These are the notes for my lectures on Kinetic Theory of Plasmas and on Magnetohydrodynamics, taught since 2014 as part of the MMathPhys programme at Oxford. Part I contains the lectures on plasma kinetics that formed part of the course on Kinetic Theory, taught jointly with Paul Dellar and James Binney (succeeded by Jean-Baptiste Fouvry). The more advanced sections cover the material first taught in the Advanced Topics in Plasma Physics course during the extraordinary Trinity Term 2020, under the coronavirus lockdown. Part II is an introduction to magnetohydrodynamics, which was part of the course on Advanced Fluid Dynamics, taught (since 2015) jointly with Paul Dellar. These notes evolved from two earlier courses: “Advanced Plasma Theory,” taught as a graduate course at Imperial College in 2008, and “Magnetohydrodynamics and Turbulence,” taught three times as a Mathematics Part III course at Cambridge in 2005-06. Extracts from these notes have also been used in (and in part written for) my lectures at successive plasma-physics sessions of ´Ecole de Physique des Houches in 2017 and 2019. Finally, Part III of these notes is dedicated to the marriage of kinetics and MHD and originates from the Les Houches lectures of 2013 and 2015. I will be grateful for any feedback from students, tutors or sympathisers.
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Magnetohydrodynamics Problem Set
Part III. Kinetic MHD
PART I

Kinetic Theory of Plasmas

1. Kinetic Description of a Plasma

We shall study a gas consisting of charged particles—ions and electrons. In general, there may be many different species of ions, with different masses and charges, and, of course, only one type of electrons.

I will index particle species by $\alpha$ ($\alpha = e$ for electrons, $\alpha = i$ for ions). Each is characterised by its mass $m_\alpha$ and charge $q_\alpha = Z_\alpha e$, where $e$ is the magnitude of the electron charge and $Z_\alpha$ is a positive or negative integer (e.g., $Z_e = -1$).

1.1. Quasineutrality

We shall always assume that plasma is neutral overall:

$$\sum_\alpha q_\alpha N_\alpha = eV \sum_\alpha Z_\alpha \bar{n}_\alpha = 0,$$

(1.1)

where $N_\alpha$ is the number of the particles of species $\alpha$, $\bar{n}_\alpha = N_\alpha / V$ is their mean number density and $V$ the volume of the plasma. This condition is known as quasineutrality.

1.2. Weak Interactions

Interaction between charged particles is governed by the Coulomb potential:

$$U(|r_i^{(\alpha)} - r_j^{(\alpha')}|) = -\frac{q_\alpha q_{\alpha'}}{|r_i^{(\alpha)} - r_j^{(\alpha')}|},$$

(1.2)

where by $r_i^{(\alpha)}$ I mean the position of the $i$-th particle of species $\alpha$. It is a safe bet that we will only be able to have a nice closed kinetic description if the gas is approximately ideal, i.e., if particles interact weakly, viz.,

$$k_B T \gg U \sim e^2 \Delta r \sim e^2 n^{1/3},$$

(1.3)

where $k_B$ is the Boltzmann constant, which will henceforth be absorbed into the temperature $T$, and $\Delta r \sim n^{-1/3}$ is the typical interparticle distance. Let us see what this condition means and implies physically.

1.3. Debye Shielding

Let us consider a plasma in thermodynamic equilibrium (as one does in statistical mechanics, I will refuse to discuss, for the time being, how exactly it got there). Take one particular particle, of species $\alpha$. It creates an electric field around itself, $E = -\nabla \varphi$; all other particles sit in this field (Fig. 1)—and, indeed, also affect it, as we will see below.

In equilibrium, the densities of these particles ought to satisfy Boltzmann’s formula:

$$n_{\alpha'}(r) = \bar{n}_{\alpha'} e^{-q_{\alpha'} \varphi(r)/T} \approx \bar{n}_{\alpha'} - \bar{n}_{\alpha'} q_{\alpha'} \varphi / T,$$

(1.4)

where $\bar{n}_{\alpha'}$ is the mean density of the particles of species $\alpha'$ and $\varphi(r)$ is the electrostatic potential, which depends on the distance $r$ from our “central” particle. As $r \to \infty$, $\varphi \to 0$ and $n_{\alpha'} \to \bar{n}_{\alpha'}$. The exponential can be Taylor-expanded provided the weak-interaction condition (1.3) is satisfied ($e \varphi \ll T$).
By the Gauss–Poisson law, we have

$$\nabla \cdot \mathbf{E} = -\nabla^2 \varphi = 4\pi q_\alpha \delta(r) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'}$$

$$\approx 4\pi q_\alpha \delta(r) + 4\pi \sum_{\alpha'} q_{\alpha'} \bar{n}_{\alpha'} - \left( \sum_{\alpha'} \frac{4\pi \bar{n}_{\alpha'} q_{\alpha'}^2}{T} \right) \varphi.$$  \hspace{1cm} (1.5)

In the first line of this equation, the first term on the right-hand side is the charge density associated with the “central” particle and the second term is the charge density of the rest of the particles. In the second line, I used the Taylor-expanded Boltzmann expression (1.4) for the particle densities and then the quasineutrality (1.1) to establish the vanishing of the second term. The combination that has arisen in the last term as a prefactor of $\varphi$ has dimensions of inverse square length, so we define the Debye length to be

$$\lambda_D \equiv \left( \sum_{\alpha} 4\pi \bar{n}_\alpha q_\alpha^2 \right)^{-1/2}.$$  \hspace{1cm} (1.6)

Using also the obvious fact that the solution of (1.5) must be spherically symmetric, we recast this equation as follows

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \varphi}{\partial r} - \frac{1}{\lambda_D^2} \varphi = -4\pi q_\alpha \delta(r).$$ \hspace{1cm} (1.7)

The solution to this that asymptotes to the Coulomb potential $\varphi \to q_\alpha / r$ as $r \to 0$ and to zero as $r \to \infty$ is

$$\varphi = \frac{q_\alpha}{r} e^{-r/\lambda_D}.$$ \hspace{1cm} (1.8)

Thus, in a quasineutral plasma, charges are shielded on typical distances $\sim \lambda_D$.

Obviously, this statistical calculation only makes sense if the “Debye sphere” has very many particles in it, viz., if

$$n \lambda_D^3 \gg 1.$$ \hspace{1cm} (1.9)

Let us check that this is the case:

$$n \lambda_D^3 \sim n \left( \frac{T}{ne^2} \right)^{3/2} = \left( \frac{T}{n^{1/3}e^2} \right)^{3/2} \gg 1,$$ \hspace{1cm} (1.10)
provided the weak-interaction condition (1.3) is satisfied. The quantity \( n\lambda_D^3 \) is called the plasma parameter.

\[ \text{1.4. Micro- and Macroscopic Fields} \]

This calculation tells us something very important about electromagnetic fields in a plasma. Let \( E^{(\text{micro})}(r, t) \) and \( B^{(\text{micro})}(r, t) \) be the exact microscopic fields at a given location \( r \) and time \( t \). These fields are responsible for interactions between particles. On distances \( l \ll \lambda_D \), these will be just the two-particle interactions—binary collisions between particles in vacuo, just like in a neutral gas (except the interparticle potential is the Coulomb potential). In contrast, on distances \( l \gg \lambda_D \), individual particles’ fields are shielded and what remains are fields due to collective influence of large numbers of particles—macroscopic fields:

\[ E^{(\text{micro})} = \langle E^{(\text{micro})} \rangle + \delta E, \quad B^{(\text{micro})} = \langle B^{(\text{micro})} \rangle + \delta B, \quad (1.11) \]

where the macroscopic fields \( E \) and \( B \) are averages over some intermediate scale \( l \) such that

\[ \Delta r \sim n^{-1/3} \ll l \ll \lambda_D. \quad (1.12) \]

Such averaging (or “coarse-graining”) is made possible by the condition (1.9).

Thus, plasma has a new feature compared to neutral gas: because the Coulomb potential is long-range \((\propto 1/r)\), the fields decay on a length scale that is long compared to the interparticle distances \([\lambda_D \gg \Delta r \sim n^{-1/3} \text{ according to (1.9)}]\) and so, besides interactions between individual particles, there are also collective effects: interaction of particles with mean macroscopic fields due to all other particles.

Before I use this approach to construct a description of the plasma as a continuum (on scales \( \gg l \)), let us check that particles travel sufficiently long distances between collisions in order to feel the macroscopic fields, viz., that their mean free path is \( \lambda_{\text{mfp}} \gg \lambda_D \). The mean free path can be estimated in terms of the collision cross-section \( \sigma \):

\[ \lambda_{\text{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4} \quad (1.13) \]

because \( \sigma \sim d^2 \) and the effective distance \( d \) by which particles have to approach each other in order to have significant Coulomb interaction is inferred by balancing the Coulomb potential energy (1.2) with the particle temperature, \( e^2/d \sim T \). Using (1.13) and (1.6), we find

\[ \frac{\lambda_{\text{mfp}}}{\lambda_D} \sim \frac{T^2}{ne^4} \left(\frac{ne^2}{T}\right)^{1/2} \sim n\lambda_D^3 \gg 1, \quad \text{q.e.d.} \quad (1.14) \]

Thus, it makes sense to talk about a particle travelling long distances experiencing the macroscopic fields exerted by the rest of the plasma collectively before being deflected by a much larger, but also much shorter-range, microscopic field of another individual particle.
1.5. Maxwell’s Equations

The exact microscopic fields satisfy Maxwell’s equations and, since Maxwell’s equations are linear, so do the macroscopic fields: by direct averaging,

\[ \nabla \cdot \langle E^{(\text{micro})} \rangle = 4\pi \langle \sigma^{(\text{micro})} \rangle, \]
\[ \nabla \cdot \langle B^{(\text{micro})} \rangle = 0, \]
\[ \nabla \times \langle E^{(\text{micro})} \rangle + \frac{1}{c} \frac{\partial \langle B^{(\text{micro})} \rangle}{\partial t} = 0, \]
\[ \nabla \times \langle B^{(\text{micro})} \rangle - \frac{1}{c} \frac{\partial \langle E^{(\text{micro})} \rangle}{\partial t} = 4\pi c \langle j^{(\text{micro})} \rangle. \]

The new quantities here are the averages of the microscopic charge density \( \sigma^{(\text{micro})} \) and the microscopic current density \( j^{(\text{micro})} \). How do we calculate them?

Clearly, they depend on where all the particles are at any given time and how fast these particles move. We can assemble all this information in one function:

\[ F_\alpha(r, v, t) = \sum_{i=1}^{N_\alpha} \delta^3(r - r^{(\alpha)}_i(t)) \delta^3(v - v^{(\alpha)}_i(t)), \]

where \( r^{(\alpha)}_i(t) \) and \( v^{(\alpha)}_i(t) \) are the exact phase-space coordinates of particle \( i \) of species \( \alpha \) at time \( t \), i.e., these are the solutions of the exact equations of motion for all these particles moving in the microscopic fields \( E^{(\text{micro})}(t, r) \) and \( B^{(\text{micro})}(t, r) \). The function \( F_\alpha \) is called the Klimontovich distribution function. It is a random object (i.e., it fluctuates on scales \( \ll \lambda_D \)) because it depends on the exact particle trajectories, which depend on the exact microscopic fields. In terms of this distribution function,

\[ \sigma^{(\text{micro})}(r, t) = \sum_\alpha q_\alpha \int d^3v \ F_\alpha(r, v, t), \]
\[ j^{(\text{micro})}(r, t) = \sum_\alpha q_\alpha \int d^3v \ v F_\alpha(r, v, t). \]

We now need to average these quantities for use in (1.15) and (1.18). We shall assume that the average over microscales (1.12) and the ensemble average (i.e., the average over many different initial conditions) are the same. The ensemble average of \( F_\alpha \) is an object familiar from the kinetic theory of gases (Dellar 2015), the so-called one-particle distribution function:

\[ \langle F_\alpha \rangle = f_{1\alpha}(r, v, t) \]

(I shall henceforth omit the subscript 1). If we learn how to compute \( f_{1\alpha} \), then we can average (1.20) and (1.21), substitute them into (1.15) and (1.18), and have the following set of macroscopic Maxwell’s equations:

\[ \nabla \cdot E = 4\pi \sum_\alpha q_\alpha \int d^3v \ f_{1\alpha}(r, v, t), \]
\[ \nabla \cdot B = 0, \]
\[ \nabla \times E + \frac{1}{c} \frac{\partial B}{\partial t} = 0, \]
\[ \nabla \times B - \frac{1}{c} \frac{\partial E}{\partial t} = 4\pi \sum_\alpha q_\alpha \int d^3v \ v f_{1\alpha}(r, v, t). \]
1.6. Vlasov–Landau Equation

We now need an evolution equation for \( f_\alpha(r, v, t) \), hopefully in terms of the macroscopic fields \( E(r, t) \) and \( B(r, t) \), so we can couple it to (1.23–1.26) and thus have a closed system of equations describing our plasma.

The process of deriving it starts with Liouville’s theorem and is a direct generalisation of the BBGKY procedure familiar from gas kinetics (e.g., Dellar 2015)\(^1\) to the somewhat more cumbersome case of a plasma:

— many species \( \alpha \);
— presence of forces due to the macroscopic fields \( E \) and \( B \);
— Coulomb potential for interparticle collisions, with some attendant complications to do with its long-range nature: in brief, use Rutherford’s cross section and cut off long-range interactions at \( \lambda_D \) (this is described in many textbooks and plasma-physics courses: see, e.g., Parra 2019\(^a\) or a shortcut in §10.2.3 of these Lectures).

The result of this derivation is

\[
\frac{\partial f_\alpha}{\partial t} + \{ f_\alpha, H_{1\alpha} \} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c.
\]

(1.27)

The Poisson bracket contains \( H_{1\alpha} \), the Hamiltonian for a single particle of species \( \alpha \) moving in the macroscopic fields \( E \) and \( B \)—all the microscopic fields \( \delta E^2 \) are gone into the collision operator on the right-hand side, of which more will be said shortly (§1.8).

Technically speaking, one ought to be working with canonical variables, but dealing with canonical momenta of charged particles in a magnetic field is an unnecessary complication, so I shall stick to the \((r, v)\) representation of the phase space. Then (1.27) takes the form of Liouville’s equation, but with microscopic fields hidden inside the collision operator [see (1.41)]:

\[
\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial r} \cdot (\dot{r} f_\alpha) + \frac{\partial}{\partial v} \cdot (\dot{v} f_\alpha) = \left( \frac{\partial f_\alpha}{\partial t} \right)_c,
\]

(1.28)

where

\[
\dot{r} = v, \quad \dot{v} = \frac{q_\alpha}{m_\alpha} \left( E + \frac{v \times B}{c} \right).
\]

(1.29)

This gives us the Vlasov–Landau equation:

\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left( E + \frac{v \times B}{c} \right) \cdot \frac{\partial f_\alpha}{\partial v} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c.
\]

(1.30)

Any other macroscopic force that the plasma might be subject to (e.g., gravity) can be added to the Lorentz force in the third term on the left-hand side, as long as its divergence in velocity space is \( (\partial / \partial v) \cdot \text{force} = 0 \). Equation (1.30) is closed by Maxwell’s equations (1.23–1.26).

1.7. Klimontovich’s Version of BBGKY

By way of a technical digression, let me outline the (beginning of the) derivation of (1.30) due to Klimontovich (1967). Consider the Klimontovich distribution function (1.19) and calculate

\(^1\)In §1.7, I will sketch Klimontovich’s version of this procedure (Klimontovich 1967).

\(^2\)\( \delta B \) turns out to be irrelevant as long as the particle motion is non-relativistic, \( v/c \ll 1 \).
its time derivative: by the chain rule,
\[
\frac{\partial F_\alpha}{\partial t} = - \sum_i \frac{d r_i^{(\alpha)}(t)}{dt} \cdot \left[ \frac{\partial}{\partial r} \delta^3(r - r_i^{(\alpha)}(t)) \delta^3(v - v_i^{(\alpha)}(t)) \right] \\
- \sum_i \frac{d v_i^{(\alpha)}(t)}{dt} \cdot \left[ \frac{\partial}{\partial v} \delta^3(r - r_i^{(\alpha)}(t)) \delta^3(v - v_i^{(\alpha)}(t)) \right].
\] (1.31)

First, because \(r_i^{(\alpha)}(t)\) and \(v_i^{(\alpha)}(t)\) obviously do not depend on the phase-space variables \(r\) and \(v\), the derivatives \(\partial/\partial r\) and \(\partial/\partial v\) can be pulled outside, so the right-hand side of (1.31) can be written as a divergence in phase space. Secondly, the particle equations of motion give us
\[
\frac{d r_i^{(\alpha)}(t)}{dt} = v_i^{(\alpha)}(t),
\] (1.32)
\[
\frac{d v_i^{(\alpha)}(t)}{dt} = \frac{q_\alpha}{m_\alpha} \left[ E^{(\text{micro})}(r_i^{(\alpha)}(t), t) + \frac{v_i^{(\alpha)}(t) \times B^{(\text{micro})}(r_i^{(\alpha)}(t), t)}{c} \right],
\] (1.33)

which are to be substituted into the right-hand side of (1.31)—after it is written in the divergence form. Since the time derivatives of \(r_i^{(\alpha)}(t)\) and \(v_i^{(\alpha)}(t)\) inside the divergence multiply delta functions identifying \(r_i^{(\alpha)}(t)\) with \(r\) and \(v_i^{(\alpha)}(t)\) with \(v\), \(r_i^{(\alpha)}(t)\) may be replaced by \(r\) and \(v_i^{(\alpha)}(t)\) by \(v\) in the right-hand sides of (1.32) and (1.33) when they go into (1.31). This gives (wrapping all the sums of delta functions back into \(F_\alpha\))
\[
\frac{\partial F_\alpha}{\partial t} = -\nabla \cdot (v F_\alpha) - \frac{\partial}{\partial v} \left[ \frac{q_\alpha}{m_\alpha} \left( E^{(\text{micro})}(r, t) + \frac{v \times B^{(\text{micro})}(r, t)}{c} \right) F_\alpha \right].
\] (1.34)

Finally, because \(r\) and \(v\) are independent variables and the Lorentz force has zero divergence in \(v\) space, \(F_\alpha\) satisfies exactly
\[
\frac{\partial F_\alpha}{\partial t} + v \cdot \nabla F_\alpha + \frac{q_\alpha}{m_\alpha} \left( E^{(\text{micro})} + \frac{v \times B^{(\text{micro})}}{c} \right) \cdot \frac{\partial F_\alpha}{\partial v} = 0.
\] (1.35)

This is the Klimontovich equation. There is no collision integral here because microscopic fields are explicitly present. The equation is closed by the microscopic Maxwell’s equations:
\[
\nabla \cdot E^{(\text{micro})} = 4\pi \sum_\alpha q_\alpha \int d^3v v F_\alpha(r, v, t),
\] (1.36)
\[
\nabla \cdot B^{(\text{micro})} = 0,
\] (1.37)
\[
\nabla \times E^{(\text{micro})} + \frac{1}{c} \frac{\partial B^{(\text{micro})}}{\partial t} = 0,
\] (1.38)
\[
\nabla \times B^{(\text{micro})} - \frac{1}{c} \frac{\partial E^{(\text{micro})}}{\partial t} = 4\pi \sum_\alpha q_\alpha \int d^3v v F_\alpha(r, v, t).
\] (1.39)

Now let us separate the microscopic fields into mean (macroscopic) and fluctuating parts according to (1.11); also
\[
F_\alpha = \langle f_\alpha \rangle + \delta f_\alpha.
\] (1.40)

Maxwell’s equations are linear, so averaging them gives the same equations for \(E\) and \(B\) in terms of \(f_\alpha\) [see (1.23–1.26)] and for \(\delta E\) and \(\delta B\) in terms of \(\delta F_\alpha\). Averaging the Klimontovich equation (1.35) gives the Vlasov–Landau equation:
\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \left( E + \frac{v \times B}{c} \right) \cdot \frac{\partial f_\alpha}{\partial v} = -\frac{q_\alpha}{m_\alpha} \left( \langle \delta E + \frac{v \times \delta B}{c} \rangle \cdot \frac{\partial \delta F_\alpha}{\partial v} \right) \equiv \left( \frac{\partial \delta F_\alpha}{\partial t} \right)_c.
\] (1.41)
The macroscopic fields in the left-hand side satisfy the macroscopic Maxwell’s equations (1.23–1.26). The microscopic fluctuating fields $\delta E$ and $\delta B$ inside the average in the right-hand side satisfy microscopic Maxwell’s equations with fluctuating charge and current densities expressed in terms of $\delta F_\alpha$. Thus, the right-hand side is quadratic in $\delta F_\alpha$. In order to close this equation, we need an expression for the correlation function $\langle \delta F_\alpha \delta F_{\alpha'} \rangle$ in terms of $f_\alpha$ and $f_{\alpha'}$. This is basically what the BBGKY procedure plus truncation of velocity integrals based on an expansion in $1/n\lambda_D^3$ achieve. The result is the Landau collision operator (or the more precise Lenard–Balescu one; see §§10.2.2 and 10.2.3).

Further details are a bit complicated (see Klimontovich 1967), but my aim here was just to show how the fields are split into macroscopic and microscopic ones, with the former appearing explicitly in the kinetic equation and the latter wrapped up inside the collision operator. The presence of the macroscopic fields and the consequent necessity for coupling the kinetic equation with Maxwell’s equations for these fields is the main mathematical difference between the kinetics of neutral gases and the kinetics of plasmas.

1.8. Collision Operator

Finally, a few words about the plasma collision operator (or “collision integral”), first derived explicitly by Landau (1936) (the same considerations apply to the more general operator due to Lenard 1960 and Balescu 1960). It describes two-particle collisions both within the species $\alpha$ and with other species $\alpha'$ and so depends both on $f_\alpha$ and on all other $f_{\alpha'}$. Its derivation is left to you as an exercise in BBGKY’ing, calculating cross sections and velocity integrals (or in googling; shortcut: see Parra 2019a or Swanson 2008).3 In these Lectures, I shall focus on collisionless aspects of plasma kinetics. Whenever a need arises for invoking the collision operator, the important things about it for us will be its properties:

- conservation of particles (within each species $\alpha$),
  \[ \int d^3v \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0; \] (1.42)

- conservation of momentum,
  \[ \sum_\alpha \int d^3v m_\alpha v \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0 \] (1.43)

(same-species collisions conserve momentum, whereas different-species collisions conserve it only after summation over species—there is friction of one species against another; e.g., the friction of electrons against the ions is the Ohmic resistivity of the plasma, known as “Spitzer resistivity”: see Parra 2019a or Helander & Sigmar 2005);

- conservation of energy,
  \[ \sum_\alpha \int d^3v \frac{m_\alpha v^2}{2} \left( \frac{\partial f_\alpha}{\partial t} \right)_c = 0; \] (1.44)

- Boltzmann’s $H$-theorem: the kinetic entropy
  \[ S = -\sum_\alpha \int d^3r \int d^3v \ f_\alpha \ln f_\alpha \] (1.45)
cannot decrease, and, as $S$ is conserved by all the collisionless terms in (1.30), the collision

---

3A somewhat unorthodox derivation of both the Lenard–Balescu and Landau operators will be given in §§10.2.2 and 10.2.3, respectively, as a by-product of a discussion of collisionless (sic) relaxation.
operator must have the property that
\[
\frac{dS}{dt} = -\sum_{\alpha} \int d^3r \int d^3v \left( \frac{\partial f_{\alpha}}{\partial t} \right)_c \ln f_{\alpha} \geq 0,
\] (1.46)
with equality obtained if and only if \( f_{\alpha} \) is a local Maxwellian;

- unlike the Boltzmann operator for neutral gases, the Landau operator expresses
  the cumulative effect of many glancing (rather than “head-on”) collisions (due to
  the long-range nature of the Coulomb interaction) and so it is a Fokker–Planck operator:
\[
\left( \frac{\partial f_{\alpha}}{\partial t} \right)_c = \frac{\partial}{\partial v} \cdot \sum_{\alpha'} \left( A_{\alpha\alpha'}[f_{\alpha'}] + D_{\alpha\alpha'}[f_{\alpha'}] \right) \cdot \frac{\partial}{\partial v} f_{\alpha},
\] (1.47)
where the drag \( A_{\alpha\alpha'} \) (vector) and diffusion \( D_{\alpha\alpha'} \) (matrix) coefficients are integral (in \( v \) space) functionals of \( f_{\alpha'} \). The Fokker–Planck form (1.47) of the Landau operator means
that it describes diffusion in velocity space and so will erase sharp gradients in \( f_{\alpha} \) with
respect to \( v \)—a property that we will find very important in §5.

1.9. So What’s New and What Now?

Let me summarise the new features that have appeared in the kinetic description of a
plasma compared to that of a neutral gas.

- First, particles are charged, so they interact via Coulomb potential. The collision
  operator is, therefore, different: the cross-section is the Rutherford cross-section, most
  collisions are glancing (with interaction on distances up to the Debye length), leading to
diffusion of the particle distribution function in velocity space. Mathematically, this is
manifested in the collision operator in (1.30) having the Fokker–Planck structure (1.47).

One can spin out of the Vlasov–Landau equation (1.30) a theory that is analogous
to what is done with Boltzmann’s equation in gas kinetics (Dellar 2015): derive
fluid equations, calculate viscosity, thermal conductivity, Ohmic resistivity, etc., of a
collisionally dominated plasma, i.e., of a plasma in which the collision frequency of the
particles is much greater than all other relevant time scales. This is done in the same
way as in gas kinetics, but now applying the Chapman–Enskog procedure to the Landau
collision operator. This is quite a lot of work—and constitutes core textbook material
(see Parra 2019a). In magnetised plasmas especially, the resulting fluid dynamics of the
plasma are quite interesting and quite different from neutral fluids—we shall see some
of this in Part II of these Lectures, while the classic treatment of the transport theory
can be found in Braginskii (1965); a great textbook on collisional transport is Helander

- Secondly, Coulomb potential is long-range, so the electric and magnetic fields have a
  macroscopic (mean) part on scales longer than the Debye length—a particle experiencing
these fields is not undergoing a collision in the sense of bouncing off another particle,
but is, rather, interacting, via the fields, with the collective of all the other particles.
Mathematically, this manifests itself as a Lorentz-force term appearing in the right-hand
side of the Vlasov–Landau kinetic equation (1.30). The macroscopic \( E \) and \( B \) fields that

\footnotesize{The simplest example that I can think of in which the collision operator is a velocity-space
diffusion operator of this kind is the gas of Brownian particles [each with velocity described
by Langevin’s equation (12.24)]. This is treated in detail in §6.9 of Schekochihin (2019). In
these Lectures, the general Fokker–Planck form (1.47) emerges in (7.35), and then again, from
a different angle, in (10.28); the Landau operator is (10.43).}
figure in it are determined by the particles via their mean charge and current densities that enter the macroscopic Maxwell’s equations (1.23–1.26).

In the case of neutral gas, all the interesting kinetic physics is in the collision operator, hence the focus on transport theory in gas-kinetic literature (see, e.g., the classic monograph by Chapman & Cowling 1991 if you want an overdose of this). In the collisionless limit, the kinetic equation for a neutral gas,

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f = 0,
\]

simply describes particles with some initial distribution individualistically flying in straight lines along their initial directions of travel. In contrast, for a plasma, even the collisionless kinetics (and, indeed, especially the collisionless—or weakly collisional—kinetics) is interesting and nontrivial because the particles, via the average properties of their distribution—charge densities and currents,—collectively modify \(E\) and \(B\), which then act on individual particles and thus modify \(f_\alpha\), etc. This “plasma democracy” is a whole new conceptual world and it is on the effects involving interactions between particles and fields that I shall focus here, in pursuit of maximum novelty.\(^5\)

I shall also be in pursuit of maximum simplicity (well, to use Einstein’s dictum, “as simple as possible, but not simpler”!) and so will mostly restrict my considerations to the “electrostatic approximation”:

\[
B = 0, \quad E = -\nabla \varphi.
\]

This, of course, eliminates a huge number of interesting and important phenomena without which plasma physics would not be the voluminous subject that it is, but I cannot do them justice in just a few lectures (so see Parra 2019\(^b\) for a course largely devoted to collisionless magnetised plasmas). Some opportunities for generalising electrostatic theory to electromagnetic one will be provided in Q2 and Q3.

Thus, we shall henceforth consider a simplified kinetic system, called the Vlasov–Poisson system:

\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial v} = 0, \quad \nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d^3v f_\alpha.
\]

Formally, considering a collisionless plasma\(^6\) would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. What are the characteristic time scales (and length scales) in a plasma and what phenomena occur on these scales? These questions bring us to our next theme.

\(^5\)Similarly interesting things happen when the field tying the particles together is gravity—an even more complicated situation because, while the potential is long-range, rather like the Coulomb potential, gravity is not shielded and so all particles feel each other at all distances. This gives rise to remarkably interesting theory (Binney 2016; Fouvry 2019).

\(^6\)Or, I stress again, a weakly collisional plasma. The collision operator is dropped in (1.50), but let us not forget about it entirely even if the collision frequency is small; it will make a come back in §5.
A displaced population of electrons will set up a quasineutrality-restoring electric field, leading to plasma oscillations.

2. Equilibrium and Fluctuations

2.1. Plasma Frequency

Consider a plasma in equilibrium, in a happy quasineutral state. Suppose a population of electrons strays from this equilibrium and upsets quasineutrality a bit (Fig. 2). If they have shifted by distance $\delta x$, the restoring force on each electron will be

$$m_e \delta \ddot{x} = -eE = -4\pi e^2 n_e \delta x \Rightarrow \delta \ddot{x} = -\frac{4\pi e^2 n_e}{m_e} \delta x,$$

so there will be oscillations at what is known as the (electron) plasma frequency:

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}.$$  \hspace{1cm} (2.1)

Thus, we expect fluctuations of electric field in a plasma with characteristic frequencies $\omega \sim \omega_{pe}$ (these are Langmuir waves; I will derive their dispersion relation formally in §3.4). These fluctuations are due to collective motions of the particles—so they are still macroscopic fields in the nomenclature of §1.4.

The time scale associated with $\omega_{pe}$ is the scale of restoration of quasineutrality. The distance an electron can travel over this time scale before the restoring force kicks in, i.e., the distance over which quasineutrality can be violated, is (using the thermal speed $v_{th} \sim \sqrt{T/m_e}$ to estimate the electron’s velocity)

$$\frac{v_{th}}{\omega_{pe}} \sim \sqrt{\frac{T}{m_e}} \sqrt{\frac{m_e}{e^2 n_e}} = \sqrt{\frac{T}{e^2 n_e}} \sim \lambda_D,$$

the Debye length (1.6)—not surprising, as this is, indeed, the scale on which microscopic fields are shielded and plasma is quasineutral (§1.3).

Finally, let us check that the plasma oscillations happen on collisionless time scales. The collision frequency of the electrons is, using (2.3) and (1.14),

$$\nu_e \sim \frac{v_{th}}{\lambda_{mfp}} = \frac{v_{th}}{\omega_{pe}} \lambda_D \lambda_{mfp} \approx \omega_{pe} \ll \omega_{pe}, \quad \text{q.e.d.}$$

2.2. Slow vs. Fast

The plasma frequency $\omega_{pe}$ is only one of the characteristic frequencies (the largest) of the fluctuations that can occur in plasmas. We will think of the scales of all these
fluctuations as short and of the associated variation in time and space as fast. They occur against the background of some equilibrium state,\(^7\) which is either constant or varies slowly in time and space. The slow evolution and spatial variation of the equilibrium state can be due to slowly changing, large-scale external conditions that gave rise to this state or, as we will discover soon, it can be due to the average effect of a sea of small fluctuations.

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function:

\[
f(r, v, t) = f_0(\epsilon^a r, v, \epsilon t) + \delta f(r, v, t),
\]

where \(\epsilon\) is some small parameter characterising the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales, hence \(\epsilon^a\)). To avoid clutter, I shall drop the species index where this does not lead to ambiguity.

For simplicity, I will abolish the spatial dependence of the equilibrium distribution altogether and consider homogeneous systems:

\[
f_0 = f_0(v, \epsilon t),
\]

which also means \(E_0 = 0\) (there is no equilibrium electric field). Equivalently, all our considerations are restricted to scales much smaller than the characteristic system size. Formally, this equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:\(^8\)

\[
f_0(v, t) = \langle f(r, v, t) \rangle \equiv \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} dt' \int \frac{d^3r}{V} f(r, v, t'),
\]

where \(\omega^{-1} \ll \Delta t \ll t_{eq}\), where \(t_{eq}\) is the equilibrium time scale.

### 2.3. Multiscale Dynamics

It is convenient to work in Fourier space:

\[
\varphi(r, t) = \sum_k e^{ik \cdot r} \varphi_k(t), \quad f(r, v, t) = f_0(v, t) + \sum_k e^{ik \cdot r} \delta f_k(v, t).
\]

Then the Poisson equation (1.51) becomes

\[
\varphi_k = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v \delta f_{k\alpha}
\]

and the Vlasov equation (1.50) written for \(k = 0\) (i.e., the spatial average of the equation) is

\[
\frac{\partial f_0}{\partial t} + \frac{\partial \delta f_{k=0}}{\partial t} = -\frac{q}{m} \sum_k \varphi_{-k} i k \cdot \frac{\partial \delta f_k}{\partial v},
\]

\(^7\)Or even just an initial state that is slow to change.

\(^8\)I use angular brackets to denote this average, but it should be clear that this is not the same thing as the average (1.11) that separated the macroscopic fields from the microscopic ones. The latter average was over sub-Debye scales, whereas the new average (2.7) is over scales that are larger than fluctuation scales but smaller than the system size; both fluctuations and equilibrium are “macroscopic” in the language of §1.4.
where we can replace \( \varphi_{-k} = \varphi_k^* \) because \( \varphi(r,t) \) is a real field. Averaging over time according to (2.7) eliminates fast variation and gives

\[
\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{k} \left\langle \varphi_k^* i \mathbf{k} \cdot \frac{\partial \delta f_k}{\partial \mathbf{v}} \right\rangle.
\]

(2.11)

The right-hand side of (2.11) describes the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations (see §§6 and 10.2). In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of \( f_0 \). This should be obtainable at least in principle because the fluctuating fields appearing in the right-hand side of (2.11) themselves depend on \( f_0 \): indeed, writing the Vlasov equation (1.50) for the \( k \neq 0 \) modes, we find the following evolution equation for the fluctuations:

\[
\frac{\partial \delta f_k}{\partial t} + i \mathbf{k} \cdot \mathbf{v} \delta f_k = \frac{q}{m} \varphi_k i \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + \frac{q}{m} \sum_{k'} \varphi_{k'} i \mathbf{k}' \cdot \frac{\partial \delta f_{k-k'}}{\partial \mathbf{v}}.
\]

(2.12)

The three terms that control the evolution of the perturbed distribution function in (2.12) represent the three physical effects that I shall focus on in these Lectures. The second term on the left-hand side describes the free ballistic motion of particles (“streaming”). It gives rise to the phenomenon of phase mixing (§5) and, in its interplay with plasma waves, to Landau damping and kinetic instabilities (§3). The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution (§3). The second term on the right-hand side captures the nonlinear interactions between the fluctuating fields and the perturbed distribution—it is negligible only when fluctuation amplitudes are small enough (which, sadly, they rarely are) and is responsible for plasma turbulence (§§7.2, 8, and 11.2) and other nonlinear phenomena (e.g., §§11.1 and 12.1).

The programme for determining the slow evolution of the equilibrium is “simple”: solve (2.12) together with (2.9), calculate the correlation function of the fluctuations, \( \left\langle \varphi_k^* \delta f_k \right\rangle \), as a functional of \( f_0 \), and use it to close (2.11); then proceed to solve the latter. Obviously, this is impossible to do in most cases. But it is possible to construct a hierarchy of approximations to the answer and learn much interesting physics in the process.

2.4. Hierarchy of Approximations

2.4.1. Linear Theory

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, (2.11) turns into \( f_0 = \text{const} \) and (2.12) becomes

\[
\frac{\partial \delta f_k}{\partial t} + i \mathbf{k} \cdot \mathbf{v} \delta f_k = \frac{q}{m} \varphi_k i \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}},
\]

(2.13)

the linearised kinetic equation. Solving this together with (2.9) allows one to find oscillating and/or growing/decaying\(^9\) perturbations of a particular equilibrium \( f_0 \). The theory

\(^9\)We shall see (§5) that growing/decaying linear solutions imply the equilibrium distribution giving/receiving energy to/from the fluctuations.
for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape (§3).

Physically, the linear solutions will describe what happens over short term, viz., on times $t$ such that

$$\omega^{-1} \ll t \ll t_{\text{eq}} \text{ or } t_{\text{nl}},$$

where $\omega$ is the characteristic frequency of the perturbations, $t_{\text{eq}}$ is the time after which the equilibrium starts getting modified by the perturbations via (2.11) [which depends on the amplitude to which they can grow; if perturbations do grow, i.e., the equilibrium is unstable, they can modify the equilibrium by this mechanism so as to render it stable], and $t_{\text{nl}}$ is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter [second term on the right-hand side of (2.12); if perturbations grow, they can saturate by this mechanism].

2.4.2. Quasilinear Theory (QLT)

Suppose

$$t_{\text{eq}} \ll t_{\text{nl}},$$

i.e., growing perturbations start modifying the equilibrium before they saturate nonlinearly. Then the strategy is to solve (2.13) [together with (2.9)] for the perturbations, use the result to calculate their correlation function needed in the right-hand side of (2.11), then work out how the equilibrium therefore evolves and hence how large the perturbations must grow in order for this evolution to turn the unstable equilibrium into a stable one. This is a classic piece of theory, important conceptually—I will describe it in detail and do one example in §6. In reality, however, it happens relatively rarely that unstable perturbations saturate at amplitudes small enough for the nonlinear interactions not to matter [i.e., for (2.15) to hold true].

2.4.3. Weak-Turbulence (WT) Theory

Sometimes, one is not lucky enough to get away with QLT (i.e., alas, $t_{\text{nl}} \ll t_{\text{eq}}$ or $t_{\text{nl}} \sim t_{\text{eq}}$), but is lucky enough to have perturbations saturating nonlinearly at a small amplitude such that\textsuperscript{10}

$$t_{\text{nl}} \gg \omega^{-1},$$

i.e., perturbations oscillate linearly faster than they interact nonlinearly (this can happen, e.g., because propagating wave packets do not stay together long enough to break up completely in one encounter). Because waves are fast compared to nonlinear evolution in this approximation, it is possible to “quantise” them, i.e., to treat a nonlinear turbulent state of the plasma as a cocktail consisting of both “true” particles (ions and electrons) and “quasiparticles” representing electromagnetic excitations (§7).

In this approximation, one can do perturbation theory treating the nonlinear term in (2.12) as small and expanding in the small parameter $(\omega t_{\text{nl}})^{-1}$. The resulting weak (or “wave”) turbulence theory is quite an analytical tour de force—but it is a lot of work to do it properly! I will provide an introduction to it in §§7.2 and 8.4. The classic texts on this are Kadomtsev (1965) (early but lucid) and Zakharov et al. (1992) (mathematically definitive); a recent textbook in Zakharov’s tradition is Nazarenko (2011), while the quasiparticle approach, with Feynman diagrams and all that, can be learned from Tsytovich (1970, 1972, 1977, 1995) (with difficulty) or from Kingsep (1996) (quite enjoyably, but in Russian).

\textsuperscript{10}Note that the nonlinear time scale is typically inversely proportional to the amplitude; see (2.12).
Figure 3. Lev Landau (1908-1968), great Soviet physicist, quintessential theoretician, author of the Book, cult figure. It is a minor feature of his scientific biography that he wrote the two most important plasma-physics papers of all time (Landau 1936, 1946). He also got a Nobel Prize (1962), but not for plasma physics. (a) Cartoon by A. A. Yuzefovich (from Landau & Lifshitz 1976); the caption says “And Dau spake...” (unto the students, also depicted). (b) Landau’s mugshot from NKVD prison (1938), where he ended up for seditious talk and from whence he was released in 1939 after Peter Kapitsa’s personal appeal to Stalin.

Note that because the nonlinear term couples perturbations at different $k$’s (scales), this theory will lead to multi-scale (usually, power-law) fluctuation spectra.

2.4.4. Strong-Turbulence Theory

If perturbations manage to grow to a level at which

$$t_{nl} \sim \omega^{-1},$$

we are facing strong turbulence. This is actually what mostly happens (including in WT systems, where turbulence often transitions into the strong regime at small enough, or large enough, scales). Theory of such regimes tends to be of phenomenological/scaling kind, often in the spirit of the classic Kolmogorov (1941) theory of hydrodynamic turbulence.\(^\text{11}\) Here are two examples, not necessarily the best or most relevant, just mine: Schekochihin et al. (2009, 2016). No one really knows how to move very far beyond this sort of approach—and not for lack of trying (a recent but historically aware review is Krommes 2015).

3. Linear Theory: Waves, Landau Damping and Kinetic Instabilities

Enough idle chatter, let us calculate! In this section, we are concerned with the linearised Vlasov–Poisson system, \eqref{eq:Vlasov} and \eqref{eq:Poisson}:

$$\frac{\partial \delta f_{k\alpha}}{\partial t} + i k \cdot \mathbf{v} \delta f_{k\alpha} = \frac{q_\alpha}{m_\alpha} \varphi_k i k \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{v}},$$

$$\varphi_k = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3 \mathbf{v} \delta f_{k\alpha}.$$  \(^\text{11}\) A kind of exception is a very special case of strong Langmuir turbulence, which was extremely popular in the 1970s and 80s. The founding documents on this are Zakharov (1972) and Kingsep et al. (1973), but there is a huge and sophisticated literature that followed. There is, alas, no particularly good review, but see Thornhill & ter Haar (1978), Rudakov & Tsytovich (1978), Goldman (1984), Zakharov et al. (1985) and Robinson (1997) (I find the first of these the most readable of the lot). I will give an introduction to this topic in §8.
For compactness of notation, I will drop both the species index \( \alpha \) and the wave number \( k \) in the subscripts, unless they are necessary for understanding.

We will discover that electrostatic perturbations in a plasma described by (3.1) and (3.2) oscillate, can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to know some complex analysis.

### 3.1. Initial-Value Problem

We shall follow Landau's original paper (Landau 1946) in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not appropriate to think of them over \( t \in [-\infty, +\infty] \) (and—NB!!!—the damped perturbations are not pure eigenvmodes; see §5.3). So we look for \( \delta f(v, t) \) satisfying (3.1) with the initial condition

\[
\delta f(v, t = 0) = g(v).
\]

It is, therefore, appropriate to use the Laplace transform to solve (3.1):

\[
\hat{\delta f}(p) = \int_0^\infty dt e^{-pt} \delta f(t).
\]

It is a mathematical certainty that if there exists a real number \( \sigma > 0 \) such that

\[
|\delta f(t)| < e^{\sigma t} \text{ as } t \to \infty,
\]

then the integral (3.4) exists (i.e., is finite) for all values of \( p \) such that \( \text{Re } p \geq \sigma \). The inverse Laplace transform, giving us back our distribution function as a function of time, is then

\[
\delta f(t) = \frac{1}{2\pi i} \int_{-\infty+i\sigma}^{i\infty+\sigma} dp e^{pt} \hat{\delta f}(p),
\]

where the integral in the complex plane is along a straight line parallel to the imaginary axis and intersecting the real axis at \( \text{Re } p = \sigma \) (Fig. 4).

Since we expect to be able to recover our desired time-dependent function \( \delta f(v, t) \) from its Laplace transform, it is worth knowing the latter. To find it, we Laplace-
transform (3.1):
\[\int_0^\infty dt e^{-pt} \frac{\partial \delta f}{\partial t} = \left[ e^{-pt} \delta f \right]_0^\infty + p \int_0^\infty dt e^{-pt} \delta f = -g + p \dot{\delta f},\]
the left-hand side (l.h.s.) and the right-hand side (r.h.s.):\[\dot{\delta f}(p) = \frac{1}{p + ik \cdot v} \left[ i \frac{q}{m} \dot{\varphi}(p) k \cdot \frac{\partial f_0}{\partial v} + g \right].\] (3.8)
Equating these two expressions, we find the solution:
\[\dot{\delta f}(p) = \frac{1}{p + ik \cdot v} \left[ i \frac{q}{m} \dot{\varphi}(p) k \cdot \frac{\partial f_0}{\partial v} + g \right].\] (3.8)
where the Laplace transform of the potential, \(\hat{\varphi}(p)\), itself depends on \(\delta \hat{f}\) via (3.2):
\[\hat{\varphi}(p) = \int_0^\infty dt e^{-pt} \varphi(t) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v \delta \hat{f}_\alpha(p)\]
\[= \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v \frac{1}{p + ik \cdot v} \left[ i \frac{q_\alpha}{m_\alpha} \dot{\varphi}(p) k \cdot \frac{\partial f_0}{\partial v} + g_\alpha \right].\] (3.9)
This is an algebraic equation for \(\hat{\varphi}(p)\). Collecting terms, we get
\[\left[ 1 - \sum_\alpha \frac{4\pi q_\alpha^2}{k^2 m_\alpha} i \int d^3v \frac{1}{p + ik \cdot v} k \cdot \frac{\partial f_0}{\partial v} \right] \hat{\varphi}(p) = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3v \frac{g_\alpha}{p + ik \cdot v}.\] (3.10)
The prefactor in the left-hand side, which I denote \(\epsilon(p, k)\), is called the dielectric function, because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, so let us write it out beautifully:
\[\epsilon(p, k) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k^2 m_\alpha} i \int d^3v \frac{1}{p + ik \cdot v} k \cdot \frac{\partial f_0}{\partial v},\] (3.11)
where the plasma frequency of species \(\alpha\) is defined by [cf. (2.2)]
\[\omega_{p\alpha}^2 = \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}.\] (3.12)
The solution of (3.10) is
\[\hat{\varphi}(p) = \frac{4\pi}{k^2 \epsilon(p, k)} \sum_\alpha q_\alpha \int d^3v \frac{g_\alpha}{p + ik \cdot v}.\] (3.13)
To calculate \(\varphi(t)\), we need to inverse-Laplace-transform \(\hat{\varphi}\): similarly to (3.6),
\[\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty+\sigma} dp e^{pt} \hat{\varphi}(p).\] (3.14)
How do we do this integral? Recall that \(\delta \hat{f}\) and, therefore, \(\hat{\varphi}\) only exists (i.e., is finite) for \(\text{Re } p \geq \sigma\), whereas at \(\text{Re } p < \sigma\), it can have singularities, i.e., poles—let us call them \(p_i\), indexed by \(i\). If we analytically continue \(\hat{\varphi}(p)\) everywhere to \(\text{Re } p < \sigma\) except those poles, the result must have the form
\[\hat{\varphi}(p) = \sum_i \frac{c_i}{p - p_i} + A(p),\] (3.15)
where \( c_i \) are some coefficients (residues) and \( A(p) \) is the analytic part of the solution. The integration contour in (3.14) can be shifted to \( \text{Re} p \to -\infty \) but with the proviso that it cannot cross the poles, as shown in Fig. 5 (this is proven by making a closed loop out of the old and the new contours, joining them at \( \pm i\infty \), and noting that this loop encloses no poles). Then the contributions to the integral from the vertical segments of the contour are exponentially small,\(^{12}\) the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy’s formula, be expressed in terms of the poles and residues:

\[
\varphi(t) = \sum_i c_i e^{p_i t}.
\]

Thus, in the long-time limit, perturbations of the potential will evolve \( \propto e^{p_i t} \), where \( p_i \) are poles of \( \hat{\varphi}(p) \). In general, \( p_i = -i\omega_i + \gamma_i \), where \( \omega_i \) is a real frequency (giving wave-like behaviour of perturbations), \( \gamma_i < 0 \) represents damping and \( \gamma_i > 0 \) growth of the perturbations (instability).

Note that we need not be particularly interested in what \( c_i \)’s are because, if we set up an initial perturbation with a given \( k \) and then wait long enough, only the fastest-growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes. Thus, a typical outcome of the linear theory is \( \varphi(t) \) oscillating at some frequency and growing or decaying at some unique rate. Since this is a solution of a linear equation, the prefactor in front of the exponential can be scaled arbitrarily and so does not matter.

