}, \hat{z}$ are orts s.t. \hat{z} is aligned with the local direction of m. field, $\hat{z} = \hat{b}$.

$$\text{Then } \left(\frac{\partial}{\partial x} \right)_{w_1, w_2} = -w_1 \sin \theta \frac{\partial}{\partial w_x} + w_1 \cos \theta \frac{\partial}{\partial w_y}$$

$$\text{but } \vec{w} \times \hat{\vec{b}} = w_L \sin \theta \hat{x} - w_L \cos \theta \hat{y}$$

So I can write eq.(8) as

$$\Omega_2 \left(\frac{\partial f_2}{\partial \omega} \right)_{W_\perp, W_\parallel} = \frac{d f_2}{dt_2} + \vec{w} \cdot \nabla f_2 + (\vec{a}_2 - \vec{w} \cdot \nabla \vec{u}_2) \cdot \frac{\partial f_2}{\partial \vec{w}} - \left(\frac{\partial f_2}{\partial t} \right)_c$$

$\underbrace{\omega}_{\text{or } k u_2}$ $\underbrace{k v_{th2}}$ $\underbrace{\omega \frac{u_2}{k v_{th2}}}$ $\underbrace{k u_2}$ $\underbrace{v_2}_{\text{or small terms as per p. 8}}$ (36)

All of this is small provided

$$\frac{w}{\Sigma_2} \ll 1 \text{ and } k_{P_2} \ll 1 \text{ and } \frac{\sqrt{2}}{\Sigma_2} \ll 1$$

Thus, in this approximation,

$$\left(\frac{\partial f_2}{\partial \vec{w}}\right)_{w_\perp, w_\parallel} = 0 \Rightarrow f_2 = f_2(t, \vec{r}, w_\perp, w_\parallel) \quad (32)$$

Independent of λ .

The eqn for f_2 is found by using (37) in (36) and averaging over ω : ~~$\int d\omega f_2(\omega)$~~ $\langle \text{rhs of (36)} \rangle_\omega = 0$.

I shall do this calculation later, but first, in pursuit of instant gratification, let me explore the consequences of (37) :

$$\begin{aligned}\hat{P}_2 &= \int d^3\vec{w} n_2 \vec{w} \vec{w} f_2(t, \vec{r}, w_L, w_{||}) = \\ &= \int dw_L w_L \int dw_{||} f_2(t, \vec{r}, w_L, w_{||}) \underbrace{\int d\vec{w} \vec{w} \vec{w}}_{2\pi \langle \vec{w} \vec{w} \rangle_{||}} = \\ &= P_{\perp 2} (\mathbb{I} - \vec{b} \vec{b}) + P_{|| 2} \vec{b} \vec{b}\end{aligned}\tag{38}$$

where

$$P_{\perp 2} = \int d^3\vec{w} \frac{w_L^2}{2} f_2$$

$$P_{|| 2} = \int d^3\vec{w} w_{||}^2 f_2$$

I will call this P'_2

Eq. (33) becomes

$$\boxed{\rho \frac{d\vec{u}}{dt} = -\nabla \left(p_{\perp} + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[\vec{b} \vec{b} \left(p_{\perp} - p_{||} - \sum_m m_2 n_2 u_{||m}^{(2)} + \frac{B^2}{4\pi} \right) \right]}$$

$\underbrace{p_{\perp}}$ $\underbrace{p_{||} - \sum_m m_2 n_2 u_{||m}^{(2)}}$
combined pressure combined parallel stress

(39)

So the difference between MHD and our "kinetic MHD" boils down to $p_{\perp} - p'_{||} \neq 0$ and to kinetic determination of p_{\perp} and $p'_{||}$. Manifestly, this difference matters if

$$|p_{\perp} - p'_{||}| \gtrsim \frac{B^2}{4\pi} \quad \text{or} \quad \frac{|p_{\perp} - p'_{||}|}{P} \gtrsim \frac{B^2}{4\pi P} = \frac{2}{\beta} \tag{40}$$

i.e., not in low- β plasmas (of which more later in this course — their physics is different!). Plasmas where (40) is satisfied are usually in space.