Going back to (3.13), we realise that the poles of \( \hat{\varphi}(p) \) are zeros of the dielectric function:

\[
\epsilon(p_i, k) = 0 \quad \Rightarrow \quad p_i = p_i(k) = -i\omega_i(k) + \gamma_i(k).
\]

\(^{12}\)They are exponentially small in time as \( t \to \infty \) because the integrand of the inverse Laplace transform (3.14) contains a factor of \( e^{\text{Re} pt} \), which decays faster than any of the “modes” in (3.16). If \( \hat{\varphi}(p) \) does not grow too fast at large \( p \), the integral along the vertical part of the contour may also vanish at any finite \( t \), but that is not guaranteed in general: indeed, looking ahead to the explicit expression (3.27) for \( \hat{\varphi}(p) \), with the Landau prescription for analytic continuation to \( \text{Re} p < 0 \) analogous to (3.20), we see that \( \hat{\varphi}(p) \) will contain a term \( \propto G_\alpha(ip/k) \), which can be large at large \( \text{Re} p \), e.g., if \( G_\alpha(v_\perp) \) is a Maxwellian.
To find the wave frequencies $\omega_i$ and the damping/growth rates $\gamma_i$, we must solve this equation, which is called the plasma dispersion relation.

### 3.2. Calculating the Dielectric Function: the “Landau Prescription”

In order to be able to solve $\epsilon(p, k) = 0$, we must learn how to calculate $\epsilon(p, k)$ for any given $p$ and $k$. Before I wrote (3.15), I said that $\hat{\varphi}$, given by (3.13), had to be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ($\text{Re}p \geq \sigma > 0$), but I did not explain how this was to be done. In order to do it, we must learn how to calculate the velocity integral in (3.11)—if we want $\epsilon(p, k)$ and, therefore, its zeros $p_i$—and also how to calculate the similar integral in (3.13) containing $g_\alpha$.

First of all, let us turn these integrals into a 1D form. Given $k$, we can always choose the $z$ axis to be along $k$.\(^{13}\) Then

$$
\int \frac{1}{p + ik \cdot v} \cdot k \cdot \frac{\partial f_0}{\partial v} = \int \frac{1}{p + ikv_z} k \frac{\partial}{\partial v_z} \left( \int \frac{dv_x}{p} \int \frac{dv_y f_0(v_x, v_y, v_z)}{v_z - ip/k} \right) \equiv F(v_z)
$$

Assuming, reasonably, that $F'(v_z)$ is a nice (analytic) function everywhere, we conclude that the integrand in (3.18) has one pole, $v_z = ip/k$. When $\text{Re}p \geq \sigma > 0$, this pole is harmless because, in the complex plane associated with the $v_z$ variable, it lies above the integration contour, which is the real axis, $v_z \in (-\infty, +\infty)$. We can think of analytically continuing the above integral to $\text{Re}p < \sigma$ as moving the pole $v_z = ip/k$ down, towards and below the real axis. As long as $\text{Re}p > 0$, this can be done with impunity, in the sense that the pole stays above the integration contour, and so the analytic continuation is simply the same integral (3.18), still along the real axis. However, if the pole moves so far down that $\text{Re}p = 0$ or $\text{Re}p < 0$, we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. 6. This is called the Landau prescription and the contour thus deformed is called the Landau contour, $C_L$.

Let me prove that this is indeed an analytic continuation, i.e., that the integral (3.18), adjusted to be along $C_L$, is an analytic function for all values of $p$. Let us cut the Landau contour at $v_z = \pm R$ and close it in the upper half-plane with a semicircle $C_R$ of radius $R > \sigma/k$ (Fig. 7). Then, with integration running along the truncated $C_L$ and

\(^{13}\)NB: This means that in what follows, $k \geq 0$ by definition.
Since analyticity is guaranteed for $\text{Re } p \geq \sigma$, the integral along $C_R$ is analytic. The right-hand side is also analytic, by assumption. Therefore, the integral along $C_L$ is analytic—this is the integral along the Landau contour if we take $R \to \infty$. Q.e.d.

With the Landau prescription, our integral is calculated as follows:

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} + \int_{C_R} dv_z \frac{F'(v_z)}{v_z - ip/k} = 2\pi i F'(\frac{ip}{k}).$$

(3.19)

The integral along the real axis is formally divergent and so we take its principal value, defined as

$$\mathcal{P} \int_{-\infty}^{\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon \to 0} \left[ \int_{-\infty}^{\omega/k-\varepsilon} dv_z \frac{F'(v_z)}{v_z - \omega/k} + \int_{\omega/k+\varepsilon}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} \right].$$

(3.21)

The difference between (3.21) and the usual Lebesgue definition of an integral is that the latter would be

$$\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon_1 \to 0} \int_{-\infty}^{\omega/k-\varepsilon_1} dv_z \frac{F'(v_z)}{v_z - \omega/k} + \lim_{\varepsilon_2 \to 0} \int_{\omega/k+\varepsilon_2}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k},$$

(3.22)

and this, with, in general, $\varepsilon_1 \neq \varepsilon_2$, diverges logarithmically, whereas in (3.21), the divergences neatly cancel.

The $\text{Re } p = 0$ case in (3.20),

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} + i\pi F'(\frac{\omega}{k}),$$

(3.23)
which tends to be of most use in analytical theory, is a particular instance of Plemelj’s formula: for a real \( \zeta \) and a well-behaved function \( f \) (no poles on or near the real axis),

\[
\lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta \pm i\varepsilon} = \mathcal{P} \int_{-\infty}^{+\infty} dx \frac{f(x)}{x - \zeta} \pm i\pi f(\zeta),
\]

also sometimes written as

\[
\lim_{\varepsilon \to +0} \frac{1}{x - \zeta \pm i\varepsilon} = \mathcal{P} \frac{1}{x - \zeta} \pm i\pi \delta(x - \zeta),
\]

Finally, armed with Landau’s prescription, we are ready to calculate. The dielectric function (3.11) becomes

\[
\epsilon(p, k) = 1 - \sum_{\alpha} \frac{\omega_{pa}^2}{k^2} \frac{1}{n_\alpha} \int_{C_L} dv_z \frac{F_\alpha'(v_z)}{v_z - ip/k},
\]

and, analogously, our Laplace-transformed solution (3.13) becomes

\[
\hat{\varphi}(p) = -\frac{4\pi i}{k^3 \epsilon(p, k)} \sum_{\alpha} q_{\alpha} \int_{C_L} dv_z \frac{G_\alpha(v_z)}{v_z - ip/k},
\]

where \( G_\alpha(v_z) = \int dv_x \int dv_y g_\alpha(v_x, v_y, v_z) \).

3.3. Solving the Dispersion Relation: the Limit of Slow Damping/Growth

A particularly analytically solvable and physically interesting case is one in which, for \( p = -i\omega + \gamma, \gamma \ll \omega \) and \( \gamma \ll kv_{th} \), i.e., the case of the damping or growth time of the waves being longer than either their period or the time particles take to cross them. In this limit, the dispersion relation (3.17) is

\[
\epsilon(p, k) \approx \epsilon(-i\omega, k) + i\gamma \frac{\partial}{\partial \omega} \epsilon(-i\omega, k) = 0.
\]

Setting the real part of (3.28) to zero gives the equation for the real frequency:

\[
\text{Re} \epsilon(-i\omega, k) = 0.
\]

Setting the imaginary part of (3.28) to zero gives us the damping/growth rate in terms of the real frequency:

\[
\gamma = -\text{Im} \epsilon(-i\omega, k) \left[ \frac{\partial}{\partial \omega} \text{Re} \epsilon(-i\omega, k) \right]^{-1}.
\]

Thus, we now only need \( \epsilon(p, k) \) with \( p = -i\omega \). Using (3.23), we get

\[
\text{Re} \epsilon = 1 - \sum_{\alpha} \frac{\omega_{pa}^2}{k^2} \frac{1}{n_\alpha} \mathcal{P} \int dv_z \frac{F_\alpha'(v_z)}{v_z - \omega/k},
\]

\[
\text{Im} \epsilon = -\sum_{\alpha} \frac{\omega_{pa}^2}{k^2} \frac{\pi}{n_\alpha} F_\alpha'(\frac{\omega}{k}).
\]

Let us consider a two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- “High-frequency” electron waves: \( \omega \gg kv_{th} \), where \( v_{th} = \sqrt{2T_e/m_e} \) is the “thermal
speed” of the electrons;\textsuperscript{14} this limit will give us Langmuir waves (§3.4), slowly damped or growing (§3.5).

- “Low-frequency” ion waves: a particularly tractable limit will be that of “hot” electrons and “cold” ions, viz., $kv_{\text{the}} \gg \omega \gg kv_{\text{thi}}$, where $v_{\text{thi}} = \sqrt{2T_i/m_i}$ is the “thermal speed” of the ions; this limit will give us the sound (“ion-acoustic waves”; §3.8), which also can be damped or growing (§3.9).

3.4. Langmuir Waves

Consider the limit

$$\frac{\omega}{k} \gg v_{\text{the}}, \quad (3.33)$$

i.e., the phase velocity of the waves is much greater than the typical velocity of a particle from the “thermal bulk” of the distribution. This means that in (3.31), we can expand in $v_z \sim v_{\text{the}}$ being small compared to $\omega/k$ (higher values of $v_z$ are cut off by the “thermal” fall-off of the equilibrium distribution function). Note that $\omega \gg kv_{\text{the}}$ also implies $\omega \gg kv_{\text{thi}}$ because $v_{\text{thi}} = \sqrt{T_i/T_e m_i} \ll 1$ (3.34) as long as $T_i/T_e$ is not huge.\textsuperscript{15} Thus, (3.31) becomes

$$\text{Re} \epsilon = 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_\alpha} \frac{k}{\omega} P \int dv_z F'_\alpha(v_z) \left[ 1 + \frac{kv_z}{\omega} + \left( \frac{kv_z}{\omega} \right)^2 + \left( \frac{kv_z}{\omega} \right)^3 + \ldots \right]$$

$$= 1 + \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k\omega} \left[ \frac{1}{n_\alpha} \int dv_z F'_\alpha(v_z) - \frac{k}{\omega} \frac{1}{n_\alpha} \int dv_z F_\alpha(v_z) \right] = 0$$

$$- 2 \frac{k^2}{\omega^2} \frac{1}{n_\alpha} \int dv_z v_z F_\alpha(v_z) - 3 \frac{k^3}{\omega^3} \frac{1}{n_\alpha} \int dv_z v_z^2 F_\alpha(v_z) + \ldots$$

$$= 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \left[ 1 + \frac{3}{2} \frac{k^2 v_{\text{the}}^2}{\omega^2} + \ldots \right], \quad (3.35)$$

where we have integrated by parts everywhere, assumed that there are no mean flows, $\langle v_z \rangle = 0$, and, in the last term, used

$$\langle v_z^2 \rangle = \frac{v_{\text{the}}^2}{2}, \quad (3.36)$$

which is indeed the case for a Maxwellian $F_\alpha$ or, if $F_\alpha$ is not a Maxwellian, can be viewed as the definition of $v_{\text{the}}$.

The ion contribution to (3.35) is small because

$$\frac{\omega_{p\alpha}^2}{\omega_{pc}^2} = \frac{Zm_e}{m_i} \ll 1, \quad (3.37)$$

\textsuperscript{14}This is a standard well-defined quantity for a Maxwellian equilibrium distribution $F_\alpha(v_z) = (n_e/\sqrt{\pi} v_{\text{the}}) \exp(-v_z^2/v_{\text{the}}^2)$, but if we wish to consider a non-Maxwellian $F_\alpha$, let $v_{\text{the}}$ be some typical speed characterising the width of the equilibrium distribution, defined by, e.g., (3.36).

\textsuperscript{15}For hydrogen plasma, $\sqrt{m_i/m_e} \approx 42$, the answer to the Ultimate Question of Life, Universe and Everything (Adams 1979).
so ions do not participate in this dynamics at all. Therefore, to lowest order, the dispersion relation (3.29) becomes

$$\text{Re} \, \epsilon \approx 1 - \frac{\omega_{pe}^2}{\omega^2} = 0 \Rightarrow \omega^2 = \omega_{pe}^2 = \frac{4\pi e^2 n_e}{m_e}.$$  \hspace{1cm} (3.38)

This is the Tonks & Langmuir (1929) dispersion relation for what is known as Langmuir, or plasma, oscillations. This is the formal derivation of the result that we already had, on less mathematically rigorous, physical grounds, in §2.1.

We can do a little better if we retain the (small) $k$-dependent term in (3.35):

$$\text{Re} \, \epsilon \approx 1 - \frac{\omega_{pe}^2}{\omega^2} \left(1 + \frac{3k^2v_{th}^2}{2\omega^2}\right) = 0 \Rightarrow \omega^2 \approx \omega_{pe}^2 \left(1 + 3k^2\lambda_{De}^2\right), \hspace{1cm} (3.39)$$

where $\lambda_{De} = v_{th}/\sqrt{2}\omega_{pe} = \sqrt{T_e/4\pi e^2 n_e}$ is the “electron Debye length” [cf. (1.6)]. Equation (3.39) is the Bohm & Gross (1949a) dispersion relation, describing an upgrade of the Langmuir oscillations to dispersive Langmuir waves, which have a non-zero group velocity (this effect is due to electron pressure; see Exercise 3.1).

Note that all this is only valid for $\omega \gg kv_{th}$, which we now see is equivalent to

$$k\lambda_{De} \ll 1 \hspace{1cm} (3.40)$$

(the wave length of the perturbation is long compared to the Debye length).

Exercise 3.1. Langmuir hydrodynamics.\footnote{This is based on the 2017 exam question.} Starting from the linearised kinetic equation for electrons and ignoring perturbations of the ion distribution function completely, work out the fluid equations for electrons (i.e., the evolution equations for the electron density $n_e$ and velocity $u_e$) and show that you can recover the Langmuir waves (3.39) if you assume that electrons behave as a 1D adiabatic fluid (i.e., have the equation of state $p_e n_e^{1-\gamma} = \text{const}$ with $\gamma = 3$). You can prove that they indeed do this by calculating their density and pressure directly from the Landau solution for the perturbed distribution function (see §§5.3 and 5.6), ignoring resonant particles. The “hydrodynamic” description of Langmuir waves will reappear in §8.

3.5. Landau Damping and Kinetic Instabilities

Now let us calculate the damping rate of Langmuir waves using (3.30), (3.38) and (3.32):

$$\frac{\partial \text{Re} \epsilon}{\partial \omega} \approx \frac{2\omega_{pe}^2}{\omega^3}, \quad \text{Im} \epsilon \approx -\frac{\omega_{pe}^2}{k^2} \frac{\pi}{n_e} F'_e\left(\frac{\omega}{k}\right) \Rightarrow \gamma \approx \frac{\pi \omega^3}{2} \frac{1}{k^2} \frac{1}{n_e} F'_e\left(\frac{\omega}{k}\right). \hspace{1cm} (3.41)$$

where $\omega$ is given by (3.39). Provided $\omega F'(\omega/k) < 0$ (as would be the case, e.g., for any distribution monotonically decreasing with $|v_z|$; see Fig. 8a), $\gamma < 0$ and so this is indeed a damping rate, the celebrated Landau damping (Landau 1946; it was confirmed experimentally two decades later, by Malmberg & Wharton 1964).

The same theory also describes a class of kinetic instabilities: if $\omega F'(\omega/k) > 0$, then $\gamma > 0$, so perturbations grow exponentially with time. An iconic example is the bump-on-tail instability (Fig. 8b), which arises when a high-energy ($v_z \gg v_{th}$) electron beam...
Figure 8. The Landau resonance (particle velocities equalling phase speed of the wave $v_z = \omega/k$) leads to damping of the wave if more particles lag just behind than overtake the wave and to instability in the opposite case.

is injected into a plasma and whose quasilinear saturation we will study in great detail in §6.

We see that the damping or growth of plasma waves occur via their interaction with the particles whose velocities coincide with the phase velocity of the wave ("Landau resonance"). Because such particles are moving in phase with the wave, its electric field is stationary in their reference frame and so can do work on these particles, giving its energy to them (damping) or receiving energy from them (instability). In contrast, other, out-of-phase, particles experience no mean energy change over time because the field that they "see" is oscillating. It turns out (§3.6) that the process works in the spirit of socialist redistribution: the particles slightly lagging behind the wave will, on average, receive energy from it, damping the wave, whereas those overtaking the wave will have some of their energy taken away, amplifying the wave. The condition $\omega F'(\omega/k) < 0$ corresponds to the stragglers being more numerous than the strivers, leading to net damping; $\omega F'(\omega/k) > 0$ implies the opposite, leading to an instability (which then leads to flattening of the distribution; see §6).

Let us note again that these results are quantitatively valid only in the limit (3.33), or, equivalently, (3.40). It makes sense that damping should be slow ($\gamma \ll \omega$) in the limit where the waves propagate much faster than the majority of the electrons ($\omega/k > v_{\text{th}}$) and so can interact only with a small number of particularly fast particles (for a Maxwellian equilibrium distribution, it is an exponentially small number $\sim e^{-\omega^2/k^2v_{\text{th}}^2}$).

If, on the other hand, $\omega/k \sim v_{\text{th}}$, the waves interact with the majority population and the damping should be strong: a priori, we might expect $\gamma \sim kv_{\text{th}}$.\(^{18}\)

Exercise 3.2. Stability of isotropic distributions. Prove that if $f_0(\nu_x, \nu_y, \nu_z) = f_0(v_x, v_y, v_z)$, i.e., if it is a 3D-isotropic distribution, monotonic or otherwise, the Langmuir waves at $k\lambda_{De} \ll 1$ are always damped (this is solved in Lifshitz & Pitaevskii 1981; the statement of stability of isotropic distributions is in fact valid much more generally than just for long-wavelength Langmuir waves: see Exercise 4.2).

\(^{17}\)Here we are dealing with the case of a "warm beam" (meaning that it has a finite width). It turns out that there exists also another instability, leading to growth of perturbations with $\omega/k$ to the right of the bump’s peak, due to a different, “fluid” kind of resonance and possible even for "cold beams" (i.e., beams of particles that all have the same velocity): see §3.7.

\(^{18}\)This is indeed correct. You can confirm it numerically using (3.82) and (3.88).
Landau’s method of working out waves and damping in collisionless plasmas has always elicited a degree of dissatisfaction in the minds of some mathematically inclined physicists and motivated them to search for alternatives. Perhaps the earliest and best known such alternative is the formalism due to van Kampen. His objective was more mathematical rigour—but even if this is of limited appeal to you, the book by van Kampen & Felderhof (1967) is still a good read and a good chance to question and re-examine your understanding of how it all works.¹⁹

Landau damping became a cause célèbre in the hard-core mathematics community, as well as in the wider science world, with the award of the Fields Medal in 2010 to Cédric Villani, who proved (with C. Mouhot) that, basically, Landau’s solution of the linearised Vlasov equation survived as a solution of the full nonlinear Vlasov equation for small enough and regular enough initial perturbations: see a “popular” account of this by Villani (2014). The regularity restriction is apparently important and the result can break down in interesting ways (Bedrossian 2016). The culprit is plasma echo, of which more will be said in §11 (without any claim to mathematical rigour).

Even in the linear approximation, Landau damping depends on mathematical assumptions that are relaxed at one’s peril: you might find further enlightenment, or at least enjoyment, in the recent paper by Ramos & White (2018), where the Landau problem is recast as a proper eigenvalue problem—and it is shown, amongst other mathematical delights, that if one fiddles cleverly with initial conditions, one can obtain solutions that do not decay at the Landau rate and, in fact, can have any time evolution that one cares to specify!

---

### 3.6. Physical Picture of Landau Damping

The following simple argument (Lifshitz & Pitaevskii 1981) illustrates the physical mechanism of Landau damping.

Consider an electron moving along the z axis, subject to a wave-like electric field:

\[
\frac{dv_z}{dt} = -\frac{e}{m_e} E(z,t) = -\frac{e}{m_e} E_0 \cos(\omega t - kz) e^{\epsilon t}.
\]

I have given the electric field a slow time dependence, \( E \propto e^{\epsilon t} \), but will later take \( \epsilon \to +0 \)—this describes the field switching on infinitely slowly from \( t = -\infty \). Let us assume that the amplitude \( E_0 \) of the electric field is so small that it changes the electron’s trajectory only a little over several wave periods. Then the equations of motion can be solved perturbatively.

The lowest-order (\( E_0 = 0 \)) solution is

\[
v_z(t) = v_0 = \text{const}, \quad z(t) = v_0 t.
\]

In the next order, let

\[
v_z(t) = v_0 + \delta v_z(t), \quad z(t) = v_0 t + \delta z(t).
\]

---

¹⁹Blithely skipping half a century of literature, let me mention a recent paper by Heninger & Morrison (2018), which (following up on Morrison 1994, 2000) recast van Kampen’s scheme as a new transform (called “G-transform”) being used instead of the Laplace transform to solve Landau’s initial-value problem.
Integrating again, one gets
\[ \frac{d\delta v_z}{dt} = -\frac{e}{m_e} E(z(t), t) \approx -\frac{e}{m_e} E(v_0 t, t) = -\frac{eE_0}{m_e} \Re e^{i(\omega - kv_0) + \varepsilon} t. \] (3.46)
Integrating this gives
\[ \delta v_z(t) = -\frac{eE_0}{m_e} \int_0^t dt' \Re e^{i(\omega - kv_0) + \varepsilon} t' - 1 \]
\[ = -\frac{eE_0}{m_e} Re \left( e^{i(\omega - kv_0) + \varepsilon} t - 1 \right) - \varepsilon t \]
\[ = -\frac{eE_0}{m_e} \left[ (\omega - kv_0)^2 \{ e^{\varepsilon t} \cos[(\omega - kv_0)t] - 1 \} + 2\varepsilon(\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t] \right] \]
\[ \approx -\frac{eE_0}{m_e} \left( \varepsilon t \right). \] (3.47)
Integrating again, one gets
\[ \delta z(t) = \int_0^t dt' \delta v_z(t') \]
\[ = -\frac{eE_0}{m_e} \int_0^t dt' \Re \left( e^{i(\omega - kv_0) + \varepsilon} t' - 1 \right) \]
\[ = -\frac{eE_0}{m_e} \left\{ Re \left( e^{i(\omega - kv_0) + \varepsilon} t - 1 \right) - \varepsilon t \right\} \]
\[ = -\frac{eE_0}{m_e} \left\{ (\omega - kv_0)^2 \{ e^{\varepsilon t} \cos[(\omega - kv_0)t] - 1 \} + 2\varepsilon(\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t] \right\} \]
\[ \approx -\frac{eE_0}{m_e} \left( \varepsilon t \right). \] (3.48)
The work done by the field on the electron per unit time, averaged over time, is the power gained by the electron:
\[ \delta P(v_0) = -e \langle E(z(t), t) \delta v_z(t) \rangle \]
\[ \approx -e \left\langle \left[ E(v_0 t, t) + \delta z(t) \frac{\partial E}{\partial z}(v_0 t, t) \right] [v_0 + \delta v_z(t)] \right\rangle \]
\[ = -eE_0 e^{\varepsilon t} \left\{ v_0 \cos[(\omega - kv_0)t] + \delta v_z(t) \cos[(\omega - kv_0)t] + v_0 \delta z(t) \sin[(\omega - kv_0)t] \right\} \]
\[ \text{vanishes under averaging} \]
\[ \text{only cos term from (3.47) survives averaging} \]
\[ \text{only sin term from (3.48) survives averaging} \]
\[ = e^2 \frac{E_0^2}{2m_e} e^{2\varepsilon t} \left\{ \frac{\varepsilon}{(\omega - kv_0)^2 + \varepsilon^2} + \frac{2kv_0 \varepsilon(\omega - kv_0)}{[(\omega - kv_0)^2 + \varepsilon^2]} \right\} \]
\[ = e^2 \frac{E_0^2}{2m_e} e^{2\varepsilon t} \frac{\varepsilon v_0}{(\omega - kv_0)^2 + \varepsilon^2} \]
\[ \equiv \chi(v_0) \] (3.49)
We see (Fig. 9) that
— if the electron is lagging behind the wave, \( v_0 \lesssim \omega/k \), then \( \chi'(v_0) > 0 \Rightarrow \delta P(v_0) > 0 \), so energy goes from the field to the electron (the wave is damped);
— if the electron is overtaking the wave, \( v_0 \gtrsim \omega/k \), then \( \chi'(v_0) < 0 \Rightarrow \delta P(v_0) < 0 \), so energy goes from the electron to the field (the wave is amplified).
Now remember that we have a whole distribution of these electrons, \( F(v_z) \). So the total power
per unit volume going into (or out of) them is

\[ P = \int dv_z F(v_z) \delta P(v_z) = \frac{e^2 E_0^2 e^{2iz}}{2m_e} \int dv_z F(v_z) \chi'(v_z) \]

\[ = -\frac{e^2 E_0^2 e^{2iz}}{2m_e} \int dv_z F'(v_z) \chi(v_z). \quad (3.50) \]

Noticing that, by Plemelj's formula (3.25), in the limit \( \varepsilon \to +0, \)

\[ \chi(v_z) = \frac{\varepsilon v_z}{(\omega - kv_z)^2 + \varepsilon^2} = -\frac{iv_z}{2} \left( \frac{1}{kv_z - \omega - i\varepsilon} - \frac{1}{kv_z - \omega + i\varepsilon} \right) \to \frac{\pi}{k^2} \frac{\omega}{\delta(v_z - \omega / k)}, \quad (3.51) \]

we conclude

\[ P = -\frac{e^2 E_0^2}{2m_e k^2} \pi \omega F'(\omega / k). \quad (3.52) \]

As in §3.5, we find damping if \( \omega F'(\omega / k) < 0 \) and instability if \( \omega F'(\omega / k) > 0. \)

Thus, around the wave-particle resonance \( v_z = \omega / k, \) the particles just lagging behind the wave receive energy from the wave and those just overtaking it give up energy to it. Therefore, qualitatively, damping occurs if the former particles are more numerous than the latter. We see that Landau's mathematics in §§3.1–3.5 led us to a result that makes physical sense.

### 3.7. Hot and Cold Beams

Let us return to the unstable situation, when a high-energy beam produces a bump on the tail of the distribution function and thus electrostatic perturbations can suck energy out of the beam and grow in the region of wave numbers where \( v_0 < \omega / k < u_b. \) Here \( v_0 \) is the point of the minimum of the distribution in Fig. 8(b) and \( u_b \) is the point of the maximum of the bump, which is the velocity of the beam; we are assuming that \( u_b \gg v_{the}. \) In view of (3.41), the instability will have a greater growth rate if the bump’s slope is steeper, i.e., if the beam is colder (narrower in \( v_z \) space).

Imagine modelling the beam by a little Maxwellian distribution with mean velocity \( u_b, \) tucked onto the bulk distribution:\(^20\)

\[ F_e(v_z) = \frac{n_e - n_b}{\sqrt{\pi} v_{the}} \exp \left( -\frac{v_z^2}{v_{the}^2} \right) + \frac{n_b}{\sqrt{\pi} v_b} \exp \left[ -\frac{(v_z - u_b)^2}{v_b^2} \right], \quad (3.53) \]

where \( n_b \) is the density of the beam, \( v_b \) is its width, and so \( T_b = m_e v_b^2 / 2 \) is its “temperature”, just like \( T_e = m_e v_{the}^2 / 2 \) is the temperature of the thermal bulk. A colder beam will have less of a thermal spread around \( u_b. \) It turns out that if the width of

---

\(^20\) The fact that we are working in 1D means that we are restricting our consideration to perturbations whose wave numbers \( k \) are parallel to the beam’s velocity. In general, allowing transverse wave numbers brings into play the transverse (electromagnetic) part of the dielectric tensor (see Q2). However, for non-relativistic beams, the fastest-growing modes will still be the longitudinal, electrostatic ones (see, e.g., Alexandrov et al. 1984, §32).
the beam is sufficiently small, another instability appears, whose origin is hydrodynamic rather than kinetic. Let us work it out.

Consider a very simple limiting case of the distribution \((3.53)\): let \(v_b \to 0\) and \(n_b \ll n_e\). Then (Fig. 10)

\[
F_e(v_z) = F_M(v_z) + n_b \delta(v_z - u_b),
\]

where \(F_M\) is the bulk Maxwellian from \((3.53)\) (with density \(\approx n_e\), neglecting \(n_b\) in comparison). Let us substitute the distribution \((3.54)\) into the dielectric function \((3.26)\), seek solutions with \(p/k \gg v_{th}\), expand the part containing \(F_M\) in the same way as we did in §3.4, \(^{21}\) neglect the ion contribution for the same reason as we did there, and deal with \(\delta'(v_z - u_b)\) in the integrand via integration by parts. The resulting dispersion relation is

\[
\epsilon \approx 1 + \frac{\omega_{pe}^2}{p^2} - \frac{n_b}{n_e} \frac{\omega_{pe}^2}{(ku_b - ip)^2} = 0.
\]

\((3.55)\)

Since \(n_b \ll n_e\), the last term can only matter for those perturbations that are close to resonance with the beam (this is called the Cherenkov resonance):

\[
p = -iku_b + \gamma, \quad \gamma \ll ku_b.
\]

\((3.56)\)

This turns \((3.55)\) into

\[
1 - \frac{\omega_{pe}^2}{k^2u_b^2} + \frac{n_b}{n_e} \frac{\omega_{pe}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \gamma = \pm \sqrt{\frac{n_b}{n_e} \left( \frac{1}{k^2u_b^2} - \frac{1}{\omega_{pe}^2} \right)^{-1/2}}.
\]

\((3.57)\)

The expression under the square root is positive and so there is a growing mode only if \(k < \omega_{pe}/u_b\). This is in contrast to the case of a hot (or warm) beam in §3.5: there, having a kinetic instability required \(\omega F_e'(\omega/k) > 0\), which was only possible at \(k > \omega_{pe}/u_b\) (the phase speed of the perturbations had to be to the left of the bump’s maximum). The new instability that we have found—the hydrodynamic beam instability—has the largest growth rate at \(ku_b = \omega_{pe}\), i.e., when the beam and the plasma oscillations are in resonance, in which case, to resolve the singularity, we need to retain \(\gamma\) in the second

\(^{21}\) We can treat the Landau contour as simply running along the real axis because we are expecting to find a solution with \(\text{Re} p > 0\) [see \((3.20)\)], for reasons independent of the Landau resonance.
term in (3.55). Doing so and expanding in $\gamma$, we get

$$\epsilon \approx 1 - \frac{\omega_{pe}^2}{(\omega_{pe} + i\gamma)^2} + \frac{n_b \omega_{pe}^2}{n_e \gamma^2} \approx \frac{2i\gamma}{\omega_{pe}} + \frac{n_b \omega_{pe}^2}{n_e \gamma^2} = 0.$$  \hspace{1cm} (3.58)

Solution:

$$\gamma = \left(\frac{\pm \sqrt{3} + i}{2}, -i\right) \left(\frac{n_b}{2n_e}\right)^{1/3} \omega_{pe}. \hspace{1cm} (3.59)$$

The unstable root (Re $\gamma > 0$) is the interesting one. The growth rate of the combined beam instability, hydrodynamic and kinetic, is sketched in Fig. 11.

**Exercise 3.3.** This instability is called “hydrodynamic” because it can be derived from fluid equations (cf. Exercise 3.1) describing cold electrons ($v_{the} = 0$) and a cold beam ($v_b = 0$). Convince yourself that this is the case.

**Exercise 3.4.** Using the model distribution (3.53), work out the conditions on $v_b$ and $n_b$ that must be satisfied in order for our derivation of the hydrodynamic beam instability to be valid, i.e., for (3.55) to be a good approximation to the true dispersion relation. What is the effect of finite $v_b$ on the hydrodynamic instability? Sketch the growth rate of unstable perturbations as a function of $k$, taking into account both the hydrodynamic instability and the kinetic one, as well as the Landau damping.

**Exercise 3.5. Two-stream instability.** This is a popular instability\footnote{It was discovered by engineers (Haeff 1949; Pierce & Hebenstreit 1949) and quickly adopted by physicists (Bohm & Gross 1949b). Buneman (1958) realised that a case with an electron and an ion stream (i.e., with plasma carrying a current) is unstable in an analogous way. The kinetic version of the latter situation is the ion-acoustic instability derived in §3.9. In §4.4, I will discuss in a more general way the stability of distributions featuring streams.} that arises, e.g., in a situation where the plasma consists of two cold counter-streams of electrons propagating against a quasineutrality-enforcing background of effectively immobile ions (Fig. 12a). Model the corresponding electron distribution by

$$F_e(v_z) = \frac{n_e}{2} [\delta(v_z - u_b) + \delta(v_z + u_b)] \hspace{1cm} (3.60)$$

and solve the resulting dispersion relation (where the ion terms can be neglected for the same reason as in §3.4). Find the wave number at which perturbations grow fastest and the corresponding growth rate. Find also the maximum wave number at which perturbations can grow. If you want to know what happens when the two streams are warm (have a finite thermal spread $v_b$; Fig. 12b), a nice fully tractable quantitative model of such a situation is the double-Lorentzian distribution (4.16). The dispersion relation for it can be solved exactly: this is done in Q4. You will again find a hydrodynamic instability, but is there also a kinetic one (due to Landau resonance)? It is an interesting and non-trivial question why not.

### 3.8. Ion-Acoustic Waves

Let us now see what happens at lower frequencies,

$$v_{the} \gg \frac{\omega}{k} \gg v_{thi}, \hspace{1cm} (3.61)$$

i.e., when the waves propagate slower than the bulk of the electron distribution but faster than the bulk of the ion one (Fig. 13). This is another regime in which we might expect to find weakly damped waves: they are out of phase with the majority of the ions, so $F'_i(\omega/k)$ might be small because $F_i(\omega/k)$ is small, while as far as the electrons are concerned, the phase speed of the waves is deep in the core of the distribution, perhaps
close to its maximum at \( v_z = 0 \) (if that is where its maximum is) and so \( F'_e(\omega/k) \) might turn out to be small because \( F_e(v_z) \) changes slowly in that region.

To make this more specific, let us consider Maxwellian electrons:

\[
F_e(v_z) = \frac{n_e}{\sqrt{\pi} v_{\text{the}}} \exp \left[ -\frac{(v_z - u_e)^2}{v_{\text{the}}^2} \right],
\]

(3.62)

where we are, in general, allowing the electrons to have a mean flow (current). We will assume that \( u_e \ll v_{\text{th}} \) but allow \( u_e \sim \omega/k \). We can anticipate that this will give us an interesting new effect. Indeed,

\[
F'_e(v_z) = -\frac{2(v_z - u_e)}{v_{\text{the}}^2} F_e(v_z).
\]

(3.63)

For resonant particles, \( v_z = \omega/k \), the prefactor will be small, so we can hope for \( \gamma \ll \omega \), as anticipated above, but note that its sign will depend on the relative size of \( u_e \) and \( \omega/k \) and so we might (we will!) get an instability (§3.9).

But let us not get ahead of ourselves: we must first calculate the real frequency \( \omega(k) \) of these waves, from (3.29) and (3.31):

\[
\text{Re} \, \epsilon = 1 - \frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} - \frac{\omega_{\text{pi}}^2}{k^2} \frac{1}{n_i} \mathcal{P} \int dv_z \frac{F'_i(v_z)}{v_z - \omega/k} = 0.
\]

(3.64)

\[
\approx \frac{\omega_{\text{pi}}^2}{\omega^2} \left( 1 + 3k^2 \lambda_{\text{Di}}^2 \right)
\]

The last (ion) term in this equation can be expanded in \( kv_z/\omega \ll 1 \) in exactly the same way as it was done in (3.35). The expansion is valid provided

\[
k\lambda_{\text{Di}} \ll 1,
\]

(3.65)

and I will retain only the lowest-order term, dropping the \( k^2 \lambda_{\text{Di}}^2 \) correction. The second (electron) term in (3.64) is subject to the opposite limit, \( v_z \gg \omega/k \), so, using (3.63),

\[
\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{F'_e(v_z)}{v_z - \omega/k} \approx -\frac{\omega_{\text{pe}}^2}{k^2} \frac{1}{n_e} \mathcal{P} \int dv_z \frac{2(v_z - u_e)}{v_{\text{the}}^2 v_z} F_e(v_z) \approx -\frac{2\omega_{\text{pe}}^2}{k^2 v_{\text{the}}^2} = -\frac{1}{k^2 \lambda_{\text{De}}^2},
\]

(3.66)

where we have neglected \( u_e \ll v_z \) because this integral is over the thermal bulk of the electron distribution.

With all these approximations, (3.64) becomes

\[
\text{Re} \, \epsilon = 1 + \frac{1}{k^2 \lambda_{\text{De}}^2} - \frac{\omega_{\text{pi}}^2}{\omega^2} = 0.
\]

(3.67)
The dispersion relation is then
\[ \omega^2 = \frac{\omega_{pi}^2}{1 + 1/k^2 \lambda_{De}^2} = \frac{k^2 c_s^2}{1 + k^2 \lambda_{De}^2} , \] (3.68)
where
\[ c_s = \omega_{pi} \lambda_{De} = \sqrt{\frac{Z T_e}{m_i}} \] (3.69)
is the sound speed, called that because, if \( k \lambda_{De} \ll 1 \), (3.68) describes a wave that is very obviously a sound, or ion-acoustic, wave:
\[ \omega = \pm k c_s . \] (3.70)

The phase speed of this wave is the sound speed, \( \omega/k = c_s \). That the expression (3.69)
for \( c_s \) combines electron temperature and ion mass is a hint as to the underlying physics
of sound propagation in plasma: ions provide the inertia, electrons the pressure (see Exercise 3.6).

We can now check under what circumstances the condition (3.61) is indeed satisfied:
\[ \frac{c_s}{v_{the}} = \sqrt{\frac{Z m_e}{2 m_i}} \ll 1, \quad \frac{c_s}{v_{thi}} = \sqrt{\frac{Z T_e}{2 T_i}} \gg 1, \] (3.71)
with the latter condition requiring that the ions should be colder than the electrons.

**Exercise 3.6. Hydrodynamics of sound waves.**
Starting from the linearised kinetic equations for ions and electrons, work out the fluid equations for the plasma, i.e., the evolution equations for its mass density and mass flow velocity. Assuming \( m_e \ll m_i \) (negligible electron inertia) and \( T_i \ll T_e \) (cold ions), show that these equations are
\[ \partial \delta n_i / \partial t + n_i \nabla \cdot \mathbf{u}_i = 0, \quad m_i m_i \partial \mathbf{u}_i / \partial t + \nabla \delta p_e = 0, \] (3.72)
and that the sound waves (3.70) with \( c_s \) given by (3.69) are recovered if electrons have the
equation of state of an isothermal fluid. Why, and under what assumptions, should they be
isothermal physically? Prove mathematically that they indeed are. Why is the equation of state
for electrons different in a sound wave than in a Langmuir wave (see Exercise 3.1)? We will
revisit ion hydrodynamics in §8.

3.9. Damping of Ion-Acoustic Waves and Ion-Acoustic Instability

Are ion acoustic waves damped? Can they grow? We have a standard protocol for
answering this question: calculate \( \text{Re} \epsilon \) and \( \text{Im} \epsilon \) and substitute into (3.30). Using (3.67)
and (3.32), we find
\[ \frac{\partial \text{Re} \epsilon}{\partial \omega} = \frac{2 \omega_{pi}^2}{\omega^3}, \quad \text{Im} \epsilon = -\frac{\omega_{pe}^2}{k^3} \frac{\pi}{n_e} F_i'\left(\frac{\omega}{k}\right) + \frac{\omega_{pi}^2}{k^3} \frac{\pi}{n_i} F_i'\left(\frac{\omega}{k}\right). \] (3.73)

The two terms in \( \text{Im} \epsilon \) represent the interaction between the waves and, respectively,
electrons and ions. The ion term is small both on account of \( \omega_{pi} \ll \omega_{pe} \) and, assuming
Maxwellian ions, of the exponential smallness of \( F_i(\omega/k) \propto \exp[-(\omega/k v_{thi})^2] \). We are then left with
\[ \gamma = -\frac{\text{Im} \epsilon}{\partial (\text{Re} \epsilon)/\partial \omega} = -\sqrt{\frac{\pi}{k^2 v_{the}^3}} \frac{m_i}{Z m_e} \left(\frac{\omega}{k} - u_e\right), \] (3.74)
Figure 13. Ion-acoustic resonance: damping ($c_s > u_e$) or instability ($c_s < u_e$). Ion Landau damping is weak because $c_s \gg v_{thi}$, so in the tail of $F_i(v_z)$; electron damping/instability is also weak because $u_e, c_s \ll v_{the}$, so close to the peak $F_e(v_z)$.

where we have used (3.63). In the long-wavelength limit, $k \lambda_{De} \ll 1$, we have $\omega = \pm kc_s$, and so, for the “+” mode,

$$\gamma = -\sqrt{\frac{\pi Z m_e}{8 m_i}} k (c_s - u_e).$$

(3.75)

If the electron flow is subsonic, $u_e < c_s$, this describes the Landau damping of ion acoustic waves on hot electrons. If, on the other hand, the electron flow is supersonic, the sign of $\gamma$ reverses\(^{24}\) and we discover the ion-acoustic instability: excitation of ion acoustic waves by a fast electron current. The instability belongs to the same general class as, e.g., the bump-on-tail instability (§3.5) in that it involves waves sucking energy from particles, but the new conceptual feature here is that such energy conversion can result from a collaboration between different particle species (electrons supplying the energy, ions carrying the wave).

There is a host of related instabilities involving various combinations of electron and ion beams, currents, streams and counter-streams—excellent treatments of them can be found in the textbooks by Krall & Trivelpiece (1973) and by Alexandrov et al. (1984) or in the review by Davidson (1983). I shall return to this topic in §4.4.

Exercise 3.7. Damping of sound waves on ions.\(^{25}\) Find the ion contribution to the damping of ion-acoustic waves. Under what conditions does it become comparable to, or larger than, the electron contribution?

3.10. Ion Langmuir Waves

Note that since

$$\frac{\lambda_{De}}{\lambda_{Di}} = \frac{v_{thi} \omega_{pi}}{v_{thi} \omega_{pe}} = \sqrt{\frac{Z T_e}{T_i}},$$

(3.76)

\(^{24}\)Recall that $k > 0$ by the choice of the $z$ axis.

\(^{25}\)The 2016 exam question was loosely based on this.
the condition \((3.65)\) need not entail \(k\lambda_{De} \ll 1\) in the limit of cold ions [see (3.71)]—in this case, the size of the Debye sphere (1.6) is set by the ions, rather than by the electrons, and so we can have perfectly macroscopic (in the language of §1.4) perturbations on scales both larger and smaller than \(\lambda_{De}\). At larger scales, we have found sound waves (3.70). At smaller scales, \(k\lambda_{De} \gg 1\), the dispersion relation (3.68) gives us ion Langmuir oscillations:

\[
\omega^2 = \omega_{pi}^2 = \frac{4\pi Z^2 e^2 n_i}{m_i},
\]

(3.77)

which are analogous to the electron Langmuir oscillations (3.38) and, like them, turn into dispersive ion Langmuir waves if the small \(k^2\lambda_{Di}^2\) correction in (3.64) is retained, leading to the Bohm–Gross dispersion relation (3.39), but with ion quantities this time.

**Exercise 3.8.** Derive the dispersion relation for ion Langmuir waves. Investigate their damping/instability.

---

### 3.11. Summary of Electrostatic (Longitudinal) Plasma Waves

We have achieved what turns out to be a complete characterisation of electrostatic (also known as “longitudinal”, in the sense that \(k \parallel E\)) waves in an unmagnetised plasma. These are summarised in Fig. 14. In the limit of short wavelengths, \(k\lambda_{De} \gg 1\) and \(k\lambda_{Di} \gg 1\), the electron and ion branches, respectively, becomes dispersive, their damping rates increase and eventually stop being small. This corresponds to waves having phase speeds that are comparable to the speeds of the particles in the thermal bulk of their distributions, so a great number of particles are available to have Landau resonance with the waves and absorb their energy—the damping becomes strong.

Note that if the cold-ion condition \(T_i \ll T_e\) is not satisfied, the sound speed is comparable to the ion thermal speed \(c_s \sim v_{thi}\), and so the ion-acoustic waves are strongly
damped at all wave numbers—it is well-nigh impossible to propagate sound through a collisionless hot plasma (in such an environment, no one will hear you scream)!


Clearly, we have entered the realm of practical calculation—it is now easy to imagine an industry of solving the plasma dispersion relation

$$\epsilon(p, k) = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_{\alpha}} \int_{C_L} dv_z \frac{F'_{\alpha}(v_z)}{v_z - ip/k} = 0$$  \hspace{1cm} (3.78)$$

and similar dispersion relations arising from, e.g., considering electromagnetic perturbations (see Q2), magnetised plasmas (see Parra 2019\textsuperscript{b}), different equilibria $F_{\alpha}$ (see Q3 and Q4), etc.

A Maxwellian equilibrium is obviously an extremely important special case because that is, after all, the distribution towards which plasma is pushed by collisions on long time scales:

$$f_{0\alpha}(v) = \frac{n_{\alpha}}{(\pi v_{\text{th}\alpha}^2)^{3/2}} e^{-v^2/v_{\text{th}\alpha}^2} \Rightarrow F_{\alpha}(v_z) = \frac{n_{\alpha}}{\sqrt{\pi} v_{\text{th}\alpha}} e^{-v_z^2/v_{\text{th}\alpha}^2}. \hspace{1cm} (3.79)$$

For this case, we would like to introduce a new “special function” that would incorporate the Landau prescription for calculating the velocity integral in (3.78) and that we could in some sense “tabulate” once and for all:\textsuperscript{26}

Taking $F_{\alpha}$ to be (3.79) and letting $u = v_z/v_{\text{th}\alpha}$ and $\zeta_{\alpha} = ip/kv_{\text{th}\alpha}$, we can rewrite the velocity integral in (3.78) as follows

$$\frac{1}{n_{\alpha}} \int_{C_L} dv_z \frac{F'_{\alpha}(v_z)}{v_z - ip/k} = -\frac{2}{\sqrt{\pi} v_{\text{th}\alpha}^2} \int_{C_L} du \frac{ue^{-u^2}}{u - \zeta_{\alpha}} = -\frac{2}{v_{\text{th}\alpha}^2} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})], \hspace{1cm} (3.80)$$

where the plasma dispersion function is defined to be

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta}. \hspace{1cm} (3.81)$$

In these terms, the plasma dispersion relation (3.78) becomes

$$\epsilon = 1 + \sum_{\alpha} \frac{1 + \zeta_{\alpha} Z(\zeta_{\alpha})}{k^2\lambda_{D\alpha}^2} = 0. \hspace{1cm} (3.82)$$

Note that if the Maxwellian distribution (3.79) has a mean flow, as it did, e.g., in (3.62), this amounts to a shift by some mean velocity $u_{\alpha}$ and all one needs to do to adjust the above results is to shift the argument of $Z$ accordingly:

$$\zeta_{\alpha} \rightarrow \zeta_{\alpha} - \frac{u_{\alpha}}{v_{\text{th}\alpha}}. \hspace{1cm} (3.83)$$

3.12.1. Some Properties of $Z(\zeta)$

It is not hard to see that

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}} \int du e^{-u^2} \frac{\partial}{\partial u} \frac{1}{u - \zeta} = - \frac{2}{\sqrt{\pi}} \int du \frac{ue^{-u^2}}{u - \zeta} = -2 [1 + \zeta Z(\zeta)]. \hspace{1cm} (3.84)$$

Let us treat this identity as a differential equation: the integrating factor is $e^{\zeta^2}$, so

$$e^{\zeta^2} Z(\zeta) = -2 \int_0^\zeta dt e^{t^2} + Z(0). \hspace{1cm} (3.85)$$

\textsuperscript{26}In the olden days, one would literally tabulate it (Fried & Conte 1961). In the 21st century, we could just teach a computer to compute it [see (3.88)] and make an app.
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We know the boundary condition at $\zeta = 0$ from (3.23): for real $\zeta$,

$$
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-u^2}}{u-\zeta} = \frac{1}{\sqrt{\pi}} P \int_{-\infty}^{+\infty} \frac{e^{-u^2}}{u-\zeta} + i \sqrt{\pi} e^{-\zeta^2} \Rightarrow Z(0) = i \sqrt{\pi}.
$$

(3.86)

Using this in (3.85) and changing the integration variable $t = -ix$, we find

$$
Z(\zeta) = e^{-\zeta^2} \left( i \sqrt{\pi} + 2i \int_{0}^{i\zeta} dx e^{-x^2} \right) = 2i e^{-\zeta^2} \int_{-\infty}^{i\zeta} dx e^{-x^2}.
$$

(3.87)

This turns out to be a uniformly valid expression for $Z(\zeta)$: our function is simply a complex erf! Here is a Mathematica script for calculating it:

$$
Z[\text{zeta}] := I \text{Sqrt[Pi]} \text{Exp[-zeta]^2}(1 + I \text{ Erfi[zeta]}).
$$

(3.88)

You can use this to code up (3.82) and explore, e.g., the strongly damped solutions ($\zeta \sim 1$, $\gamma \sim \omega$).

3.12.2. Asymptotics of $Z(\zeta)$

If you are a devotee of the ancient art of asymptotic theory, you will find most useful (as, effectively, we did in §§3.4–3.9) various limiting forms of $Z(\zeta)$. At small argument $|\zeta| \ll 1$, the Taylor series is

$$
Z(\zeta) = i \sqrt{\pi} e^{-\zeta^2} - 2\zeta \left( 1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - \frac{8\zeta^6}{105} + \ldots \right)
$$

(3.89)

At large argument, $|\zeta| \gg 1$, $|\text{Re}\zeta| \gg |\text{Im}\zeta|$, the asymptotic series is

$$
Z(\zeta) = i \sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta} \left( 1 - \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \ldots \right)
$$

(3.90)

All the results (for a Maxwellian equilibrium) that I derived in §§3.4–3.10 can be readily obtained from (3.82) by using the above limiting cases (see Q1). It is, indeed, a general practical strategy for studying this and similar plasma dispersion relations to look for solutions in the limits $\zeta_\alpha \to 0$ or $\zeta_\alpha \to \infty$, then check under what physical conditions the solutions thus obtained are valid (i.e., that they indeed satisfy $|\zeta_\alpha| \ll 1$ or $|\zeta_\alpha| \gg 1$, $|\text{Re}\zeta| \gg |\text{Im}\zeta|$), and then fill in the non-asymptotic blanks in the same way that an experienced hunter espying antlers sticking out above the shrubbery can reconstruct, in contour outline, the rest of the hiding deer.

**Exercise 3.9.** Work out the Taylor series (3.89). A useful step might be to prove this interesting formula (which also turns out to be handy in other calculations; see, e.g., Q7):

$$
\frac{d^n Z}{d\zeta^m} = (-1)^m \sqrt{\pi} \int_{C_\zeta} du \frac{H_m(u) e^{-u^2}}{u-\zeta},
$$

(3.91)

where $H_m(u)$ are Hermite polynomials [defined in (12.33)].

**Exercise 3.10.** Work out the asymptotic series (3.90) using the Landau prescription (3.20) and expanding the principal-value integral similarly to the way it was done in (3.35). Work out also (or look up; e.g., Fried & Conte 1961) other asymptotic forms of $Z(\zeta)$, relaxing the condition $|\text{Re}\zeta| \gg |\text{Im}\zeta|$.

4. Linear Theory: General Stability Theory

In §§, we learned how to perturb some given equilibrium distribution $f_{0\alpha}$ infinitesimally and work out whether this perturbation will decay, grow, oscillate, and how quickly. Let
Now pose the question in a more general way. In a collisionless plasma, there can be infinitely many possible equilibria, including quite complicated ones. If we set one up, will it persist, i.e., is it stable? If it is not stable, what modification do we expect it to undergo in order to become stable? Other than solving the dispersion relation (3.17) to answer the first question and developing various types of nonlinear theories to answer the second (along the lines advertised in §2.4 and developed in §6 and subsequent sections), both of which can be quite complicated and often intractable technical challenges, do we have at our disposal any general principles that allow us to pronounce on stability? Is there a general insight that we can cultivate as to what sort of distributions are likely to be stable or unstable and to what sorts of perturbations?

We have had glimpses of such general principles already. For example, in §3.5, it was ascertained, by an explicit calculation, that one could encounter a situation with a (small) growth rate if the equilibrium distribution had a positive derivative somewhere along the direction (z) of the wave number of the perturbation, viz., \( v_z F'(v_z) > 0 \). I developed this further in §3.7, finding that not only hot but also cold beams and streams triggered instabilities. In Exercise 3.2, I dropped a hint that general statements could perhaps be made about certain general classes of distributions: 3D-isotropic equilibria could be proven stable (we shall prove this again, by a different method, in Exercise 4.2). How general are such statements? Are they sufficient or also necessary criteria? Is there a universal stability litmus test? Let us attack the problem of kinetic stability with an aspiration to generality—although still, for now, for electrostatic perturbations only. We shall also, for now, limit our ambition to determining linear stability of generic equilibria, i.e., their stability against infinitesimal perturbations. Nonlinear stability will have to wait till §9.

### 4.1. Nyquist’s Method

The problem of linear stability comes down to the question of whether the dispersion relation (3.17) has any unstable solutions: roots with growth rates \( \gamma_i(k) > 0 \).

It is going to be useful to write the dielectric function (3.26) as follows

\[
\epsilon(p, k) = 1 - \frac{\omega_p^2}{k^2} \int_{C_L} dv_z \frac{\bar{F}'(v_z)}{v_z - ip/k},
\]

\[
\bar{F} = \frac{1}{n_e} \sum_\alpha Z_\alpha Z_{e\alpha} m_e F_\alpha = \frac{F_e}{n_e} + \frac{Z m_e F_i}{n_i},
\]

where the last expression in (4.2) is for the case of a two-species plasma. Thus, the distribution functions of different species come into the linear problem additively, weighted by their species’ charges and (inverse) masses.

Let us develop a method (due to Nyquist 1932) for counting zeros of \( \epsilon(p) \) (I will henceforth suppress \( k \) in the argument) in the half-plane \( \text{Re} \, p > 0 \) (the unstable roots of the dispersion relation). Observe that \( \epsilon(p) \) is analytic (by virtue of our efforts in §3.2 to make it so) and that if \( p = p_i \) is its zero of order \( N_i \), then in its vicinity,

\[
\epsilon(p) = \text{const} \, (p - p_i)^{N_i} + \ldots \Rightarrow \frac{\partial \ln \epsilon(p)}{\partial p} = \frac{N_i}{p - p_i} + \ldots,
\]

so zeros of \( \epsilon(p) \) are poles of \( \partial \ln \epsilon(p)/\partial p \); the latter function has no other poles because \( \epsilon(p) \) is analytic. If we now integrate this function over a closed contour \( C_R \) running along the imaginary axis (and just to the right of it: \( p = -i \omega + 0 \)) in the complex \( p \) plane from \( iR \) to \( -iR \) and then along a semicircle of radius \( R \) back to \( iR \) (Fig. 15), we will, in the
limit $R \to \infty$, capture all these poles:

$$\lim_{R \to \infty} \int_{C_R} dp \frac{\partial \ln \epsilon(p)}{\partial p} = 2\pi i \sum_i N_i = 2\pi i N,$$

(4.4)

where $N$ is the total number of zeros of $\epsilon(p)$ in the half-plane $\Re p > 0$. It turns out (as I shall prove in a moment) that the contribution to the integral over $C_R$ from the semicircle vanishes at $R \to \infty$ and so we need only integrate along the imaginary axis:

$$2\pi i N = \int_{-\infty}^{\infty} dp \frac{\partial \ln \epsilon(p)}{\partial p} = \ln \frac{\epsilon(-\infty)}{\epsilon(+\infty)}.$$  

(4.5)

Proof. All we need to show is that

$$|p| \frac{\partial \ln \epsilon(p)}{\partial p} \to 0 \quad \text{as} \quad |p| \to \infty, \quad \Re p > 0.$$

(4.6)

Indeed, using (4.1) and the Landau integration rule (3.20), we have in this limit:

$$\epsilon(p) = 1 - \frac{\omega^2}{\kappa^2} \int_{-\infty}^{+\infty} dv_z \bar{F}'(v_z) \frac{ik}{p} \left(1 - \frac{ikv_z}{p} + \ldots \right) \approx 1 + \frac{1}{p^2} \sum_\alpha \omega^2_{p\alpha},$$

(4.7)

where I have integrated by parts and used $\int dv_z F_\alpha = n_\alpha$. Manifestly, the condition (4.6) is satisfied.

Note that, along the imaginary axis $p = -i\omega$, by the same expansion and using also the Plemelj formula (3.23), we have

$$\epsilon(-i\omega) \approx 1 - \frac{1}{\omega^2} \sum_\alpha \omega^2_{p\alpha} - i\pi \frac{\omega^2}{\kappa^2} \bar{F}'\left(\frac{\omega}{\kappa}\right) \to 1 \mp i0 \quad \text{as} \quad \omega \to \mp\infty.$$  

(4.8)

This is going to be useful shortly.

In view of (4.8) and of our newly proven formula (4.5), as the function $\epsilon(-i\omega)$ runs along the real line in $\omega$, it changes from

$$\epsilon(i\infty) = 1 - i0 \quad \text{at} \quad \omega = -\infty,$$

(4.9)

where I have arbitrarily fixed its phase, to

$$\epsilon(-i\infty) = e^{2\pi i N} + i0 \quad \text{at} \quad \omega = +\infty,$$

(4.10)
(a) Single-maximum, stable equilibrium   (b) Strange but stable equilibrium

**Figure 16.** Two examples of Nyquist diagrams showing stability (because failing to circle zero): (a) the case of a monotonically decreasing distribution (§4.2, Fig. 17a); (b) another stable case, even though very complicated (it also illustrates the argument in §4.3).

where $N$ is the number of times the function

$$
\epsilon(-i\omega) = 1 - \frac{\omega^2_{pe}}{k^2} \left[ P \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - \omega/k} + i\pi F'(\omega/k) \right]
$$

(4.11)
circles around the origin in the complex $\epsilon$ plane. Since $N$ is also the number of unstable roots of the dispersion relation, this gives us a way to count these roots by sketching $\epsilon(-i\omega)$—this sketch is called the Nyquist diagram. Two examples of Nyquist diagrams implying stability are given in Fig. 16: the curve $\epsilon(-i\omega)$ departs from $1 - i\omega$ and comes back to $1 + i\omega$ via a path that, however complicated, never makes a full circle around zero. Two examples of unstable situations appear in Fig. 18(b,d); in these cases, zero is circumnavigated, implying that the equilibrium distribution $\bar{F}$ is unstable (at a given value of $k$).

In order to work out whether the Nyquist curve circles zero (and how many times), all one needs to do is find $\text{Re}\,\epsilon(-i\omega)$ at all points $\omega$ where $\text{Im}\,\epsilon(-i\omega) = 0$, i.e., where the curve intersects the real line, and hence sketch the Nyquist diagram. We shall see in a moment, with the aid of some important examples, how this is done, but let us do a little bit of preparatory work first.

It follows immediately from (4.11) that these crossings happen whenever $\omega/k = v_*$ is a velocity at which $\bar{F}(v_z)$ has an extremum, $\bar{F}'(v_*) = 0$. At these points, the dielectric function (4.11) is real and can be expressed so:

$$
\epsilon(-ikv_*) = 1 + \frac{\omega^2_{pe}}{k^2} P(v_*)
$$

(4.12)

Here $P(v_*)$ is (minus) the principal-value integral in (4.11), which can be manipulated as follows:

$$
P(v_*) = -P\int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - v_*} = -P\int_{-\infty}^{+\infty} dv_z \frac{1}{v_z - v_*} \frac{\partial}{\partial v_z} [\bar{F}(v_z) - \bar{F}(v_*)]
$$

$$
= \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_*) - \bar{F}(v_z)}{(v_z - v_*)^2},
$$

(4.13)

where I have integrated by parts; the additional term $\bar{F}(v_*)$ was inserted under the
derivative in order to eliminate the boundary terms arising in this integration by parts around the pole $v_z = v_\ast$.\textsuperscript{27}

Now we are ready to analyse particular (and, as we shall see, also generic) equilibrium distributions $\bar{F}(v_z)$.

### 4.2. Stability of Monotonically Decreasing Distributions

Consider first a distribution function that has a single maximum at $v_z = v_0$ and monotonically decays in both directions away from it (Fig. 17a): $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) < 0$. This means that, besides at $\omega = \mp\infty$, $\text{Im} \epsilon(-i\omega) \propto \bar{F}'(\omega/k)$ also vanishes at $\omega = kv_0$. It is then clear that

$$\epsilon(-ikv_0) = 1 + \frac{\omega^2}{k^2} P(v_0) > 1$$

(4.14)

because $\bar{F}(v_0) > \bar{F}(v)$ for all $v_z$ and so $P(v_0) > 0$. Thus, the Nyquist curve departs from $1 - i0$ at $\omega = -\infty$, intersects the real line once at $\omega = kv_0$ and then comes back to $1 + i0$ without circling zero; the corresponding Nyquist digram is sketched in Fig. 16(a).

Conclusion:

**Monotonically decreasing distributions are stable against electrostatic perturbations.**

We do not in fact need all this mathematical machinery just to prove the stability of monotonically decreasing distributions (in §9.2, we shall see that this is a very robust result)—but it will come handy when dealing with less simple cases. Parenthetically, let us work out some direct proofs of stability.

**Exercise 4.1. Direct proof of linear stability of monotonically decreasing distributions.** (a) Consider the dielectric function (4.1) with $p = -i\omega + \gamma$ and assume $\gamma > 0$ (so the Landau contour is just the real axis). Work out the real and imaginary parts of the dispersion relation $\epsilon(p) = 0$ and show that it can never be satisfied if $v_z \bar{F}'(v_z) \leq 0$, i.e., that any equilibrium distribution that has a maximum at zero and decreases monotonically on both sides of it is stable against electrostatic perturbations.\textsuperscript{28}

(b) What if the maximum is at $v_z = v_0 \neq 0$?

\textsuperscript{27}Note that in the final expression in (4.13), there is no longer a need for principal-value integration because, $v_\ast$ being a point of extremum of $\bar{F}$, the numerator of the integrand is quadratic in $v_z - v_\ast$ in the vicinity of $v_\ast$.

\textsuperscript{28}This kind of argument can also be useful in stability considerations applying to more complicated situations, e.g., magnetised plasmas (Bernstein 1958).
Exercise 4.2. Direct proof of linear stability of isotropic distributions. (a) Recall Exercise 3.2 and show that all homogeneous, 3D-isotropic (in velocity) equilibria are stable against electrostatic perturbations (with no need to assume long wave lengths).

(b) Prove, in the same way, that isotropic equilibria are also stable against electromagnetic perturbations. You will need to derive the transverse dielectric function in the same way as in Q2 or Q3, but for a general equilibrium distribution \( f_0(\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) \); failing that, you can look it up in a book, e.g., Krall & Trivelpiece (1973) or Davidson (1983).

4.3. Penrose’s Instability Criterion

It would be good to learn how to test for stability generic distributions that have multiple minima and maxima: the simplest of them is shown in Fig. 17b, evoking the bump-on-tail situation discussed in §3.5 and thus posing a risk (but, as we are about to see, not a certainty!) of being unstable.

The Nyquist curve \( \epsilon(-i\omega) \) departs from \( 1 - i0 \) at \( \omega = -\infty \), then crosses the real line for the first time at \( \omega = kv_1 \), corresponding to the leftmost maximum of \( \bar{F} \).\(^{29}\) This crossing is upwards, from the lower to the upper half-plane, and it is not hard to see that a maximum will always correspond to such an upward crossing and a minimum to a downward one, from the upper to the lower half-plane: this follows directly from the change of sign of \( \text{Im} \epsilon \) in Eq. (4.11) because \( F'(\omega/k) \) goes from positive to negative at any point of maximum and vice versa at any minimum. After a few crossings back and forth,

\(^{29}\)For the distribution sketched in Fig. 17(b), this maximum is global, so \( P(v_1) > 0 \) and, therefore, \( \epsilon(-ikv_1) > 1 \). This is the rightmost such crossing when \( v_1 \) is the global maximum.
corresponding to local minima and maxima (if any), the Nyquist curve will come to the the downward crossing corresponding to the global minimum (other than at \( v_z = \pm \infty \)) of the distribution function at, say, \( \omega = kv_0 \). If at this point \( P(v_0) > 0 \), then \( \epsilon(-ikv_0) > 1 \) and the same is true at all other crossing points \( v_* \) because \( v_0 \) is the global minimum of \( \bar{F} \) and so \( P(v_*) > P(v_0) > 0 \) for all other extrema. In this situation, illustrated in Fig. 18(a), the Nyquist curve never circumnavigates zero and, therefore, \( P(v_0) > 0 \) is a sufficient condition of stability. It is also the necessary one, which is proved in the following way.

Suppose \( P(v_0) < 0 \). Then, in (4.12), we can always find a range of \( k \) that are small enough that \( \epsilon(-ikv_0) < 0 \), so the downward crossing at \( v_0 \) happens on the negative side of zero in the \( \epsilon \) plane. After this downward crossing, the Nyquist curve will make more crossings, until it finally comes to rest at \( 1 + i0 \) as \( \omega = +\infty \). Let us denote by \( v_2 > v_0 \) the point of extremum for which the corresponding crossing occurs at a point on the \( \text{Re} \epsilon \) axis that is closest to (but always will be to the right of) \( \epsilon(-ikv_0) < 0 \). If \( \epsilon(-ikv_2) > 0 \), then there is no way back, zero has been fully circumnavigated and so there must be at least one unstable root (see Fig. 18b,d). If \( \epsilon(-ikv_2) < 0 \), there is in principle some wiggle room for the Nyquist curve to avoid circling zero (see Fig. 18c for a single-minimum distribution of Fig. 17b—or Fig. 16b for some serious wiggles). However, since \( P(v_2) > P(v_0) \) for any \( v_2 \) (because \( v_0 \) is the global minimum of \( \bar{F} \)), we can always increase \( k \) in (4.12) just enough so \( \epsilon(-ikv_2) > 0 \) even though \( \epsilon(-ikv_0) < 0 \) still (this corresponds to turning Fig. 18c into Fig. 18d). Thus, if \( P(v_0) < 0 \), there will always be some range of \( k \) inside which there is an instability.

We have obtained a sufficient and necessary condition of instability of an equilibrium \( \bar{F}(v_z) \) against electrostatic perturbations: if \( v_0 \) is the point of global minimum of \( \bar{F} \), then

\[
P(v_0) = \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_0) - \bar{F}(v_z)}{(v_z - v_0)^2} < 0 \iff \bar{F} \text{ is unstable} .\quad (4.15)
\]

This is the famous Penrose’s instability criterion (the famous criterion, not the famous Penrose; it was proved by Oliver Penrose 1960, in a stylistically somewhat different way than I did it here). Note that considerations of the kind presented above can be used to work out the wave-number intervals, corresponding to various troughs in \( \bar{F} \), in which instabilities exist.

Intuitively, the criterion (4.15) says that, in order for a distribution to be unstable, it needs to have a trough and this trough must be deep enough. Thus, if \( F(v_0) = 0 \), i.e., if the distribution has a “hole”, it is always unstable (an extreme example of this is the two-stream instability; see Exercise 3.5). Another corollary is that you cannot stabilise a distribution by just adding some particles in a narrow interval around \( v_0 \), as this would create two minima nearby, which, the filled interval being narrow, are still going to render the system unstable. To change that, you must fill the trough substantially with particles—hence the tendency to flatten bumps into plateaux, which we will discover in §6 (this answers, albeit in very broad strokes, the question posed at the beginning of §4 about the types of stable distributions towards which the unstable ones will be pushed as the instabilities saturate).

**Exercise 4.3.** Consider a single-minimum distribution like the one in Fig. 17(b), but with the global maximum on the right and the lesser maximum on the left of the minimum.

---

\(^{30}\) Another way of putting this is: a distribution \( \bar{F} \) is unstable iff it has a minimum at some \( v_0 \) for which \( P(v_0) < 0 \). Obviously, if \( P(v_0) < 0 \) at some minimum, it is also negative at the global minimum.
Draw various possible Nyquist diagrams and convince yourself that Penrose’s criterion works. If you enjoy this, think of a distribution that would give rise to the Nyquist diagram in Fig. 16(b).

**Exercise 4.4.** What happens if the distribution function $\bar{F}$ has an inflection point, i.e., $\bar{F}(v_0) = 0, \bar{F}'(v_0) = 0, \bar{F}''(v_0) = 0$?

**Exercise 4.5.** What happens if the distribution function has a trough with a flat bottom (i.e., a flat minimum over some interval of velocities)?

### 4.4. Bumps, Beams, Streams and Flows

An elementary example of the use of Penrose’s criterion is the two-stream instability, first introduced in Exercise 3.5. The case of two cold streams, represented by (3.60) and Fig. 12(a), is obviously unstable because there is a gaping hole in this distribution. What if we now give these streams some thermal width? This can be modeled by the double-Lorentzian distribution (Fig. 12b)

$$F_e(v_z) = \frac{n_e v_b}{2\pi} \left[ \frac{1}{(v_z - u_b)^2 + v_b^2} + \frac{1}{(v_z + u_b)^2 + v_b^2} \right], \quad (4.16)$$

which is particularly easy to handle analytically. For the moment, we will consider the ions to be infinitely heavy, so $\bar{F} = F_e$.

Since the distribution (4.16) is symmetric, it can only have its minimum at $v_0 = 0$. Asking that it should indeed be a minimum, rather than a maximum, i.e., $\bar{F}'(0) > 0$, one finds that the condition for this is

$$u_b > \frac{v_b}{\sqrt{3}}. \quad (4.17)$$

Otherwise, the two streams are too wide (in velocity space) and the distribution is monotonically decreasing, so, according to §4.2, it is stable.

If the condition (4.17) is satisfied, the distribution has two bumps, but is this enough to make it unstable? Substituting this distribution into Penrose’s criterion (4.15) and doing the integral exactly, we get the necessary and sufficient instability condition:

$$P(0) = -\frac{u_b^2 - v_b^2}{(u_b^2 + v_b^2)^2} < 0 \iff u_b > v_b. \quad (4.18)$$

Thus, if the streams are sufficiently fast and/or their thermal spread is sufficiently narrow, an instability will occur, but it is not quite enough just to have a little trough. Note, by the way, that Penrose’s criterion does not differentiate between hydrodynamic (cold) and kinetic (hot) instability mechanisms (§3.7).

**Exercise 4.6.** Use Nyquist’s method to work out the range of wave numbers at which perturbations will grow for the two-stream instability (you will find the answer in Jackson 1960—yes, that Jackson). Convince yourself that this is all in accord with the explicit solution of the dispersion relation obtained in Q4.

It is obvious how these considerations can be generalised to more complicated situations, e.g., to cases where the streams have different velocities, where one of them is, in fact, the thermal bulk of the distribution and the other is a little bump on its tail (§3.7), where there are more than two streams, etc. The streams also need not be composed

---

31 The easiest way to do it is to turn the integration path along the real axis into a loop by completing it with a semicircle at positive or negative complex infinity, where the integrand vanishes, and use Cauchy’s formula.
of the particles of the same species: indeed, as we saw in (4.1), in the linear theory, the distributions of all species are additively combined into $\bar{F}$ with weights that are inversely proportional to their masses [see (4.2)]. Thus, the ion-acoustic instability (§3.9) is also just a kind of of two-stream—or, if you like, bump-on-tail—instability, with the entire hot and mighty electron distribution making up a magnificent bump on the tail of the cold, $m_e/m_i$-weighted ion one (Fig. 19).32 When the streams/beams have thermal spreads, they are more commonly thought of as mean flows—or currents, if the electron flows are not compensated by the ion ones.

**Exercise 4.7.** Construct an equilibrium distribution to model your favorite plasma system with flows and/or beams and investigate its stability: find the growth rate as a function of wave number, instability conditions, etc.

### 4.5. Anisotropies

So we have found that various holes, bumps, streams, beams, flows, currents and other such nonmonotonic features in the (combined, multispecies) equilibrium distribution present an instability risk, unless they are sufficiently small, shallow, wide and/or close enough to the thermal bulk. All of these are, of course, anisotropic features—indeed, as we saw in Exercise 4.2, 3D-isotropic distributions are harmless, instability-wise. It turns out that anisotropies of the distribution function in velocity space are dangerous even when the distribution decays monotonically in all directions.33 However, the instabilities that occur in such situations are electromagnetic, rather than electrostatic, and so require an investigation into the properties of the transverse dielectric function of the kind derived in Q2 or Q3, but for a general equilibrium. A nice treatment of anisotropy-driven instabilities can be found in Krall & Trivelpiece (1973) and an even more thorough one in Davidson (1983). In §§9.2.2 and 9.4.1, I will show in quite a simple way that, at least in principle, there is always energy available to be extracted from anisotropic distributions.

**Exercise 4.8. Criterion of instability of anisotropic distributions.** This is an independent-study topic. Consider linear stability of general distribution functions to electromagnetic perturbations and work out the stability criterion in the spirit of §4. You should discover that anisotropic distributions such as, e.g., the bi-Maxwellian (12.6), tend to be unstable. Krall & Trivelpiece (1973, §9.10) would be a good place to read about it, but do range beyond.

32 In fact, when the two species’ temperatures are the same, there is still an instability, whose criterion can again be obtained by the Nyquist-Penrose method: see Jackson (1960).

33 In Q3, you have an opportunity to derive the most famous of all instabilities triggered by anisotropy.
5. Energy, Entropy, Heating, Irreversibility, and Phase Mixing

While we are done with the “calculational” part of linear theory (calculating whether the field perturbations oscillate, decay or grow, and at what rates), we are not yet done with the “conceptual” part: what exactly is going on, mathematically and physically?

The plan of addressing this question in this section is as follows.

- I will show that Landau damping of perturbations of a plasma in thermal equilibrium leads to the heating of this equilibrium—basically, that energy is conserved. This is not a surprise, but it is useful to see explicitly how this works (§5.1).
- I will then ask how it is possible to have heating (an irreversible process) in a plasma that was assumed collisionless and must conserve entropy. In other words, why, physically, is Landau damping a damping? This will lead us to consider entropy evolution in our system and to introduce an important concept of free energy (§5.2).
- In the above context, we will examine (§§5.3 and 5.6) the Laplace-transform solution (3.8) for the perturbed distribution function and establish the phenomenon of phase mixing—emergence of fine structure in velocity (phase) space. This will allow collisions and, therefore, irreversibility back in (§5.5). We will also see that the Landau-damped solutions are not eigenmodes (while growing solutions can be), and so conclude that it made sense to insist on using an initial-value-problem formalism.

5.1. Energy Conservation and Heating

Let us go back to the full, nonlinear Vlasov–Poisson system, with the collision term restored:

\[
\frac{\partial f_\alpha}{\partial t} + v \cdot \nabla f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial v} = \left( \frac{\partial f_\alpha}{\partial t} \right)_c, \tag{5.1}
\]

\[- \nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d^3v f_\alpha. \tag{5.2}\]

Let us calculate the rate of change of the electric energy:

\[
\frac{d}{dt} \int d^3r \frac{E^2}{8\pi} = \int d^3r \frac{\nabla \varphi}{4\pi} \cdot \frac{\partial (\nabla \varphi)}{\partial t} = - \int d^3r \frac{\varphi}{4\pi} \frac{\partial \nabla^2 \varphi}{\partial t} = \sum_\alpha q_\alpha \int \int d^3r \int d^3v \varphi \frac{\partial f_\alpha}{\partial t} \tag{5.3}
\]

by parts

use (5.2)

\[
= \sum_\alpha q_\alpha \int \int d^3r \int d^3v \varphi \left[ -v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} (\nabla \varphi) \cdot \frac{\partial f_\alpha}{\partial v} \right] \]

by parts

vanishes because

\[
f(\pm \infty) = 0
\]

vanishes

because number of particles is conserved

\[
= \sum_\alpha q_\alpha \int \int d^3r \int d^3v f_\alpha v \cdot \nabla \varphi = - \int d^3r E \cdot j,
\]

where \(j\) is the current density. So the rate of change of the electric field is minus the rate at which electric field does work on the charges, a.k.a. Joule heating—not a surprising result. The energy goes into accelerating particles, of course: the rate of change of their
kinetic energy is 
\[
\frac{dK}{dt} = \sum_\alpha \int \int d^3r \int d^3v \frac{m_\alpha v^2}{2} \frac{\partial f_\alpha}{\partial t}
\]
use (5.1)

\[
= \sum_\alpha \int \int d^3r \int d^3v \frac{m_\alpha v^2}{2} \left[ -v \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha} \nabla \phi \cdot \nabla f_\alpha \right] + \frac{\partial f_\alpha}{\partial t} \phi
\]
vanishes because full divergence by parts in \(v\)

\[
= -\sum_\alpha q_\alpha \int \int d^3r \int d^3v f_\alpha v \cdot \nabla \phi = \int d^3r E \cdot j.
\]

Combining (5.3) and (5.4) gives us the law of energy conservation:

\[
\frac{d}{dt} \left( K + \int d^3r \frac{E^2}{8\pi} \right) = 0.
\]

Exercise 5.1. Demonstrate energy conservation for the more general case in which magnetic-field perturbations are also allowed.

Thus, if the perturbations are damped, the energy of the particles must increase—this is usually called heating. Strictly speaking, however, heating is a slow, irreversible increase in the mean temperature of the thermal equilibrium. Let us make this statement quantitative. Consider a Maxwellian plasma, homogeneous in space but possibly with some slow dependence on time (cf. §2):

\[
f_{0\alpha} = \frac{n_\alpha}{(\pi v_{th\alpha}^2)^{3/2}} e^{-v^2/v_{th\alpha}^2} = n_\alpha \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} e^{-m_\alpha v^2/2T_\alpha}.
\]

In a homogeneous system with a fixed volume, the density \(n_\alpha\) is constant in time because the number of particles is constant: \(d(Vn_\alpha)/dt = 0\). The temperature, however, is allowed to change: \(T_\alpha = T_\alpha(t)\). The total kinetic energy of the particles is

\[
K = V \sum_\alpha \int d^3v \frac{m_\alpha v^2}{2} f_{0\alpha} + \sum_\alpha \int \int d^3r \int d^3v \frac{m_\alpha v^2}{2} \delta f_\alpha.
\]

Let us average this over time, as per (2.7): the perturbed part vanishes and we have

\[
\langle K \rangle = V \sum_\alpha \frac{3}{2} n_\alpha T_\alpha.
\]

Averaging also (5.5) and using (5.8), we get

\[
\sum_\alpha \frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = -\frac{d}{dt} \frac{1}{V} \int d^3r \langle E^2 \rangle \frac{1}{8\pi},
\]
so the heating rate of the equilibrium equals the rate of decrease of the mean energy of the perturbations.
We saw that the perturbations evolve according to (3.16). If we wait for a while, only the slowest-damped mode will matter, with all others exponentially small in comparison. Let us call its frequency and its damping rate \(\omega_k\) and \(\gamma_k\), respectively, so \(E_k \propto e^{-i\omega_k t - \gamma_k t}\). If \(|\gamma_k| \ll \omega_k\), the time average (2.7) can be defined in such a way that \(\omega_k - 1/\Delta t \ll |\gamma_k|^{-1}\). Then (5.9) becomes
\[
\sum_{\alpha} \frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = - \sum_k 2\gamma_k \frac{|E_k|^2}{8\pi} > 0.
\]
(5.10)
The Landau damping rate of the electric-field perturbations is the heating rate of the equilibrium.\(^{34}\)

This result, while at first glance utterly obvious, might, on reflection, appear to be paradoxical: surely, the heating of the equilibrium implies increasing entropy—but the damping that is leading to the heating is collisionless and, in a collisionless system, in view of the \(H\) theorem, how can the entropy change?

### 5.2. Entropy and Free Energy

The kinetic entropy for each species of particles is defined to be
\[
S_\alpha = - \int \int d^3r d^3v f_\alpha \ln f_\alpha.
\]
(5.11)
This quantity [or, indeed, the full-phase-space integral of any quantity that is a function only of \(f_\alpha\); see (9.8)] can only be changed by collisions and, furthermore, the plasma-physics version of Boltzmann’s \(H\) theorem says that
\[
\frac{d}{dt} \sum_\alpha S_\alpha = - \sum_\alpha \int \int d^3r d^3v \left( \frac{\partial f_\alpha}{\partial t} \right) \ln f_\alpha \geq 0,
\]
(5.12)
where equality is achieved iff all \(f_\alpha\) are Maxwellian with the same temperature \(T_\alpha = T\).

Thus, if collisions are ignored, the total entropy stays constant and everything that happens is, in principle, reversible. So how can there be net damping of waves and, worse still, net heating of the equilibrium particle distribution?! Presumably, any damping solution can be turned into a growing solution by reversing all particle trajectories—so should the overall perturbation level not stay constant?

As I already noted in §5.1, strictly speaking, heating is the increase of the equilibrium temperature—and, therefore, of the equilibrium entropy. Indeed, for each species, the equilibrium entropy is
\[
S_0 = - \int \int d^3r d^3v f_0 \ln f_0 = - \int \int d^3r d^3v f_0 \left\{ \ln \left[ n \left( \frac{m}{2\pi} \right)^{3/2} \right] - \frac{3}{2} \ln T - \frac{mv^2}{2T} \right\}
\]
\[= V \left\{ -n \ln \left[ n \left( \frac{m}{2\pi} \right)^{3/2} \right] + \frac{3}{2} n \ln T + \frac{3}{2} n \right\},
\]
(5.13)
where I have used \(\int d^3v (mv^2/2)f_0 = (3/2)nT\). Since \(n = \text{const}\),
\[
T \frac{dS_0}{dt} = V \frac{3}{2} n \frac{dT}{dt},
\]
(5.14)
so heating is indeed associated with an increase of \(S_0\).

Since, according to (5.10), this can be successfully accomplished by collisionless damping and since entropy overall can only increase due to collisions, we must search for the

\(^{34}\)Obviously, the damping of waves on particles of species \(\alpha\) increases only the temperature of that species.
“missing entropy” (or, rather, for the missing decrease of entropy) in the perturbed part of the distribution. The mean entropy associated with the perturbed distribution is \[ \langle \delta S \rangle = \langle S - S_0 \rangle = - \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \, \langle (f_0 + \delta f) \ln(f_0 + \delta f) - f_0 \ln f_0 \rangle \]
\[ = - \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \, \left( (f_0 + \delta f) \left[ \ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2 f_0^2} + \cdots \right] - f_0 \ln f_0 \right) \]
\[ \approx - \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \, \frac{\delta f^2}{2 f_0}, \tag{5.15} \]
after expanding to second order in small $\delta f / f_0$ and using $\langle \delta f \rangle = 0$. The total mean entropy of each species, $\langle S \rangle = S_0 + \langle \delta S \rangle$, can only be changed by collisions, so, if collisions are ignored, any heating of a given species, i.e., any increase in its $S$ of each species, after expanding to second order in small decrease "missing entropy" (or, rather, for the missing $\delta S$ in its $\langle S \rangle$). The latter can only be achieved by increasing $\langle \delta f^2 \rangle$: indeed, using (5.14) and (5.15), we find
\[ T \left( \frac{dS_0}{dt} + \frac{d\langle \delta S \rangle}{dt} \right) = V \frac{3}{2} n \frac{dT}{dt} - \frac{d}{dt} \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \frac{T \langle \delta f^2 \rangle}{2 f_0} \]
\[ = - \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \, T \left( \left\langle \frac{\partial f}{\partial t} \right\rangle_c \ln f \right), \tag{5.16} \]
If the right-hand side is ignored, $T$ can only increase if $\langle \delta f^2 \rangle$ increases too.

It is useful to work out the collision term in (5.16) in terms of $f_0$ and $\delta f$: using the fact that $\langle \delta f \rangle = 0$ by definition and that the number of particles is conserved by the collision operator, we get
\[ \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \, T \left( \left\langle \frac{\partial f}{\partial t} \right\rangle_c \ln f \right) \approx \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \left[ T \left( \frac{\partial f_0}{\partial t} \right)_c \ln f_0 + \left\langle \frac{T \delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \right\rangle \right] \]
\[ = V \int \! d^3 \mathbf{v} \frac{m v^2}{2} \left( \frac{\partial f_0}{\partial t} \right)_c + \int \! d^3 \mathbf{r} \, d^3 \mathbf{v} \left\langle \frac{T \delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \right\rangle. \tag{5.17} \]

The second term is the collisional damping of $\delta f$, of which more will be said soon. The first term is the collisional energy exchange between the equilibrium distributions of different species (intraspecies collisions conserve energy, but inter-species ones do not, because there is friction between species). If the species under consideration is $\alpha$, this energy exchange can be represented as $\sum_{\alpha} \nu_{\alpha \alpha'} (T_{\alpha} - T_{\alpha'})$ (see, e.g., Helander & Sigmar 2005) and will act to equilibrate temperatures between species as the system strives toward thermal equilibrium. If the collision frequencies $\nu_{\alpha \alpha'}$ are small, this is a slow effect. Due to overall energy conservation, the energy-exchange terms vanish exactly if (5.17) is summed over species.

Finally, let us sum (5.16) over species and use (5.9) to relate the total heating to the rate

\[ 35 \text{As an aside, note that this piece of the calculation is entirely independent of what } f_0 \text{ is. It simply demonstrates that the entropy of an averaged distribution } f_0 \text{ is always larger than that of the exact distribution } f, \text{ as long as } \delta f \ll f_0. \text{ If the average is reinterpreted as a coarse graining over phase space, this argument is sometimes viewed as a kind of “proof” (or illustration) of the second law of thermodynamics. Indeed, take } f_0 \text{ and } f \text{ to be the same at some initial time } t. \text{ Then } S_0(t) = \langle S(t) \rangle. \text{ Now advance to } t + \delta t. \text{ Some small } \delta f \text{ arises, but coarse graining “deletes” the information contained in it and (5.15) shows that } S_0(t + \delta t) > \langle S(t + \delta t) \rangle (\text{cf. the general statistical-mechanical argument to the same effect: Schekochihin 2019, §§12.4 and 13.4}). \]

\[ 36 \text{In the second term, } T \text{ can be brought inside the time derivative because } \langle \delta f^2 \rangle / f_0 \ll f_0. \]
of change of the electric-perturbation energy:

\[
\frac{d}{dt} \left[ \sum_\alpha \int\int d^3r \ d^3v \ \frac{T_\alpha \langle \delta f_\alpha^2 \rangle}{2 f_\alpha} \right] + \int d^3r \ \frac{\langle E^2 \rangle}{8\pi} = \sum_\alpha \int\int d^3r \ d^3v \ \left\langle \frac{T_\alpha}{f_\alpha} \left( \frac{\partial \delta f_\alpha}{\partial t} \right)_c \right\rangle \leq 0 ,
\]

(5.18)

where we used (5.17) in the right-hand side (with the total equilibrium collisional energy-exchange terms vanishing upon summation over species). The right-hand side must be non-positive-definite because collisions cannot decrease entropy [see (5.12)].

Equation (5.18) is a way to express the idea that, except for the effect of collisions, the change in the electric-perturbation energy (\(= \) heating) must be compensated by the change in \(\langle \delta f^2 \rangle\), in terms of a conservation law of a quadratic positive-definite quantity, \(W\), that measures the total amount of perturbation in the system (a quadratic norm of the perturbed solution). It is not hard to realise that this quantity is the free energy of the perturbed state, comprising the entropy of the perturbed distribution and the energy of the electric field:

\[
W = \mathcal{E} - \sum_\alpha T_\alpha \langle \delta S_\alpha \rangle , \quad \mathcal{E} = \int d^3r \ \frac{\langle E^2 \rangle}{8\pi} .
\]

(5.19)

It is quite a typical situation in non-equilibrium systems that there is an energy-like (quadratic in the relevant fields and positive definite) quantity, which is conserved except for dissipation. For example, in hydrodynamics, the motions of a fluid are governed by the Navier–Stokes equation:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} ,
\]

(5.20)

where \(\mathbf{u}\) is velocity, \(\rho\) mass density (\(\rho = \text{const}\) for an incompressible fluid), \(p\) pressure and \(\mu\) the dynamical viscosity of the fluid. The conservation law is

\[
\frac{d}{dt} \int d^3r \ \rho u^2/2 = -\mu \int d^3r \ |\nabla \mathbf{u}|^2 \leq 0 .
\]

(5.21)

The conserved quadratic quantity is kinetic energy and the negative-definite dissipation (leading to net entropy production) is viscous heating.

**Exercise 5.2. Free energy and kinetic energy of mean plasma flow.** Suppose the perturbation \(\delta f\) contains a mean flow of particles, with velocity \(\mathbf{u}\). Show that it is then always formally possible to decompose

\[
\delta f = \frac{2\mathbf{u} \cdot \mathbf{v}}{v_{th}^2} f_0 + h ,
\]

(5.22)

where \(\int d^3\mathbf{v} \ v h = 0\). Hence show that

\[
\int\int d^3r \ d^3v \ \frac{T\langle \delta f^2 \rangle}{2 f_0} = \int d^3r \ \frac{mn(u^2)}{2} + \int\int d^3r \ d^3v \ \frac{T\langle h^2 \rangle}{2 f_0} ,
\]

(5.23)

\[\text{37Note that the existence of such a quantity implies that the Maxwellian equilibrium is stable: if a quadratic norm of the perturbed solution cannot grow, clearly there cannot be any exponentially growing solutions. This is known as Newcomb’s theorem, first communicated to the world in the paper by Bernstein (1958, Appendix I). A generalisation of this principle to isotropic distributions is the subject of Q5(c) and of §9.3, where the conserved quantity \(W\) will reemerge in a different way, confirming its status as a Platonic entity that cannot be avoided.}\]

\[\text{38You will find a similar conservation law for incompressible MHD if, in §13.10, you work out the time evolution of \(\int d^3r \ (\rho u^2/2 + B^2/8\pi)\) assuming \(\rho = \text{const}\) and \(\nabla \cdot \mathbf{u} = 0\) [cf. (14.70)].}\]
i.e., the entropic part of the free energy is equal to the kinetic energy of the mean plasma flow plus the free entropy associated with the part of the perturbed distribution that has no mean velocity. This means that “fluid” energy budgets such as (5.21) and (14.70) are not just analogs, but particular cases, of the free-energy budget (5.18). In §5.7, I will show how these ideas play out for Langmuir waves.

Thus, as the electric perturbations decay via Landau damping, the perturbed distribution function must grow. This calls for going back to our solution for it (§3.1) and analysing carefully the behaviour of $\delta f$.

### 5.3. Structure of Perturbed Distribution Function

Start with our solution (3.8) for $\delta f(p)$ and substitute into it the solution (3.15) for $\varphi(p)$:

$$
\delta \hat{f}(p) = \frac{1}{p + ik \cdot v} \left\{ \frac{q}{m} \left[ \sum_i \frac{c_i}{p - p_i} + A(p) \right] k \cdot \frac{\partial f_0}{\partial v} + g \right\}.
$$

To compute the inverse Laplace transform (3.6), we adopt the same method as in §3.1 (Fig. 5), viz., shift the integration contour to large negative Re $p$ as shown in Fig. 20 and
use Cauchy’s formula:

\[
\delta f(t) = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} dp e^{pt} \delta \hat{f}(p) = i \frac{q}{m} \sum_i \frac{c_i e^{p_i t}}{p_i + i k \cdot v} k \cdot \frac{\partial f_0}{\partial v}
\]

\[
+ e^{-ik \cdot v t} \left\{ g - \frac{m}{q} \left[ \sum_i c_i \left( p_i + i k \cdot v \right) + A(-ik \cdot v) \right] k \cdot \frac{\partial f_0}{\partial v} \right\}.
\]

A perceptive reader has spotted that this formula does not seem to satisfy \(\delta f(t = 0) = g\) unless \(A(-ik \cdot v) = 0\). This is because, as explained in footnote 12, the method for calculating the inverse Laplace transform that involves discarding the integral along the vertical part of the shifted contour in Fig. 20 only works in the limit of long times. It is an amusing exercise in complex analysis to show that, in the (overly restrictive) case of \(\hat{\varphi}(p)\) decaying quickly at \(\text{Re} p \to -\infty\), the solution (5.25) is also valid at finite \(t\) and, accordingly, \(A(-ik \cdot v) = 0\), i.e., \(A(p)\) vanishes for any purely imaginary \(p\).

The solution (5.25) teaches us two important things.

1) First, the Landau-damped solution is not an eigenmode. Even though the evolution of the potential, given by (3.16), does look like a sum of damped eigenmodes of the form \(\varphi \propto e^{p_i t}, \text{Re} p_i < 0\), the full solution of the Vlasov–Poisson system does not decay: there is a part of \(\delta f(t)\), the “ballistic response” \(\propto e^{-ik \cdot v t}\), that oscillates without decaying—in fact, we shall see in §5.6 that \(\delta f\) even has a growing part! It is this part that is responsible for keeping free energy conserved, as per (5.18) without collisions (§5.7). Thus, you may think of Landau damping as a process of transferring (free) energy from the electric-field perturbations to the perturbations of the distribution function.

In contrast to the case of damping, a growing solution (\(\text{Re} p_i > 0\)) can be viewed as an eigenmode because, after a few growth times, the first term in (5.25) will be exponentially larger than the ballistic term. This will allow us to ignore the latter in our treatment of QLT (§6.1)—a handy, although not necessary (see Q8), simplification. Note that reversibility is not an issue for the growing solutions: so, there may be (and often are) damped solutions as well, so what? We only care about the growing modes because they will be all that is left if we wait long enough.

2) Secondly, the \(\delta f\) perturbations have fine structure in velocity (phase) space. This structure gets finer with time: roughly speaking, if \(\delta f \propto e^{-ik \cdot v t}\), then

\[
\frac{1}{\delta f} \frac{\partial \delta f}{\partial v} \sim ik t \to \infty \text{ as } t \to \infty.
\]

This phenomenon is called phase mixing. You can think of the basic mechanism responsible for it as a shearing in phase space: the homogeneous part of the linearised kinetic equation,

\[
\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial z} = \ldots,
\]

(5.27)

describes advection of \(\delta f\) by a linear shear flow in the \((z, v)\) plane. This turns any \(\delta f\) structure in this plane into long thin filaments, with large gradients in \(v\) (Fig. 21).
5.4. Landau Damping Is Phase Mixing

Phase mixing helps us make sense of the notion that, even though $\varphi$ is the velocity integral of $\delta f$, the former can be decaying while the latter is not:

$$\varphi = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \int d^3v \, \delta f_{\alpha} \propto e^{-\gamma t} \to 0.$$  (5.28)

The velocity integral over the fine structure increasingly cancels as time goes on—a perturbation initially “visible” as $\varphi$ phase-mixes away, disappearing into the negative entropy associated with the fine velocity dependence of $\delta f$ [see (5.15)].

More generally speaking, one can similarly argue that the refinement of velocity dependence of $\delta f$ causes lower velocity moments of $\delta f$ (density, flow velocity, pressure, heat flux, and so on) to decrease with time, transferring free energy to higher moments (ever higher as time goes on). One way to formalise this statement neatly is in terms of Hermite moments: since Hermite polynomials are orthogonal, the free energy of the perturbed distribution can be written as a sum of “energies” of the Hermite moments [see (12.36)]. It is then possible to represent the Landau-damped perturbations as having a broad spectrum in Hermite space, with the majority of the free energy residing in high-order moments—infinitely high in the formal limit of zero collisionality and infinite time (see Q7 and Kanekar et al. 2015).\(^{39}\)

Since the $m$th-order Hermite moment can, for $m \gg 1$, be asymptotically represented as a cosine function in $v$ space oscillating with the “frequency” $\sqrt{2m/v_{th}}$ [see (12.37)], (5.26) implies that the typical order of the moment in which the free energy resides grows with time as $m \sim (kv_{th}t)^2$.

Taking Hermite (or other kind of) moments of the kinetic equation is essentially the procedure for deriving “fluid” equations for the plasma—or, rather, plasma becomes a fluid if this procedure can be stopped after a few moments (e.g., in the limit of strong collisionality, this happens at the third moment; see Dellar 2015 and Parra 2019a). Since Landau damping is a long-time effect of this phase-mixing process, it cannot be captured by any fluid approximation to the kinetic system involving a truncation of the hierarchy of moment equations at some finite-order moment—it is an essentially kinetic effect “beyond all orders”.

One useful way to see this is by examining the structure of Langmuir hydrodynamics, which was\(^{39}\)

An approach involving a Fourier, rather than Hermite, transform in velocity space will be presented in §11.2.2.
the subject of Exercise 3.1. The moment hierarchy can be truncated by assuming $kv_{th}/\omega \gg 1$, but one can never capture Landau damping however many moments one keeps; indeed, the Landau damping rate (3.41) for, say, a Maxwellian plasma will be $\gamma \propto \exp(-\omega^2/k^2 v_{th}^2)$, all coefficients in the Taylor expansion of which in powers of $kv_{th}/\omega$ are zero.

5.5. Role of Collisions

As ever larger velocity-space gradients emerge, it becomes inevitable that at some point they will become so large that collisions can no longer be ignored. Indeed, the Landau collision operator is a Fokker–Planck (diffusion) operator in velocity space [see (1.47)] and so it will eventually wipe out the fine structure in $v$, however small is the collision frequency $\nu$. Let us estimate how long this takes.

The size of the velocity-space gradients of $\delta f$ due to ballistic response is given by (5.26). Then the collision term is

$$\left( \frac{\partial \delta f}{\partial t} \right)_c \sim \nu v_{th}^2 \frac{\partial^2 \delta f}{\partial v^2} \sim -\nu v_{th}^2 k^2 t^2 \delta f. \quad (5.29)$$

Solving for the time evolution of the perturbed distribution function due to collisions, we get

$$\frac{\partial \delta f}{\partial t} \sim -\nu (kv_{th}t)^2 \delta f \quad \Rightarrow \quad \delta f \sim \exp\left(-\frac{1}{3} \nu k^2 v_{th}^2 t^3\right) \equiv e^{-(t/t_c)^3}. \quad (5.30)$$

Therefore, the characteristic collisional decay time is

$$t_c \sim \frac{1}{\nu^{1/3} (kv_{th})^{2/3}}. \quad (5.31)$$

Note that $t_c \ll \nu^{-1}$ provided $\nu \ll kv_{th}$, i.e., $t_c$ is within the range of times over which our “collisionless” theory is valid. After time $t_c$, “collisionless” damping becomes irreversible because the part of $\delta f$ that is fast-varying in velocity space is lost (entropy has grown) and so it is no longer possible, even in principle, to invert all particle trajectories, have the system retrace back its steps, “phase-unmix” and thus “undamp” the damped perturbation.

In a sufficiently collisionless system, phase unmixing is, in fact, possible if nonlinearity is allowed—giving rise to the beautiful phenomenon of plasma echo, in which perturbations can first appear to be damped away but then come back from phase space (§11.1). This effect is a source of much preoccupation to pure mathematicians (Villani 2014; Bedrossian 2016): indeed the validity of the linearised Vlasov equation (3.1) as a sensible approximation to the full nonlinear one (2.12) is in question if the velocity derivative $\partial \delta f/\partial v$ in the last term of the latter starts growing uncontrollably. Phase unmixing has also recently turned out to have interesting consequences for the role of Landau damping in plasma turbulence (§11.2).
Some rather purist theoreticians sometimes choose to replace collisional estimates of the type discussed above by a stipulation that $\delta f(v)$ must be “coarse-grained” beyond some suitably chosen scale in $v$ (Fig. 22)—this is equivalent to saying that the formation of the fine-structured phase-space part of $\delta f$ constitutes a loss of information and so leads to growth of entropy (i.e., loss of negative entropy associated with $\langle \delta f^2 \rangle$). Somewhat non-rigorously, this means that we can just consider the ballistic term in (5.25) to have been wiped out and use the coarse-grained (i.e., velocity-space-averaged) version of $\delta f$:  

$$\delta f = i \frac{q}{m} \sum_i c_i e^{p_i t} \frac{k \cdot \partial f_0}{\partial v}.$$  

(5.32)

We can check that the correct solution (3.16) for the potential can be recovered from this:

$$\varphi = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int d^3 v \delta f_\alpha = \sum_i c_i e^{p_i t} \left[ \sum_\alpha \frac{4\pi q_\alpha^2}{m_\alpha k^2} \int d^3 v \frac{1}{p_i + ik \cdot v} k \cdot \frac{\partial f_0}{\partial v} - 1 + 1 \right] = \sum_i c_i e^{p_i t}. \tag{5.33}$$

If you are wondering how this works without the coarse-graining kludge, read on.

5.6. Further Analysis of $\delta f$: Case–van Kampen Mode

Having given a rather qualitative analysis of the structure and consequences of the solution (5.25), I anticipate a degree of dissatisfaction from a perceptive reader. Yes, there is a non-decaying piece of $\delta f$. But conservation of free energy in a collisionless system in the face of Landau damping in fact requires $\langle \delta f^2 \rangle$ to grow, not just to fail to decay [see (5.18)]. How do we see that this does indeed happen? The analysis that follows addresses this question. These considerations are not really necessary for most practical plasma-physics calculations (see, however, Q8), but it may be necessary for your peace of mind and greater comfort with this whole conceptual framework.

Let us rearrange the solution (5.25) as follows:

$$\delta f(t) = i \frac{q}{m} \sum_i c_i e^{p_i t} - e^{-ik \cdot vt} \frac{1}{p_i + ik \cdot v} k \cdot \frac{\partial f_0}{\partial v} + (g + \ldots) e^{-ik \cdot vt}. \tag{5.34}$$

The second term is the ballistic evolution of perturbations (particles flying apart in straight lines at different velocities)—a homogeneous solution of the kinetic equation (3.1). This develops a lot of fine-scale velocity-space structure, but obviously does not grow. The first term, a particular solution arising from the (linear) wave-particle interaction, is more interesting, especially around the resonances $Re p_i + k \cdot v = 0$.

Consider one of the modes, $p_i = -i\omega + \gamma$, and assume $\gamma \ll k \cdot v \sim \omega$. This allows us to introduce “intermediate” times:

$$\frac{1}{k \cdot v} \ll t \ll \frac{1}{\gamma}. \tag{5.35}$$

With an understanding that any integral involving the resonant denominator must be taken along the Landau contour (see Q8). If you adopt this shorthand, you can, nonrigorously but often expeditiously, use Fourier transforms into frequency space, rather than Laplace transforms.
relevant piece of the perturbed distribution (5.34),

\[ \delta f \propto \frac{e^{p_i t} - e^{-ik \cdot vt}}{p_i + ik \cdot v} = -ie^{-i\omega t}e^{\gamma t} - e^{-i(k \cdot v - \omega)t} \approx -ie^{-i\omega t}\frac{1 - e^{-i(k \cdot v - \omega)t}}{k \cdot v - \omega}, \quad (5.36) \]

with the last, approximate, expression valid at the intermediate times (5.35), assuming also that, even though we might be close to the resonance, we shall not come closer than \( \gamma \), viz., \(|k \cdot v - \omega| \gg \gamma\). Respecting this ordering, but taking \(|k \cdot v - \omega| \ll 1/t\), we find

\[ \delta f \propto te^{-i\omega t}. \quad (5.37) \]

Thus, \( \delta f \) has a peak that grows with time, emerging from the sea of fine-scale but constant-amplitude structures (Fig. 23). The width of this peak is obviously \(|k \cdot v - \omega| \sim 1/t\) and so \( \delta f \) around the resonance develops a sharp structure, which, in the formal limit \( t \to \infty \) (but respecting \( \gamma t \ll 1 \), i.e., with infinitesimal damping), tends to a delta function:

\[ \delta f \propto -ie^{-i\omega t}\frac{1 - e^{-i(k \cdot v - \omega)t}}{k \cdot v - \omega} \to e^{-i\omega t}\pi\delta(k \cdot v - \omega) \quad \text{as} \quad t \to \infty. \quad (5.38) \]

Here is a “formal” proof:

\[ \frac{1 - e^{-ixt}}{x} = \frac{1 - \cos xt}{x} + \frac{i \sin xt}{x} \approx \frac{e^{ixt} - e^{-ixt}}{2x} = \frac{i}{2} \int_{-t}^{t} dt' e^{ixt'} \to i\pi \delta(x) \quad \text{as} \quad t \to \infty. \quad (5.39) \]

The delta-function solution (5.38) is an instance of a Case–van Kampen mode (van Kampen 1955; Case 1959)—an object that belongs to the mathematical realms briefly alluded to at the end of §3.5. Note that writing the solution in the vicinity of the resonance in this form is tantamount to stipulating that any integral taken with respect to \( v \) (or \( k \)) and involving \( \delta f \) must always be done along the Landau contour, circumventing the pole from below [cf. (3.23)]. We will find the representation (5.38) of \( \delta f \) useful in working out the quasilinear theory of Landau damping in Q8.

If we restore finite damping, all this goes on until \( t \sim 1/\gamma \), with the delta function reaching the height \( \propto 1/\gamma \) and width \( \propto \gamma \). In the limit \( t \gg 1/\gamma \), the damped part of the solution decays, \( e^{\gamma t} \to 0 \), and we are left with just the ballistic part, the second term in (5.25).
5.7. Free-Energy Conservation for Landau-Damped Langmuir Waves

Finally, let us convince ourselves that, if we ignore collisions, we can recover (5.18) with a zero right-hand side from the full collisionless Landau-damped solution given by (3.16) and (5.34). For simplicity, let us consider the case of electron Langmuir waves and prove that

\[
\frac{d}{dt} \int d^3v \frac{T|\delta f_k|^2}{2f_0} = -2\gamma_k \frac{|E_k|^2}{8\pi} = -\frac{d}{dt} \frac{|E_k|^2}{8\pi}.
\]  

(5.40)

In (5.34), let the relevant root of the dispersion relation be \( p_i = -i\omega_p + \gamma_k \), where \( \gamma_k \) is given by (3.41), and assume a Maxwellian \( f_0 \). Based on the discussion \( \S 5.6 \), we should expect \( \delta f_k \) to develop a growing delta-like peak around the resonance \( k \cdot v \approx \omega_p \). In this region of velocity space, the distribution function (5.34) for electrons (\( q = -e \)) is

\[
\delta f_k^{(res)} \approx \frac{e}{m_e c_i} \frac{1 - e^{-i(k \cdot v - ip_i)t}}{k \cdot v - ip_i} \frac{2k \cdot v}{v_{th}^2} f_0 \approx \frac{e\varphi_k}{T} \frac{k \cdot v}{k \cdot v - \omega_p} f_0 \approx \frac{i\pi\delta(k \cdot v - \omega_p)}{f_0}.
\]

(5.41)

We are going to have to compute \( |\delta f_k|^2 \) and squaring delta functions is a dangerous game belonging to the class of games that one must play very carefully.\(^{41}\) Here is how:

\[
\frac{\partial}{\partial t} \left| \frac{1 - e^{-ixt}}{x} \right|^2 = \frac{\partial}{\partial t} \left| \frac{4}{x^2} \sin^2 \frac{x}{2} \right| = \left. \frac{2\sin \frac{x}{t} - 2\pi\delta(x) }{\left. \frac{x}{t} \rightarrow \infty \right.} \right| \left. \frac{1 - e^{-ixt}}{x} \right|^2 \rightarrow 2\pi t\delta(x).
\]

(5.42)

Using this prescription,

\[
\int d^3v \frac{T|\delta f_k^{(res)}|^2}{2f_0} = \int d^3v \frac{e^2|\varphi_k|^2}{2T} (k \cdot v)^2 2\pi t\delta(k \cdot v - \omega_p) f_0
\]

\[
= t|\varphi_k|^2 \frac{2\pi e^2\omega_p^2}{m_e v_{th}^2 k} F\left( \frac{\omega_p}{k} \right)
\]

\[
= 4t \frac{k^2|\varphi_k|^2}{8\pi} \frac{\omega_p^4}{k^3 n_e v_{th}^2} F\left( \frac{\omega_p}{k} \right)
\]

\[
= -4\gamma_k \frac{|E_k|^2}{8\pi}.
\]

(5.43)

Thus, the entropic part of the free energy grows secularly with time (assuming still \( \gamma_k t \ll 1 \)). Its time derivative is

\[
\frac{d}{dt} \int d^3v \frac{T|\delta f_k^{(res)}|^2}{2f_0} \approx -4\gamma_k \frac{|E_k|^2}{8\pi} = -\frac{d}{dt} \frac{|E_k|^2}{8\pi}.
\]

(5.44)

Despite what it looks like, the extra factor of 2 in (5.44) compared to (5.40) is a feature, not a bug. If you have done Exercise 3.1 (or even just paid attention in \( \S 2.1 \)), you know that a Langmuir oscillations involve some mean (oscillating) flows of the plasma, and so a sloshing of energy between potential, \( |E_k|^2/8\pi \), and kinetic, \( n_e m_e |u_k|^2/2 \), where \( u_k = (1/n_e) \int d^3v \frac{v|\delta f_k|}{f_0} \). These flows are contained in the non-resonant (“thermal”) part of \( \delta f_k \), i.e., in \( \delta f_k \) at velocities such that \( k \cdot v \ll \omega_p \). In this region of velocity space, let us write the distribution function (5.34) as follows:

\[
\delta f_k^{(th)} = \frac{e}{m_e c_i} \frac{1}{k \cdot v - ip_i} \frac{2k \cdot v}{v_{th}^2} f_0 + e^{-i(k \cdot v)t} h_k \approx \frac{e\varphi_k}{T} \frac{k \cdot v}{\omega_p} f_0 + e^{-i(k \cdot v)t} h_k,
\]

(5.45)

where \( h_k \) denotes everything in (5.34) that multiplies \( e^{-i(k \cdot v)t} \). This should remind you of

\(^{41}\)I am grateful to Glenn Wagner for making me practice what I preach and work out this derivation correctly, with all the meaningful factors of 2.
Exercise 5.2. The first term in (5.45) is precisely the plasma flow:

\[ u_k = \frac{1}{n_e} \int d^3 v \mathbf{v} \delta f_k^{(th)} = -\frac{e\varphi_k}{T\omega_p n_e} \mathbf{k} \cdot \int d^3 v \mathbf{v} f_0 = \frac{eE_k}{i \omega_p m_e} \]  

(5.46)

The contribution from the second term in (5.45) to the velocity integral has vanished in the limit \( \mathbf{k} \cdot \mathbf{v t} \gg 1 \). Note that (5.46) just says that \( m_e \dot{u_k} = -eE_k \), as indeed is the case in a Langmuir oscillation. It is not hard to check that \( n_e m_e |u_k|^2 / 2 = |E_k|^2 / 8\pi \).

Let us now work out the contribution of (5.45) to the free energy [cf. (5.23)]:

\[ \int d^3 v \frac{T|\delta f_k^{(th)}|^2}{2f_0} = \int d^3 v \left[ \frac{e^2 |\varphi_k|^2}{2T\omega_p^2} (\mathbf{k} \cdot \mathbf{v})^2 f_0 + \frac{T|h_k|^2}{2f_0} + e^{-i\mathbf{k} \cdot \mathbf{v} t} (\ldots) + e^{i\mathbf{k} \cdot \mathbf{v} t} (\ldots)^* \right] \]

\[ = \frac{|E_k|^2}{8\pi} + \int d^3 v \frac{T|h_k|^2}{2f_0}, \]  

(5.47)

where the velocity integral has been done in the same way as in (5.46) and the contribution from the terms that oscillate in \( \mathbf{v} \) has been integrated away. The salient property of \( h_k \) is that it does not depend on time. Therefore, its contribution to the time derivative of the free energy vanishes and we get

\[ \frac{d}{dt} \int d^3 v \frac{T|\delta f_k^{(th)}|^2}{2f_0} = \frac{d}{dt} \frac{|E_k|^2}{8\pi}, \]

(5.48)

i.e., the kinetic energy of the Langmuir oscillations decays at the same rate as their potential (electric) energy.

Finally, adding (5.44) and (5.48), we get (5.40), q.e.d.

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Exercise 5.3. Free-energy conservation for sound waves. Consider an ion-acoustic wave (§3.8) damped on electrons according to (3.75) (with \( u_e = 0 \)). Work out the contributions to free energy from ions and from electrons. Check that free energy is conserved.

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6. Quasilinear Theory

6.1. General Scheme of QLT

In §§3 and 5, I discussed at length the structure of the linear solution corresponding to a Landau-damped initial perturbation. This could be adequately done for a Maxwellian plasma distribution function, and the result was that, after some interesting transient time-dependent phase-space dynamics, perturbations damped away and their energy turned into heat, increasing somewhat the temperature of the equilibrium (see, however, Q8).

Let us now turn to a different problem: an unstable (and so decidedly non-Maxwellian) equilibrium distribution giving rise to exponentially growing perturbations. The specific example on which we shall focus is the bump-on-tail instability, which involves generation of unstable Langmuir waves with phase velocities corresponding to instances of positive derivative of the equilibrium distribution function (Fig. 24). The energy of the waves grows exponentially:

\[ \frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2, \quad \gamma_k = \frac{\pi \omega_p^3}{2k^2} \frac{1}{n_e} F'(\frac{\omega_p k}{k}) \]  

(6.1)

where \( F(v_z) = \int dv_x \int dv_y f_0(v) \) [see (3.41)]. In the absence of collisions, the only way for the system to achieve a nontrivial steady state (i.e., such that \( |E_k|^2 \) is not just zero everywhere) is by adjusting the equilibrium distribution so that

\[ \gamma_k = 0 \iff F'(\frac{\omega_p k}{k}) = 0 \]  

(6.2)
at all \( k \) where \( |E_k|^2 \neq 0 \), say, \( k \in [k_2,k_1] \). If we translate this range into velocities, \( v = \omega_{pe}/k \), we see that the equilibrium must develop a flat spot:

\[
F'(v) = 0 \quad \text{for} \quad v \in [v_1,v_2] = \left[ \frac{\omega_{pe}}{k_1}, \frac{\omega_{pe}}{k_2} \right].
\]

(6.3)

This is called a quasilinear plateau (§6.4). Obviously, the rest of the equilibrium distribution may (and will) also be modified in some, to be determined, way (§§6.6, 6.7).

These modifications of the original (initial) equilibrium distribution can be accomplished by the growing fluctuations via the feedback mechanism already discussed in §2.3, namely, the equilibrium distribution will evolve slowly according to (2.11):

\[
\frac{\partial f_0}{\partial t} = -q \sum_k \left< \varphi_k i k \cdot \partial f_k / \partial v \right>.
\]

(6.4)

The time averaging here [see (2.7)] is over \( \omega_{pe}^{-1} \ll \Delta t \ll \gamma_k^{-1} \).

The general scheme of QLT is:

- start with an unstable equilibrium \( f_0 \),
- use the linearised equations (3.1) and (3.2) to work out the linear solution for the growing perturbations \( \varphi_k \) and \( \delta f_k \) in terms of \( f_0 \),
- use this solution in (6.4) to evolve \( f_0 \), leading, if everything works as it is supposed to, to an ever less unstable equilibrium.

We shall keep only the fastest growing mode (all others are exponentially small after a while), and so the solution (3.16) for the electric perturbations is

\[
\varphi_k = c_k e^{(-i\omega_k + \gamma_k) t}.
\]

(6.5)

In the solution (5.25) for the perturbed distribution function, we may ignore the ballistic term because the exponentially growing piece (the first term) will eventually leave all this velocity-space structure behind,\(^{42}\) so

\[
\delta f_k = i q c_k e^{(-i\omega_k + \gamma_k) t} k \cdot \frac{\partial f_0}{\partial v} - q c_k \left[ \frac{\varphi_k k}{m k \cdot v - \omega_k - i\gamma_k} \right] k \cdot \frac{\partial f_0}{\partial v}.
\]

(6.6)

\(^{42}\)See, however, Q8 on how to avoid having to wait for this to happen: in fact, the results below are valid for \( \gamma_k t \lesssim 1 \) as well.
Substituting (6.6) into (6.4), we get

\[
\frac{\partial f_0}{\partial t} = -\frac{q^2}{m^2} \sum_k |\varphi_k|^2 i k \cdot \frac{1}{k \cdot v - \omega_k - i \gamma_k} k \cdot \frac{\partial f_0}{\partial v} = \frac{\partial}{\partial v} \cdot D(v) \cdot \frac{\partial f_0}{\partial v}.
\] (6.7)

This is a diffusion equation in velocity space, with a velocity-dependent diffusion matrix

\[
D(v) = -\frac{q^2}{m^2} \sum_k i k k |\varphi_k|^2 \frac{1}{k \cdot v - \omega_k - i \gamma_k} k \cdot v - \omega_k - i \gamma_k
\]

\[
= -\frac{q^2}{m^2} \sum_k i k k |\varphi_k|^2 \left(\frac{1}{2} \left(\frac{1}{k \cdot v - \omega_k - i \gamma_k} + \frac{1}{-k \cdot v - \omega_k - i \gamma_k}\right)\right)
\]

\[
= -\frac{q^2}{m^2} \sum_k k k |E_k|^2 \frac{1}{2} \left(\frac{1}{k \cdot v - \omega_k - i \gamma_k} - \frac{1}{-k \cdot v - \omega_k + i \gamma_k}\right)
\]

\[
= \frac{q^2}{m^2} \sum_k \frac{k k}{k^2} |E_k|^2 \frac{1}{2} \left(\frac{1}{k \cdot v - \omega_k - i \gamma_k} - \frac{1}{-k \cdot v - \omega_k + i \gamma_k}\right)
\]

\[
= \frac{q^2}{m^2} \sum_k \frac{k k}{k^2} |E_k|^2 \frac{1}{(k \cdot v - \omega_k)^2 + \gamma_k^2}.
\] (6.8)

To obtain these expressions, I used the fact that the wave-number sum could just as well be over \(-k\) instead of \(k\) and that \(\omega_{-k} = -\omega_k, \gamma_{-k} = \gamma_k\) [because \(\varphi_{-k} = \varphi_k^*,\) where \(\varphi_k\) is given by (6.5)]. The matrix \(D\) is manifestly positive definite—this adds credence to our a priori expectation that a plateau will form: diffusion will smooth the bump in the equilibrium distribution function.

The question of validity of the QL approximation is quite nontrivial and rife with subtle issues, all of which I have swept under the carpet. They mostly have to do with whether coupling between waves [the last term in (2.12)] truly remains unimportant throughout the quasilinear evolution, especially as the plateau regime is approached and the growth rate of the waves becomes infinitesimally small. If you wish to investigate further—and in the process gain a finer appreciation of nonlinear plasma theory,—the article by Besse et al. (2011) (as far as I know, the most recent substantial contribution to the topic) is a good starting point, from which you can follow the paper trail backwards in time and decide for yourself whether you trust the QLT.

### 6.2. Conservation Laws

When we get to the stage of solving a specific problem (§6.3), we shall see that paying attention to energy and momentum budgets leads one to important discoveries about the QL evolution of the particle distribution. With this prospect in mind, as well as by way of a consistency check, let us check that the quasilinear kinetic equation (6.7) conserves energy and momentum.
6.2.1. Energy Conservation

The rate of change of the particle energy associated with the equilibrium distribution is

\[
\frac{dK}{dt} = \frac{d}{dt} \sum_\alpha \int d^3 r d^3 v \frac{m_\alpha v^2}{2} f_{\alpha 0} = \sum_\alpha \int d^3 r d^3 v \frac{m_\alpha v^2}{2} \frac{\partial}{\partial v} D_\alpha(v) \cdot \frac{\partial f_{\alpha 0}}{\partial v}
\]

\[
= - \sum_\alpha \int d^3 r d^3 v m_\alpha v \cdot D_\alpha(v) \cdot \frac{\partial f_{\alpha 0}}{\partial v}
\]

\[
= - V \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \sum_k \frac{|E_k|^2}{k^2} \int d^3 v \text{Im} \left[ \frac{k \cdot v}{k \cdot v - \omega_k - i\gamma_k} k \cdot \frac{\partial f_{\alpha 0}}{\partial v} \right]
\]

add and subtract \( \omega_k + i\gamma_k \) in the numerator

\[
= 1 - \epsilon (-i\omega_k + i\gamma_k) = 1
\]

because \(-i\omega_k + i\gamma_k \) is a solution of dispersion relation \( \epsilon = 0 \)

\[
= - V \sum_k 2\gamma_k \frac{|E_k|^2}{8\pi} = - \frac{d}{dt} \int d^3 r \frac{E^2}{8\pi}, \text{ q.e.d.}, \quad (6.9)
\]

viz., the total energy \( K + \int d^3 r E^2/8\pi = \text{const} \). This will motivate §6.6.

6.2.2. Momentum Conservation

Since unstable distributions like the one with a bump on its tail can carry net momentum, it is useful to calculate its rate of change:

\[
\frac{dP}{dt} = \frac{d}{dt} \sum_\alpha \int d^3 r d^3 v m_\alpha v f_{\alpha 0} = \sum_\alpha \int d^3 r d^3 v m_\alpha v \frac{\partial}{\partial v} D_\alpha(v) \cdot \frac{\partial f_{\alpha 0}}{\partial v}
\]

\[
= - \sum_\alpha \int d^3 r d^3 v m_\alpha D_\alpha(v) \cdot \frac{\partial f_{\alpha 0}}{\partial v}
\]

\[
= - V \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \sum_k \frac{k |E_k|^2}{k^2} \int d^3 v \text{Im} \left[ \frac{1}{k \cdot v - \omega_k - i\gamma_k} k \cdot \frac{\partial f_{\alpha 0}}{\partial v} \right]
\]

\[
= - V \sum_k \frac{k |E_k|^2}{4\pi} \text{Im} \sum_\alpha \frac{\omega_\alpha^2}{k^2} \frac{1}{n_\alpha} \int d^3 v \frac{1}{k \cdot v - \omega_k - i\gamma_k} k \cdot \frac{\partial f_{\alpha 0}}{\partial v} = 0, \text{ q.e.d.}, \quad (6.10)
\]

so momentum can only be redistributed between particles. This will motivate §6.7.

6.3. Quasilinear Equations for the Bump-on-Tail Instability in 1D

What follows is the iconic QL calculation due to Vedenov et al. (1962) and Drummond & Pines (1962).

These two papers, published in the same year, are a spectacular example of the “great minds think alike” principle. They both appeared in the Proceedings of the 1961 IAEA conference in Salzburg, one of those early international gatherings in which the Soviets (grudgingly allowed out) and the Westerners (eager to meet them) were telling each other about their achievements in the recently declassified controlled-nuclear-fusion research. The entire Proceedings are now online (http://www-naweb.iaea.org/napc/physics/FEC/1961.pdf)—a remarkable historical document and a great read, containing, besides the papers (in three languages), a record of the discussions that were held. The Vedenov et al. (1962) paper is in Russian, but you will
find a very similar exposition in English in the review by Vedenov (1963) published shortly thereafter. Two other lucid accounts of quasilinear theory belonging to the same historical (and historic!) period are in the books by Kadomtsev (1965) and by Sagdeev & Galeev (1969).

As promised in §6.1, I shall consider electron Langmuir oscillations in 1D, triggered by the bump-on-tail instability, so \( k = k\hat{z}, \omega_k = \omega_{pe}, \gamma_k \) is given by (6.1), and the QL diffusion equation (6.7) becomes

\[
\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v},
\]

where \( F(v) \) is the 1D version of the distribution function, \( v = v_z \) and the diffusion coefficient, now a scalar, is given by

\[
D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \text{Im} \left( \frac{1}{kv - \omega_{pe} - i\gamma_k} \right).\]

(6.12)

As I explained when discussing (6.1), if the fluctuation field has reached a steady state, it must be the case that

\[
\frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2 = 0 \iff |E_k|^2 = 0 \quad \text{or} \quad \gamma_k = 0,
\]

(6.13)
i.e., either there are no fluctuations or there is no growth (or damping) rate. The result is a non-zero spectrum of fluctuations in the interval \( k \in [k_2, k_1] \) and a plateau in the distribution function in the corresponding velocity interval \( v \in [v_1, v_2] = [\omega_{pe}/k_1, \omega_{pe}/k_2] \) [see (6.3) and Fig. 25]. The particles in this interval are resonant with Langmuir waves; those in the ("thermal") bulk of the distribution outside this interval are non-resonant.

We will have solved the problem completely if we find

- \( F_{\text{plateau}} \), the value of the distribution function in the interval \([v_1, v_2]\),
- the extent of the plateau \([v_1, v_2]\),
- the functional form of the spectrum \( |E_k|^2 \) in the interval \([k_2, k_1]\),
- any modifications of the distribution function \( F(v) \) of the nonresonant particles.

### 6.4. Resonant Region: QL Plateau and Spectrum

Consider first the velocities \( v \in [v_1, v_2] \) for which \( |E_k = \omega_{pe}/v|^2 \neq 0 \). If \( L \) is the linear size of the system, the wave-number sum in (6.12) can be replaced by an integral according to

\[
\sum_{k} \Delta k = \sum_{k} \frac{\Delta k}{2\pi/L} = \frac{L}{2\pi} \int dk.
\]

(6.14)
Defining the continuous energy spectrum of the Langmuir waves

\[ W(k) = \frac{L}{2\pi} \frac{|E_k|^2}{4\pi}, \tag{6.15} \]

we rewrite the QL diffusion coefficient (6.12) in the following form:

\[ D(v) = \frac{e^2}{m_e^2 v} \cdot \text{Im} \int \frac{4\pi W(k)}{k - \omega_{pe}/v - i\gamma_k/v} = \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right). \tag{6.16} \]

The last expression is obtained by applying Plemelj's formula (3.25) to the wave-number integral taken in the limit \( \gamma_k/v \to +0 \).

Substituting now this expression into (6.11) and using also (6.1) to express \( \gamma_k = \frac{\pi}{2} \omega_{pe}^3 \frac{1}{n_e} F'\left(\frac{\omega_{pe}}{v}\right) \Rightarrow \frac{\partial F}{\partial v} = \left[ \frac{2}{\pi} \frac{k^2}{\omega_{pe}^3} \frac{n_e}{v} \gamma_k \right]_{k = \omega_{pe}/v}, \tag{6.17} \]

we get

\[ \frac{\partial F}{\partial t} = \frac{\partial}{\partial v} \left[ \frac{\omega_{pe}}{v} W\left(\frac{\omega_{pe}}{v}\right) \right] = \frac{\omega_{pe}}{m_e v^3} 2 \frac{\gamma_{pe}}{v} W\left(\frac{\omega_{pe}}{v}\right). \tag{6.18} \]

Rearranging, we arrive at

\[ \frac{\partial}{\partial t} \left[ F - \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \right] = 0. \tag{6.19} \]

Thus, during QL evolution, the expression in the square brackets stays constant in time. Since at \( t = 0 \), there are no waves, \( W = 0 \), we find

\[ F(0, v) + \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) = F(t, v) \rightarrow F_{\text{plateau}} \text{ as } t \to \infty. \tag{6.20} \]

In the saturated state \( (t \to \infty) \), \( W(\omega_{pe}/v) = 0 \) outside the interval \( v \in [v_1, v_2] \).

Therefore, (6.20) gives us two implicit equations for \( v_1 \) and \( v_2 \):

\[ F(0, v_1) = F(0, v_2) = F_{\text{plateau}} \tag{6.21} \]

and, after integration over velocities, also an equation for \( F_{\text{plateau}} \):

\[ \int_{v_1}^{v_2} dv \left[ F_{\text{plateau}} - F(0, v) \right] = 0 \Rightarrow F_{\text{plateau}} = \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} dv F(0, v). \tag{6.22} \]

Finally, integrating (6.20) with respect to \( v \) and using the boundary condition \( W(\omega_{pe}/v_1) = 0 \), we get, at \( t \to \infty \),

\[ W\left(\frac{\omega_{pe}}{v}\right) = \frac{m_e v^3}{\omega_{pe}} \int_{v_1}^{v} dv' \left[ F_{\text{plateau}} - F(0, v') \right]. \tag{6.23} \]

\[ ^{43} \text{Why the prefactor is } 1/4\pi, \text{ rather than } 1/8\pi, \text{ will become clear at the end of §6.5.} \]

\[ ^{44} \text{In fact, the wave-number integral must be taken along the Landau contour (i.e., keeping the contour below the pole) regardless of the sign of } \gamma_k; \text{ see Q8, where you get to work out the QL theory for Landau-damped, rather than growing, perturbations.} \]

\[ ^{45} \text{This is somewhat reminiscent of the “Maxwell construction” in thermodynamics of real gases: the plateau sits at such a level that the integral under it, i.e., the number of particles involved, stays the same as it was for the same velocities in the initial state; see Fig. 24.} \]
Hence the spectrum is

\[ W(k) = \frac{m_e \omega_{pe}^2}{k^3} \int_{v_1}^{\omega_{pe}/k} dv \left[ F^{\text{plateau}} - F(0, v) \right] \text{ for } k \in \left[ \frac{\omega_{pe}}{v_2}, \frac{\omega_{pe}}{v_1} \right] \]  

(6.24)

and \( W(k) = 0 \) everywhere else (Fig. 26).

Thus, we have completed the first three items of the programme formulated at the end of §6.3. What about the particle distribution outside the resonant region? How is it modified by the quasilinear evolution? Is it modified at all? The following calculation shows that it must be.

### 6.5. Energy of Resonant Particles

Since feeding the instability requires extracting energy from the resonant particles, their energy must change. We calculate this change by taking the \( m_e v^2/2 \) moment of (6.20):

\[
K_{\text{res}}(\infty) - K_{\text{res}}(0) = \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} \left[ F^{\text{plateau}} - F(0, v) \right] \\
= \int_{v_1}^{v_2} dv \frac{m_e v^2}{2} \frac{\partial}{\partial v} \frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right) \\
= -\omega_{pe} \int_{v_1}^{v_2} dv \frac{1}{v^2} W\left(\frac{\omega_{pe}}{v}\right) \\
= -\int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk W(k) = -2 \sum_k \frac{|E_k|^2}{8\pi} \equiv -2 \mathcal{E}(\infty). 
\]  

(6.25)

Thus, only half of the energy lost by the resonant particles goes into the electric-field energy of the waves,

\[
\mathcal{E}(\infty) = \frac{K_{\text{res}}(0) - K_{\text{res}}(\infty)}{2}.
\]  

(6.26)

Since the energy must be conserved overall [see (6.9)], we must account for the missing half: this is easy to do physically, as, obviously, the electric energy of the waves is their potential energy, which is half of their total energy—the other half being the kinetic energy of the oscillatory plasma motions associated with the wave (in §5.7, this was worked out explicitly). These oscillations are enabled by the non-resonant, “thermal-bulk” particles, and so we must be able to show that, as a result of QL evolution, these particles pick up the total of \( \mathcal{E}(\infty) \) of energy—one might say that the plasma is heated.
6.6. Heating of Non-Resonant Particles

Consider the thermal bulk of the distribution, \( v \ll v_1 \) (assuming that the bump is indeed far out in the tail of the distribution). The QL diffusion coefficient (6.12) becomes, assuming now \( \gamma_k, kv \ll \omega_{pe} \) and using the last expression in Eq. (6.8),

\[
D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \left( \frac{\gamma_k}{(kv - \omega_{pe})^2 + \gamma_k^2} \right) \approx \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2},
\]

independent of \( v \). The QL evolution equation (6.11) for the bulk distribution is then

\[
\frac{\partial F}{\partial t} = \frac{1}{m_e n_e} \frac{d\epsilon}{dt} \frac{\partial^2 F}{\partial v^2}. \tag{6.28}
\]

Equation (6.28) describes slow diffusion of the bulk distribution, i.e., as the wave field grows, the bulk distribution gets a little broader (which is what heating is). Namely, the “thermal” energy satisfies

\[
\frac{dK_{th}}{dt} = \frac{1}{m_e n_e} \frac{d\epsilon}{dt} \frac{\partial^2 F}{\partial v^2}
\]

Integrating this with respect to time, we find that the missing half of the energy lost by the resonant particles indeed goes into the thermal bulk:

\[
K_{th}(\infty) - K_{th}(0) = \epsilon(\infty) = \frac{K_{res}(0) - K_{res}(\infty)}{2}. \tag{6.30}
\]

Overall, the energy is, of course, conserved:

\[
K_{th}(\infty) + K_{res}(\infty) + \epsilon(\infty) = K_{th}(0) + K_{res}(0), \tag{6.31}
\]
as it should be, in accordance with (6.9).

Equation (6.28) can be explicitly solved: changing the time variable to \( \tau = \epsilon(t)/m_e n_e \) turns it into a simple diffusion equation

\[
\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial v^2}. \tag{6.32}
\]

If we let the initial distribution be a Maxwellian and ignore the bump on its tail, the solution is

\[
F(\tau, v) = \int dv' F(0, v') \frac{e^{-(v-v')^2/4\tau}}{\sqrt{4\pi \tau}} = \int dv' \frac{n_e}{\sqrt{\pi v_{the}^2}} \exp \left\{ - \frac{v^2}{v_{the}^2} - \frac{(v-v')^2}{4\tau} \right\}
\]

\[
= \frac{n_e}{\sqrt{\pi (v_{the}^2 + 4\tau)}} \exp \left[ - \frac{v^2}{v_{the}^2 + 4\tau} \right]. \tag{6.33}
\]

Since

\[
v_{the}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{4\epsilon(t)}{m_e n_e} = \frac{2}{m_e} \left[ T_e + \frac{2\epsilon(t)}{n_e} \right], \tag{6.34}
\]
one concludes that an initially Maxwellian bulk stays Maxwellian but its temperature grows as

\[\text{Note that this implies } d\int dv F(v)/dt = 0, \text{ so the number of these particles is conserved, there is no exchange between the non-resonant and resonant populations.}\]
the wave energy grows, reaching in saturation

\[ T_e(\infty) = T_e(0) + \frac{2\bar{E}(\infty)}{n_e}. \]  

(6.35)

6.7. Momentum Conservation

The bump-on-tail configuration is in general asymmetric in \( v \) and so the particles in the bump carry a net mean momentum. Let us find out whether this momentum changes. Taking the \( m_e v \) moment of (6.20), we calculate the total momentum lost by the resonant particles:

\[
P_{\text{res}}(\infty) - P_{\text{res}}(0) = \int_{v_1}^{v_2} dv \, m_e v \left[F_{\text{plateau}} - F(0, v)\right]
= \int_{v_1}^{v_2} dv \, m_e v \frac{\partial}{\partial v} \frac{\omega_{pe}}{v} W\left(\frac{\omega_{pe}}{v}\right)
= -\omega_{pe} \int_{v_1}^{v_2} dv \frac{1}{v^3} W\left(\frac{\omega_{pe}}{v}\right)
= -\int_{\omega_{pe}/v_1}^{\omega_{pe}/v_2} dk \frac{kW(k)}{\omega_{pe}} < 0.
\]

This is negative, so momentum is indeed lost. Since it cannot go into electric field [see (6.10)], it must all get transferred to the thermal particles. Let us confirm this.

Going back to the QL diffusion equation (6.28) for the non-resonant particles, at first glance, we have a problem: the diffusion coefficient is independent of \( v \) and so momentum is conserved. However, one should never take zero for an answer when dealing with asymptotic expansions—indeed, it turns out here that we ought to work to higher order in our calculation of \( D(v) \). Keeping next-order terms in (6.27), we get

\[
D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{pe})^2 + \gamma_k^2} = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2} \left(1 + \frac{2kv}{\omega_{pe}} + \ldots\right)
\approx \frac{4\pi e^2}{m_e^2 \omega_{pe}^2} \frac{d}{dt} \left[ \sum_k \frac{|E_k|^2}{8\pi} + v \sum_k \frac{k|E_k|^2}{4\pi \omega_{pe}} \right] = \frac{1}{m_e n_e} \frac{d}{dt} \left[ \bar{E} + v \int dk \frac{kW(k)}{\omega_{pe}} \right].
\]

(6.37)

Thus, there is a wave-induced drag term in the QL diffusion equation (6.11), which indeed turns out to impart to the thermal particles the small additional momentum
that, according to (6.36), the resonant particles lose when rearranging themselves to produce the QL plateau:

\[
\frac{dP_{th}}{dt} = \frac{d}{dt} \int dv \, m_e v F = \int dv \, m_e v \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v} = -m_e \int dv \, D(v) \frac{\partial F}{\partial v}
\]

whence, integrating and comparing with (6.36),

\[
P_{th}(\infty) - P_{th}(0) = \int dk \, kW(k) = P_{res}(0) - P_{res}(\infty) \quad (6.39)
\]

This means that the thermal bulk of the final distribution is not only slightly broader (hotter) than that of the initial one (§6.6), but it is also slightly shifted towards the plateau (Fig. 27).

In a collisionless plasma, this is the steady state. However, as this steady state is approached, \( \gamma_k \to 0 \), so the QL evolution becomes ever slower and even a very small collision frequency can become important. Eventually, collisions will erode the plateau and return the plasma to a global Maxwellian equilibrium—which is the fate of all things.

In what remains of these kinetic-theory lectures, I will try to make some inroads into nonlinear theory beyond QLT—this means not (entirely) neglecting the last term in (2.12), which is responsible for the nonlinear interactions between the perturbed electric field \( \varphi_{k'} \) and the perturbed distribution function \( \delta f_{k-k'} \). Nonlinear theory of anything is, of course, hard—indeed, in most cases, intractable. These days, an impatient researcher’s answer to being faced with a hard question is to outsource it to a computer. This sometimes leads to spectacular successes, but also, somewhat more frequently, to spectacular confusion about how to interpret the output. In dealing with a steady stream of data produced by ever more powerful machines, one is often helped by the residual memory of analytical results obtained in the prehistoric era when computation was harder than theory and plasma physicists had to find ingenious ways to solve nonlinear problems “by hand”—which usually required finding ingenious ways of posing problems that were solvable. These could be separated into three broad categories: interesting particular cases of nonlinear behaviour involving usually just a few interacting waves (two very pretty examples of this are §§11.1 and 12.1), looking at systems of very many waves amenable to some approximate statistical treatment (§§7, 8, and 11.2), and asking for general criteria of certain kinds of behaviour, such as stability or otherwise (§9).

7. Quasiparticle Kinetics

7.1. QLT in the Language of Quasiparticles

First I would like to outline a neat way of reformulating the QL theory, which both sheds some light on the meaning of what was done in §6 and opens up promising avenues for theorising further about nonlinear plasma states.

Let us re-imagine our system of particles and waves as a mixture of two interacting gases: “true” particles (electrons) and quasiparticles, or plasmons, which will be the “quantised” version of Langmuir waves. If each of these plasmons has momentum \( \hbar k \)
and energy $\hbar \omega_k$, we can declare

$$N_k = \frac{V|E_k|^2}{\hbar \omega_k}$$  \hspace{1cm} (7.1)$$

to be the mean occupation number of plasmons with wave number $k$ in a box of volume $V$. The total energy of these plasmons is then

$$\sum_k \hbar \omega_k N_k = V \sum_k \frac{|E_k|^2}{4\pi},$$ \hspace{1cm} (7.2)$$
twice the total electric energy in the system (twice because it includes the energy of the mean oscillatory motion of electrons within a wave; see discussion at the end of §6.5 and/or in §5.7). Similarly, the total momentum of the plasmons is

$$\sum_k \hbar k N_k = V \sum_k \frac{k|E_k|^2}{4\pi \omega_k}.$$ \hspace{1cm} (7.3)$$

This is indeed in line with our previous calculations [see (6.39)]. Note that the role of $\hbar$ here is simply to define a splitting of wave energy into individual plasmons—this can be done in an arbitrary way, provided $\hbar$ is small enough to ensure $N_k \gg 1$. Since there is nothing quantum-mechanical about our system, all our results will in the end have to be independent of $\hbar$, so we will use $\hbar$ as an arbitrarily small parameter, in which it will be convenient to expand, while expecting it eventually to cancel out in all physically meaningful relationships.

We may now think of the QL evolution (or indeed generally of the nonlinear evolution) of our plasma in terms of interactions between plasmons and electrons. These are resonant electrons; the thermal bulk only participates via its supporting role of enabling oscillatory plasma motions associated with plasmons. The electrons are described by their distribution function $f_0(v)$, which we can, to make our formalism nicely uniform, recast in terms of occupation numbers: if the wave number corresponding to velocity $v$ is $p = m_e v / \hbar$, then its occupation number is

$$n_p = \left(\frac{2\pi \hbar}{m_e}\right)^3 f_0(v) \Rightarrow \sum_p n_p = \frac{V}{(2\pi)^3} \int d^3p n_p = V \int d^3v f_0(v) = V n_e. \hspace{1cm} (7.4)$$

It is understood that $n_p \ll 1$ (our electron gas is non-degenerate).

The QL evolution of the plasmon and electron distributions is controlled by two processes: absorption or emission of a plasmon by an electron (known as Cherenkov absorption/emission). Diagrammatically, these can be depicted as shown in Fig. 28. As we know from §6.2, they are subject to momentum conservation, $p = k + (p - k)$, and energy conservation:

$$0 = \varepsilon^e_p - \varepsilon^l_k - \varepsilon^e_{p - k} = \frac{\hbar^2 p^2}{2m_e} - \hbar \omega_k - \frac{\hbar^2 |p - k|^2}{2m_e} = \hbar \left(-\omega_k + \frac{\hbar p \cdot k}{m_e} - \frac{\hbar k^2}{2m_e}\right) = \hbar (k \cdot v - \omega_k) + O(\hbar^2). \hspace{1cm} (7.5)$$

This is the familiar resonance condition $k \cdot v - \omega_k = 0$. The superscripts $e$ and $l$ stand for electrons and (Langmuir) plasmons.
7.1.1. Plasmon Distribution

We may now write an equation for the evolution of the plasmon occupation number:

$$\frac{\partial N_k}{\partial t} = \sum_p \left[ -w(p - k, k \rightarrow p) \delta(\varepsilon_{p-k}^e + \varepsilon_{k}^l - \varepsilon_{p}^e) n_{p-k} N_k \right]$$

Fig. 28(a)

$$+ w(p \rightarrow k, p - k) \delta(\varepsilon_{p}^e - \varepsilon_{k}^l - \varepsilon_{p-k}^e) n_p (N_k + 1),$$

Fig. 28(b)

(7.6)

where \( w \) are the probabilities of absorption and emission and must be equal:

$$w(p - k, k \rightarrow p) = w(p \rightarrow k, p - k) \equiv w(p, k).$$

(7.7)

The first term in the right-hand side of (7.6) describes the absorption of one of (indistinguishable) \( N_k \) plasmons by one of \( n_p \) electrons, the second term describes the emission by one of \( n_p \) electrons of one of \( N_k + 1 \) plasmons. The +1 is, of course, a small correction to \( N_k \gg 1 \) and can be neglected, although sometimes, in analogous but more complicated calculations, it has to be kept because lowest-order terms cancel (see, e.g., §§7.2.1 and 7.2.2). Using (7.7), (7.5) and (7.4), we find

$$\frac{\partial N_k}{\partial t} \approx \sum_p w(p, k) \delta(\varepsilon_{p}^e - \varepsilon_{k}^l - \varepsilon_{p-k}^e) (n_p - n_{p-k}) N_k$$

$$= V \int \delta^3 v w\left(\frac{m_e v}{\hbar}, k\right) \delta(\omega_k - \omega_v) \left[ f_0(v) - f_0\left(v - \frac{\hbar k}{m_e}\right) \right] N_k$$

$$\approx V \int \delta^3 v w\left(\frac{m_e v}{\hbar}, k\right) \frac{1}{\hbar} \delta(\omega_k - \omega_v) \frac{\hbar k}{m_e} \cdot \frac{\partial f_0}{\partial v} N_k$$

$$= \frac{V}{m_e} w\left(\frac{m_e \omega_{pe}}{\hbar k}, k\right) F'(\frac{\omega_{pe}}{k}) N_k \equiv 2\gamma_k N_k$$

(7.8)

Note that, as expected, \( \hbar \) has disappeared from our equations, after having been used as an expansion parameter.

Since \( N_k \propto |E_k|^2 \) [see (7.1)], the prefactor in (7.8) is clearly just the (twice) growth or damping rate of the waves. Comparing with (6.1), we read off the expression for the

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47Technically speaking, the first term in this equation should be multiplied by \((1 - n_p)\) and the second by \((1 - n_{p-k})\) because electrons are fermions and if our electron, having absorbed or emitted a plasmon, tries to emerge with a momentum that another electron already has, the probability of such an event must be zero. However, since our plasma is classical, \( n_p \ll 1 \) and so these extra factors can be approximated by unity. The same comment applies to calculations in §§7.1.2, 7.2.4, and 7.2.5.
absorption/emission probability:

\[ w \left( \frac{m \omega_{pe}}{\hbar k}, k \right) = \frac{\pi m \omega_{pe}^3}{V n_{e} k^2} \tag{7.9} \]

Thus, our calculation of Landau damping in §3.5 could be thought of as a calculation of this probability. Whether there is damping or an instability is decided by whether it is absorption or emission of plasmons that occurs more frequently—and that depends on whether, for any given \( k \), there are more electrons that are slightly slower or slightly faster than the plasmons with wave number \( k \). Note that getting the correct sign of the damping rate is automatic in this approach, since the probability \( w \) must obviously be positive.

7.1.2. Electron Distribution

The evolution equation for the occupation number of electrons can be derived in a similar fashion, if we itemise the processes that lead to an electron ending up in a state with a given wave number \( p = m_e v / \hbar \) or moving from this state to one with a different wave number. The four relevant diagrams are the two in Fig. 28 and the additional two shown in Fig. 29. The absorption and emission probabilities are the same as before and so are the energy conservation conditions. Therefore,

\[
\frac{\partial n_p}{\partial t} = \sum_k \left[ w(p - k, k \rightarrow p) \delta(\varepsilon_p - \varepsilon_{p-k} - \varepsilon_{e} p) n_{p-k} N_k \right] 
+ \frac{w(p + k \rightarrow k, p)}{N_k + 1} \delta(\varepsilon_{p+k} - \varepsilon_{e} p) n_{p+k} N_k 
- \frac{w(p, k \rightarrow p + k)}{N_k} \delta(\varepsilon_p + \varepsilon_{e} p - \varepsilon_{p+k}) n_p N_k 
- \frac{w(p \rightarrow k, p - k)}{N_k + 1} \delta(\varepsilon_p - \varepsilon_{e} p - \varepsilon_{p-k}) n_p N_k 
\approx \sum_k [w(p + k, k)(n_{p+k} - n_p) - w(p, k)(n_p - n_{p-k})] N_k
\approx \frac{\partial}{\partial p} \cdot \sum_k w(p, k) N_k k k \cdot \frac{\partial n_p}{\partial p}, \tag{7.10}
\]

where I have expanded twice in small \( k \) (i.e., in \( \hbar \)) and abbreviated \( w(p, k) = w(p, k) \delta(\varepsilon_p - \varepsilon_{e} p - \varepsilon_{p-k}) \). This is a diffusion equation in \( p \) (or, equivalently, \( v = \hbar p / m_e \))

Figure 29. Additional diagrams for (7.10).
space. In view of (7.4), (7.10) has the same form as (6.7), viz.,
\[
\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \cdot D(v) \cdot \frac{\partial f_0}{\partial v},
\]
(7.11)
where the diffusion matrix is
\[
D(v) = \sum_k k \frac{\hbar N_k}{m_e} \frac{m_e v}{\hbar} \left( \frac{m_e v}{\hbar}, k \right) \delta(k \cdot v - \omega_k) = \frac{e^2}{m_e} \sum_k \frac{k k}{k^2} |E_k|^2 \pi \delta(k \cdot v - \omega_k).
\]
(7.12)
The last expression is identical to the resonant form of the QL diffusion matrix (6.8) [cf. (6.16) and (12.49)]. To derive it, we used the definition (7.1) of $N_k$ and the absorption/emission probability (7.7), already known from linear theory.

Thus, we are able to recover the (resonant part of the) QL theory from our new electron-plasmon interaction approach. There is more to this approach than a pretty “field-theoretic” reformulation of already-derived earlier results. The diagram technique and the interpretation of the nonlinear state of the plasma as arising from interactions between particles and quasiparticles can be readily generalised to situations in which the nonlinear interactions in (2.12) cannot be neglected and/or more than one type of waves is present. In this new language, the nonlinear interactions would be manifested as interactions between plasmons (rather than only between plasmons and electrons) contributing to the rate of change of $N_k$. There are many possibilities: four-plasmon interactions, interactions between plasmons and phonons (sound waves), as well as between the latter and electrons and/or ions, etc. A comprehensive monograph on this subject is Tsytovich (1995).\footnote{It is a terrifying remake of his two earlier, equally deadly, books Tsytovich (1970, 1977). He also has a short book of lectures, Tsytovich (1972), which are useful as a kind of extended abstract of his canon, but are not fully self-contained. Two excellent texts on WT that speak a slightly different mathematical language are Zakharov et al. (1992) and Nazarenko (2011) (my §§7.3, 8.3 and 8.4 were largely inspired by the first of these and the papers that it was based on). Two early-day classics written by Founding Fathers are Kadomtsev (1965) and Sagdeev & Galeev (1969), documenting how it all started. Specifically on weak turbulence of Langmuir waves, there is a long, mushy review by Musher et al. (1995).}

I have introduced the language of kinetics of quasiparticles and their interactions with “true” particles as a reformulation of QLT for plasmas. The method is much more general and originates, as far as I know, from condensed-matter physics, the classic problem being the kinetics of electrons and phonons in metals—the founding texts on this subject are Peierls (1955) and Ziman (1960).

7.2. Weak Turbulence

If we can think about plasmon-electron interactions in terms of occupation numbers and probabilities of absorption/emission, we can also think this way of interactions between plasmons, not necessarily involving electrons or ions directly.

7.2.1. Three–Wave Interactions

A plasmon with energy $\varepsilon^a_k = \hbar \omega^a_k$ and momentum $\hbar k$ might “decay” into two other plasmons, with energies and momenta $\varepsilon^b_p = \hbar \omega^b_p$, $\hbar p$ and $\varepsilon^c_q = \hbar \omega^c_q$, $\hbar q$ (Fig. 30a), or, conversely, the latter two plasmons might merge into one (Fig. 30b). The superscripts $a,$
(a) decay of plasmon \( a \) into \( b \) and \( c \)  
(b) merger of plasmons \( b \) and \( c \) into \( a \)

Figure 30. Diagrams for (7.15) (three-wave interaction).

\( b \), and \( c \) are there to indicate that these plasmons can be of different kinds: e.g., longitudinal Langmuir waves considered in §7.1, sound waves (phonons; see §3.8), transverse electromagnetic waves (see Q2). Both the decay and merger processes must conserve momentum and energy, viz.,

\[
k = p + q, \quad \omega_k^a = \omega_p^b + \omega_q^c.
\]  

and occur with equal probabilities

\[
w^{a \rightarrow b+c}(k \rightarrow p, q) = w^{b+c \rightarrow a}(p, q \rightarrow k) \equiv w^{a \leftrightarrow b+c}(k, p).
\]  

These constraints are enough for us to infer the general architecture of the evolution equations for the plasmon occupation numbers. Assuming \( a \), \( b \), and \( c \) are all different, the rate of decrease of \( N_k^a \) comes from the first of these processes and the rate of its increase from the second, so, analogously to (7.6),

\[
\frac{\partial N_k^a}{\partial t} = \sum_{p, q} \delta_{k, p+q} \delta(\omega_k^a - \omega_p^b - \omega_q^c) \left[ -w^{a \rightarrow b+c}(k \rightarrow p, q) N_k^a N_p^b (N_q^c + 1) \right. \\
+ w^{b+c \rightarrow a}(p, q \rightarrow k) N_p^b N_q^c (N_k^a + 1) \right].
\]  

The rates are proportional to the products of the occupation numbers of all the participants in the three-plasmon interaction, so they are cubic. The plasmon numbers are not conserved, and, just like in (7.6), the +1’s in (7.15) appear where the occupation numbers are incremented as a result of the interaction. However, unlike in (7.6), these small corrections are not negligible because if they were neglected, the right-hand of (7.15) would vanish. In view of (7.14), we have (assuming all occupation numbers to be \( \gg 1 \))

\[
\frac{\partial N_k^a}{\partial t} \approx \sum_p w^{a \leftrightarrow b+c}(k, p) \delta(\omega_k^a - \omega_p^b - \omega_{k-p}^c) \left( N_p^b N_k^a - N_k^a N_p^b - N_k^a N_{k-p}^c \right).
\]  

This is a kinetic equation for plasmons of species \( a \) in a system where three-wave interactions are permitted between this species and two others.

**Exercise 7.1.** (a) Derive the kinetic equations for \( N_p^b \) and \( N_q^c \), still assuming that \( a \), \( b \), \( c \) are all different. Why does this assumption matter? What are the kinetic equations if \( a = b \neq c \)?

---

\[49\] Note that if \( N_k^a \) is much larger than all other occupation numbers, the first term in (7.16) can be neglected and the equation takes the form \( \partial N_k^a / \partial t = -\gamma_k N_k^a \). This is known as a “decay instability”.

(b) Confirm that the total energy and momentum of this gas of plasmons, defined analogously
to (7.2) and (7.3), respectively, are conserved by the kinetic equations that you have derived.

When would a contribution to the rate of change of $N_k^a$ such as the right-hand side
of (7.16) be important, in comparison with, e.g., the quasilinear rate (7.8)? Obviously,
when the quasilinear rate is small in comparison, e.g., when the waves interact with very
few resonant particles in the tail of the distribution, $\gamma_k$ in (7.8) is exponentially small,
and/or $N_k^a$ fails to saturate quasilinearly before the right-hand side of (7.16) gets larger
than $\gamma_k N_k^a$. This should be checked in each particular case—I shall talk about what kind
of particular cases are possible later on.

It is clear that the right-hand side of (7.16) must originate from the nonlinear term
in the kinetic equation (2.12) because it involves coupling between field perturbations
at different wave numbers—indeed, the momentum conservation law (7.13) is just the
restriction on wave numbers arising in the Fourier representation of a quadratic nonlin-
erarity.

How does one complete the derivation of (7.16), i.e., calculate $w^{a\leftrightarrow b+c}(k,p)$? I am
afraid the only way to do it is perturbatively, starting from the Vlasov–Poisson system
(2.12) and (2.9) (or some approximation thereto), expanding $\varphi_k$ into the linear part plus
at least one iteration involving the nonlinear term, then working out the rate of change
of $N_k^a \propto k^2|\varphi_k|^2$, comparing the result with (7.16) and reading off $w^{a\leftrightarrow b+c}(k,p)$—just as
we did in (7.9). What then, might you ask, is the point of the quasiparticle formalism if
in the end we must revert back to laborious perturbation theory? Well, apart from giving
one copious amounts of physical insight and a pleasing impression of doing field-theoretic
calculations, this approach actually reduces the amount of hard labour involved in that
perturbation theory: having applied the constraints imposed by the conservation laws
and by the principle of equal probabilities (7.14), we were able to establish the general
structure of the answer, viz., that the coefficients in front of the three terms in (7.16)
must be the same—the same coefficients also turn up in the kinetic equations for $N_b^P$
and $N_c^q$ derived in Exercise 7.1. This means that, in ploughing through the perturbation
theory, you only need to keep track of contributions that give rise to just one of these
terms, thus cutting down the amount of algebra by a factor of 9. Once you try these
calculations, you will be very grateful for such a concession!\footnote{I will provide one such calculation in §8.4 and ask you to do another one in Exercises 8.7
and 8.9 (if you are brave, do also the open-ended Exercise 8.10). My vehicle for those will be a
simplified set of equations describing the coupled dynamics of plasmons, phonons, and ions. An
ultimate repository (one might say graveyard) of many such calculations, done in a fully general
setting, is Tsytovich (1995).}

Since the nonlinearity in (2.12) is quadratic, does this mean that the rate of change
of $N_k^a$ must always be quadratic in plasmon occupation numbers? Not at all, because
the probability $w^{a\leftrightarrow b+c}(k,p)$ of any particular three-wave process can easily turn out
to be zero. Indeed, the conservation laws (7.13) are not necessarily that easy to satisfy.
For example, consider the case of three long-wavelength Langmuir plasmons, all with
$k\lambda_D << 1$. Their frequency is given by (3.39), viz.,

$$\omega_k \approx \omega_{pe} \left(1 + \frac{3}{4} k^2 \lambda_D^2 \right).$$

(7.17)

Clearly, the frequency condition in (7.13) is unfulfillable—you cannot couple two long-
wavelength Langmuir plasmons and get a third such plasmon. What happens then, in a
gas of Langmuir plasmons? There are two ways of getting a non-zero rate of change of
$N_k$ in such a situation.
Exercise 7.2. It turns out that a three-plasmon process involving three ion acoustic waves (§3.8) is also impossible. Why is this?

7.2.2. Four–Wave Interactions

The first possibility is to go to next order: while three plasmons cannot couple, four can:

\[ k + k' = p + p', \quad \omega_k + \omega_{k'} = \omega_p + \omega_{p'} . \]

(7.18)

The probability of this “plasmon-scattering” process, or its inverse, is

\[ w(k, k' \to p, p') = w(p, p' \to k, k') \equiv w(k, p, p') . \]

(7.19)

The kinetic equation is a sum of the two diagrams in Fig. 31:

\[
\frac{\partial N_k}{\partial t} = \sum_{k', p, p'} \delta_{k + k' + p + p'} \delta(\omega_k + \omega_{k'} - \omega_p - \omega_{p'}) w(k, p, p') \\
\quad \times \left[ -N_k N_{k'} (N_p + 1)(N_{p'} + 1) + N_p N_{p'} (N_k + 1)(N_{k'} + 1) \right] .
\]

(7.20)

The lowest-order terms have again cancelled out. The algebra discount for doing the perturbation theory is buy one—get three free (see Exercise 8.7).

7.2.3. Langmuir–Sound Turbulence

The second possibility is that a Langmuir plasmon decays into something else. One obviously attractive, and realisable, option is that it should decay into a phonon \((s)\) and another, slightly lower-frequency Langmuir plasmon \((l)\) (Fig. 32a):

\[ \omega_l p = \omega_s k + \omega_l p - k . \]

(7.21)

Indeed this turns out to be the only three-wave process allowed for longitudinal plasma waves. Even this is only allowed—obviously—provided sound waves can actually exist, i.e., if \(T_i \ll T_e\) (see §3.8).\(^{51}\)

Note that this process is formally very similar to the one that was considered in §7.1: there, an electron emitted or absorbed a plasmon; here, a plasmon emits or absorbs a phonon. The difference is that in §7.1, we could assume that the electron occupation numbers were small but electron momenta large in comparison with the plasmons, whereas here the occupation numbers and momenta of the plasmons and the phonons

\(^{51}\)If sound waves are strongly damped (e.g., at \(T_i = T_e\)), four-wave interactions (§7.2.2) and/or induced-scattering processes of the kind considered in §7.2.4 have to be invoked to describe Langmuir turbulence: see §8.2.7 and Exercises 8.7 and 8.10.
Exercise 7.3. (a) Work out under what assumption the energy conservation condition (7.21) can be turned into the wave-resonance condition
\[ \omega_s^k = \mathbf{k} \cdot \mathbf{v}_l^k, \]
where \( \mathbf{v}_l^k \) is the group velocity of the Langmuir wave.

(b) Show, in fact, that the “optimal” phonon for a plasmon with momentum \( \hbar \mathbf{p} \) to emit has \( \mathbf{k} \approx 2\mathbf{p} \). Thus, phonons are pushed to larger \( \mathbf{k} \)'s (smaller scales).

Let the probability of the plasmon-decay/phonon-emission process, or its inverse (Fig. 32b), be \( w(p, k) \), and, for compactness of notation, let
\[ w(p, k) = w(p, k)\delta(\omega_l^p - \omega_s^k - \omega_l^{p-k}). \] (7.22)

Tallying up the relevant diagrams gets us the results that were, in fact, already anticipated in Exercise 7.1.

The two diagrams in Fig. 32 give us the rate of change of the number of phonons:
\[ \frac{\partial N_s^k}{\partial t} = \sum_p w(p, k) \left( N_l^p N_s^{p-k} + N_l^k N_s^p - N_l^{p-k} N_s^k \right) \equiv S_k + \gamma_s^k N_s^k. \] (7.23)

Thus, in the initial absence of phonons, there is a source of them \( (S_k > 0) \), due to plasmon decay. In their presence, there is a certain rate at which they might be reabsorbed. The sign \( \gamma_s^k \) is not obviously fixed, but it is clear that it should be negative if there is to be a steady state. If we assumed \( N_l^p \ll N_s^k \) and \( k \ll p \), we would end up with a very similar situation to (7.8): a “Landau damping” of phonons on plasmons—or an instability, depending on the sign of \( \gamma_s^k \), which is determined by the functional form of the plasmon spectrum \( N_l^p \).

To work out the rate of change of \( N_l^p \), we need to take into account the diagrams that went into (7.23) plus two more, shown in Fig. 33,—entirely analogously to the derivation of the QL equation for electrons in §7.1.2 (cf. Fig. 29). The difference now is that, in general, the occupation numbers of plasmons are not much smaller than those of the phonons, and also their momenta \( \mathbf{p} \) and \( \mathbf{k} \) are of the same order, rather than \( k \ll p \).
result is
\[ \frac{\partial N_p^l}{\partial t} = \sum_k \left[ w(p, k) \left( N_p^{l-k} N_k^s - N_p^{l-k} N_k^s - N_k^s N_p^l \right) \right] + w(p + k, k) \left( N_p^{l+k} N_k^s + N_p^{l+k} N_k^s - N_k^s N_p^l \right) \equiv M_p + \gamma_p^l N_p^l. \]  

(7.24)

In the initial absence of phonons, we find an exponential change, at the rate \( \gamma_p^l \), in the number of plasmons at a given \( p \), due to emission of phonons—the counterpart to the source term in (7.23). In the presence of phonons, this rate is modified (downwards—the sign of this contribution is definite), to account for the reabsorption of the phonons, and there is also a mode-coupling term \( M_p \) (containing the \( N_p^{l-k} N_k^s \) and \( N_p^{l+k} N_k^s \) contributions). Again, assuming \( N_p^l \ll N_k^s \) and \( k \ll p \) would turn (7.24) into something very similar to the QL diffusion equation (7.10), but this is not the right regime this time.

The Langmuir–sound system will be my workhorse example, so, to reduce clutter, let me simplify notation a bit:
\[ N_p^l = N_p, \quad N_k^s = n_k, \quad \omega_p^l = \omega_p, \quad \omega_k^s = \Omega_k. \]  

(7.25)

The kinetic equations (7.23) and (7.24) can be written rather compactly as follows
\[ \frac{\partial n_k}{\partial t} = \sum_p T_{p,k}, \quad \frac{\partial N_p}{\partial t} = \sum_k \left( T_{p+k,k} - T_{p,k} \right), \]  

(7.26)

where \( T_{p,k} \) is the expression under the wave-number sum in (7.23), viz.,
\[ T_{p,k} = w(p, k) \delta(\omega_p - \omega_{p-k} - \Omega_k) [N_p N_{p-k} + (N_p - N_{p-k}) n_k]. \]  

(7.27)

I will call \( T_{p,k} \) the transfer function as it quantifies the shuffling of energy (and of plasmon number) between wave numbers.

**Exercise 7.4.** Confirm by direct calculation the fact (obvious from the diagrams) that the kinetic equations for the Langmuir–sound turbulence (§7.2.3) and for the Langmuir turbulence with four-wave interactions (§7.2.2) conserve the total number of plasmons \( N = \sum_p N_p \), as well as the total energy and the total momentum of the quasiparticle gas.

The quasiparticle formalism expounded above gives “kinetic theory” an extended meaning—kinetics both of “real” particles and the waves. Interestingly, for the interactions that involve exclusively waves, the underlying dynamics are in fact fluid, at least approximately: one does not need to worry about weak kinetic damping effects on Langmuir or sound waves in order to derive the “kinetic equations” (7.23) and (7.24) (I will show this formally in §8). But not all nonlinear effects, and not all WT, are fluid-dynamical in this sense. There are nonlinear interactions that can be described by WT and that involve “true” kinetic effects—i.e., wave-particle resonances. The best known example of these is presented in the next section.

**7.2.4. Wave–Particle Interactions: Induced Scattering**

In §7.1, we saw how to handle the interactions involving one plasmon and one “real” particle (an electron). There is a nonlinear version of these interactions, involving two plasmons: a plasmon with momentum \( \hbar k \) meets a particle with momentum \( \hbar p = m v \), gets absorbed, particle gets excited, cannot contain itself, immediately emits another plasmon with momentum \( \hbar k' \), and speeds away with momentum \( \hbar p' \) (Fig. 34a). This is called induced scattering (or stimulated
emission). The conservation laws are [cf. (7.5)]\textsuperscript{52}
\[ k + p = k' + p' \Rightarrow p' = p + k - k', \]
\[ 0 = \varepsilon_k^l + \varepsilon_p^e - \varepsilon_{k'}^l - \varepsilon_{p'}^e = h\omega_k + \frac{h^2p^2}{2m} - h\omega_{k'} - \frac{h^2|p + k - k'|^2}{2m} \approx h \left[ \omega_k - \omega_{k'} - (k - k') \cdot v \right]. \]

The probability of this process, and of its inverse (Fig. 34b), is \( w(p, k, k') \), and I shall again abbreviate \( w(p, k, k') = w(p, k, k')\delta(\varepsilon_k^l + \varepsilon_p^e - \varepsilon_{k'}^l - \varepsilon_{p'}^e + k - k'). \)

We are ready to calculate. The diagrams in Fig. 34(a,b) give us
\[
\frac{\partial N_k}{\partial t} = \sum_{p,k'} w(p,k,k') \left[ -N_k n_p(N_{k'} + 1) + N_{k'} n_{p+k-k'}(N_k + 1) \right]
\approx \sum_{p,k'} w(p,k,k')(k - k') \cdot \frac{\partial n_p}{\partial p} N_{k'} N_k \equiv 2\gamma^1_{nk} N_k. \tag{7.30}
\]

This is rather similar to the Landau-damping equation (7.8), except with a new rate, which, expressed in terms of \( f_0(v) \) using (7.4), works out to be
\[
2\gamma^1_{nk} = \sum_{k'} N_{k'} \int d^3V \frac{V}{m} w \left( \frac{mv}{h}, k, k' \right) \delta(\omega_k - \omega_{k'} - (k - k') \cdot v)(k - k') \cdot \frac{\partial f_0}{\partial v}. \tag{7.31}
\]

This is sometimes called “nonlinear Landau damping”—confusingly, because it has nothing to do with the subject of §12.1.

Finally, adding up the diagrams in Fig. 34(a,b) and two similar ones in Fig. 34(c,d), we get the equation for the electron distribution function: denoting \( k'' = k - k' \),
\[
\frac{\partial n_p}{\partial v} \approx \sum_{k',k''} N_k N_{k'} [w(p,k,k')(n_{p+k''} - n_p) - w(p-k'',k,k')(n_p - n_{p+k''})]
\approx \partial \frac{\partial n_p}{\partial p} \cdot \sum_{k',k''} w(p,k,k') N_k N_{k'} k'' \cdot \frac{\partial n_p}{\partial p} \Leftrightarrow \frac{\partial f_0}{\partial t} = \frac{\partial f_0}{\partial v} \cdot D^1_n(v) \cdot \frac{\partial f_0}{\partial v}. \tag{7.32}
\]

This is a diffusion equation, like (7.11), but now with a “nonlinear” diffusion coefficient that depends quadratically on the electric-field spectrum. Note that since this effect is second order

\textsuperscript{52}There is nothing in this consideration that requires the particle to be an electron, it can perfectly well be an ion. Indeed, for Langmuir waves, the latter possibility is a more relevant one, and will be considered in Exercise 8.10.
in $N_k$, it should, formally, be subdominant to the QL diffusion (7.11), which is first order.
However, the resonance (7.29) is much easier to achieve because, rather than matching $\omega_k$ to $\mathbf{k} \cdot \mathbf{v}$, which might push you far into the tail of the distribution, here all you need to do is match the frequency difference $\omega_k - \omega_{k'}$ for some pair of wave numbers to $(\mathbf{k} - \mathbf{k'}) \cdot \mathbf{v}$ (so, roughly, you are matching the particle’s velocity not to the wave’s phase velocity but to its group velocity). This accommodating nature of induced scattering is its claim to ubiquitous relevance.

There is an important observation that one can make without knowing $w(\mathbf{p}, \mathbf{k}, \mathbf{k'})$. Since (7.32) is a diffusion equation, it should cause $f_0(v)$ to broaden with time, i.e., the particle distribution will heat up (cf. Q10)—this is called turbulent heating. Since energy is conserved, the plasmon population should on average lose energy, shifting to lower frequencies and lower wave numbers—this is corroborated by quantitative calculations (see Tsytovich’s books or the original papers by Pikelner & Tsytovich 1969 and Liperovskii & Tsytovich 1970, as well as Musher et al. 1995, who review further interesting complications). Therefore, WT will push waves into a region of $\mathbf{k}$ space where the WT approximation ceases to be valid and the turbulence becomes strong—I will come back to this point in §7.4.

Working out $w(\mathbf{p}, \mathbf{k}, \mathbf{k'})$ for this problem is a bit of a nightmare—or, depending on your attitude, a character-building endeavour (Exercise 8.10). You will find the answer in Kingssep (1996), or, alongside 1001 similar calculations of probabilities of WT processes, in Tsytovich (1995),

7.2.5. A Digression: “Real” Collisions

Finally, let me give you another example of the use of the diagram technique, which, while not properly a WT calculation, illustrates the unifying nature of the formalism. If we can talk about interactions between waves and particles, why not use the same language for interactions between particles and thus work out the general form of the collision integral previewed in (1.47).

A binary Coulomb collision can be thought of as two particles meeting, exchanging a plasmon (a) (b) (Fig. 35). Let the probability of this process be $w(\mathbf{p}, \mathbf{p}', \mathbf{k})$ and abbreviate $w(\mathbf{p}, \mathbf{p}', \mathbf{k}) = w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \delta(\varepsilon_p + \varepsilon_{p'} - \varepsilon_{p-k} - \varepsilon_{p+k})$. Letting the diagrams in Fig. 35, and their inverses, we get

$$0 = \varepsilon_p + \varepsilon_{p'} - \varepsilon_{p-k} - \varepsilon_{p+k} = \frac{\hbar^2}{2m} \left(p^2 + p'^2 - |p - k|^2 - |p' + k|^2\right) \approx \frac{\hbar^2}{m} \mathbf{k} \cdot (\mathbf{p} - \mathbf{p}'),$$

(7.33)

assuming at the last step that $k \ll p, p'$ (glancing collisions). Let the probability of this process be $w(\mathbf{p}, \mathbf{p}', \mathbf{k})$ and abbreviate $w(\mathbf{p}, \mathbf{p}', \mathbf{k}) = w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \delta(\varepsilon_p + \varepsilon_{p'} - \varepsilon_{p-k} - \varepsilon_{p+k})$. Adding up the diagrams in Fig. 35, and their inverses, we get

$$\frac{\partial n_p}{\partial t} = \sum_{k, p'} \left[w(\mathbf{p}, \mathbf{p}', \mathbf{k}) (-n_p n_{p'} + n_p n_{p'+k}) + w(\mathbf{p} + \mathbf{k}, \mathbf{p}', \mathbf{k}) (n_p n_{p+k} n_{p'} - n_p n_{p'+k})\right]$$

$$\approx \sum_{k, p'} \left[w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \left(n_p \mathbf{k} \cdot \frac{\partial n_{p'}}{\partial \mathbf{p'}} - n_{p'} \mathbf{k} \cdot \frac{\partial n_p}{\partial \mathbf{p}}\right)ight.$$

$$\left.+ w(\mathbf{p} + \mathbf{k}, \mathbf{p}', \mathbf{k}) \left(n_{p'} \mathbf{k} \cdot \frac{\partial n_{p+k}}{\partial \mathbf{p}} - n_{p+k} \mathbf{k} \cdot \frac{\partial n_{p'}}{\partial \mathbf{p'}}\right)\right]$$

$$\approx \frac{\partial}{\partial \mathbf{p}} \cdot \sum_{k, p'} w(\mathbf{p}, \mathbf{p}', \mathbf{k}) \mathbf{k} \cdot \left(n_{p'} \frac{\partial n_p}{\partial \mathbf{p}} - n_p \frac{\partial n_{p'}}{\partial \mathbf{p'}}\right).$$

(7.34)

Figure 35. Diagrams for (7.34): binary collisions.
Rewriting this in terms of \( f_0(v) \), we get
\[
\frac{\partial f_0(v)}{\partial t} = \frac{\partial}{\partial v'} \int d\mathbf{v}' \sum_k \frac{V\hbar}{m^2} w \left( \frac{mv}{\hbar}, \frac{mv'}{\hbar} \right) \delta \left( \mathbf{k} \cdot (v - v') \right) k k \cdot \left[ f_0(v') \frac{\partial f_0(v)}{\partial v} - f_0(v) \frac{\partial f_0(v')}{\partial v'} \right].
\] (7.35)

This is indeed the general Fokker–Planck form (1.47) of the collision integral. The probability \( w \) will be worked out in §10.2.2 via a QL-style calculation.

**Exercise 7.5. Multispecies collisions.** Generalise this construction to binary collisions between particles of different species.

### 7.3. Statistical Mechanics of Quasiparticles

#### 7.3.1. Entropy

The expressions for the rates of change of quasiparticle occupation numbers are a kind of collision integrals for the quasiparticles (the analogy reinforced by §7.2.5). It is not hard to see that they have an entropy and an \( H \)-theorem. The quasiparticles are indistinguishable bosons (no exclusion principle), so their entropy ought to be
\[
S = - \sum_k [N_k \ln N_k - (1 + N_k) \ln(1 + N_k)] \approx \sum_k \ln N_k.
\] (7.36)

The simple approximate expression arises because all occupation numbers are \( N_k \gg 1 \). In other words, the entropy of the quasiparticle gas is just the sum of Boltzmann entropies associated with each \( k \).

Let us prove that this entropy cannot decrease: e.g., for the Langmuir–sound system (7.26),
\[
\frac{dS}{dt} = \sum_k \frac{1}{n_k} \frac{\partial n_k}{\partial t} + \sum_p \frac{1}{N_p} \frac{\partial N_p}{\partial t} = \sum_{k,p} \left( \frac{T_{p,k}}{n_k} + \frac{T_{p+k,k}}{N_p} \right)
\]
\[
= \sum_{k,p} T_{p,k} \left( \frac{1}{n_k} + \frac{1}{N_{p-k}} - \frac{1}{N_p} \right)
\]
\[
= \sum_{k,p} w(p, k) \left[ N_p N_{p-k} + (N_p - N_{p-k})n_k \right]^2 \geq 0, \quad \text{q.e.d.} \] (7.37)

**Exercise 7.6.** Prove that the \( H \)-theorem holds for the four-wave kinetic equation (7.20).

You might be wondering—legitimately—how it happened that, treating a collisionless and, therefore, reversible, system, we have ended up with an \( H \)-theorem and, therefore, irreversibility. Obviously, an implicit assumption has been made that must be the quasiparticle equivalent of the hypothesis of molecular chaos—in WT, this is the so called “random-phase approximation”, which I will introduce in §8.4.2, once I have a specific example of underlying dynamical equations to work with.

#### 7.3.2. Thermodynamical Equilibrium Distributions

Where there is entropy increase, there is thermodynamical equilibrium. The standard operating procedure is to maximise entropy subject to holding constant whatever invariants the system has. The Langmuir–sound system has two: the total energy \( \mathcal{E} \)
and the total number of plasmons $N$ (see Exercise 7.4). Therefore, we must solve the maximisation problem

$$S - \beta \left( \sum_k \hbar \Omega_k n_k + \sum_p \hbar \omega_p N_p - \varepsilon \right) + \beta \mu \left( \sum_p N_p - N \right) \to \text{max},$$

where $\beta = 1/T$ and $\mu$ are Lagrange multipliers (temperature and chemical potential). Varying the above expression with respect to $n_k$ and $N_p$, we get the equilibrium distributions of the plasmon and phonon fields:

$$n_k = \frac{T}{\hbar \Omega_k}, \quad N_p = \frac{T}{\hbar \omega_p - \mu}.$$  \hspace{1cm} (7.39)

The distributions (7.39) are Rayleigh–Jeans distributions, familiar from the statistical mechanics of radiation.

**Exercise 7.7.** (a) Verify by direct calculation that the Rayleigh–Jeans distributions (7.39) are indeed stationary solutions of the kinetic equations (7.26).  

(b) An attempt to integrate these distributions to find $T$ and $\mu$ in terms of $\varepsilon$ and $N$ produces divergent results. Is this a problem? How can it be fixed?

### 7.3.3. Stationary Nonequilibrium Distributions

I have kept using the word “turbulence”, but I actually I have not done any turbulence so far—just derived generic kinetic equations for weakly interacting quasiparticles (§7.2) and shown that they support some fairly bland thermal equilibria. The interesting applications are indeed to turbulence—and that means to systems that are not in thermal equilibrium.

Since our kinetic equations conserve energy, that means that their right-hand sides cannot either generate or dissipate any net energy—they can only move energy around between different wave numbers or exchange it between different species of waves. Not worrying about the latter for a moment, assuming wave-number isotropy, and defining the $k$-shell-integrated total-energy spectrum for the Langmuir–sound system as

$$\bar{E}_k = \frac{V k^2}{2 \pi^2} (\hbar \Omega_k n_k + \hbar \omega_k N_k), \quad \int dk \bar{E}_k = \varepsilon,$$

it must be possible to write the evolution equation for it in terms of an energy flux $\varepsilon_k$:

$$\frac{\partial \bar{E}_k}{\partial t} = \frac{V k^2}{2 \pi^2} \sum_p \left[ \hbar \Omega_k T_{p,k} + \hbar \omega_k (T_{k} + p, p - T_{k,p}) \right] \equiv - \frac{\partial \varepsilon_k}{\partial k}. \hspace{1cm} (7.41)$$

Since the plasmon number $N = \sum_p N_p \equiv \int dp \bar{N}_p$ is also conserved, $\bar{N}_p = (V p^2 / 2 \pi^2) N_p$ satisfies

$$\frac{\partial \bar{N}_p}{\partial t} = \frac{V p^2}{2 \pi^2} \sum_k \left[ T_{p+k,k} - T_{p,k} \right] \equiv - \frac{\partial \Gamma_p}{\partial p}. \hspace{1cm} (7.42)$$

where $\Gamma_p$ is the plasmon-number flux (also known as the “wave-action flux”). Stationary solutions can be of two flavours: those with zero flux and those with a constant flux. The

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53 As usual in statistical mechanics, we do not need to worry about the total momentum because we can always trasfer ourselves into the reference frame moving with the total momentum of our system.

54 The prefactor comes from $\sum_k = V \int d^3k/(2\pi)^3 = V \int dk 4\pi k^2/(2\pi)^3 = (V/2\pi^2) \int dk k^2$. 

Rayleigh–Jeans solutions (7.39) are of the former kind: nothing flows anywhere because there are no sources or sinks of either energy or particles. If there were, there would not be an equilibrium, and a (statistically) stationary state would have to feature the invariants constantly flowing from sources to sinks. This is exactly what a turbulent system is: a chaotic system in which energy and other invariants, if they are there, are constantly injected at some scales (i.e., some \( k \)’s) and flow to other scales where they can be dissipated. The sources and sinks are not in the kinetic equations that we have derived—they must be added as they represent some additional physics, not covered by nonlinear interactions, which are conservative. Injection can be (and usually is in plasma physics) done by some instability; dissipation is ultimately always by collisions, but, in “collisionless” (i.e., weakly collisional) systems, those are usually accessed via phase mixing (Landau damping)—this was discussed quite thoroughly in §5, although to what extent that discussion applies to turbulent systems is a tricky question (see §11).

If the injection and dissipation are concentrated at sufficiently disparate wave numbers, there will be scale-invariant \( k \)-space intervals in between that energy and other invariants must flow through in order to get dissipated—it is in these intervals that constant-flux solutions will materialise. Working out these solutions, by setting

\[
\varepsilon_k = \varepsilon = \text{const}, \quad \Gamma_p = \Gamma = \text{const},
\]

and the directions of the corresponding fluxes (the signs of \( \varepsilon \) and \( \Gamma \)) is what we shall need the kinetic equations for (see §§8.4.3–8.4.6).

There are three further nuances. First, valid constant-flux solutions will only be obtainable from kinetic equations without the need to include sources and sinks if the interactions are local in \( k \), i.e., if the sources and the sinks cannot couple to each other directly (otherwise energy might just “leap” directly from sources to sinks). There will be a scheme for checking whether that is true (see the end of §8.4.6). Secondly, stationary solutions of the kinetic equations are only of any use if they are stable. That too must be, and can be, checked within the WT formalism (I shall not deal with this; see Zakharov et al. 1992). Thirdly, the \( k \)-space isotropy assumption is not always good, and there is an interesting zoo of non-isotropic possibilities (I will ignore them all; see, e.g., Musher et al. 1995).

Let us keep all this in mind, but this is about as far as one can get without actually calculating \( w(p, k) \). In §8.4, I shall calculate it for the Langmuir-sound turbulence (§7.2.3), after I have derived the dynamical equations for it. Once this is done, I will show you how one can solve the WT kinetic equations and obtain steady-state, constant-flux, nonequilibrium distributions and spectra.

7.4. Validity of WT Approximation

Since I have not yet fully revealed what the WT approximation consists of, I cannot just yet give you a full account of the conditions for its validity. These will emerge piecemeal here and in §§8.1 and 8.4, as some specific meat is put on the generic bones that have emerged so far.

What should be clear so far is that we have confined ourselves to a situation where we must be able to think of our plasma as a gas of monocromatic waves with definite wave numbers (momenta) and frequencies (energies), which interact subject to conservation rules on both, e.g., (7.13) or (7.18). Matching wave numbers is a straightforward requirement that mathematically is traceable to the simple fact that the nonlinear terms in the underlying dynamical equations [e.g., (2.12)], when Fourier transformed, turn into convolution sums over wave numbers. In contrast, the matching of frequencies is a more nontrivial proposition. At the very least, in order to be meaningful, it requires that the
wave frequencies be identifiable, i.e., that the waves survive (much) longer than their own period of oscillation. In other words, the typical nonlinear interaction time is long compared to the wave period:

$$\omega_k t_{nl} \gg 1,$$

(7.44) as promised in §2.4.3. It is perhaps intuitive then that the delta functions constraining the WT interaction probabilities, such as $\delta(\omega_p - \omega_{p-k} - \Omega_k)$ in (7.27), in fact have an effective width of order $\sim t_{nl}^{-1}$, which is assumed small compared to any of the frequencies.

How well this assumption is satisfied depends on the amplitudes of the perturbations (i.e., on the overall fluctuation level), which must remain sufficiently small. This has to be checked for each particular system—see, e.g., §8.1.5.

It happens quite often—indeed, more often than not—that a WT system drives itself towards such (low) wave numbers, frequencies, and/or (high) amplitudes that the WT applicability condition (7.44) gets broken, i.e., the frequency-matching delta functions are broadened to the point that they no longer constrain interactions (there was a glimpse of this at the end of §7.2.4). This turns the system from a weakly coupled to a strongly coupled one, from weak turbulence to strong turbulence. In such a regime, it is no longer true that monochromatic plasma waves are the natural elementary building blocks of the nonlinear system, and some other entities, each formally composed of many Fourier modes, emerge to claim pre-eminence. What they are is not always (indeed, not usually) clear. I will attempt to illustrate this point in a specific way in §§8.4.7–8.7.

8. Langmuir Turbulence

This section is dedicated to one particular kind of plasma turbulence—probably the best-, or at any rate the earliest-, studied kind: turbulence of coupled Langmuir and sound waves. This is not the most relevant situation for most fusion (except perhaps inertial-fusion) or astrophysical plasmas, but it has the advantage of involving the two most basic plasma waves that we have already studied above, at length, and it also gives me an opportunity to illustrate some concepts and methods that find their application in many other plasma problems: how separation of scales enables one to derive reduced systems of equations (§§8.1 and 8.2), how they turn out to have Hamiltonian structure (§8.3), how one can use them to calculate probabilities of basic WT processes and thus complete, and solve, the kinetic equations derived in §7.2 (§8.4), how the WT approximation breaks down and waves (quasiparticles) cease to be the basic actors of plasma dynamics, ceding ground to essentially nonlinear structures (§§8.5 and 8.6), and how then a strongly turbulent regime might be tackled, with great difficulty but also great ingenuity (§§8.7 and 8.8).

8.1. Zakharov’s Equations

So let me pick up where I left off in §7.2.3 and ask what dynamical equations are satisfied by the plasmons and the phonons—so I can use these equations to calculate $w(p, k)$ (and, it will turn out, do much more besides). First, I would like to derive these equations in a physically intuitive, if non-rigorous, way—I will clean up my act in §8.2.

55 Another key assumption that enables one to write the kinetic equations in the form we did above is the random-phase approximation, already alluded to at the end of §7.3.1 to excuse entropy nonconservation and introduced properly in §8.4.2.

56 In a different setting, this theme will be taken up again in §§14.2 and 14.3.
8.1.1. **Langmuir Waves**

In Exercise 3.1, you had an opportunity to derive the linearised fluid equations for Langmuir waves—they are just the density and velocity moments of the electron kinetic equation, ignoring ions entirely:

$$\frac{\partial \delta \tilde{n}_e}{\partial t} + \nabla \cdot (\tilde{n}_e \tilde{u}_e) = 0, \quad (8.1)$$

$$m_e \tilde{n}_e \frac{\partial \tilde{u}_e}{\partial t} = -e \tilde{n}_e \tilde{E} - \nabla \delta \tilde{p}_e. \quad (8.2)$$

The overtildes are there to mark the quantities that will oscillate as a Langmuir wave. The overbars will designate the fields that are much slower than that—in the above, $\bar{n}_e$ so far just means the equilibrium density over which the perturbations occur. The last term in (8.2) contains the electron pressure perturbation, which, in Exercise 3.1, you found to be

$$\delta \tilde{p}_e = \frac{3}{2} T_e \delta \tilde{n}_e = \frac{3}{2} m_e v_{\text{th}}^2 \delta \tilde{n}_e \quad (8.3)$$

at long wavelengths, i.e., if $k v_{\text{th}} \ll \omega_{pe}$ (and so kinetic effects could be ignored). The electric field is $\tilde{E} = -\nabla \tilde{\varphi}$, and the Poisson equation tells us that $4\pi e \delta \tilde{n}_e = \nabla^2 \tilde{\varphi}$. Taking the time derivative of (8.1) and using (8.2) and (8.3), we thus arrive at a closed equation for $\tilde{\varphi}$:

$$\frac{\partial^2}{\partial t^2} \nabla^2 \tilde{\varphi} + \nabla \cdot \left( \frac{4\pi e^2 \tilde{n}_e}{m_e} \nabla \tilde{\varphi} \right) - \frac{3}{2} v_{\text{th}}^2 \nabla^4 \tilde{\varphi} = 0. \quad (8.4)$$

In the linear problem, the dimensional factor in the second term is just $\omega_{pe}^2$, and then from (8.4) we can promptly read off the dispersion relation (3.39) for a Langmuir wave:

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{\text{th}}^2. \quad (8.5)$$

Let me now posit that in a nonlinear system where the dominant interactions are between Langmuir and sound waves, the nature of these interactions must be “modulational”: sound waves are much slower than Langmuir waves and so the latter will “see” the former simply as slow density modulation on top of the equilibrium. Mathematically this is captured if we let

$$\bar{n}_e = n_{0e} + \delta \tilde{n}_e, \quad (8.6)$$

where $n_{0e}$ is the homogeneous equilibrium density and $\delta \tilde{n}_e$ the density perturbation associated with sound. This gives us just the right kind of nonlinearity: (8.4) becomes

$$\left( \frac{\partial^2}{\partial t^2} + \omega_{pe}^2 - \frac{3}{2} v_{\text{th}}^2 \nabla^2 \right) \nabla^2 \tilde{\varphi} = -\omega_{pe}^2 \nabla \cdot \left( \frac{\delta \tilde{n}_e}{n_{0e}} \nabla \tilde{\varphi} \right). \quad (8.7)$$

In order to close the system, we need an equation for $\delta \tilde{n}_e$, the electron-density perturbation associated with sound waves.

8.1.2. **Sound Waves**

Let us work in the limit of cold ions, $T_i \ll T_e$, when sound waves can actually thrive without being bothered by Landau damping (§3.8). In this limit, they too can...
be described by simple linearised fluid equations, derived in Exercise 3.6:

\[ \frac{\partial \delta n_i}{\partial t} + n_0 i \nabla \cdot u_i = 0, \quad (8.8) \]
\[ m_i n_0 i \frac{\partial u_i}{\partial t} = -\nabla \delta \bar{p}_e. \quad (8.9) \]

Since ions are cold, only the electron pressure matters. We do not need to have bars over the ion quantities because they do not have Langmuir-frequency variation, but the electron pressure here is the slow, averaged part that the ions can “see”. If you did Exercise 3.6, you know that electrons are isothermal at this slow time scale, viz., \( \delta \bar{p}_e = T_e \delta \bar{n}_e \). Also, the Poisson equation for these perturbations just turns into a statement of quasineutrality:

\[ Z \delta n_i = \delta \bar{n}_e \] (there cannot be any deviations from that on time scales longer than the inverse plasma frequency; see §2.1); the same relationship is also of course true for the equilibrium densities: \( Z n_0 i = n_0 e \). The equation for sound waves follows immediately:

\[ \frac{\partial^2 \delta \bar{n}_e}{\partial t^2} = c_s^2 \nabla^2 \delta \bar{n}_e. \quad (8.10) \]

Now we need to find a way to modify this equation to account for the emission (or absorption) of sound waves by Langmuir waves. The answer is intuitive: the average effect of a fast-oscillating electric field on particles amounts to an additional effective pressure equal to the energy density of the field—this is known as the \textit{ponderomotive force}. Here is a quick derivation. An electron’s equation of motion is

\[ m_e \ddot{\bar{r}} = -e \bar{E}(\bar{r}). \quad (8.11) \]

Now split this motion into fast-oscillating and slowly drifting parts, \( \bar{r} = \bar{\bar{r}} + \tilde{r} \), and assume \( |\tilde{r}| \ll |\bar{\bar{r}}| \). The fast-oscillating part satisfies

\[ m_e \ddot{\bar{\bar{r}}} \approx -e \bar{E}(\bar{\bar{r}}) \quad \Rightarrow \quad \bar{\bar{r}} \approx \frac{e}{m_e \omega_{pe}^2} \bar{E}(\bar{\bar{r}}), \quad (8.12) \]

assuming that \( \bar{E} \) oscillates at the single frequency \( \omega_{pe} \). Time-averaging (8.11) over the fast oscillations and expanding \( \bar{E}(\bar{\bar{r}} + \tilde{r}) \) in small \( \tilde{r} \), we get

\[ m_e \ddot{\tilde{r}} \approx -e \left[ \bar{E}(\bar{\bar{r}}) + \bar{\bar{r}} \cdot \nabla \bar{E}(\bar{\bar{r}}) \right] = -e^2 \frac{\bar{E} \cdot \nabla E}{m_e \omega_{pe}^2} = -\frac{1}{n_0 e} \nabla \frac{|\nabla \varphi|^2}{8\pi}. \quad (8.13) \]

Adding this effective electron pressure \( |\nabla \varphi|^2 / 8\pi \) (or, equivalently, an effective electric potential) to (8.9) and retracing the route that led to (8.10), we get the desired equation featuring a nonlinear coupling of sound waves to Langmuir waves:

\[ \left( \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 \right) \delta \bar{n}_e = c_s^2 \nabla \frac{|\nabla \varphi|^2}{8\pi T_e}. \quad (8.14) \]

Together with (8.7), this forms a closed system of equations.

\[ \text{Exercise 8.1. Is there a ponderomotive contribution to the effective ion pressure and need it be taken into account?} \]
8.1.3. Final Form of Zakharov’s Equations

We are one step away from the final form of the Zakharov (1972) equations. A further simplification is achieved if, in (8.16), we take out the $\omega_{pe}$ oscillations:

$$\tilde{\phi} = \frac{1}{2} \left( \psi e^{-i \omega_{pe} t} + \psi^* e^{i \omega_{pe} t} \right),$$  \hspace{1cm} (8.15)

where $\psi$ varies on a time scale much longer than $\omega_{pe}^{-1}$. Substituting (8.15) into (8.7) and neglecting $\partial_t^2 \psi \ll i \omega_{pe} \partial_t \psi$, we get Zakharov’s first equation:

$$\nabla^2 \left( i \omega_{pe}^{-1} \frac{\partial \psi}{\partial t} + \frac{3}{2} \lambda^2_{De} \nabla^2 \psi \right) = \frac{1}{2} \nabla \cdot \left( \frac{\delta n_e}{n_0e} \nabla \psi \right).$$  \hspace{1cm} (8.16)

Zakharov’s second equation is the same as (8.14), but with the substitution of (8.15):

$$\left( \frac{\partial^2}{\partial t^2} - c^2_s \nabla^2 \right) \delta n_e = c^2_s \nabla^2 \left| \nabla \psi \right|^2 \frac{16 \pi T_e}{16 \pi T_e}.$$  \hspace{1cm} (8.17)

We have our closed system—and even a cursory glance at the structure of these equations confirms that they should capture just the kind of interactions anticipated in §7.2.3: the right-hand side of (8.17) is quadratic in $\psi$ and so describes two plasmons coupling to generate a phonon (or, rather, a plasmon emitting/absorbing a phonon and becoming a different plasmon); the right-hand side of (8.16) contains a product of $\delta n_e$ and $\psi$ and so describes the modification of a plasmon by the emission or absorption of a phonon.

**Exercise 8.2. Secondary instabilities of a Langmuir wave.** That plasmons emit phonons—i.e., that decay instabilities exist—can be demonstrated directly from the dynamical equations (8.16) and (8.17). Consider a monochromatic Langmuir wave as the lowest-order solution of Zakharov’s equations in the absence of nonlinearity, then perturb around it, linearise, and work out the secondary instabilities of the perturbations: what are the growth rates, the peak-growth wave numbers, various parameter regimes, etc.? The best treatment of this topic in the literature that I am aware of is the review by Thornhill & ter Haar (1978, §3) (although the calculation already appears in Zakharov 1972). This is not really kinetic theory, more fluid dynamics, but it is a challenging yet doable calculation that one can sink one’s teeth into and enjoy.

8.1.4. Quasistatic Limit

The linear frequency of the variation of $\psi$ in (8.16) is

$$\delta \omega_k = \frac{3}{2} k^2 \lambda^2_{De} \omega_{pe} = \frac{3}{4} k^2 \frac{v^2_{the}}{\omega_{pe}}.$$ \hspace{1cm} (8.18)

This frequency will still be greater than the sound frequency, enabling the ponderomotive source term in the right-hand side of (8.17) to excite sound waves, provided

$$\frac{\delta \omega_k}{k c_s} \sim k \lambda_{De} \frac{v_{the}}{c_s} \gg 1 \iff k \lambda_{De} \gg \sqrt{\frac{m_e}{m_i}}.$$ \hspace{1cm} (8.19)

In the opposite limit,

$$k \lambda_{De} \ll \sqrt{\frac{m_e}{m_i}},$$ \hspace{1cm} (8.20)
the source term is quasistatic, no sound waves can be excited and (8.17) simply enforces a balance between the electron pressure and the ponderomotive force:

$$\delta \bar{n}_e = -\frac{\nabla^2 \psi}{16\pi T_e}.$$  (8.21)

Putting this into (8.16) gets us a closed equation for $\psi$:

$$\nabla^2 \left( i\omega_{pe}^{-1} \frac{\partial \psi}{\partial t} + \frac{3}{2} \lambda_{De}^2 \nabla^2 \psi \right) = -\nabla \cdot \left( \frac{\left|\nabla \psi\right|^2 \nabla \psi}{32\pi n_{0e} T_e} \right).$$  (8.22)

The nonlinearity is cubic, so this is actually an example of a situation where four-wave interactions will dominate nonlinear dynamics (see §7.2.2).

Let us check that the nonlinear term in (8.22) does not spoil the assumption that $\psi$ changes slowly compared to sound. The characteristic nonlinear time is

$$t_{nl}^{-1} \sim \omega_{pe} A^2, \quad A^2 = \frac{|\tilde{E}|^2}{8\pi n_e T_e},$$  (8.23)

where $A^2$ is the nondimensionalised mean square amplitude of the Langmuir oscillations. Then

$$t_{nl} k c_s \gg 1 \iff A^2 \ll k \lambda_{De} \sqrt{m_e/m_i}.$$  (8.24)

If we wanted to break this restriction, i.e., if $\psi$ wanted to saturate at higher amplitudes than this, we would have to go back to (8.17)—and, as we are about to see, also to range outside the WT approximation.

8.1.5. Weak Turbulence or Strong Turbulence?

Finally, let us check whether our equations can take us beyond the WT regime, and when. Consider first the quasistatic limit (§8.1.4): as argued in §7.4, the WT approximation requires

$$t_{nl} \delta \omega_k \gg 1 \iff A^2 \ll k^2 \lambda_{De}^2.$$  (8.25)

This is a slightly more stringent condition than (8.24), so there is a range of amplitudes in which (8.22) describes strong turbulence.

If we now consider $\sqrt{m_e/m_i} \ll k \lambda_{De} \ll 1$, when sound waves exist, the nonlinearities in both (8.17) and (8.16) will be weak and WT applicable if

$$A^2 \ll \frac{\delta \bar{n}_e}{n_{0e}} \ll k^2 \lambda_{De}^2.$$  (8.26)

Note that since $\tilde{E} = -\nabla \tilde{\phi}$, the condition $A^2 \ll k^2 \lambda_{De}^2$ is equivalent to

$$\frac{e \tilde{\phi}}{T_e} \ll 1,$$  (8.27)

whereas our equations will be perfectly valid when $e \tilde{\phi}/T_e \sim 1$ [see (8.41)].

If you are completely happy with the derivation in §§8.1.1 and 8.1.2, skip the next

57 One might call this the incompressible limit of Zakharov’s equations.

58 A mathematically identical equation, but with $T_e$ replaced by $T_e + T_i/Z$ in the right-hand side, arises in the limit of hot ions—hot enough to stream quickly and thus cause the ion density to have a simple Boltzmann response. This approximation is derived in §8.2.7.
section, but if you feel that the approximations and assumptions hoisted upon you are
difficult to accept at this level of (non-)rigour, read on: in §8.2, I provide a formal
perturbative derivation of the Zakharov (1972) equations (8.16) and (8.17), which is
surprisingly difficult to find in the literature.

8.2. Formal Derivation of Zakharov’s Equations

8.2.1. Scale Separations

The problem has four characteristic timescales: the plasma oscillation frequency, the
electron streaming rate, the ion sound frequency and the ion streaming rate:

\[ \omega_{pe} \gg kv_{th} \gg kcs \sim kv_{th}1. \]  (8.28)

The relative size of these frequencies is controlled by the following three independent parameters:

\[ \frac{kv_{th}}{\omega_{pe}} \ll 1, \quad \frac{kcs}{kv_{th}} \sim \sqrt{\frac{me}{m_i}} \ll 1, \quad \frac{kv_{th}}{kcs} \sim \sqrt{\frac{T_i}{T_e}} \sim 1. \]  (8.29)

The scale separation between ions and electrons is non-negotiable as the mass ratio is always
small. As long as \( k\lambda_{De} \ll 1 \), which we will assume here, the electron Landau damping is
exponentially small and the electrons will be fluid (as we will see shortly; it is no surprise, given
what we know from §3.5). Ions too behave as a fluid if they are cold (\( T_i \ll T_e \); cf. §3.8), which
is the limit most often considered in the context of Zakharov’s equations, if not necessarily one
that is most relevant physically.

8.2.2. Electron Kinetics and Ordering

As anticipated in §8.1, let us split the electron distribution function and the electrostatic
potential into two parts: the time-averaged (“slow”, denoted by overbars) and fluctuating (“fast”,
denoted by overtildes):

\[ f_e = \bar{f}_e + \tilde{f}_e, \ \ \varphi = \bar{\varphi} + \tilde{\varphi}. \]  (8.30)

The time average is taken over time scales longer than both \( \omega_{pe}^{-1} \) and \( (kv_{th})^{-1} \) but shorter
than \( (kcs)^{-1} \) or \( (kv_{th})^{-1} \), i.e., \( \bar{f}_e \) and \( \bar{\varphi} \) are the electron distribution and potential that the ions can “see”. The slow part of the electron distribution is assumed to consist of a homogeneous
Maxwellian equilibrium (5.6) plus a perturbation:

\[ \bar{f}_e = f_{e0} + \delta f_e. \]  (8.31)

The slow and fast distribution functions satisfy the following equations, which are obtained
by time averaging the Vlasov equation (1.50) for electrons (\( \alpha = e \), \( q_\alpha = -e \)) and subtracting
the average from the exact equation:

\[ \mathbf{v} \cdot \nabla \delta f_e + \frac{e}{m_e} (\nabla \bar{\varphi}) \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}} + \frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}} = 0, \]  (8.32)

\[ \frac{\partial \bar{f}_e}{\partial t} + \mathbf{v} \cdot \nabla \bar{f}_e + \frac{e}{m_e} (\nabla \bar{\varphi}) \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{v}} + \frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \mathbf{v}} = 0, \]  (8.33)

where all time evolution on ion scales is neglected because \( kv_{th} \) is large. The slow and fast parts of the Poisson equation (1.51) are

\[ -\nabla^2 \bar{\varphi} = 4\pi e(Z\delta n_i - \delta \bar{n}_e) = 4\pi e \left( Z \int d^3 \mathbf{v} \delta f_i - \int d^3 \mathbf{v} \delta \tilde{f}_e \right), \]  (8.34)

\[ -\nabla^2 \tilde{\varphi} = -4\pi e \tilde{n}_e = -4\pi e \int d^3 \mathbf{v} \tilde{f}_e, \]  (8.35)

where \( \delta f_i \) is the perturbed ion distribution function and \( \delta n_i \) its density. We shall solve (8.33)
and (8.35) for \( \tilde{\varphi} \) and \( \tilde{f}_e \), use that to calculate the last term in (8.32), which will give rise to
the ponderomotive force, then solve (8.32) for \( \tilde{f}_e \) in terms of \( \tilde{\varphi} \), and finally use that solution in
(8.34) to get an expression for \( \tilde{\varphi} \) in terms of \( f_i \). The latter can then be coupled with the ion
Vlasov–Landau equation (5.1) \((\alpha = i, q_\alpha = Ze)\), giving rise to a closed “hybrid” system for kinetic ions and “fluid” electrons.

In order to implement this plan, we shall carry out a perturbation expansion of the above equations in the small parameter

\[
\varepsilon = k\lambda_{De}.
\]

The algebra becomes more compact if we first make the following ansatz, designed to remove the third (the largest, as we will see) term in (8.33):

\[
\tilde{f}_e = -\mathbf{u} \cdot \frac{\partial f_e}{\partial \mathbf{v}} + h,
\]

where \(\mathbf{u}\) is, by definition, the velocity associated with the plasma oscillation [cf. (5.46)]:

\[
\frac{\partial \mathbf{u}}{\partial t} = \frac{e}{m_e} \nabla \tilde{\phi}.
\]

Then the fast electron kinetic equation (8.33) becomes

\[
\frac{\partial \mathbf{h}}{\partial t} = \mathbf{v} \cdot \nabla \left( \mathbf{u} \cdot \frac{\partial f_e}{\partial \mathbf{v}} \right) - \mathbf{v} \cdot \nabla h + \frac{e}{m_e} (\nabla \tilde{\phi}) \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{v}} \frac{\partial f_e}{\partial \mathbf{v}} - \frac{e}{m_e} (\nabla \tilde{\phi}) \cdot \frac{\partial h}{\partial \mathbf{v}}
\]

\[
+ \frac{e}{m_e} (\nabla \tilde{\phi}) \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{v}} \frac{\partial f_e}{\partial \mathbf{v}} - \frac{e}{m_e} (\nabla \tilde{\phi}) \cdot \frac{\partial h}{\partial \mathbf{v}},
\]

where the ordering of each term in the small parameter (8.36) has been determined, as indicated above, based on the following assumptions. The plasma-oscillation velocity (8.38) is

\[
\frac{\mathbf{u}}{v_{th}} \sim \frac{k e \tilde{\phi}}{m_e v_{th} \omega_{pe}} \sim k\lambda_{De} \frac{e \tilde{\phi}}{T_e} \sim \varepsilon,
\]

if, in general,

\[
\frac{e \tilde{\phi}}{T_e} \sim 1.
\]

Anticipating that the ponderomotive “potential” [see (8.13)] will enter on equal footing with the slow potential and that the slow perturbed electron distribution will express the Boltzmann response to the latter modified by the former, let us adopt the ordering

\[
\frac{\delta \tilde{f}_e}{f_{0e}} \sim \frac{e \tilde{\phi}}{T_e} \sim \frac{e^2 |\mathbf{E}|^2}{m_e \omega_{pe}^2 T_e} \sim (k\lambda_{De})^2 \left( \frac{e \tilde{\phi}}{T_e} \right)^2 \sim \varepsilon^2.
\]

Since the inhomogeneous terms in (8.39) are, thus, \(O(\varepsilon^2)\), it follows that \(h \sim \varepsilon^2 f_{0e}\) and, since the first term in (8.37) has no density moment, \(\tilde{n}_e \sim \varepsilon^2 n_{0e}\).

From (8.39), to lowest order,

\[
\frac{\partial f_{0e}}{\partial t} = \mathbf{v} \cdot \nabla \left( \mathbf{u} \cdot \frac{\partial f_{0e}}{\partial \mathbf{v}} \right) + \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\partial f_{0e}}{\partial \mathbf{v}}
\]

\[
= -2 \frac{v_{th}}{v_{th}} \left[ v_i v_j \delta_{ij} u_j + \left( \delta_{ij} - \frac{2 v_i v_j}{v_{th}} \right) \frac{\partial u_i}{\partial t} u_j \right] f_{0e},
\]

where I have used (8.38) and the fact that \(f_{0e}\) is a Maxwellian.

This is equivalent to splitting the electron distribution function into fast and slow parts using as the velocity variable of \(f_e\) the peculiar velocity of the particle around a centre oscillating with velocity \(\mathbf{u}\) (cf. DuBois et al. 1995): namely, set \(f_e = f_e (\mathbf{r}, \mathbf{v} - \mathbf{u}(t, \mathbf{r})) + h(t, \mathbf{r}, \mathbf{v})\) and expand in small \(\mathbf{u}\).

Note that this ordering means that the equations that are being derived will be, in principle, suitable both for weak- and strong-turbulence regimes (see §8.1.5).
8.2.3. Ponderomotive Response

With (8.43) in hand, we are now in a position to calculate the last term in (8.32). First, using (8.37) and (8.38) and keeping terms of order $\varepsilon^2$ and $\varepsilon^3$,

$$
\frac{e}{m_e} (\nabla \tilde{\varphi}) \cdot \frac{\partial f_e}{\partial \mathbf{v}} = \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial}{\partial \mathbf{v}} \left( -\mathbf{u} \cdot \frac{f_{0e}}{\partial \mathbf{v}} + h^{(2)} \right)
$$

$$
= \frac{2}{v_{\text{the}}^2} \left( \delta_{ij} \frac{2v_i v_j}{v_{\text{the}}^2} \right) \frac{\partial u_i}{\partial t} u_j f_{0e} + \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial h^{(2)}}{\partial \mathbf{v}}
$$

$$
= \frac{\partial}{\partial t} \left\{ \frac{2}{v_{\text{the}}^2} \left( \frac{u_i^2}{2} \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right) f_{0e} + \mathbf{u} \cdot \frac{\partial h^{(2)}}{\partial \mathbf{v}} \right\} - \mathbf{u} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\partial h^{(2)}}{\partial t}.
$$

(8.44)

The first term is a full time derivative and so vanishes under averaging, whereas the second term can be calculated using (8.43):

$$
-\mathbf{u} \cdot \frac{\partial}{\partial \mathbf{v}} \frac{\partial h^{(2)}}{\partial t} = \frac{2}{v_{\text{the}}^2} \left[ \frac{u_i u_j}{v_{\text{the}}} \frac{\partial u_i}{\partial \mathbf{v}} u_j + v_i \frac{\partial u_i}{\partial \mathbf{v}} u_j - \frac{2v_i v_j v_l}{v_{\text{the}}^2} u_l \frac{\partial u_i}{\partial \mathbf{v}} u_j \right] f_{0e}
$$

$$
= \frac{2}{v_{\text{the}}^2} \left\{ \mathbf{u} \cdot \nabla u \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \left[ \frac{u_i^2}{2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right] \right\} f_{0e}
$$

$$
= \frac{2}{v_{\text{the}}^2} \left\{ \mathbf{u} \cdot \nabla \left[ \frac{u_i^2}{2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right] \right\} f_{0e}.
$$

(8.45)

The last expression was obtained after noticing that any full time derivative vanishes under averaging and that, $\mathbf{u}$ defined by (8.38) being a potential field, one could rewrite $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla |\mathbf{u}|^2/2$.

Note that (8.45) is $O(\varepsilon^3)$, as are the other two terms in (8.32). Inserting (8.45) into (8.32), we obtain the following solution for the slow part of the perturbed electron distribution:

$$
\delta \tilde{f}_e = \left\{ \frac{e\tilde{\varphi}}{T_e} - \frac{2}{v_{\text{the}}^2} \left[ \frac{u_i^2}{2} - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{v_{\text{the}}^2} \right] \right\} f_{0e}.
$$

(8.46)

The first term is the Boltzmann response, the second the ponderomotive one. The resulting electron density perturbation is

$$
\frac{\delta n_e}{n_{0e}} = \frac{e\tilde{\varphi}}{T_e} \frac{u_i^2}{v_{\text{the}}^2}.
$$

(8.47)

8.2.4. Electron Fluid Dynamics

In order to obtain the evolution equation for $\tilde{\varphi}$, we will need to solve (8.39), coupled to (8.35), to higher order than the lowest, namely, up to $\varepsilon^4$. Rather than solving the kinetic equation (8.39) order by order, it turns out to be a faster procedure to take moments of it exactly and then close the resulting hierarchy by calculating the second moment using $h^{(2)}$ given by (8.43).

The zeroth (density) moment of (8.39) is

$$
\frac{\partial \tilde{n}_e}{\partial t} + \nabla \cdot (\tilde{n}_e \mathbf{u}) + \nabla \cdot \int d^3 \mathbf{v} \mathbf{v} \mathbf{h} = 0,
$$

(8.48)

the continuity equation. The first moment is

$$
\frac{\partial}{\partial t} \int d^3 \mathbf{v} \mathbf{v} \mathbf{h} = -\nabla \cdot \int d^3 \mathbf{v} (\mathbf{u} \mathbf{v} + \mathbf{v} \mathbf{u}) \tilde{f}_e - \nabla \cdot \int d^3 \mathbf{v} \mathbf{v} \mathbf{h} + \frac{e}{m_e} \tilde{n}_e \nabla \tilde{\varphi} + \frac{e}{m_e} \tilde{n}_e \nabla \nabla \tilde{\varphi}.
$$

(8.49)

The first term on the right-hand side is zero to all orders up to at least $\varepsilon^4$ because, according to
To get \( \bar{\omega} \) we substitute (8.15) and (8.56) into (8.55), we get Zakharov’s first equation (8.16) in the same side of (8.55), not captured in (8.7), are nonlinear interactions between Langmuir waves, which will disappear in a moment.

Note that, in order to maintain this level of precision, we need to keep the lowest-order contribution to \( \bar{h} \) in the second velocity moment. This satisfies (8.43), which it is now convenient to rewrite as

\[
\frac{\partial h^{(2)}}{\partial t} = \frac{2}{v_{\text{th}}^2} \left[ v_{ij} \partial_i u_j + \frac{\partial}{\partial t} \left( \frac{u_i u_j v_i v_j}{v_{\text{th}}^2} \right) \right] f_{0e}.
\]

The stress tensor satisfies

\[
\frac{\partial}{\partial t} \int d^3 \mathbf{v} v_i v_j h = -\frac{n_{0e} v_{\text{th}}^2}{2} (\partial_t u_j + \partial_j u_i + \delta_{ij} \nabla \cdot \mathbf{u}) + \frac{\partial}{\partial t} n_{0e} \nabla \nabla : \mathbf{u} u.
\]

Inserting (8.54) into (8.50) and using also the fast Poisson equation (8.35) to express \( \bar{n}_e \) via \( \phi \), we obtain

\[
\frac{\partial^2}{\partial t^2} \nabla^2 \bar{\phi} + \nabla \cdot \left( \omega_{pe}^2 \frac{\bar{n}_e}{n_{0e}} \nabla \bar{\phi} \right) - \frac{3}{2} v_{\text{th}}^2 \nabla^4 \bar{\phi} = 4 \pi e n_{0e} \nabla \nabla : \bar{\mathbf{u}} u - \nabla \cdot \left( \frac{e}{m_e} \nabla \bar{\phi} \nabla \bar{\phi} \right). \tag{8.55}
\]

The left-hand side of this equation is what was intuited in (8.7). The terms on the right-hand side of (8.55), not captured in (8.7), are nonlinear interactions between Langmuir waves, which will disappear in a moment.

The dominant \( \omega_{pe} \) oscillation in (8.55) can now be taken out in the way that was already anticipated in §8.1.3, by introducing, via (8.15), the slow-varying complex amplitude \( \psi \). Then, from (8.38), to lowest order in \( \varepsilon \),

\[
\mathbf{u} = i \frac{e}{2 m_e \omega_{pe}} \nabla \left( \psi e^{-i \omega_{pe} t} - \psi^* e^{i \omega_{pe} t} \right).
\]

Substituting (8.15) and (8.56) into (8.55), we get Zakharov’s first equation (8.16) in the same way as we did in §8.1.3. The right-hand side of (8.55) disappears because we can average out the oscillatory terms with frequencies \( \omega_{pe} \) and \( 2 \omega_{pe} \). Finally, substituting (8.56) into (8.47), we have, for the slow density perturbation,

\[
\frac{\delta n_e}{n_{0e}} = \frac{e \bar{\phi}}{T_e} - \frac{|\nabla \psi|^2}{16 \pi n_{0e} T_e}.
\]

To get \( \bar{\phi} \), we need to bring in the ions.

8.2.5. Ion Kinetics

Since the left-hand side of the slow Poisson equation (8.34) is \( O(\varepsilon^4) \), while its right-hand side is \( O(\varepsilon^2) \), (8.34) predictably turns into the quasineutrality equation

\[
\delta n_e = Z \delta n_i. \tag{8.58}
\]

Combined with (8.57), this becomes

\[
\frac{e \bar{\phi}}{T_e} = \frac{|\nabla \psi|^2}{16 \pi n_{0e} T_e} + \frac{1}{n_{0i}} \int d^3 \mathbf{v} \delta f_i. \tag{8.59}
\]
where \( \psi \) obeys (8.16). The first term in (8.59) is the effective ponderomotive “potential” (in the fluid-dynamical language, it can also be thought of, perhaps more physically, as the effective ponderomotive “pressure”: see §8.1.2). The ion distribution function \( f_i = f_{0i} + \delta f_i \) is found from the ion Vlasov–Landau equation (5.1) with the slow potential \( \bar{\psi} \):

\[
\frac{\partial f_i}{\partial t} + v \cdot \nabla f_i - \frac{Ze}{m_i} \left( \nabla \bar{\psi} \right) \frac{\partial f_i}{\partial v} = \left( \frac{\partial f_i}{\partial t} \right)_c .
\]

(8.60)

Together with (8.16), (8.60) and (8.59) make up a closed hybrid system describing kinetic ions and fluid electrons. The electrons affect the ions via the ponderomotive nonlinearity in (8.59), while the ions modulate the plasma frequency and thereby the dynamics of the electrons.

### 8.2.6. Ion Fluid Dynamics

Let me now show how ions can become fluid, giving rise to the second equation (8.17) in the classic Zakharov (1972) system.

The zeroth and first moments of (8.60) are

\[
\frac{\partial \delta n_i}{\partial t} + v \cdot \nabla \int d^3 v \, \delta f_i = 0 ,
\]

(8.61)

\[
\frac{\partial}{\partial t} \int d^3 v \, v \delta f_i + \nabla \cdot \int d^3 v \, v v \delta f_i = - \frac{Ze}{m_i} n_i \nabla \bar{\psi} = -c_s^2 n_i \nabla \left( \frac{\delta n_i}{n_{0i}} + \frac{\left| \nabla \psi \right|^2}{16 \pi n_{0e} T_e} \right) ,
\]

(8.62)

where the last expression was obtained with the aid of (8.59). Combining these two equations and keeping only the lowest-order terms, both in \( \varepsilon \) and in \( T_i/T_e \), which is now assumed small so as to allow the ion pressure (stress) tensor in the left-hand side of (8.62) to be neglected, and replacing \( \delta n_i/n_{0i} \) with \( \delta n_e/n_{0e} \) (by quasineutrality), we get Zakharov’s second equation (8.17), describing sound waves excited by the ponderomotive force.

### 8.2.7. Boltzmann Quasistatics

When the ions are not cold \( (T_i/T_e \) is not small), (8.17) regains the ion pressure term, via which it couples to the rest of the moments of \( \delta f_i \). This is a dissipation channel for the sound waves, via Landau damping, at a typical rate \( \sim k_{\text{thi}} \). In the WT language, this is a regime in which induced scattering of plasmons by ions (§7.2.4) should become the dominant process.

There is, however, a very simple limiting case where kinetic physics again becomes irrelevant. The characteristic frequency \( \delta \omega_k \) of \( \psi \) is given by (8.18). Consider the limit in which the ion streaming rate greatly exceeds that frequency:

\[
k_{\text{thi}} \gg \delta \omega_k \sim k^2 \lambda_{De}^2 \omega_{pe} \iff k \lambda_{De} \ll \sqrt{\frac{T_i}{m_i/T_e}}
\]

(8.63)

This is best achievable (obviously) when ions are hot, the opposite limit to the one considered in §8.2.6. In this limit, the \( \partial f_i/\partial t \) term in the ion kinetic equation (8.60) can be neglected. It is then easy to verify, by direct substitution, that the resulting equation is solved by the Maxwell–Boltzmann distribution:

\[
f_i = \frac{n_{0i}}{(\pi v_{\text{thi}}^2)^{3/2}} \exp \left( -\frac{v^2 - Ze\bar{\psi}}{2 v_{\text{thi}}^2} \right) \quad \Rightarrow \quad \delta f_i = -\frac{Ze\bar{\psi}}{T_i} f_{0i} \quad \Rightarrow \quad \frac{\delta n_i}{n_{0i}} = -\frac{Ze\bar{\psi}}{T_i} .
\]

(8.64)

Thus, the ion response is simply Boltzmann. So is the (slow) electron response, except with an additional effective potential associated with the ponderomotive force—we already know this from (8.57). Combining (8.64) with that and with quasineutrality, via (8.59), we get

\[
\delta n_e = -\frac{|\nabla \psi|^2}{16 \pi (T_e + T_i/Z)} .
\]

(8.65)

Interestingly, this is mathematically the same result as (8.21) was for cold ions and \( k \lambda_{De} \ll \sqrt{m_e/m_i} \). Just like in that cold-ion quasistatic limit, in this, hot-ion one, everything is now wrapped up in a single Zakharov equation—(8.22) with \( T_e \) replaced by \( T_e + T_i/Z \),—which describes Langmuir dynamics controlled by four-wave interactions (§7.2.2).
8.3. Hamiltonian Form of Zakharov’s Equations

In order to make a transition from the dynamical equations for the fields $\psi$ and $\delta n_e$ to kinetic equations for the occupation numbers of plasmons and phonons, we need to recast the former in terms of variables that we can directly connect to the energies and, therefore, occupation numbers, of our “particles”, in the way it was done in (7.2) for the quasiparticle formulation of QLT. Thus, for the plasmons, we again have the total energy\footnote{If the appearance of $\hbar$ in a purely classical calculation triggers you, you may safely set $\hbar = 1$ everywhere. I carry it in order to keep track of dimensions, and for the uniformity of the formalism between here and \S7.}

$$H^t = \sum_p \hbar \omega_p N_p = V \sum_p \frac{|\bm{E}_p|^2}{4\pi} = V \sum_p \frac{p^2 |\psi_p|^2}{8\pi} \equiv \sum_p \hbar \omega_p a_p^* a_p, \quad (8.66)$$

where a new variable

$$a_p = p \sqrt{\frac{V}{\hbar \omega_p}} \psi_p \quad (8.67)$$

has been introduced. It makes (8.66) look like a Hamiltonian of a system of particles created or annihilated by the “operators” $a_p^*$ and $a_p$. We shall see in a moment that this is not a coincidence.

Let us now write the first Zakharov equation (8.16) in wave-number space:

$$p^2 \left( i \omega_p^{-1} \frac{\partial}{\partial t} - \frac{3}{2} p^2 \lambda^2_{De} \right) \psi_p = \frac{1}{2} p \sum_k (p - k) \psi_{p - k} \xi_k, \quad \xi_k \equiv \left( \frac{\delta \bar{n}_e}{n_0} \right)_k, \quad (8.68)$$

or, in terms of $a_p$,

$$i \frac{\partial a_p}{\partial t} - \delta \omega_p a_p = \frac{\omega_p}{2} \sum_{k,q} \delta_{p,k+q} \frac{p \cdot q}{pq} a_q \xi_k. \quad (8.69)$$

I have made use of the previously introduced notation (8.18) for the “residual” frequency $\delta \omega_p$ and also approximated $\omega_p \approx \omega_{p - k} \approx \omega_{pe}$ in the nonlinear term.

Now let us turn to the second Zakharov equation (8.17), describing sound waves. In fact, I need to walk back to the two fluid equations from which it originated: the ion continuity equation (8.8) and the ion momentum equation (8.9), the former recast in terms of $\xi = \delta \bar{n}_e/n_0$ using quasineutrality and the latter with $\delta \bar{p}_e = T_e \delta \bar{n}_e$ + an “effective pressure” term representing the ponderomotive force [see (8.13)]:

$$\frac{\partial \xi_k}{\partial t} = -i k \cdot \bm{u}_{i,k}, \quad \frac{\partial \bm{u}_{i,k}}{\partial t} = -i k \left( c_s^2 \xi_k + \chi_k \right), \quad (8.70)$$

where, expanding the quadratic expression for the ponderomotive force as a Fourier convolution and using (8.67),

$$\chi_k = \frac{\langle |\nabla \psi|^2 \rangle_k}{16\pi m_in_0} = -\sum_p \frac{p \cdot (k - p) \overline{\psi_p \psi_{k-p}}}{16\pi m_in_0} = \frac{\hbar \omega_{pe}}{2m_in_0} V \sum_{p,q} \delta_{k,p-q} \frac{p \cdot q}{pq} a_p a_q. \quad (8.71)$$

At the last step, I made use of the fact that $\psi_q = \psi_{-q}$—this follows from (8.15) and the reality condition $\tilde{\psi}_{-q} = \tilde{\psi}_q^*$. Now, clearly, the flow is potential, $\bm{u}_{i,k} = i k \phi_k$, so our equations become

$$\frac{\partial \xi_k}{\partial t} = k^2 \phi_k, \quad \frac{\partial \phi_k}{\partial t} = -c_s^2 \xi_k - \chi_k. \quad (8.72)$$
The linear part of these equations is Hamiltonian, in canonical variables \((\xi_k, \phi_k)\), with the Hamiltonian being (up to a multiplicative constant) the energy of the sound waves (cf. Exercise 5.3):

\[
H_s = \int d^3r \left( \frac{m_i n_0 u_i^2}{2} + \frac{n_0 e T_e \delta n_e^2}{2 n_0 e} \right) = \frac{m_i n_0 V}{2} \sum_k \left( k^2 \phi_k^2 + c_s^2 \xi_k^2 \right) = \sum_k \hbar \Omega_k b_k^* b_k,
\]

where \(\Omega_k = kc_s\). The last step is accomplished by the standard transformation

\[
b_k = \sqrt{\frac{m_i n_0 V}{2 \hbar c_s}} \left( \frac{c_s}{k} \xi_k + i \sqrt{\frac{k}{c_s}} \phi_k \right).
\]

Note that, while \(\xi_k\) and \(\phi_k\) are Fourier transforms of real fields (so \(\xi_k^* = \xi_k^\dagger\) and \(\phi_k^* = \phi_k^\dagger\)), the new variable \(b_k\) is genuinely complex and so contains full information about both fields, which can be extracted back from it: e.g.,

\[
\xi_k = \sqrt{\frac{\hbar k}{2m_i n_0 Vc_s}} (b_k + b_k^*).
\]  

The two equations (8.72) can, therefore, be wrapped into one:

\[
i \frac{\partial b_k}{\partial t} - \omega_s^2 b_k = \sqrt{\frac{m_i n_0 V}{2 \hbar c_s}} \chi_k = \sum_{p,q} M_{kpq} \delta_{p,k+q} a_p^* a_q^*,
\]

where the mode-coupling coefficients (“matrix elements”) are

\[
M_{kpq} = \frac{\omega_{pe}}{2} \sqrt{\frac{\hbar k}{2m_i n_0 Vc_s}} \frac{p \cdot q}{pq}.
\]

Using (8.75) in (8.69), we can get the first Zakharov equation into a form that features the same coupling coefficients:

\[
i \frac{\partial a_p}{\partial t} - \delta \omega_p a_p = \sum_{k,q} M_{kpq} \delta_{p,k+q} a_q (b_k + b_k^*),
\]

where the mode-coupling coefficients ("matrix elements") are

\[
M_{kpq} = \omega_{pe} \sqrt{\frac{\hbar k}{2m_i n_0 Vc_s}} \frac{p \cdot q}{pq}.
\]

It is now manifest that (8.78) and (8.76) are Hamiltonian equations in the form they are usually written for creation-annihilation variables:

\[
i \hbar \frac{\partial a_p}{\partial t} = \partial H \frac{\partial}{\partial a_p^*}, \quad i \hbar \frac{\partial b_k}{\partial t} = \partial H \frac{\partial}{\partial b_k^*},
\]

where the Hamiltonian consists of the free-plasmon energy (8.66) (with the constant part of the frequency \(\omega_{pe}\) taken out), the free-phonon energy (8.73), and the interaction Hamiltonian describing the emission/absorption of phonons by plasmons:

\[
H = \hbar \sum_k (\delta \omega_k a_k^* a_k + \Omega_k b_k^* b_k) + \hbar \sum_{k,p,q} \delta_{p,k+q} M_{kpq} (b_k^* a_q^* a_p + a_p^* a_q b_k).
\]

The two terms in the interaction Hamiltonian manifestly correspond to the two diagrams in Fig. 32.

**Exercise 8.3.** Prove (and note) that our equations also conserve the total number of plasmons \(N = \sum_p a_p^* a_p\) (which is why it was OK to take the constant-frequency part out of the
Hamiltonian). In Exercise 7.4, you already showed that the kinetic equation for \( N_p \) also had this property.

That our system ought to be Hamiltonian, and that the Hamiltonian for the three-wave interactions described by the diagrams in Fig. 32 should have the general form (8.80), could have been argued \textit{a priori}, so the added value from the above derivations is that we have also worked out the coupling coefficients \( M_{kpq} \) and the correspondence between the creation-annihilation variables \( a_k, b_k \) and the physical fields \( \varphi, \delta n_e \). If you like the Hamiltonian approach, you will find a very extensive treatment of its application to plasma turbulence in the review by Zakharov \textit{et al.} (1985)—and an even more general (not specially focused on plasma physics), and more pedagogical, exposition in the now-classic book by Zakharov \textit{et al.} (1992).

Exercise 8.4. Hamiltonian for four-wave interactions. Consider the quasistatic version (8.22) of the Zakharov system. By deriving the evolution equation for \( a_k \), or otherwise, show that the plasmon Hamiltonian with four-wave interactions is

\[
H = \hbar \sum_k \delta \omega_k a_k^* a_k + \hbar \sum_{k,k',p,p'} \delta \omega_{k+k'-p-p'} M_{kk'pp'} a_p^* a_p^* a_k a_{k'}^* ,
\]

and calculate the coupling coefficients \( M_{kk'pp'} \).

You will find the answers to this Exercise, as well as to Exercises 8.5, 8.7, and 8.9, in the literature cited above, or in the original, famous paper by Zakharov (1972)—but following his derivations will not necessarily be easier than working them out by yourself.

8.4. Weak Langmuir–Sound Turbulence

Let me now show you how to derive (§§8.4.1–8.4.2) and solve (§§8.4.3–8.4.6) the kinetic equations (7.23) and (7.24) starting from the dynamical equations (8.76) and (8.78). As far as I know, the example of WT considered here was first worked out by Zakharov & Kuznetsov (1978) and became a minor classic of the WT genre.

Note that because our starting point is a fairly generic Hamiltonian system (with the nature of the specific problem hidden in \( M_{kpq} \)), the basic strategies and ideas laid out below will have much more general applicability than to the specific, narrow, and perhaps even irrelevant, problem of Langmuir–sound turbulence at \( T_i \ll T_e \).

8.4.1. Perturbation Theory

We need evolution equations for \( N_p = \langle |a_p|^2 \rangle \) and \( n_k = \langle |b_k|^2 \rangle \), so let us multiply (8.78) and (8.76) by \( a_k^* \) and \( b_k^* \), respectively, subtract the complex conjugates of the same, average\(^{62}\) and see what happens. In fact, we only need to do one equation, e.g., the sound-wave one (8.76), because we know that it will have the structure (7.23) and, once we work out \( w(p,k) \), we will be able to write the Langmuir-wave kinetic equation (7.24) without any further effort.

Thus, we have for the phonon occupation number:\(^{63}\)

\[
\frac{\partial n_k}{\partial t} = 2 \text{Im} \sum_{p,q} \delta_{k,p-q} M_{kpq} \langle b_k^* a_p a_q^* \rangle .
\]

As always in nonlinear problems, the second-order correlator depends on the third-order

\(^{62}\)What “average” means will be explained in §8.4.2.

\(^{63}\)I have removed the time averaging (denoted by overbars) that was needed to iron out variation of the Langmuir electric fields on time scales shorter than the sound frequency. Any sensible averaging here will take care of that automatically.
one—the so-called closure problem. The perturbative solution to this problem is to assume the nonlinearity to be weak and so to truncate the expansion for the fields in the higher-order correlator at the linear level. As we shall see shortly, if we do this at the level of the third-order correlator, everything will vanish, so we need to iterate once more.

Using (8.78) for \(a_p\) and \(a_q^*\) and (8.76) for \(b_k^*\), we get

\[
\frac{\partial}{\partial t} \left[ i (\omega_p - \omega_q - \Omega_k) \right] \langle b_k^* a_p a_q^* \rangle = i \sum_{k', k''} \left[ M_{kk' k''} \delta_{k, k' - k''} \langle a_{k'}^* a_{k''} a_p a_q^* \rangle \right]
- M_{k'p k''} \delta_{p, k' - k} \langle b_{k'}^* b_{k''} a_{k'}^* a_q^* \rangle
+ M_{k''q k'} \delta_{q, k'' - k} \langle b_{k''}^* b_{k'} a_{p} a_{k''}^* \rangle \equiv i A_{kpq}. \tag{8.83}
\]

Note that, since only the difference of Langmuir frequencies enters, I have, for brevity of notation, replaced \(\Delta \omega_p\) with \(\omega_p\). We do not need to go any further in perturbation theory and so can solve the above equation treating the right-hand side as a constant (being an average, it changes slowly in time compared to any of the waves’ oscillations):

\[
\langle b_k^* a_p a_q^* \rangle = \frac{1 - e^{-i(\omega_p - \omega_q - \Omega_k)t}}{\omega_p - \omega_q - \Omega_k} A_{kpq} \to i \pi \delta(\omega_p - \omega_q - \Omega_k) A_{kpq} \tag{8.84}
\]

as \(t \to \infty\) (i.e., after many wave periods), by the same token as in (5.39)—there will be a lot of these delta functions in kinetic theory!

Now we need to calculate the fourth-order correlation functions inside \(A_{kpq}\). That will require another conceptual step.

### 8.4.2. Random-Phase Approximation

Our dynamical equations (8.76) and (8.78), with their small nonlinearities, describe the evolution of amplitudes and phases of the dynamical variables: \(a_k = |a_k|(t)e^{i\phi_k(t)}\) and similarly for \(b_k\). The amplitudes evolve slowly (to lowest order in the nonlinearity, not at all) and phases quickly: \(\phi_k(t) \approx \omega_k t\). This is not unlike (indeed, exactly analogous) to ballistic motion of particles in an ideal gas. When nonlinearity is introduced, even if small, the trajectories are perturbed and phases quickly stochasticised—this too is analogous to molecular chaos arising even from infrequent collisions between particles.\(^{64}\)

Thus, we shall assume the randomness of phases and consequent vanishing (to lowest order in the perturbation theory) of correlations between the dynamical variables except when their phases manage to cancel exactly. To wit:

\[
\langle a_k \rangle = |a_k|e^{i\phi_k} = 0, \tag{8.85}
\]
\[
\langle a_k a_{k'} \rangle = |a_k| |a_{k'}| e^{i(\phi_k + \phi_{k'})} = 0, \tag{8.86}
\]
\[
\langle a_k a_{k'}^* \rangle = |a_k| |a_{k'}| e^{i(\phi_k - \phi_{k'})} = N_k \delta_{k, k'}. \tag{8.87}
\]

By the same token, all odd-order correlators vanish, as do all even-order ones that do not have an equal number of occurrences of \(a_k\) and \(a_{k'}^*\). Thus, taking, e.g., one of the fourth-order correlators from (8.83):

\[
\langle a_{k'}^* a_{k''} a_p a_q^* \rangle = N_{k'} N_p \delta_{k', k''} \delta_{p, q} + \delta_{p, k'} \delta_{q, k''}. \tag{8.88}
\]

Also, in a system where frequencies of different types of waves are well separated (\(\omega_p \gg \Omega_k\) always), only correlations between the same species of waves survive because otherwise

\(^{64}\)The fact that the wave frequencies are dispersive helps: as time goes on, different \(k\)’s decorrelate—this is similar to phase mixing.
phases cannot be matched. Let us do two other correlators in (8.83):\
\[ \langle b_k^* b_{k'} a_{k''} a_q^* \rangle = n_k N_q \delta_{k,k'} \delta_{q,q''}, \quad \langle b_k^* b_{k'} a_{p} a_{k''}^* \rangle = n_k N_p \delta_{k,k'} \delta_{p,k''}. \] (8.89)

The rest of the correlators in (8.83) are zero.

**Exercise 8.5.** Work out, under the same assumptions, the 6th-order correlator \( \langle a_k a_p a_q a_{k'} a_{p'} a_{q'} \rangle \) in terms of the second-order ones (\( N_k = \langle |a_k|^2 \rangle \)). This will come useful in Exercise 8.7.

We are done: the right-hand side of (8.83) is

\[
A_{kpq} = M_{kpq} N_p N_q - M_{kpq} n_k N_q + M_{kpq} n_k N_p
= M_{kpq} [N_p N_q + (N_p - N_q) n_k].
\] (8.90)

Note that the first term in (8.88) did not contribute because \( M_{0k'/k'} = 0 \), and that \( M_{kpq} = M_{kqp} \) was used to simplify things. Putting (8.90) back into (8.82), via (8.84), we recover, triumphantly, the anticipated kinetic equation in the form (7.23):

\[
\frac{\partial n_k}{\partial t} = 2\pi \sum_p M_{k,p,p-k}^2 \delta(\omega_p - \omega_{p-k} - \Omega_k) [N_p N_{p-k} + (N_p - N_{p-k}) n_k] \equiv \sum_p T_{p,k}.
\] (8.91)

Manifestly, by comparison with (7.23) [or (7.27)],

\[
w(p,k) = 2\pi M_{k,p,p-k}^2 = w_0 k \left( \frac{p}{p} - \frac{p - k}{|p|} \right)^2, \quad w_0 = \frac{\pi \hbar \omega_p^2}{4m_1 n_0 V_c}.
\] (8.92)

In fact, to calculate this, we did not even need to follow all of the terms in (8.83), it was enough just to work out one of them, e.g., \( \langle a_{k''}^* a_{k'} a_p a_{q}^* \rangle \propto N_p N_{p-k} \). With (8.92) in hand, we also have the kinetic equation (7.24) for the plasmons: to write it in a short form anticipated in (7.26),

\[
\frac{\partial N_p}{\partial t} = \sum_k (T_{p+k,k} - T_{p,k}),
\] (8.93)

where \( T_{p,k} \) is the transfer function (7.27), now fully specified by the expression under the wave-number sum in (8.91).

The rules (8.85–8.89), and similar, for calculating correlators are generally known as the *random-phase approximation (RPA)*. Since they amount to splitting higher-order correlators into second-order ones, they are mathematically equivalent to assuming the fields to be Gaussian to lowest order in the perturbation theory (the dynamical variables are an accumulation of a sequence of random, independent kicks due to nonlinear interactions with each other—an instance of Central Limit Theorem, one might argue). Recalling the calculations in §7.2, it is now obvious that this assumption was implicitly already made when the rates of change of occupation numbers were written as products of these numbers—a quasiparticle version of Boltzmann’s (in fact, Maxwell’s and Ehrenfest’s) *Stosszahlansatz* (no wonder then that the standard form of the collision integral emerged so easily in §7.2.5). Just how valid, or otherwise, this assumption might be, is, as often with such things, a subtler question than it looks. I will sweep it resolutely under the carpet, while referring you to extended discussions of the matter in the textbooks by Zakharov *et al.* (1992) and Nazarenko (2011), and move on happily to play with the kinetic equations (8.91) and (8.93).
Exercise 8.6. To practice perturbation theory and the use of RPA, write, starting from (8.78), the evolution equation for $N_p$ in terms of third-order correlators, then solve for the latter in terms of fourth-order ones, split the averages, and verify that the kinetic equation for the plasmons is indeed (8.93).

Exercise 8.7. Kinetic equation for weak Langmuir turbulence with four-wave interactions. Starting from the Hamiltonian derived in Exercise 8.4, work out, along the same lines as done above, the kinetic equation for the plasmon occupation number in Langmuir turbulence with four-wave interactions.

8.4.3. Further Simplifications and Approximations

As I promised in §7.3.3, I shall look for isotropic solutions of the kinetic equations (8.91) and (8.93), so $N_p = N_p$, $n_k = n_k$. This is sensible because the frequencies only depend on the magnitudes of the wave numbers, $\omega_p = \omega_p$, $\Omega_k = \Omega_k$—there are no special directions in the system.\(^65\) Since both $\omega_p$ and $\Omega_k$ are simple power functions,

$$\omega_p = \omega_{pe} + \frac{3\nu_\text{th}^2}{4\omega_{pe}} p^2 \equiv \omega_{pe} + \alpha p^2, \quad \Omega_k = c_s k,$$

(8.94)

there are one-to-one correspondences between wave numbers and frequencies, a feature that will greatly simplify calculations. Indeed, we can consider the occupation numbers to be functions of the frequencies only: $N_p = N(\omega_p)$ and $n_k = n(\Omega_k)$. Then the transfer function is also a function of frequencies only:

$$T_{p,k} = w_0 k \left( \frac{p^2 - p \cdot k}{|p - k|^2} \right)^2 \delta(\omega_p - \omega|p - k| - \Omega_k) \left[ N_p|p-k| + (N_p - N|p-k|) n_k \right]$$

$$\equiv \delta(\omega_p - \omega|p-k| - \Omega_k) \bar{T}(\omega_p, \Omega_k).$$

(8.95)

Everything inside $\bar{T}(\omega_p, \Omega_k)$ that depends on $\omega|p-k|$ is reconstructible in terms of $\omega_p$ and $\Omega_k$ via the delta function. This includes $p \cdot k$, which appears in the prefactor: indeed, the argument of the delta function is\(^66\)

$$\alpha p^2 - \alpha|p - k|^2 - \Omega_k = \alpha(2p \cdot k - k^2) - \Omega_k = 0 \implies p \cdot k = \frac{k^2}{2} + \frac{\Omega_k}{\alpha},$$

(8.96)

A very attractive further simplification is now possible if we assume $\delta \omega_p \gg \Omega_k$ (which is true):

$$T_{p+k,k} = \delta(\omega|p+k| - \omega_p - \Omega_k) \bar{T}(\omega|p+k|, \Omega_k)$$

$$\approx \delta(\omega|p+k| - \omega_p - \Omega_k) \left[ \bar{T}(\omega_p, \Omega_k) + \left( \omega|p+k| - \omega_p \right) \frac{\partial \bar{T}}{\partial \omega_p} \right]$$

(8.97)

In the same vein, expanding $N|p-k| = N(\omega|p-k|)$ in (8.95) in $\omega_p - \omega|p-k|$ = $\Omega_k$ and using (8.96) in the prefactor, we find

$$\bar{T}(\omega_p, \Omega_k) \approx w_0 k \left( 1 - \frac{k^2}{2\omega_p^2} \right)^2 \left( N_p^2 + \Omega_k n_k \frac{\partial N_p}{\partial \omega_p} \right).$$

(8.98)

Note that $\Omega_k \partial N_p/\partial \omega_p = (c_s k/2\alpha p) \partial N_p/\partial p$.

From (8.98), we see immediately that for the Rayleigh–Jeans distributions (7.39), $\bar{T}(\omega_p, \Omega_k) = 0$, so they are (still) legitimate stationary solutions of our equations.

\(^65\)This will become spectacularly untrue once magnetic field is introduced: see §§14.3 and 14.4.
\(^66\)So an “optimal” phonon for a plasmon to emit is one with $k \approx 2p$ (this was Exercise 7.3b).
makes sense that these solutions should be there, but they are not what we are after—we should be looking for the constant-flux solutions promised in §7.3.3.

8.4.4. Plasmon Flux

Let us prepare to use (8.97) and (8.95) in the plasmon kinetic equation (8.93), where we can simplify the \( k \) sum by noticing that the dependence on the direction of \( k \) (with respect to \( p \)) is only left inside the delta functions:

\[
\sum_k \delta(\alpha(2p \cdot k \pm k^2) - c_sk) = \frac{V}{(2\pi)^3} \int_0^\infty dk k^2 2\pi \int_{-1}^1 d\mu \delta(\alpha(2pk\mu \pm k^2) - c_sk),
\]

\[
= \frac{V}{4\pi^2} \int_0^{2p+cs/\alpha} \frac{dk k^2}{2\alpha p}, \tag{8.99}
\]

where “+” applies to the integral of (8.97) and “−” to the integral of (8.95). Note that \( c_s/\alpha \ll 2p \) because \( \Omega_p \ll \delta \omega_p \). We can now turn the plasmon kinetic equation (8.93) into the conservative form (7.42) by defining, as was done in §7.3.3, \( \bar{N}_p = (Vp^2/2\pi^2)N_p \) and using (8.99):

\[
\frac{\partial \bar{N}_p}{\partial t} \approx \left( \frac{V}{2\pi^2} \right)^2 \frac{p}{4\alpha} \int_0^{2p+cs/\alpha} dk \left[ k\bar{T}(\omega_p, \Omega_k) + k\Omega_k \frac{\partial \bar{T}}{\partial \omega_p} \right] - \int_0^{2p-cs/\alpha} dk k\bar{T}(\omega_p, \Omega_k)
\]

\[
\approx \left( \frac{V}{2\pi^2} \right)^2 \frac{p}{4\alpha} \left( \frac{2c_s}{\alpha} 2p\bar{T} + \int_0^{2p} dk \frac{c_sk^2}{2\alpha p} \frac{\partial \bar{T}}{\partial p} \right)
\]

\[
= \left( \frac{V}{2\pi^2} \right)^2 \frac{c_s}{8\alpha^2} \left( 8p^2\bar{T} + \int_0^{2p} dk k^2 \frac{\partial \bar{T}}{\partial p} \right) = -\frac{\partial \bar{\Gamma}_p}{\partial p}, \tag{8.100}
\]

where the plasmon-number flux is

\[
\bar{\Gamma}_p = -\left( \frac{V}{2\pi^2} \right)^2 \frac{c_s}{8\alpha^2} \int_0^{2p} dk k^2\bar{T}(\omega_p, \Omega_k)
\]

\[
= -\left( \frac{V}{2\pi^2} \right)^2 \frac{w_0c_s}{8\alpha^2} \int_0^{2p} dk k^3 \left( 1 - \frac{k^2}{2p^2} \right)^2 \left( N_p^2 + \frac{c_sk\Omega_k}{2\alpha p} \frac{\partial N_p}{\partial p} \right). \tag{8.101}
\]

I have used the approximate expression (8.98) for \( \bar{T}(\omega_p, \Omega_k) \).

So this is what I promised you in §7.3.3: a specific expression for a flux, setting which constant will yield stationary nonequilibrium distributions. Here is how to find them. Let us look for power-law solutions:

\[
N_p = Ap^{-x}, \quad n_k = Bk^{-y}. \tag{8.102}
\]

Putting these into (8.101) and rescaling \( p \) out of the integral by introducing a new integration variable \( \xi = k/p \), we get

\[
\bar{\Gamma}_p = -\left( \frac{V}{2\pi^2} \right)^2 \frac{w_0c_sA^2}{8\alpha^2} p^{4-2x} \int_0^2 d\xi \xi^3 \left( 1 - \frac{\xi^2}{2} \right)^2 \left( 1 - \frac{c_sBx}{2\alpha A} \xi^{y+1} \right) \left( 1 - \frac{c_sBx}{2\alpha A} \xi^{-y-1} \right). \tag{8.103}
\]

This can only be independent of \( p \) if the powers of \( p \) in both terms vanish, giving \( x = 2 \) and \( y = x - 1 = 1 \), so

\[
N_p = Ap^{-2}, \quad n_p = Bp^{-1}. \tag{8.104}
\]
Substituting this back into (8.103), we get, after doing the integral $I_1 = 4/3$:

$$\Gamma = -\left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_s A^2}{6\alpha^2} \left(1 - \frac{c_s B}{\alpha A}\right).$$  \hspace{1cm} (8.105)$$

If we knew the sign of this expression, we would be able to tell whether the plasmon number flows to smaller or larger $p$, but for that we need to calculate $A$ and $B$. We are not yet in a position to do this because what we would like to calculate them in terms of are the plasmon and energy fluxes $\Gamma$ and $\varepsilon$, so we could see what signs of $\Gamma$ and $\varepsilon$ gave viable solutions. So, back to work calculating $\varepsilon$.

### 8.4.5. Energy Flux

Let us turn to the phonon kinetic equation (8.91). This contains a similar integral to those we encountered above:

$$\frac{\partial n_k}{\partial t} = \sum_p \delta(\omega_p - \omega|p-k| - \Omega_k) \tilde{T}(\omega_p, \Omega_k)$$

$$= \frac{V}{(2\pi)^3} \int_0^\infty dp \int_{-1}^1 d\mu \delta(\alpha(2pk\mu - k^2) - c_s k)$$

$$= \frac{V}{2\pi^2} \frac{w_0}{4\alpha} k^{2-2x} \int_{1/2}^\infty d\zeta \zeta^{1-2x} \left(1 - \left(1 - \frac{1}{2\zeta^2}\right)^2 \left(1 - \frac{c_s B}{\alpha A} \zeta^{-x} k^{-x-1}\right) \right).$$  \hspace{1cm} (8.106)$$

At the last step, I substituted the power-law solutions (8.102) and rescaled $k$ out of the integral by changing the integration variable to $\zeta = p/k$.

We are now ready to construct an equation for the evolution of the total-energy spectrum

$$\bar{E}_k = \frac{V k^2}{2\pi^2} (\hbar \Omega_k n_k + \hbar \delta \omega_k N_k).$$  \hspace{1cm} (8.107)$$

This was already defined in (7.40), but here I have taken out the bit corresponding to the constant part of the plasmon frequency, $\hbar \omega_{pe} N_k$, because that has the same evolution as the plasmon number, governed by (8.100). Using (8.106), (8.100) and (8.103), we get

$$\frac{\partial \bar{E}_k}{\partial t} = \left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 A^2}{4\alpha} h\Omega_k k^{4-2x} I_2(k) - h\delta \omega_k \frac{\partial \Gamma_k}{\partial k}$$

$$= \left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_s A^2}{4\alpha} \left[ k^{5-2x} I_2(k) + \frac{1}{2} k^2 \frac{\partial}{\partial k} k^{4-2x} I_1(k) \right] \equiv - \frac{\partial \varepsilon_k}{\partial k}.$$  \hspace{1cm} (8.108)$$

Examining (8.108), one must recognise that it can only be scale-invariant if the $k$ dependence in $I_1$ and $I_2$ disappears. The condition for that is

$$y = x - 1.$$  \hspace{1cm} (8.109)$$

In fact, this could have perhaps been guessed a priori: it is just the statement that the contributions to $\bar{E}_k$ from the phonons and the plasmons have the same scaling: $\Omega_k n_k \sim \delta \omega_k N_k$. 


Adopting this assumption, we conclude immediately that a stationary solution requires

\[ I_2 + (2 - x)I_1 = 0, \quad (8.110) \]

where the integrals, which now depend only on the exponent \( x \) and on the constants \( A \) and \( B \), are

\[ I_1 = \int_0^2 \, d\xi \, \xi^3 \left( 1 - \frac{\xi^2}{2} \right)^2 \left( 1 - \frac{c_s B x}{2\alpha A} \xi^{-x+2} \right), \quad (8.111) \]

\[ I_2 = \int_{1/2}^\infty \, d\zeta \, \zeta^{1-2x} \left( 1 - \frac{1}{2\zeta^2} \right)^2 \left( 1 - \frac{c_s B x}{2\alpha A} \zeta^{-x+2} \right) \]

\[ = \int_0^2 \, d\xi \, \xi^{2x-3} \left( 1 - \frac{\xi^2}{2} \right)^2 \left( 1 - \frac{c_s B x}{2\alpha A} \xi^{-x+2} \right). \quad (8.112) \]

The last transformation of \( I_2 \) was accomplished by changing the integration variable to \( \xi = 1/\zeta \), mapping the integration domain of \( I_2 \) onto that of \( I_1 \). The two integrals become identical at \( x = 3 \), which then conveniently solves (8.110). Thus, we have obtained the constant-energy-flux solutions:

\[ N_k = Ak^{-3}, \quad n_k = Bk^2 \quad . \quad (8.113) \]

We are now ready to do what was promised at the end of §8.4.4: work out \( A \) and \( B \) in terms of \( \Gamma \) and \( \varepsilon \) and decide which invariant flows in which direction.

From (8.108), by direct integration,

\[ \varepsilon_k = - \left( \frac{V}{2\pi^2} \right)^2 \frac{hw_0 c_s A^2}{4\alpha} \frac{I_2 + (2 - x)I_1}{6 - 2x} k^{6-2x}. \quad (8.114) \]

The \( k \) dependence does indeed disappear at \( x = 3 \), but the remaining expression has a 0/0 indeterminacy. By L’Hospital’s rule,

\[ - \left[ \frac{I_2 + (2 - x)I_1}{6 - 2x} \right]_{x=3} = \frac{1}{2} \left[ \frac{\partial I_2}{\partial x} + (2 - x) \frac{\partial I_1}{\partial x} - I_1 \right]_{x=3} = c_1 + c_2 \frac{c_s B}{\alpha A}, \quad (8.115) \]

where the numerical coefficients, arising from the ugly mêlée of fractions and logs that is the exact calculation of \( I_1, I_2 \), and their derivatives, are

\[ c_1 = \frac{4(3\ln 2 - 1)}{9} \approx 0.035, \quad c_2 = \frac{3754}{3675} - \frac{44\ln 2}{35} \approx 0.150. \quad (8.116) \]

For the record,

\[ I_1 = \frac{4}{3} + \frac{2^{5-x} x [32 + x(x - 10)] c_s B}{(x - 6)(x - 8)(x - 10) \alpha A}, \quad (8.117) \]

\[ I_2 = \frac{2^{2x-3} [4 + x(x - 3)]}{x(x^2 - 1)} - \frac{2^{x-1} [8 + x(x - 2)] c_s B}{(x + 2)(x + 4) \alpha A}, \quad (8.118) \]

hence, after differentiation and substitution of \( x = 3 \), the values (8.116).

The useful takeaway is that the coefficients (8.116) are positive, and so then is the energy flux (8.114), always:

\[ \varepsilon = \left( \frac{V}{2\pi^2} \right)^2 \frac{hw_0 c_s A^2}{4\alpha} \left( c_1 + c_2 \frac{c_s B}{\alpha A} \right) > 0 \quad . \quad (8.119) \]
8.4.6. Direct and Inverse Cascades

Thus, if energy is injected into a plasmon–phonon gas at some wave number \( k_0 \) and at the rate \( \varepsilon \), it will be cascaded directly, i.e., flow to larger wave numbers (smaller scales) \( k > k_0 \), leaving in its wake the distributions (8.113). Note that by “energy” I mean (8.107), from which the constant part of the plasmon frequency has been taken out.

The latter (in fact, dominant) part of the energy, \( \sum_p \hbar \omega_{pe} N_p \), is proportional to the plasmon number \( N \) and conserved separately. It cannot also have a direct cascade. Indeed, the plasmon flux is given by (8.103) and we can easily calculate it for the spectra (8.113), i.e., for \( x = 3 \) and \( y = 2 \): using (8.117),

\[
\Gamma_k = -\left( \frac{V}{2\pi^2} \right)^2 \frac{\omega_0 c_A^2}{8\alpha^2} \left( \frac{4}{3} - \frac{44}{35} \frac{c_s B}{\alpha A} \right) k^{-2} \rightarrow 0 \text{ as } k \rightarrow \infty.
\]

So it peters out at small scales. The natural conclusion is, therefore, that the plasmon number will cascade inversely, i.e., flow to smaller wave numbers (larger scales) \( k < k_0 \).

The expression (8.105) must then be negative—not being sign-definite, it can be, implying \( c_s B < \alpha A \). The corresponding distributions are (8.104).

The dual-cascade picture just described is illustrated in Fig. 36.

Exercise 8.8. What is the energy flux at \( k < k_0 \)?

There is a very simple dimensional argument, invented by Fjortoft (1953) in the context of 2D turbulence, that allows one to predict which invariant will flow to small scales and which to large. Dimensionally, the constant-flux distributions must satisfy

\[
\frac{k N_k}{\tau_k} \sim \Gamma, \quad \frac{k \tilde{E}_k}{\tau_k} \sim \frac{\hbar \omega_{pe} N_k}{\tau_k} \sim \varepsilon,
\]

where \( \tau_k \) is the effective rate at which nonlinear interactions happen at the scale \( k^{-1} \). At the injection wave number \( k_0 \), this implies that

\[
\varepsilon \sim \hbar \omega_{pe} k_0 \Gamma.
\]

If there were a direct cascade of plasmon number, this would imply, at \( k \gg k_0 \), an arriving energy flux

\[
\frac{k \tilde{E}_k}{\tau_k} \sim \hbar \omega_{pe} k^2 \Gamma \gg \varepsilon.
\]

No such supply of energy is available from the injection scale, so this scenario cannot be realised. In contrast, if the plasmon number cascades inversely, to \( k \ll k_0 \), the amount of energy arriving
there is minuscule, so everything is fine. The basic conclusion is that the invariant that has more powers of $k$ cascades directly and the one that has fewer inversely.

To summarise formal progress, we have found the scaling of the distributions, the direction of cascades, and, for any given injected fluxes $\varepsilon$ and $|\Gamma|$, can solve for the constants $A$ and $B$ from (8.105) and (8.119). Thus, our solution is complete. Note that the fact that the integrals (8.111) and (8.112) that went into the calculation of the fluxes converged is the confirmation of locality of fluxes—a blow up would have indicated that the whole approach of solving in scale-invariant ranges of $k$ away from injection and dissipation scales had, in fact, been inapplicable.

This is as much as I shall say about WT. The subject is much vaster than this, and full of deep insights, subtle nuances, clever tricks, twists and turns. I have chosen to work through one complete calculation, which features many of the key ideas and methods of the WT theory, but if you wish to become an expert, read the textbooks by Zakharov et al. (1992) and Nazarenko (2011) and the rest of the literature cited above. Either of Exercises 8.9 or 8.10 offers a proactive independent-study path that should prove both challenging and exciting.

**Exercise 8.9. Spectra of Langmuir turbulence with four-wave interactions.** Starting from the kinetic equation derived in Exercise 8.7, work out the Langmuir-wave spectra arising from four-wave interactions. Your conclusion should be that plasmons will flow to larger scales, leaving behind a distribution $N_k \propto k^{-7/3}$. The answers are in Zakharov (1972) or Zakharov et al. (1992). In getting there, you will learn about Zakharov transformations, an inspired feat of analytical skulduggery, of which the mapping $\zeta = 1/\xi$ in §8.4.5 was an almost trivial example. You will also experience locality not being always guaranteed.

**Exercise 8.10. Induced scattering of Langmuir waves on ions.** This is an independent-study topic. Consider the regime ($T_i \sim T_e$) in which sound waves are strongly damped but not so strongly as for the hot-ion quasistatic limit (§8.2.7) to apply. That is, the system is described by the first Zakharov equation (8.16) coupled via $\delta n_e$ to the ion kinetic equation (8.60). In this regime, the dominant nonlinear process is the induced scattering of plasmons on ions (analogous to what was considered in §7.2.4). Work out what the scattering probability is, what, therefore, are the kinetic equations for the plasmons and the ions, and study what kind of solutions they might have. In the literature, the common monicker for this regime is “isothermal Langmuir turbulence” ("isothermal" in the sense that $T_i = T_e$; the Langmuir–sound turbulence discussed in §8.4 is, accordingly, “nonisothermal Langmuir turbulence”). In navigating the literature, you may choose to follow the Tsytovich approach, featuring heavy-duty perturbative kinetic calculations (see the books cited at the end of §7.1 or the original papers cited in §7.2.4), or (or and?) the Zakharov one, with a more Hamiltonian flavour (see reviews by Zakharov et al. 1985 and Musher et al. 1995, the latter focussing specially on WT). If you plough through even some of it, you have my respect, but you may well emerge with an enduring hatred of the Soviet scientific-writing style.

8.4.7. *From Weak to Strong Turbulence*

8.5. *Langmuir Collapse*

See Zakharov (1972).

8.6. *Solitons and Cavitons*


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67 The example that I have chosen is not, in fact, (in my view) particularly clearly presented in Zakharov et al. (1992), or in the original paper by Zakharov & Kuznetsov (1978), so I hope this section has provided some added value.
A. A. Schekochihin

8.7. Kingsep–Rudakov–Sudan Turbulence

See Kingsep et al. (1973).

8.8. Pelletier’s Equilibrium Ensemble

See Pelletier (1980).

9. Nonlinear Stability

Let me now go back to the generalist agenda first articulated at the beginning of §4: What kind of equilibria are stable? Are there universal distributions to which a collisionless plasma will relax? This time I shall ask the stability question while forbidding myself any recourse to linear theory. Later on, this will push us towards certain distributions that will turn out to make some sense statistical-mechanically (§10.1) and that I will then show can be obtained via QLT (§10.2).

9.1. Nonlinear Stability Theory: Thermodynamic Method

The general idea of the method is to find, for a given initial equilibrium distribution $f_0$, an upper bound on the amount of energy that might be transferred into electromagnetic perturbations (not necessarily small). If that bound is zero, the system is stable; if it is not zero but is sharp enough to be nontrivial, it gives us a constraint on the amplitude of the perturbations in the saturated state.

Here is how it is done:

\[ H = \int d^3r \left( \frac{E^2 + B^2}{8\pi} \right) + \int d^3r d^3v [A(r, v, f_0) - A(r, v, f_*)] = \mathcal{E} + A[f, f_*], \quad (9.1) \]

where $f_*$ is some trial distribution, which will represent our best guess about the properties of the stable distribution towards which the system wants to evolve and/or in the general vicinity of which we are interested in investigating stability. The function $A(r, v, f)$ is chosen in such way that for any $f$,

\[ A[f, f_*] \geq 0. \quad (9.2) \]

If it is also chosen so that $H$ is conserved by the (collisionless) Vlasov–Maxwell equations, then $H(t) = H(0)$ and the inequality (9.2) gives us an upper bound on the field energy at time $t$:

\[ \mathcal{E}(t) - \mathcal{E}(0) = A[f_0, f_*] - A[f(t), f_*] \leq A[f_0, f_*], \quad (9.3) \]

where $f_0$ is the initial ($t = 0$) equilibrium whose stability is under investigation.

The bound (9.3) implies stability if $A[f_0, f_*] = 0$, i.e., certainly for $f_0 = f_*$. This

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68 In Q5, isotropic, monotonically decreasing equilibria were found to be stable not just against infinitesimal (linear), electrostatic perturbations, but also against small but finite electromagnetic ones, giving us a taste of a powerful nonlinear constraint.

69 These ideas appear to have crystallised in the papers by T. K. Fowler in the early 1960s (see his review, Fowler 1968; his reminiscences and speculations on the subject 50 years later can be found in Fowler 2016), although a number of founding fathers of plasma physics were thinking along these lines around the same time (references are given in opportune places below).

70 This statement is based on the assumption that if the total electromagnetic energy decreases,
guarantees stability of any $f_*$ for which a functional $A[f, f_*]$ satisfying (9.2) and giving a conserved $H$ can be produced.

Physically, the above construction is nontrivial if the bound (9.3) is smaller than the total initial kinetic energy of the particles:

$$A[f_0, f_*] < \sum_\alpha \int d^3r \int d^3v \frac{m_\alpha v^2}{2} f_\alpha \equiv K(0).$$

(9.4)

It is obvious that one cannot extract from a distribution more energy than $K(0)$, but the above tells us that, in fact, one might only be able to extract less. $A[f_0, f_*]$ is an upper bound on the available energy of the distribution $f_0$. The sharper it can be made, the closer we are to learning something useful. Thus, the idea is to identify some suitable functional $A[f, f_*]$ for which $H$ is conserved, and some class of trial distributions $f_*$ for which (9.2) holds, then minimise $A[f_0, f_*]$ within that class, subject to whatever physical constraints one can reasonably expect to hold: e.g., conservation of particles, momentum, and/or any other (possibly approximate) invariants of the system (e.g., its adiabatic invariants; see Helander 2017).

To make some steps towards a practical implementation of this programme, let us investigate how to choose $A$ in such way as to ensure conservation of $H$:

$$\frac{dH}{dt} = \frac{d\mathcal{E}}{dt} + \sum_\alpha \int d^3r \int d^3v \frac{\partial A}{\partial f_\alpha} \frac{\partial f_\alpha}{\partial t} = \sum_\alpha \int d^3r \int d^3v \left( \frac{\partial A}{\partial f_\alpha} - \frac{m_\alpha v^2}{2} \right) \frac{\partial f_\alpha}{\partial t} = 0.$$  

(9.5)

The second equality was obtained by using the conservation of total energy,

$$\frac{d}{dt}(\mathcal{E} + K) = 0, \quad K = \sum_\alpha \int d^3r \int d^3v \frac{m_\alpha v^2}{2} f_\alpha,$$

(9.6)

where $K$ is the kinetic energy of the particles. Now (9.5) tells us how to choose $A$:

$$A(r, v, f) = \sum_\alpha \left[ \frac{m_\alpha v^2}{2} f_\alpha + G_\alpha(f_\alpha) \right],$$

(9.7)

where $G_\alpha(f_\alpha)$ are arbitrary functions of $f_\alpha$. These can be added to $A$ because Vlasov’s equation has an infinite number of invariants: for any (sufficiently smooth) $G_\alpha(f_\alpha)$,

$$\frac{d}{dt} \int d^3r \int d^3v G_\alpha(f_\alpha) = 0.$$  

(9.8)

This follows from the fact that, in the absence of collisions, the kinetic equation (1.30) expresses the conservation of phase volume in $(r, v)$ space (the flow in this phase space is divergence-free).

that corresponds to initial perturbations decaying. You might wonder what happens if $\mathcal{E}(0)$ contains some equilibrium magnetic field and if that equilibrium is unstable: can the equilibrium field’s energy be tapped and transferred partially into unstable perturbations of kinetic energy in such a way that $\mathcal{E}(t) < \mathcal{E}(0)$ even though the system is unstable? I do not know how to isolate formally the set of conditions under which this is impossible (you may wish to think about this question; §16 might help). To avoid this problem, we could just restrict applicability of all considerations in this section to unmagnetised initial equilibria.

71Krall & Trivelpiece (1973) comment with a slight air of resignation that, with the rules of the game much vaguer than in linear theory, the thermodynamical approach to stability is “more art than science”. In the Russian translation of their textbook, this statement provokes a disapproving footnote from the scientific editor (A. M. Dykhne), who observes that the right way to put it would be “more art than craft”.


Exercise 9.1. Prove the conservation law (9.8), assuming that the system is isolated.

The existence of an infinite number of conservation laws suggests that the evolution of a collisionless system in phase space is much more constrained than that of a collisional one. In the latter case, the evolution is constrained only by conservation of particles, momentum and energy and the requirement (5.12) that entropy must not decrease. I shall return shortly to the question of how available energy might be related to entropy.

A quick sanity check is to try $G_\alpha(f_\alpha) = 0$. The inequality (9.2) is then certainly satisfied for $f_* \propto \delta(v)$ and the bound (9.3) becomes

$$\mathcal{E}(t) - \mathcal{E}(0) \leq K(0), \quad (9.9)$$

i.e., one cannot extract any more energy than the total energy contained in the distribution—indeed, one cannot. Let us now move on to more nontrivial results.

9.2. Gardner’s Theorem

Gardner (1963), in a classic two-page paper, proved that if the equilibrium distributions of all species are isotropic and decrease monotonically as functions of the particle energy $\varepsilon_\alpha = m_\alpha v^2/2$, the system is stable:

$$\frac{\partial f_0}{\partial \varepsilon_\alpha} < 0 \Rightarrow \text{stability} \quad (9.10)$$

Proof. For every species (suppressing species indices), let me again take $G(f) = 0$ in (9.7), but construct a nontrivial $f_*$ that satisfies (9.2) for $f(t)$ at every time $t$ since the beginning of its evolution from the initial distribution $f_0$.

For any given $f_0$, define $f_*$ to be a monotonically decreasing function of $v^2$ (i.e., energy), such that for any $\Lambda > 0$, the volume of the region in the phase space $(r, v)$ where $f_* > \Lambda$ is the same as the volume of the phase-space region where $f_0 > \Lambda$. Then $f_*$ is the distribution with the smallest kinetic energy, denoted here by $K_*$, that can be reached.

The stability of Maxwellian equilibria against small perturbations was first proved by W. Newcomb, whose argument was published as Appendix I of Bernstein (1958) (and followed by Fowler 1963, who proved stability against large perturbations). Gardner (1963) attributes the first appearance of the stability condition (9.10) to an obscure 1960 report by M. N. Rosenbluth, although the same condition was derived also by Kruskal & Oberman (1958), more or less in the manner described in §9.3. Many great minds were clearly thinking alike in those glory days of plasma physics.
from \(f_0\) while preserving phase-space volume:

\[
K(t) \geq K_*.
\] (9.11)

Indeed, while the phase-space volume occupied by any given value of the probability density is the same for \(f_0\) and for \(f_*\), the corresponding energy is always lower for \(f_*\) than for \(f_0\) or for any other \(f\) that can evolve from it, because in \(f_*\), the values of the probability density are rearranged in such a way as to put the largest of them at the lowest values of \(v^2\), thus minimising the velocity integral in (9.6). A vivid analogy is to think of the evolution of \(f\) under the collisionless kinetic equation (1.28) as the evolution of a mixture of “fluids” of different densities (values of \(f\)) advected in a 6D phase-space \((r, v)\) by a divergence-free flow \((\dot{r}, \dot{v})\). The lowest-energy state is the one in which these fluids are arranged in layers of density decreasing with increasing \(v^2\), the heaviest at the bottom, the lightest at the top (Fig. 37).

In view of (9.11), and since \(\mathcal{A}\) is given by (9.7) with \(G(f) = 0\),

\[
A[f, f_*] = K(t) - K_* \geq 0,
\] (9.12)

so (9.2) holds and (9.3) follows. When \(f_0 = f_*\), i.e., the equilibrium distribution satisfies (9.10), the system is stable, q.e.d. The available energy is \(A[f_0, f_*] = K(0) - K_*\).

Note that the condition (9.10) is sufficient, but not necessary, as we already know from, e.g., Exercise 4.2.

### 9.2.1. Helander’s Take on Gardner

In a recent paper, Helander (2017) developed an elegant scheme for calculating “ground states” (the states of minimum energy) of Vlasov’s equation, i.e., for determining Gardner’s \(f_*\) and then calculating \(K_*\) to work out specific values of the available energy.

The idea is to look for a distribution \(f_*\), such that the kinetic energy of any distribution evolving from it cannot increase. So, let us set \(f(t = 0) = f_*\) and evolve \(f(t)\) forward a short time \(\delta t\). The collisionless kinetic equation can be written simply as [cf. (1.28)]

\[
\frac{\partial f}{\partial t} + \dot{q} \cdot \frac{\partial f}{\partial q} = 0 \quad \Rightarrow \quad f(\delta t) \approx f_* - \delta t \dot{q} \cdot \frac{\partial f_*}{\partial q},
\] (9.13)

where \(q = (r, v)\) is the phase-space variable. The first-order (in \(\delta t\)) kinetic-energy change from \(f_*\) to \(f(\delta t)\) is then

\[
\delta K[\delta q] = -\int d^6q \varepsilon(q) \delta q \cdot \frac{\partial f_*}{\partial q},
\] (9.14)

where \(\varepsilon(q) = mv^2/2\) and \(\delta q = \delta t \dot{q}\). We want to minimise \(K\), so we need \(\delta K = 0\). This will be achieved for \(f_*\) such that \(\delta K[\delta q] = 0\) for any phase-space vector \(\delta q\) that behaves appropriately (vanishes) at the boundaries and satisfies \((\partial/\partial q) \cdot \delta q = 0\)—the latter condition is imposed because the phase-space velocity field in (9.13) must be divergence-free: \((\partial/\partial q) \cdot \dot{q}\) (the system is Hamiltonian). This last condition can be enforced by means of a Langrange multiplier \(\lambda(q)\):

\[
\delta K[\delta q] - \int d^6q \lambda(q) \frac{\partial}{\partial q} \cdot \delta q = 0 \quad \Leftrightarrow \quad \varepsilon(q) \frac{\partial f_*}{\partial q} = \frac{\partial \lambda}{\partial q} \Rightarrow \quad f_* = f_*(\varepsilon(q)),
\] (9.15)

i.e., the desired minimum-energy distribution must be a function of the particle energy only, as anticipated by Gardner.

Thus, any \(f_*(\varepsilon)\) is a minimum-energy state, but we now must find one that is accessible from a given initial distribution \(f_0\) via collisionless evolution, i.e., conserving phase-space volumes. This condition can be written in the form of the conservation law (9.8) with \(G(f) = H(f(q) - \Lambda)\), where \(H\) is the Heaviside function, picking out the volume of phase space where \(f > \Lambda\):

\[
\Gamma[f, \Lambda] = \int d^6q H(f(q) - \Lambda) = \text{const}.
\] (9.16)

Since this is conserved, for any \(f_*\) accessible from a given \(f_0\), \(\Gamma[f_*, \Lambda] = \Gamma[f_0, \Lambda]\). Notice now
that if \( \partial f_\ast / \partial \varepsilon < 0 \), as it should be if \( f_\ast \) is a Gardner function, \( H(f_\ast(\varepsilon) - \Lambda) = H(\varepsilon_A - \varepsilon) \), where \( \varepsilon_A \) is such an energy that \( f_\ast(\varepsilon_A) = \Lambda \). Therefore,

\[
\Gamma[f_\ast, \Lambda] = \int d^6 q \ H(\varepsilon_A - \varepsilon(q)) \equiv \Omega(\varepsilon_A),
\]

(9.17)

where the function \( \Omega(\varepsilon_A) \) is entirely independent of \( f_\ast \), being just the integrated density of states corresponding to the energy \( \varepsilon_A \): since \( \varepsilon(q) = mv^2/2 \),

\[
\Omega(\varepsilon) = \frac{4\pi V}{3} \left( \frac{2\varepsilon}{m} \right)^{3/2}.
\]

(9.18)

Collecting all these relations, we conclude that \( \Omega(\varepsilon_A) = \Gamma[f_\ast, \Lambda] = \Gamma[f_0, \Lambda] = \Gamma[f_0, f_\ast(\varepsilon_A)] \), or, for any \( \varepsilon \),

\[
\Gamma[f_0, f_\ast(\varepsilon)] = \Omega(\varepsilon).
\]

(9.19)

This is an integral equation for the Gardner function \( f_\ast(\varepsilon) \) accessible from the initial distribution \( f_0 \).

### 9.2.2. Anisotropic Equilibria

Let me give an example of the use of Helander’s scheme for an anisotropic initial distribution—the case that, at the end of §4, I had to relegate to Exercise 4.8 as it needed substantial extra work if it were to be handled by the method developed there.

Consider a bi-Maxwellian distribution, a useful and certainly the simplest model for anisotropic equilibria:

\[
f_0 = C \exp\left(-\frac{mv_\perp^2}{2T_\perp} - \frac{mv_\parallel^2}{2T_\parallel}\right), \quad C = n \left(\frac{m}{2\pi T}\right)^{3/2},
\]

(9.20)

where \( T = T_\perp^{2/3} T_\parallel^{1/3} \) and \( T_\perp \) and \( T_\parallel \) are the “temperatures” of particle motion perpendicular and parallel to some special direction. Is this distribution unstable? (Yes: see Q3.) To work out the Gardner distribution corresponding to it, observe that the volume \( \Gamma[f_0, \Lambda] \) of the part of phase space where \( f_0 > \Lambda \) is \( V \) times the volume of the velocity-space ellipsoid

\[
\frac{mv_\perp^2}{2T_\perp} + \frac{mv_\parallel^2}{2T_\parallel} = \ln \frac{C}{\Lambda} \quad \Rightarrow \quad \Gamma[f_0, \Lambda] = \frac{4\pi V}{3} \left( \frac{2T}{m} \ln \frac{C}{\Lambda} \right)^{3/2}.
\]

(9.21)

Letting \( \Lambda = f_\ast(\varepsilon) \) and, according to (9.19), equating \( \Gamma[f_0, f_\ast(\varepsilon)] \) to (9.18), we find

\[
f_\ast(\varepsilon) = C \exp\left(-\frac{\varepsilon}{\overline{T}}\right).
\]

(9.22)

This is an interesting, if perhaps somewhat rigged, example of a system “wanting” to go to a Maxwellian equilibrium even in the absence of collisions.

The upper bound on the available energy is

\[
A[f_0, f_\ast] = K(0) - K_\ast = \frac{3}{2} V n \left( \frac{2}{3} T_\perp + \frac{1}{3} T_\parallel - T_\perp^{2/3} T_\parallel^{1/3} \right).
\]

(9.23)

The bound is zero when \( T_\perp = T_\parallel \) and is always positive otherwise (because it is the difference between an arithmetic and a geometric mean of the two temperatures). We do not, of course, have any way of knowing how good an approximation this is to the true saturated level of whatever instability (if any) might exist here in any particular physical regime,

\footnote{In the form \( \Omega'(\varepsilon) = f'_\ast(\varepsilon) \partial \Gamma[f_0, f_\ast(\varepsilon)]/\partial f_\ast \), it was first derived by Dodin & Fisch (2005).}

\footnote{Nominally, this calculation applies with equal validity to many different instabilities that can be triggered by temperature anisotropy in both unmagnetised and magnetised plasmas—and indeed also to some anisotropic distributions that can, in fact, be proved stable (which is common in magnetised plasmas where the externally imposed magnetic field is sufficiently large).}

but this does suggest that temperature anisotropy is a viable source of free energy.

In Helander (2017), you will find other examples, in particular, a nice demonstration that Maxwellian equilibria with spatially dependent density and temperature have available energy.
9.3. Thermodynamics of Small Perturbations

There is a neat development (due, it seems, to Kruskal & Oberman 1958 and Fowler 1963) of the formalism presented at the beginning of this section that leads again to Gardner’s result (9.10), but also puts us in contact with some familiar themes from §5.

Let us investigate the stability of isotropic distributions with respect to small (but not necessarily infinitesimal) perturbations, i.e., take \( f(t) = f_0 + \delta f \), \( \delta f \ll f_0 \), and also \( f_\ast = f_0 \), so the bound (9.3) will imply stability if we can find \( G(f) \) such that (9.2) holds.

In (9.7), we expand \( G(f) = G(f_0) + G'(f_0)\delta f + G''(f_0)\frac{\delta f^2}{2} + \ldots \) (9.24)
and use this to obtain, keeping terms up to second order,

\[
A[f(t), f_0] = \sum_\alpha \int \int d^3r \int d^3v \left\{ \left[ \frac{m_\alpha v^2}{2} + G'_\alpha(f_0) \right] \delta f_\alpha + G''_\alpha(f_0)\frac{\delta f^2_\alpha}{2} \right\}.
\]

(9.25)

Suppose we contrive to pick \( G_\alpha(f_0) \) in such a way that \( G'_\alpha(f_0) = -\frac{m_\alpha v^2}{2} \equiv -\varepsilon_\alpha \) (9.26)
oblittering the first-order term in (9.25). Then, since \( f_\alpha = f_{0\alpha}(\varepsilon_\alpha) \) by assumption (it is isotropic), differentiating the above condition with respect to \( f_\alpha \) gives

\[
G''_\alpha(f_{0\alpha}) = -\frac{1}{\partial f_{0\alpha}/\partial \varepsilon_\alpha} \Rightarrow A[f(t), f_0] = \sum_\alpha \int \int d^3r \int d^3v \frac{\delta f^2_\alpha}{2(-\partial f_{0\alpha}/\partial \varepsilon_\alpha)}.
\]

(9.27)

We see that \( A[f(t), f_0] \geq 0 \) and, therefore, (9.3) with \( f_\ast = f_0 \) implies stability if, again, \( f_{0\alpha}(\varepsilon_\alpha) \) is monotonically decreasing for all species.

Besides stability, this construction has given us an interesting quadratic conserved quantity for our system:

\[
H = \mathcal{E} + A[f, f_0] = \int d^3r \frac{E^2 + B^2}{8\pi} + \sum_\alpha \int \int d^3r \int d^3v \frac{\delta f^2_\alpha}{2(-\partial f_{0\alpha}/\partial \varepsilon_\alpha)}.
\]

(9.28)

The condition (9.10) makes \( H \) positive definite and so no wonder the system is stable: perturbations around \( f_0 \) have a conserved norm! For a Maxwellian equilibrium, \( -\partial f_{0\alpha}/\partial \varepsilon_\alpha = f_{0\alpha}/T_\alpha \), so this \( H \) is none other than \( W \), (the electromagnetic version of) the free energy (5.19), and so it is tempting to think of (9.28) as providing a natural generalisation of free energy to non-Maxwellian plasmas.

In Q5, the results of this section are obtained in a more straightforward way, directly from the Vlasov–Maxwell equations.

This style of thinking has been having a revival lately: see, e.g., the discussion of firehose and mirror stability of a magnetised plasma in Kunz et al. (2015). Generalised energy invariants like \( H \) are important not just for stability calculations, but also for theories of kinetic turbulence in weakly collisional environments, e.g., the solar wind (see, e.g., Schekochihin et al. 2009).
One might wonder at this point whether the condition (9.26) is fulfillable and also whether anything can be done without assuming small perturbations. An answer to both questions is provided by the following argument.

The realisation in §9.3 that our conserved quantity $H$ is a generalisation of free energy nudges us in the direction of a particular choice of functions $G_\alpha(f_\alpha)$ and trial equilibria $f_{*\alpha}$, fully inspired by conventional thermodynamics. Namely, in (9.7), let

$$G_\alpha(f_\alpha) = T_\alpha f_\alpha \left( \ln \frac{f_\alpha}{C_\alpha} - 1 \right), \quad f_{*\alpha} = C_\alpha \exp \left( -\frac{m_\alpha v^2}{2T_\alpha} \right),$$

(9.29)

where $C_\alpha$ and $T_\alpha$ are constants independent of space. It is then certainly true that $G_\alpha'(f_{*\alpha}) = -\varepsilon_\alpha$. It is also straightforward to show that the inequality (9.2) is always satisfied: essentially, this follows from the fact that the Maxwellian distribution $f_{*\alpha}$ maximises the entropy $-\int \int d^3r d^3v f_\alpha \ln f_\alpha$, subject to fixed energy, $1/T_\alpha$ being the corresponding Lagrange multiplier.

**Exercise 9.2.** Prove formally that if $G_\alpha$ and $f_{*\alpha}$ are given by (9.29), then

$$A[f, f_*] = \sum_\alpha \int \int d^3r d^3v \left[ \frac{m_\alpha v^2}{2} (f_\alpha - f_{*\alpha}) + G_\alpha(f_\alpha) - G_\alpha(f_{*\alpha}) \right] \geq 0 \quad (9.30)$$

for any values of $C_\alpha$ and $T_\alpha$.

Thus, Eq. (9.3) provides an upper bound on the energy of the electromagnetic fields that can be extracted from any given initial distribution $f_0\alpha$. In order to make this bound as sharp as possible, one picks the constants $C_\alpha$ and $T_\alpha$ (and, therefore, determines $f_{*\alpha}$) so as to minimise $A[f_0, f_\alpha]$ subject to constraints that cannot change: e.g., freezing the number of particles of each species gives

$$C_\alpha = \left( \frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \frac{1}{V} \int \int d^3r d^3v f_0\alpha.$$

(9.31)

9.4.1. **Anisotropic Equilibria**

To test-drive this method, let us go back to the bi-Maxwellian distribution (9.20) and assume it is the initial distribution for every species $\alpha$. To obtain an upper bound on the energy available for extraction from it, substitute this distribution into (9.29), use also (9.31), and find

$$A[f_0, f_*] = V \sum_\alpha n_\alpha \left[ \frac{3}{2} T_\alpha \left( \ln \frac{T_\alpha}{T_{*\alpha}} - 1 \right) + T_{\perp \alpha} + \frac{T_{|| \alpha}}{2} \right].$$

(9.32)

This is minimised by $T_\alpha = T_{*\alpha}$, resulting in the following estimate of the available energy:

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \min_{T_\alpha} A[f_0, f_*] = \frac{3}{2} V \sum_\alpha n_\alpha \left( \frac{2}{3} T_{\perp \alpha} + \frac{1}{3} T_{|| \alpha} - T_{*\alpha} \right).$$

(9.33)

This is the same result as (9.23), because, in this case, the target distribution $f_{*\alpha}$ was also the Gardner distribution (not, generally speaking, an absolute requirement).

Further examples of such calculations can be found in Krall & Trivelpiece (1973, §9.14) and Fowler (1968). A certain further development of the methodology discussed above allows one to derive upper bounds not just on the energy of perturbations but also on their growth rates (Fowler 1964, 1968).
10. Collisionless Relaxation

10.1. Statistical Mechanics of Collisionless Relaxation

In §9.2.2, and again in §9.4.1, I considered a simple example in which an initially anisotropic distribution appeared keen to evolve towards a Maxwellian, even though its relaxation was assumed completely collisionless. Is there any fundamental physical reason for collisionless plasmas to privilege Maxwellian distributions? Perhaps, in the same way that the Maxwellian emerges in statistical mechanics as a universal equilibrium state. Let me work though a statistical argument to that effect, proposed originally by Lynden-Bell (1967) in the context of collisionless relaxation of kinetic systems of mutually gravitating objects (e.g., stars in a galaxy).

Let us start by discretising the phase space into a very large number of micro-cells, each with phase volume $\delta \Gamma$. Let us assume also (in what is a rather drastic simplifying step) that the exact distribution function in each of these micro-cells is equal to either zero or some constant (the same constant in all micro-cells):

$$f(q) = \eta \text{ or } 0 \quad (10.1)$$

(this is known as a waterbag distribution—a constant probability density in a finite subvolume of phase space). Then

$$N = \int d^6q f = \eta \delta \Gamma N, \quad (10.2)$$

where $N$ is the number of particles and $N$ is the number of micro-cells with non-zero density. We are going to think of our plasma as a statistical-mechanical system of $N$ phase-density elements, which are allowed, under collisionless evolution, to move around phase space subject to the usual constraints: conservation of energy and conservation of phase volume. The latter constraint in this language means that phase-density elements can never occupy the same micro-cell, i.e., that they are subject to the Pauli-like exclusion principle. Thus, they are fermions, except that they are distinguishable (by their initial position in phase space).

Let us now coarse-grain our phase space into macro-cells, each containing $M$ micro-cells. We can represent the coarse-grained distribution $\bar{f}$ as a set of occupation numbers $N_i \leq M$ of the $i$-th macro-cell, namely, the density in the $i$-th macro-cell is

$$\bar{f}_i = \frac{\eta N_i}{M} \leq \eta. \quad (10.3)$$

The total number of ways of setting up a particular distribution $\{N_i\}$ is

$$W = \frac{N!}{\prod_i N_i !} \prod_i W_i, \quad W_i = \frac{M!}{(M - N_i)!}. \quad (10.4)$$

Here the first factor is the number of ways of distributing $N$ phase-density elements amongst the macro-cells and $W_i$ is the number of ways to distribute $N_i$ distinguishable elements between the micro-cells in the $i$-th macro-cell. Assuming that $N, N_i, M > N_i$ are all large, we can use Stirling’s formula ($\ln N! \approx N \ln N - N$) to find the Boltzmann entropy for our system:

$$S = \ln W \approx N (\ln N - 1) - M \sum_i \left[ \frac{N_i}{M} \ln \frac{N_i}{M} + \left( 1 - \frac{N_i}{M} \right) \ln \left( 1 - \frac{N_i}{M} \right) \right]$$

$$= N (\ln N - 1) - \frac{1}{\delta \Gamma} \int d^6q \left[ \frac{\bar{f}}{\eta} \ln \frac{\bar{f}}{\eta} + \left( 1 - \frac{\bar{f}}{\eta} \right) \ln \left( 1 - \frac{\bar{f}}{\eta} \right) \right], \quad (10.5)$$
where $\int d^6q = M \delta \Gamma \sum_i$. This is to be maximised under the constraints of a fixed number of particles $N$ and energy $K$ in the distribution.\(^{75}\) The problem is exactly the same as for a Fermi gas\(^{76}\) and its solution is the Fermi–Dirac distribution:

$$\bar{f} = \frac{n}{e^{(\varepsilon - \mu)/T} + 1}, \quad (10.6)$$

where $\varepsilon$ is the particle energy corresponding to the given macro-cell, and $T$ ("temperature") and $\mu$ ("chemical potential") are Lagrange multipliers that are determined by fixing $N$ and $K$:

$$N = \int d^6q \bar{f}, \quad K = \int d^6q \varepsilon \bar{f}. \quad (10.7)$$

10.1.1. Non-Degenerate Limit

Just like in the case of Fermi–Dirac statistics, the Maxwellian (non-degenerate) limit is recovered when the initial waterbag distribution is sparse in phase space—i.e., when it is sufficiently spread out around the part of phase space that is available subject to given energy $K$. Mathematically, this limit is achieved by letting

$$e^{-\mu/T} \gg 1 \quad \Rightarrow \quad \bar{f} \approx \frac{n}{e^{\mu/T} e^{-\varepsilon/T}}. \quad (10.8)$$

Then, from (10.7), after doing the integrals (with $d^6q = d^3r d^3v$ and $\varepsilon = mv^2/2$),

$$T = \frac{2}{3} \frac{K}{N}, \quad e^{\mu/T} = \frac{n}{(2\pi T/m)^{3/2} \eta}. \quad (10.9)$$

Thus, the non-degenerate, Maxwellian limit (10.8) is

$$\frac{n}{(2\pi T/m)^{3/2}} \ll \eta \quad \Rightarrow \quad \bar{f} = \frac{n}{(2\pi T/m)^{3/2}} e^{-\varepsilon/T}. \quad (10.10)$$

Of course it is not a particular surprise that a Maxwellian distribution emerges from a statistical mechanics of completely randomised objects with fixed overall energy. This might appear to be a argument in favour of universality in collisionless plasmas—but that hinges on the validity of the assumption that the phase-density elements of which the distribution consists are independent and free to sample the entirety of phase space without fear or prejudice.\(^{77}\)

---

\(^{75}\) For simplicity, we consider a (statistically) homogeneous system. Then let each phase-space macro-cell contain the entire position space and, consequently, $\bar{f}$ be independent of position. In an electrostatic system, $\bar{f}$ then gives rise to no fields and so $K$ is kinetic energy. A more general consideration is possible in which the energy is the total energy, kinetic plus electric—the latter is included in manner similar to how Lynden-Bell (1967) handles gravitational energy, except now with multiple particle species that have different charges.

\(^{76}\) The distinguishability of the phase-density elements turns out not to matter: in (10.4), the factor of $1/N!$ that would appear in $W_i$ for indistinguishable fermions is recovered in the prefactor that expresses the number of ways of populating the macro-cells. For a tutorial on Fermi gases, see Schekochihin (2019, §§16–17).

\(^{77}\) One way in which this might not be true would be if our system had further invariants that constrained its dynamics (at least on the time scales of interest). An example is a magnetised plasma in which particles conserve adiabatic invariants (cf. Helander 2017). Another is galactic dynamics—Lynden-Bell’s original preoccupation—where stars are approximately tied to the Keplerian ellipses they follow around the central black hole (Binney 2016).
is, however, not hard to remove. Let us discretise the values that the distribution function can take and index them by \( J \), so a general distribution function is represented as a superposition of waterbags:

\[
    f(q) = \sum_J f_J(q), \quad f_J(q) = \eta_J \text{ or } 0. \tag{10.11}
\]

If there are \( N_J \) phase elements with density \( \eta_J \), then \( \delta \Gamma N_J \) is the phase-space volume occupied by the \( J \)-th waterbag, i.e., the phase-space volume where \( f = \eta_J \). This is conserved by collisionless evolution of \( f \). The corresponding number of particles is \( N_J = \eta_J \delta \Gamma N_J \). As before, we may now coarse-grain \( f \) over groups (macro-cells) of \( M \) microcells and represent the resulting \( \bar{f} \) in terms of occupation numbers \( N_iJ \) of the \( i \)-th macro-cell by elements of phase density \( \eta_J \). Show that

\[
    \bar{f} = \sum_J \bar{f}_J, \quad \bar{f}_J = \frac{\eta_J e^{-\beta \eta_J (\varepsilon - \mu_J)}}{1 + \sum_J e^{-\beta \eta_J (\varepsilon - \mu_J)}}, \tag{10.12}
\]

where \( \mu_J \) and \( \beta \) are determined by

\[
    N_J = \int d^6q \bar{f}_J, \quad K = \int d^6q \varepsilon \bar{f}. \tag{10.13}
\]

Thus, the more general equilibrium distribution is a kind of superposition of many Fermi–Dirac distributions or, in the non-degenerate limit, of Maxwellians, with effective temperatures \( 1/\beta \eta_J \). Considering the Maxwellian limit, or otherwise, propose a natural way to define the overall temperature (Lynden-Bell 1967 has an answer to this, which is, in fact, not the most natural one).

**Exercise 10.2.** **Gardner distribution is a Lynden-Bell distribution.** Consider a Gardner distribution, i.e., a distribution function that is a monotonically decreasing function of \( \varepsilon \) alone (cf. §9.2). Discretise it to be a multi-waterbag distribution (10.11). Show that, under an appropriate coarse-graining, the corresponding Lynden-Bell equilibrium (10.12) is, in fact, the same distribution, i.e., that Lynden-Bell’s entropy-maximisation procedure does not change Gardner’s distribution.\(^{78}\) Thinking of Vlasov–Maxwell equations, explain why that is.

**10.1.2. Warm and Cold Beams**

Let me offer a very simple example of a physically plausible waterbag distribution: a pair of beams (streams) homogeneous in position space with density \( n_b \), with velocities \( \pm u_b \) in some direction, and constant velocity-space density over a region of width \( v_b \) around \( \pm u_b \)—similar to what is illustrated in Fig. 12b but with square-top (\( \Pi \)-shaped) humps, in 3D. This is a waterbag with

\[
    \eta \sim \frac{n_b}{v_b^3}, \quad K \sim n_b Vmu_b^2. \tag{10.14}
\]

It is non-degenerate if \( v_b \ll u_b \), i.e., if the beams are “cold”—as is obvious from (10.10). One therefore expects to see a distribution close to a Maxwellian with \( T \sim mu_b^2 \) at the end of the relaxation process. In contrast, “warm” beams, \( v_b \sim u_b \), should tend to a Fermi–Dirac flat-topped equilibrium with Fermi energy \( \varepsilon_F \sim mu_b^2 \). Remarkably, there appears to be some recent numerical evidence that something like this might indeed be happening (Skoutnev et al. 2019).

**Exercise 10.3.** For some convenient specific model of the \( v_b \)-sized beam shape in velocity space, work out and plot (or at least sketch) \( T/mu_b^2 \) and \( \mu/mu_b^2 \) as functions of \( v_b/u_b \), and hence work out the Lynden-Bell equilibria. In the limits \( v_b \ll u_b \) or \( T \ll mu_b^2 \), everything should be doable analytically (in the latter case, via Sommerfeld expansion; see, e.g., §17.3.3 of Schekochihin 2019).

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\(^{78}\)Intuitively, this should be the case, but it is, in fact, neither obvious nor trivial to prove. I owe this result to a conversation with Per Helander (2019).
10.2. QL Relaxation

I am now going to go back to the QL scheme and use it to derive a kind of “collisionless collision integral” that evolves the mean distribution function of a collisional, homogeneous, electrostatic plasma towards some equilibria that could perhaps be argued to be universal—for an initial waterbag distribution, we will recover Fermi–Dirac statistics (10.6) as a solution of our collision integral.

So the equilibrium (slow, space- and time-averaged) distribution \( f_0 \) again evolves according to (2.11), which I now would like to rewrite as follows

\[
\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \left[ -\frac{q}{m} \sum_k ik \langle \varphi_k^* \delta f_k \rangle \right] = \frac{\partial}{\partial v} \left[ -\frac{e}{m} \sum_k k \text{Im} \langle \varphi_k^* \delta f_k \rangle \right],
\]

(10.15)

where the imaginary part was distilled out of the \( k \) integral by splitting the latter in two equal parts and replacing \( k \rightarrow -k \) in one of them [cf. (6.8)]; I have also specialised to electrons (\( q = -e \)) and will assume, for simplicity, that ions play no role apart from providing a homogeneous neutralising background. The perturbation \( \delta f = f - f_0 \) satisfies (2.12) exactly and its linear part (2.13) approximately, within the QL approximation.

The Laplace-transformed solution of this equation is given by (3.8) and (3.13), which, assuming for simplicity that only electrons have \( \delta f \), are

\[
\hat{\delta f}_k(p) = -i \frac{e}{m} \frac{\hat{\varphi}(p)}{p + i k \cdot v} k \cdot \frac{\partial f_0}{\partial v} + \hat{h}_k(p),
\]

(10.16)

\[
\hat{h}_k(p) = \frac{g_k(v)}{p + i k \cdot v},
\]

(10.17)

\[
\hat{\varphi}_k(p) = -i \frac{4\pi e}{k^2 \epsilon(p, k)} \int d^3v \hat{h}_k(p),
\]

(10.18)

\[
\epsilon(p, k) = 1 - i \frac{4\pi e^2}{mk^2} \int d^3v \frac{1}{p + i k \cdot v} k \cdot \frac{\partial f_0}{\partial v}.
\]

(10.19)

The time-dependent solution is recovered via the inverse Laplace transform (3.14), where
I now wish to change the integration variable to $p = -i\omega + \sigma$:

$$\delta f_k(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{(-i\omega+\sigma)t} \delta f_{k\omega}, \quad \varphi_k(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{(-i\omega+\sigma)t} \varphi_{k\omega}, \quad (10.20)$$

where the Laplace transformed functions have been redefined to look almost like Fourier transformed ones:

$$\delta f_{k\omega} = \delta f_k(-i\omega + \sigma) = \frac{e}{m} \frac{\varphi_{k\omega}}{\omega - k \cdot v + i\sigma} k \cdot \frac{\partial f_0}{\partial v} + h_{k\omega}, \quad (10.21)$$

$$h_{k\omega} = \hat{h}_k(-i\omega + \sigma) = \frac{ig_k(v)}{\omega - k \cdot v + i\sigma}, \quad (10.22)$$

$$\hat{\varphi}_{k\omega} = \hat{\varphi}_k(-i\omega + \sigma) = -\frac{4\pi e}{k^2 \epsilon_{k\omega}} \int d^3v h_{k\omega}, \quad (10.23)$$

$$\epsilon_{k\omega} = \epsilon(-i\omega + \sigma, k) = 1 + \frac{4\pi e^2}{mk^2} \int d^3v \frac{1}{\omega - k \cdot v + i\sigma} k \cdot \frac{\partial f_0}{\partial v}. \quad (10.24)$$

So far I have not introduced any extra assumptions, but the reason for this rearrangement is that I now want to assume that $\hat{\varphi}_k(p)$ and, therefore, $\delta f_k(p)$ have no poles at $Re \sigma > 0$, i.e., $f_0$ supports no instabilities, so I may let $\sigma \to +0$ and integrate along the real line in $\omega$, only needing the infinitesimal $\sigma$ in the denominators to tell me how to circumvent the ballistic pole $\omega = k \cdot v$.

In the context of collisional relaxation of $f_0$, this assumption is justified as follows. The initial distribution may well be unstable, but we can always wait for its instabilities to get excited, saturate and change $f_0$ in such a way to shut themselves down (as they did in §6). After that, we have a distribution function that is stable or, at worst, marginally stable, i.e., $\epsilon_{k\omega}$ might have zeros on the real-$\omega$ line and certainly at $Im \omega < 0 (Re \sigma \leq 0)$, but not at $Im \omega > 0 (Re \sigma > 0)$.

We are ready to roll. Substituting (10.20) and (10.21) into (10.15), we get\(^79\)

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \sum_k k \operatorname{Im} \int \int \frac{d\omega d\omega'}{(2\pi)^2} e^{-i(\omega-\omega')t} \left( -\frac{e^2}{m^2} \frac{\langle \varphi_{k\omega} \varphi_{k'\omega}^* \rangle}{\omega - k \cdot v + i\sigma} k \cdot \frac{\partial f_0}{\partial v} - \frac{e}{m} \langle h_{k\omega} \varphi_{k'\omega}^* \rangle \right). \quad (10.25)$$

The first term is the familiar QL diffusion (here written for a general-form $\varphi$) whereas the second term turns out to be a kind of drag [cf. (1.47)], whose form we shall now elaborate a little further. Using (10.23), we get

$$-\frac{e}{m} \langle h_{k\omega}(v) \varphi_{k\omega}^* \rangle = \frac{4\pi e^2}{mk^2 \epsilon_{k\omega}} \int d^3v' \langle h_{k\omega}(v) h_{k\omega}^*(v') \rangle = \frac{4\pi e^2}{mk^2 \epsilon_{k\omega} \epsilon_{k'\omega}} \int d^3v' \left[ C_{k'\omega}(v, v') + \frac{4\pi e^2}{mk^2} \int d^3v'' \frac{C_{k'\omega}(v, v')}{\omega - k \cdot v'' + i\sigma} k \cdot \frac{\partial f_0(v'')}{\partial v''} \right], \quad (10.26)$$

where $C_{k'\omega}(v, v') = \langle h_{k\omega}(v) h_{k\omega}^*(v') \rangle$ and the last expression was obtained by multiplying and dividing by $\epsilon_{k\omega}$. The reason for this seemingly gratuitous manipulation is two-fold. First, the first term in (10.26) in fact vanishes after it is plugged into (10.25), integrated over frequencies and taken the imaginary part of.

**Exercise 10.4.** Show that this is true. An assumption needed for that is that the correlation

\(^79\)A conscientious reader might at this point become worried about what exactly is meant by averages in (10.25). She will find some succour at the beginning of §10.2.1.
function $\langle h(\mathbf{v})h(\mathbf{v}') \rangle$ is symmetric with respect to swapping velocities, $\mathbf{v} \leftrightarrow \mathbf{v}'$. Hint: it is useful to introduce $\hat{h}_k(t) = (1/2\pi) \int d\omega \ e^{-i\omega t} h_{k\omega}/\epsilon_{k\omega}$.

Secondly, the first term in (10.25), upon insertion of (10.23), turns into an expression of a similar form to the second term in (10.26):

$$
g = \frac{e^2}{m^2} \langle \varphi_{k\omega} \varphi^*_{k'\omega'} \rangle \frac{\partial f_0}{\partial \mathbf{v}} = -\frac{16\pi^2 e^4}{m^2 k^3 \epsilon_{k\omega} \epsilon_{k'\omega'}} \int d^3 \mathbf{v}' d^3 \mathbf{v}'' \frac{C_{k\omega'}(\mathbf{v}'', \mathbf{v}')}{\omega - k \cdot \mathbf{v} + i\sigma} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}.
$$

Putting all this together, we end up with (10.25) in the following form:

$$
\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \int d^3 \mathbf{v}'' \left[ D(\mathbf{v}, \mathbf{v}'') \cdot \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} - D(\mathbf{v}'', \mathbf{v}) \cdot \frac{\partial f_0(\mathbf{v}'')}{\partial \mathbf{v}''} \right],
$$

where the “diffusion kernel” is

$$
D(\mathbf{v}'', \mathbf{v}) = -\frac{16\pi^2 e^4}{m^2} \text{Im} \sum_k \frac{k k^4 \epsilon^*_{k\omega}}{k^4} \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{e^{-i(\omega - \omega') t}}{\epsilon_{k\omega} \epsilon_{k'\omega'}} \int d^3 \mathbf{v}' C_{k\omega'}(\mathbf{v}, \mathbf{v}') \frac{\omega - k \cdot \mathbf{v} + i\sigma}{(\omega - k \cdot \mathbf{v} + i\sigma)(\omega' - k \cdot \mathbf{v}' - i\sigma)(\omega - k \cdot \mathbf{v} + i\sigma)}.
$$

Note that the QL diffusion coefficient similar to the one appearing in (6.7) is then $\int d^3 \mathbf{v}'' D(\mathbf{v}, \mathbf{v}'')$. There is a further simplification available. Using (10.22), we can rewrite

$$
C_{k\omega'}(\mathbf{v}, \mathbf{v}') = \frac{C_k(\mathbf{v}, \mathbf{v}')}{(\omega - k \cdot \mathbf{v} + i\sigma)(\omega' - k \cdot \mathbf{v}' - i\sigma)}, \quad C_k(\mathbf{v}, \mathbf{v}') = \langle g_k(\mathbf{v}) g_k^*(\mathbf{v}') \rangle,
$$

and, therefore,

$$
D(\mathbf{v}'', \mathbf{v}) = -\frac{16\pi^2 e^4}{m^2} \text{Im} \sum_k \frac{k k^4}{k^4} \int d^3 \mathbf{v}' C_k(\mathbf{v}, \mathbf{v}') I_k(\mathbf{v}, \mathbf{v}', \mathbf{v}''),
$$

$$
I_k(\mathbf{v}, \mathbf{v}', \mathbf{v}'') = \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{e^{-i(\omega - \omega') t}}{\epsilon_{k\omega} \epsilon_{k'\omega'}(\omega - k \cdot \mathbf{v} + i\sigma)(\omega' - k \cdot \mathbf{v}' - i\sigma)(\omega - k \cdot \mathbf{v} + i\sigma)}.
$$

The remaining work is now in calculating the double integral (10.32). If $\epsilon_{k\omega}$ has no poles at real $\omega$ (i.e., $f_0$ is a stable distribution), the integral is done by shifting the $\omega'$ integration contour upwards to $\text{Im} \omega \to +\infty$, but snagging on the pole at $\omega' = k \cdot \mathbf{v}'$ and the $\omega$ contour downwards to $\text{Im} \omega \to -\infty$, snagging on the poles at $\omega = k \cdot \mathbf{v}$ and $\omega = k \cdot \mathbf{v}''$ (see Fig. 39). The result is

$$
I_k(\mathbf{v}, \mathbf{v}', \mathbf{v}'') = \frac{e^{-i k \cdot (\mathbf{v} - \mathbf{v}')}}{\epsilon_{k,k \cdot v} \epsilon^*_{k,k \cdot v'}} \frac{1}{\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')} \left( 1 - e^{-i k \cdot (\mathbf{v} - \mathbf{v}'') t} \frac{\epsilon_{k,k \cdot v}}{\epsilon_{k,k \cdot v}^*} \right),
$$

$$
\to -i\pi \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')) \text{ as } t \to \infty
$$

where the $\delta$-function approximation is obtained in the same way as in (5.39). Thus, (10.31) becomes

$$
D(\mathbf{v}'', \mathbf{v}) = \frac{16\pi^3 e^4}{m^2} \text{Re} \sum_k \frac{k k^4}{k^4} \delta(\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}'')) \int d^3 \mathbf{v}' C_k(\mathbf{v}, \mathbf{v}') e^{-i k \cdot (\mathbf{v} - \mathbf{v}') t}.
$$

10.2.1. Kadomtsev–Pogutse “Collision” Integral

In order to make further progress, we must discuss what is meant by the “initial” distribution $g_k(\mathbf{v})$ and its correlation function $C_k(\mathbf{v}, \mathbf{v}')$. Let us imagine that we start the
evolution of our collisionless system with some initial distribution—generally speaking, unstable—and let it proceed for a while. Instabilities might flare up and saturate, particles will stream and phase-mix the distribution, etc., so fairly quickly the exact $f$ will become stable but extremely chopped up and fine-structured in phase space.\footnote{This is the part of the evolution that Lynden-Bell (1967) calls “violent relaxation”.} It is at such a point that we pick it up and treat it as an “initial” state, from which we then examine its further evolution. This involves the evolution of $f_0$, which is an average over space and “fast” times (meaning frequencies associated with any plasma processes, e.g., $\sim \omega_{pe}$, and with particle streaming, $\sim k \cdot v$), and the evolution of $\delta f = f - f_0$, whose “initial” state is $g$ and which contains all the fine structure in phase space. It is then reasonable to think of $g$ as essentially random.

The right-hand side of (10.28) encodes further evolution of $\delta f$ and its effect on $f_0$ over time $t$ that is long compared to streaming times (\(k \cdot v t \gg 1\)) but short compared to the times over which $f_0$ changes significantly.\footnote{Note that the integrand of the $v'$ integral in (10.34) is the correlation function of $g_k(v)e^{-ik \cdot v}/\epsilon_{k,k'}$, which is simply the “initial” distribution evolved ballistically (see §5.3) and “attenuated” by the dielectric response.} In order to calculate (10.34) and, therefore, (10.28), we need to know the correlation function $C_k(v,v')$ of $g$. In a somewhat bold move, let us approximate

$$
\langle g(r,v)g(r',v') \rangle \approx \langle g(r,v)^2 \rangle \Delta \Gamma \delta(r-r')\delta(v-v'),
$$

(10.35)

where $\Delta \Gamma$ is the phase-space volume representing the “width” of the two delta functions. A way to make (10.35) almost true by construction is to redefine our average as coarse-graining over phase-space macro-cells of at least the volume $\Delta \Gamma = \Delta r^3 \Delta v^3$, where $\Delta r$ and $\Delta v$ are, respectively, the position- and velocity-space correlation scales of $g$. I said “almost” because the whole scheme depends on it being possible also to make both $\Delta r$ and $\Delta v$ sufficiently small:

$$
\Delta r \ll V^{1/3}, \quad \Delta v \ll \left( \frac{K}{mN} \right)^{1/2},
$$

(10.36)

where $V$ is the system’s volume and $K$ its total energy. In other words, we are still assuming that, as a result of phase mixing, $g$ would lose any system-scale correlations in either position or velocity space.

With this assumption and this definition of averaging, $f_0$ is the same as $\bar{f}$ in §10.1 and the macro-cells over which it is coarse-grained are the same randomly filled macro-cells as in the Lynden-Bell statistics (so, in the language of §10.1, $\Delta \Gamma = M \delta \Gamma$). Unsurprisingly, the same result is about to pop out. Indeed, let us, for simplicity (and for lack of a better...
idea), assume that the exact distribution is a waterbag (10.1). Then
\[ \langle g^2 \rangle = \langle (f - f_0)^2 \rangle = \langle f^2 \rangle - f_0^2 = (\eta - f_0)f_0, \] (10.37)
because, for a waterbag distribution, \( \langle f^2 \rangle = \langle \eta f \rangle = \eta f_0 \).

Finally, using (10.35) and then (10.37), we get
\[ C_k(v, v') = \langle g_k(v)g'_k(v') \rangle = \int \int \frac{d^3r d^3r'}{V^2} e^{-k \cdot (r-r')} \langle g(r, v)g(r', v') \rangle = (\eta - f_0(v))f_0(v) \delta(v - v'). \] (10.38)

With this model of phase-space correlations, (10.28) combined with (10.34) turns into
\[ \frac{\partial f_0}{\partial t} = \frac{16\pi^3 e^4 \Delta \Gamma}{m^2 V} \frac{\partial}{\partial v} \cdot \int d^3v'' \sum_k \frac{kk}{k^4} \frac{\delta(k \cdot (v - v''))}{|\epsilon_{k,k,v}|^2} \cdot \left[ (\eta - f_0(v''))f_0(v'') \frac{\partial f_0(v)}{\partial v} - (\eta - f_0(v))f_0(v) \frac{\partial f_0(v'')}{\partial v'} \right]. \] (10.39)

This is the “collisionless collision integral” of Kadomtsev & Pogutse (1970) (whose derivation I have more or less followed). It is not hard to show that it has an H-theorem and that this entropy is maximised—and the right-hand-side of (10.39) is annihilated—by the Fermi–Dirac distribution (10.6).

**Exercise 10.5.** Show this.

**Exercise 10.6.** Generalise the above calculation to multiple species (electrons and ions).

**Exercise 10.7.** Generalise the above calculation to multi-waterbag statistics of Exercise 10.1.

**Exercise 10.8.** Is the Hermite spectrum (12.40) obtained in Q7 consistent with the approximation (10.35)?

### 10.2.2. Lenard–Balescu Collision Integral

Just as in §10.1, the non-degenerate limit is \( f_0 \ll \eta \). This turns (10.39) into
\[ \frac{\partial f_0}{\partial t} = \frac{16\pi^3 e^4 \eta \Delta \Gamma}{m^2 V} \frac{\partial}{\partial v} \cdot \int d^3v'' \sum_k \frac{kk}{k^4} \frac{\delta(k \cdot (v - v''))}{|\epsilon_{k,k,v}|^2} \cdot \left[ f_0(v'') \frac{\partial f_0(v)}{\partial v} - f_0(v) \frac{\partial f_0(v'')}{\partial v'} \right]. \] (10.40)

This has a functional form that is identical to the standard Lenard–Balescu collision integral (Lenard 1960; Balescu 1960) and its solution is, obviously, a Maxwellian.

How is it possible that collisionless and collisional behaviour turns out to be the same? Mathematically, this is not hard to understand. Let us recall what is meant

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82It has not escaped my perceptive reader that a potential mathematical illegality is being perpetrated: is coarse-grain average of a coarse-grain average equal to itself, and is it true that \( \langle f f_0 \rangle = \langle f_0 \rangle = f_0^2 \)？ In general, no, not when the coarse-graining is done by convolution with some continuous shape function. However, it is fine if we formally do this as we did in §10.1, by grouping fixed discrete micro-cells into fixed discrete macro-cells. Alternatively, we may argue that the differences between \( f_0 \) and \( \langle f_0 \rangle \) and similar quantities are small enough to be ignored or, cleaner still, resort to an ensemble average over many possible realisations of \( f \), assuming it will do the coarse-graining job automatically. Note that our time average (2.7) in fact suffered from the same problem. Again, we may do an ensemble average and assume that that will automatically have slow time evolution and no spatial dependence (in a homogeneous system).
by collisions in plasma physics. When a plasma (or, rather, a collection of individual particles) is described by its Klimontovich distribution function (1.19), the latter satisfies a “collisionless” Vlasov equation involving microscopic electromagnetic field—the Klimontovich equation (1.35). Collisions acquire a specific mathematical meaning when this equation is averaged (coarse-grained) over the Debye scale, leading to (1.41), where the collision integral is defined as the correlation function of the differences between the averaged (“macroscopic”) and the exact (“microscopic”) fields and distribution. This is basically the same procedure as the one that led to (10.15) and hence to the “collisionless collision integral” (10.28) and its descendants (10.39) and (10.40). If we interpret the exact distribution \( f \) as the Klimontovich distribution, then the difference between it and \( f_0 \) will satisfy (10.35) automatically, because the Klimontovich distribution is only non-zero at the exact positions of the particles in phase space and so the correlation function of \( \delta f \) must needs be a delta function. In this interpretation, \( \Delta \Gamma \) is the effective width of the delta functions associated with individual particles, while \( \eta \) is their height (so the Klimontovich distribution is a kind of waterbag). Clearly, they are related by

\[
\eta \Delta \Gamma = 1, \tag{10.41}
\]

which is all we need to know in (10.40), finally turning it into the Lenard–Balescu collision integral. It is not surprising that the solution is a Maxwellian because the Klimontovich distribution is extremely non-degenerate: all these delta functions always occupy a negligible fraction of the available phase space.

### 10.2.3. Landau’s Collision Integral

While we are at it, let me show for completeness how the Landau (1936) collision integral is recovered from the Lenard–Balescu one. If we want to use (10.40) (with \( \eta \Delta \Gamma = 1 \)) as an expression for \textit{bona fide} collisions, i.e., if \( \delta f \) and \( \varphi \) are interpreted as fluctuating fields below the Debye scale, we must restrict the \( k \) summation to \( k \lambda \gg 1 \). In this approximation, \( |\epsilon_k | k \cdot v |^2 \approx 1 \).

The \( k \) sum then becomes tractable: denoting \( v - v'' = w \), we get

\[
\frac{1}{V} \sum_k \frac{kk}{k^3} \delta(k \cdot (v - v')) = \int \frac{d^3k}{(2\pi)^3} \frac{kk}{k^3} \delta(k \cdot w) = \frac{1}{w} \int \frac{d^3k_\perp}{(2\pi)^3} \frac{k_\perp \cdot k_\perp}{k_\perp^4} = \frac{1}{8\pi^2 w} \left(1 - \frac{ww}{w^2}\right) \int \frac{dk_\perp}{k_\perp},
\]

(10.42)

where \( k_\perp = k \cdot (1 - ww/w^2) \). The divergent integral is the Coulomb logarithm \( \Lambda = \ln(\lambda_D/d) \) if the integration cut off at the distance of closest approach \( d \) [see text after (1.13)] and the Debye scale \( \lambda_D \). Consequently, (10.40) becomes

\[
\frac{\partial f_0}{\partial t} = \frac{2\pi e^4 \Lambda}{m^2} \frac{\partial}{\partial v} \int \frac{d^3v''}{w} \left(1 - \frac{ww}{w^2}\right) \left[ f_0(v'') \frac{\partial f_0}{\partial v} - f_0(v) \frac{\partial f_0}{\partial v''} \right]. \tag{10.43}
\]

This is the Landau (1936) collision integral [cf. (1.47)]. Note that both it and the Lenard–Balescu integral (10.40) have the functional form anticipated in (7.35), which is as it should be.

### 10.2.4. Relation to Standard QL Calculations

Coming soon...
more generally, a kind of multi-temperature Fermi–Dirac distribution (10.12) (provided Exercise 10.7 is a success).

Suspending for now further doubts about the assumptions that have gone into this, let us note that the progress achieved in comparison with the statistical-mechanical calculations of §10.1 is that we can now follow the relaxation of \( f_0 \) in time, at least after some initial period during which all the instabilities of an initial state sort themselves out and \( f_0 \) becomes stable. The rate at which \( f_0 \) tends to the Fermi–Dirac distribution can be read off from (10.39): ignoring any factors of order unity and using (10.42) to estimate the size of the \( k \) sum,\(^{83}\) we get

\[
\nu_{\text{eff}} \sim \nu \eta \Delta \Gamma ,
\]  

(10.44)

where \( \nu \) is the “true” collision frequency. In the “true” collisional limit, \( \eta \Delta \Gamma = 1 \) [see (10.41)], but in the collisionless regime, \( \eta \Delta \Gamma \) (the number of particles there would be in a fully occupied macro-cell whose size is the correlation volume of \( g \)) must be large, so the “collisionless collision frequency” \( \nu_{\text{eff}} \) is generally speaking much larger than \( \nu \).

Can we estimate \( \nu_{\text{eff}} \) in terms of some physical (measurable) parameters?

Since we are dealing with collisionless dynamics, initial distribution will matter and determine \( \eta \), assuming of course that a waterbag is at all a good model for it (it usually will not be, but let us ignore that). One good physical example is some collection of spatially homogeneous beams, all with the same phase density—then, in terms of the beam number density \( n_b \) and width \( v_b \), \( \eta \sim n_b/v_b^3 \) [see (10.14)].

The correlation volume \( \Delta \Gamma = \Delta r^3 \Delta v^3 \) is a trickier quantity to make sense of. As the evolution of a collisionless system proceeds, one might argue that \( g \) would be getting ever more fine-scaled (“phase-mixed”; cf. §5.3), i.e., that \( \Delta v \) (and possibly also \( \Delta r \)) would be decreasing with time (eventually, \( \Delta r \) would be limited by \( \sim \lambda_D \) and \( \Delta v \) by collisions, however small they are—see §5.5—but for now let us assume that collisionless dynamics can continue as long as we like). Under this scenario, \( \Delta \Gamma(t) \) will decrease with time and so the convergence of \( f_0 \) towards equilibrium will slow down as time goes on.

It remains an open research topic exactly how to calculate \( \Delta v(t) \) and \( \Delta r(t) \), or, indeed, more generally, \( \langle g(r,v)g(r',v') \rangle \), i.e., whether (10.35) is at all adequate. A key challenge in this context is to go beyond the QL approximation. One could reinterpret \( h_{kw} \) in (10.21) as containing not just the ballistic evolution (10.22) but also the “nonlinear part” of \( \delta f_{kw} \). The question is what is its correlation function \( C_{kw}(v,v') \), which then goes into the general collision integral (10.28). You will find some thoughts on this in the classic papers by Kadamtev & Pogutse (1971) and Dupree (1972), although I recommend them with some hesitation: neither is particularly transparent, so you might end up with less clarity rather than more.\(^{84}\) The basic idea put forward in these papers is that the correlation scales \( \Delta r \) and \( \Delta v \) will decrease in time much less quickly than you might imagine based on naïve phase-mixing estimates—long-time correlated phase-space “clumps” (or “granulations”) will form. The “collisionless collision integral” (10.28) with an appropriate clump correlation function \( C_{kw}(v,v') \) then represents some effective scattering of particles by these clumps. The status of all this is rather uncertain: there was a lot of analytical work done in the 1970s and 1980s along and around these lines,

\(^{83}\) This includes the \( 1/|\epsilon_{k,k,v}|^2 \) factor. For \( k\lambda_D \gg 1, |\epsilon_{k,k,v}| \approx 1 \) and the contribution to the \( k \) sum from these wave numbers is \( \sim A \); for \( k\lambda_D \ll 1, |\epsilon_{k,k,v}| \sim 1/(k\lambda_D)^2 \) and the contribution from these wave numbers is \( \sim 1 \).

\(^{84}\) The textbook by Diamond et al. (2010, Chapter 8) is on an ideological continuum from Dupree (1972), and, while also not a transparent read (imho), is probably the most up-to-date exposition of the status of that line of thinking.
but most of it has remained hypothetical, due to the difficulty of nonlinear theory and impossibility of kinetic numerical simulations capable of resolving anything at the time. The latter impediment to progress appears set to be lifted in the near future, so now probably is a good time to revisit the old theories and attempt new ones.

11. Plasma Echo, Phase Unmixing, and Phase-Space Turbulence

In §5.3, we saw that the perturbed distribution function (5.25) associated with a Landau-damped electric perturbation contained an undamped ballistic term proportional to $e^{-ik\cdot v t}$. As time went on, this term became ever more oscillatory in $v$ and so no longer contributed to $\varphi$, but the information contained in it was not destroyed until it hit collisions. It turns out that this information can be retrieved.

Let us think in 1D and imagine sending an electric pulse into plasma (i.e., setting up some initial perturbation) at time $t_1$, with wave number $k_1$. It will be Landau damped, but the undamped ballistic part of the resulting perturbed distribution function will be

\[ \delta f_{k_1} = a_1 e^{-ik_1 v (t-t_1)}, \]

where $a_1$ is the contents of the curly bracket in (5.25) and $v$ the component of the velocity parallel to the direction of the wave vector. Now let us wait for a while and send in another pulse, with $k_2 > k_1$ at $t_2 > t_1$. This too will be Landau damped, leaving behind

\[ \delta f_{k_2} = a_2 e^{-ik_2 v (t-t_2)}. \]

In the linear approximation, these are independent solutions, which simply add up. But if nonlinearity is allowed, they can couple to produce

\[ \delta f_{k_2-k_1} \propto \delta f_{k_2} \delta f_{k_1}^* = a_1^* a_2 e^{-ik_2 v (t-t_2) + ik_1 v (t-t_1)} = a_1^* a_2 e^{-i[(k_2-k_1) t-(k_2 t_2-k_1 t_1)] v}. \]

Around the time

\[ t_{\text{echo}} = \frac{k_2 t_2 - k_1 t_1}{k_2 - k_1}, \]

the distribution function (11.3) does not oscillate strongly in $v$ and so can have an associated electric perturbation $\varphi_{k_2-k_1} \propto \int dv \delta f_{k_2-k_1}$, which will pop out of nowhere some time after the second pulse has gone in (e.g., if $k_2 = 2k_1$, $t_{\text{echo}} = 2t_2 - t_1 > t_2$; Fig. 40). Physically what happens is that the second pulse “catches up” with the first pulse in phase space couples to it and effectively reverses the direction of phase mixing, producing “phase unmixing” and bringing information back into the electric field. This is called plasma echo and is a strange but real thing: having been derived by Gould et al. (1967), it was promptly produced in a laboratory by Malmberg et al. (1968).

The derivation is a nice exercise in perturbation theory, which I will reproduce in §11.1.
But this is not a mere curiosity: in nonlinear plasma systems with many interacting modes (i.e., for plasma turbulence), the implication may be that Landau damping and phase mixing are suppressed, or even altogether disabled, thus rendering nonlinear plasma more “fluid-like” than the linear one is. This is a research frontier—I will discuss it in §11.2.

11.1. Textbook Derivation of Plasma Echo

Let us consider a situation in which only the electron distribution function is perturbed (i.e., frequencies are high) and its perturbation satisfies, as usual [cf. (2.12)],

$$\frac{\partial \delta f_k}{\partial t} + i k \cdot v \delta f_k + i \frac{e}{m} \varphi_k k' \cdot \frac{\partial f_0}{\partial v} = -i \frac{e}{m} \sum_{k'} \varphi_{k'} k' \cdot \frac{\partial \delta f_{k-k'}}{\partial v} \equiv N_k(v, t),$$

(11.5)

where $\varphi_k$ contains both the self-consistent field arising from $\delta f_k$ and an externally applied field $\chi_k$ [cf. (2.9) and Q6]:

$$\varphi_k = \chi_k - \frac{4\pi e}{k^2} \int d^3v \delta f_k.$$  

(11.6)

We shall solve an initial-value problem with $\delta f_k(t = 0) = 0$ and use $\chi_k$ to set up our two pulses:

$$\chi(t, r) = A_1 \delta(t - t_1) \cos k_1 z + A_2 \delta(t - t_2) \cos k_2 z,$$

(11.7)

where $z$ is the axis along which they propagate and $t_2 > t_1 > 0$, $k_2 > k_1$ (the case of $A_2 = 0$, $t_1 = 0$, and $N_k = 0$ is the standard Landau-damping problem solved in §3).

Laplace transforming everything, we get

$$\delta f_k(p) = \frac{1}{p + ik \cdot v} \left[ -i \frac{e}{m} \varphi_k(p) k \cdot \frac{\partial f_0}{\partial v} + \hat{N}_k(p) \right].$$

(11.8)

This is not really a solution, just a rewriting of (11.5), because the Laplace-transformed nonlinear term $\hat{N}_k(p)$ contains $\delta f_k$:

$$\hat{N}_k(p) = -i \frac{e}{m} \sum_{k'} k' \cdot \frac{\partial}{\partial v} \int_0^\infty dt e^{-pt} \int_{-i\infty + \sigma'}^{i\infty + \sigma'} \int_{-i\infty + \sigma''}^{i\infty + \sigma''} \frac{dp dp'' dp'}{(2\pi i)^2} e^{(p' + p'' + p)t} \hat{\varphi}_{k'}(p') \hat{\delta f}_{k-k'}(p''),$$

(11.9)

where $Re p > \sigma' + \sigma''$ and I will drop, but remember, the integration limits. Keeping this expression in mind, let us combine (11.6) and (11.8):

$$\hat{\varphi}_k(p) = \frac{\hat{\chi}_k(p)}{\epsilon(p, k)} - \frac{4\pi e}{k^2 \epsilon(p, k)} \int d^3v \frac{\hat{N}_k(p)}{p + ik \cdot v},$$

(11.10)

where $\epsilon(p, k)$ is the usual dielectric function, most recently written out in (10.19). The first term in (11.10) is the linear response [cf. (12.18) in Q6], while the second one is the nonlinear one. Assuming the latter to be small, we can calculate it perturbatively, i.e.,

$$\hat{N}_k(p) = \frac{e^2}{m^2} \sum_{k'} k' \cdot \frac{\partial}{\partial v} \int \frac{dp dp'' dp'}{(2\pi i)^2} \frac{\hat{\chi}_{k'}(p') \hat{\chi}_{k-k'}(p'')}{\epsilon(p', k') \epsilon(p'', k - k') [p'' + i(k - k') \cdot v] (p' + p'' - p)} \times (k - k') \cdot \frac{\partial f_0}{\partial v}.$$  

(11.11)

This is as far as we can go without inputting specific information about $\hat{\chi}_k(p)$, which
we shall need inorder to do the $p'$ and $p''$ integrals. From (11.7),

$$\hat{\chi}(p) = \frac{A_1}{2} e^{-pt_1} (\delta_{k,k_1} + \delta_{k,-k_1}) + \frac{A_2}{2} e^{-pt_2} (\delta_{k,k_2} + \delta_{k,-k_2}).$$

(11.12)

There will be a lot of terms in $\hat{\chi}_k'(p')\hat{\chi}_k''(p'')$, but we only want the ones that are proportional to $\delta_{k,k_2-k_1}$ (it is a tedious, but character-building exercise to confirm that there are no echoes in the rest of them). They are

$$\frac{A_1 A_2}{4} \delta_{k,k_2-k_1} \left( e^{-p't_1-p''t_2} \delta_{k',-k_1} + e^{-p't_2-p''t_1} \delta_{k',k_2} \right).$$

(11.13)

Putting this into (11.11) and swapping integration variables $p' \leftrightarrow p''$ in the second term, we get, for $k = k_2 - k_1$,

$$\hat{N}_k(p) = -\frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} \int dp' dp'' \frac{e^{-p't_1-p''t_2}}{(2\pi i)^2 \epsilon(p',-k_1)\epsilon(p'',k_2)(p''+p''-p)} \times \left( \frac{1}{p''+ik_2 v_z} + \frac{1}{p'-ik_1 v_z} \right) \frac{\partial f_0}{\partial v_z}$$

$$= \frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} \int dp' \frac{e^{p'(t_2-t_1)-pt_2}(p+ik_2 v_z)}{2\pi i \epsilon(p',-k_1)\epsilon(p'-p'',k_2)(p'-p'+ik_2 v_z)(p'-ik_1 v_z)} \frac{\partial f_0}{\partial v_z}$$

$$= \frac{e^2 A_1 A_2 k_1 k_2}{4m^2} \frac{\partial}{\partial v_z} e^{ik_1 v_z(t_2-t_1)-pt_2} \frac{\partial f_0}{\partial v_z}.$$  

(11.14)

In the last two lines, I did the the $p''$ and $p'$, in that order. The $p''$ integral is done by pushing the integration contour rightwards to Re $p'' \to \infty$, but circumventing the pole at $p'' = p - p'$ clockwise; only the pole’s contribution survives because $e^{-p''t_2} \to 0$ at Re $p'' \to \infty$ (Fig. 41). The $p'$ integral ix done by pushing the contour leftwards to Re $p' \to -\infty$, but circumventing the ballistic pole at $p' = ik_1 v_z$ anticlockwise; again only the pole’s contribution survives because $e^{p'(t_2-t_1)} \to 0$ at Re $p' \to -\infty$ (Fig. 41). The dielectric function is assumed to have no poles at Re $p' > 0$ (i.e., there is only damping, no instabilities) and the Landau-damping poles it might have at Re $p' < 0$ are ignored because they will give rise to damped solutions, not echoes.

Finally, let us put (11.14) into (11.10), integrate out $v_x$ and $v_y$ dependence, so $f_0(v) \to$
\( F(v_z) \) [cf. (3.18)], and inverse Laplace transform, to get, for the echo (with \( k = k_2 - k_1 \)),

\[
\varphi_k^{\text{echo}}(t) = -\frac{4\pi e}{k^2} \int dv_z \int \frac{dp}{2\pi i} \frac{e^{ip}\tilde{N}_k(p)}{\epsilon(p,k)(p + ikv_z)}
\]

\[
= -i\frac{e^3A_1A_2k_1k_2}{m^2k} \int dv_z \int \frac{dp}{2\pi i} e^{ip(t-t_2)+ik_1v_z(t_2-t_1)} \epsilon(p,k)\epsilon(-ikv_z,k) \epsilon(p-ikv_z,k_2)(p + ikv_z)^2 \frac{\partial F}{\partial v_z}
\]

\[
= -i(t-t_2)\frac{e^3A_1A_2k_1k_2}{m^2k} \int dv_z \int \frac{dp}{2\pi i} e^{-i[k(t-t_2)-k_1(t_2-t_1)]v_z} \epsilon(-ikv_z,k) \epsilon(ikv_z,k_2)\epsilon(-ikv_z,k_2) \frac{\partial F}{\partial v_z}.
\]

In the second line, I integrated by parts with respect to \( v_z \), producing a second-order ballistic pole at \( p = -ikv_z \), and in the third line, I profited from this pole by pushing the \( p \)-integration contour leftwards to \( \text{Re} \, p \to -\infty \), using the fact that \( e^{ip(t-t_2)} \to 0 \) there, for \( t > t_2 \), and thus being left only with the contribution from the pole, circumnavigated anticlockwise (Fig. 41). Note that, since the pole is second order, I had to differentiate the expression under the integral with respect to \( p \) to obtain the residue—in doing so, I ignored the derivatives of the dielectric functions in comparison with the derivative \( \frac{\partial F}{\partial v_z} \), because the latter was large in \( t - t_2 \). Note that the exponential in (11.15) that oscillates in \( v_z \) does not in fact oscillate very much around precisely the time \( t = t_{\text{echo}} \) given by (11.4), so we have picked up the effect we were looking for!

There is a nice, simple answer in the limit of \( k\lambda_{\text{De}} \gg 1 \), when all dielectric functions can be approximated by unity (and Landau damping is strong). Assuming \( F(v_z) \) is a Maxwellian, we get

\[
\varphi_k^{\text{echo}}(t) \approx (t-t_{\text{echo}})(t-t_2)\frac{e^{3A_1A_2k_1k_2}}{m^2k} e^{-k^2v_{\text{th}}^2(t-t_{\text{echo}})^2/4}.
\]

Thus, the echo pulse arises at \( t = t_{\text{echo}} \), and then phase mixes away again.

**Exercise 11.1. Plasma echo for weakly damped waves.** Work out \( \varphi_k^{\text{echo}}(t) \) in the opposite limit, \( k\lambda_{\text{De}} \ll 1 \), when the Landau-damping rates are exponentially small and the \( v_z \) integral in (11.15) can be done by picking up the poles associated with the zeros of the dielectric functions in the denominator. Show in particular that the echo pulse first grows exponentially at the rate \( \gamma_{k_1}(k_2 - k_1)/k_1 \) and then decays at the rate \( \gamma_{k_2-k_1} \), where \( \gamma_k \) is the Landau-damping rate at the wave number \( k \). You will find the solution in Gould et al. (1967).

11.2. Phase-Space Turbulence and Stochastic Echo

Let me now show how perturbations being dragged back from phase space by nonlinear interactions can be described formally. In the process, we will learn a new way of looking at phase mixing—and unmixing—and gain some insight into the nature of “phase-space turbulence”.

In §10.2, we saw that a rather crucial role in the “collisionless collision integral” (10.28) was played by the correlation function \( C_k(v,v') \) of the perturbed distribution function \( g_k(v) \). In that calculation, I treated \( g_k(v) \) as an initial condition from which we took only a short step quasilinearly. But, as I argued in §10.2.1, this function had to be viewed as containing the stochastic phase-space structure that emerged from nonlinear evolution.\(^{85}\) So let me use this background as a reason to be interested in the correlation function

\[
C_k(v,v') = \langle \delta f_k(v)\delta f_k^*(v') \rangle.
\]

\(^{85}\)Another way to think of this is by noticing the formal similarity between (10.16–10.17) and (11.8), except in the latter equation, the nonlinearity \( \tilde{N}_k(p) \) appears where \( g_k \) did in the former.
Phase mixing should manifest itself as $C_k(v, v')$ developing ever smaller-scale dependence on $v - v'$, and phase unmixing the opposite.

Since
\[
\frac{\partial C_k(v, v')}{\partial t} = \left\langle \delta f^*_k(v') \frac{\partial \delta f_k(v)}{\partial t} + \delta f_k(v) \frac{\partial \delta f^*_k(v')}{\partial t} \right\rangle,
\] (11.18)
to derive the evolution equation for $C_k(v, v')$, we must multiply (11.5) by $\delta f^*_k(v')$ and add to the resulting equation its complex conjugate with $v$ and $v'$ swapped:
\[
\frac{\partial C_k(v, v')}{\partial t} + ik \cdot (v - v')C_k(v, v') = -i \frac{e}{m} \left[ \left\langle \varphi_k \delta f^*_k(v') \right\rangle k \cdot \frac{\partial f_0}{\partial v} \right.
\]
\[
+ \sum_{k'} k' \cdot \frac{\partial}{\partial v} \left\langle \varphi_{k'} \delta f_{k-k'}(v) \delta f^*_k(v') \right\rangle \left. \right\rangle + (v \leftrightarrow v')^* \equiv S_k(v, v') + N_k(v, v').
\] (11.19)

11.2.1. Hacking the Closure Problem

With 3rd-order correlators appearing in the evolution equation for the 2nd-order one, we are faced with the usual “closure problem” of nonlinear theory. In order to make the points I wish to make, I am going to have to make some rather brutal simplifications. Namely, I will treat the field $\varphi_k$ as a known field, i.e., ignore any self-consistent contribution that $\delta f_k$ makes to it. This can be interpreted as approximating $\varphi_k \approx \chi_k$ and ignoring the second term in (11.6), or as using just the linear approximation in (11.10), or as hoping, non-rigorously, that the effect of correlations between $\delta f_k(v)$ under the velocity integral in (11.6) and $\delta f$’s appearing in the right-hand side of (11.19) can be ignored. Furthermore, I will assume that $\varphi_k(t)$ is a random and rapidly decorrelating field—more rapidly than $\delta f_k(t, v)$ evolves either linearly or nonlinearly. This is almost certainly not a great assumption unless we are literally imposing a white-noise field onto our system, but it will have to do if we want some kind of closure.

Thus, the assumption is [cf. (12.21) in Q6]
\[
\frac{e^2}{m^2} \left\langle \varphi_k(t) \varphi_{k'}^*(t') \right\rangle = 2D_k \delta_{k, k'} \delta(t - t'),
\] (11.20)
where $D_k$ is a known function.\(^{86}\) The correlators in the right-hand side of (11.19) can then be computed as follows. Let us formally integrate (11.5) up to time $t$:
\[
\delta f_k(t) = -i \int^t dt' \left\{ k \cdot v \delta f_k(t') + \frac{e}{m} \left[ \varphi_k(t') k \cdot \frac{\partial f_0}{\partial v} + \sum_{k'} \varphi_{k'}(t') k' \cdot \frac{\partial \delta f_{k-k'}(t')}{\partial v} \right] \right\}
\] (11.21)
It does not matter from when in the past we integrate, because only a very short period immediately preceding $t$ will contribute to correlations between $\delta f(t)$ and $\varphi(t)$. Let us now substitute (11.21) into the right-hand side of (11.19). Since $\delta f(t)$ can only depend on the values of $\varphi(t')$ in the past, $t' < t$, we may split any correlators between $\delta f(t')$ and

\(^{86}\)Here $k = |k|$, so statistical isotropy is assumed, as is statistical spatial homegeneity: $\delta_{k, k'}$ enforces the latter, viz., the requirement that $\left\langle \varphi(r, t) \varphi(r', t') \right\rangle$ is a function only of $r - r'$ (it is a simple exercise to confirm this).
\( \varphi(t') \) or \( \varphi(t) \). Namely:

\[
-\frac{i}{m} \frac{e}{\varphi_k(t)} \langle \varphi_k(t) \delta f_{k'}^*(t, v') \rangle = \frac{e}{m} \int_0^t dt' \left\{ k \cdot v \left( \frac{\delta f_{k'}^*(t', v')}{\varphi_k(t')} \right) \right\} \\
+ \frac{e}{m} \left[ \frac{\varphi_k(t) \varphi_k(t')}{k'} k \cdot \frac{\partial f_0(v')}{\partial v'} \right. \\
+ \sum_{k'} \left( \varphi_k(t) \varphi_{k'}^*(t') \right) k' \cdot \frac{\partial \left( \delta f_{k-k'}^*(t', v') \right)}{\partial v'} \right] = D_k \cdot \frac{\partial f_0(v')}{\partial v'}, \\
(11.22)
\]

so the source term in (11.19) is

\[
S_k(v, v') = 2 D_k k' : \frac{\partial f_0(v)}{\partial v} \frac{\partial f_0(v')}{\partial v'}. \\
(11.23)
\]

In the same vein, to calculate \( N_k(v, v') \), we may construct an evolution equation for \( \delta f_{k-k'}(t, v) \delta f_{k'}^*(t, v') \) and formally integrate it:

\[
\delta f_{k-k'}(t, v) \delta f_{k'}^*(t, v') = -i \int_0^t dt' \left\{ \frac{e}{m} \sum_{k''} \left[ \varphi_{k''}(t') k'' \cdot \frac{\partial \delta f_{k-k''}(t', v)}{\partial v} \delta f_{k'}^*(t', v') \right. \\
- \left. \varphi_{k''}^*(t') k'' \cdot \frac{\partial \delta f_{k-k''}(t', v')}{\partial v'} \delta f_{k-k'}(t', v) \right] + \ldots \right\}. \\
(11.24)
\]

The terms hiding behind “...” will not matter because all odd-order correlators of \( \varphi \) vanish when we work out

\[
-\frac{i}{m} \frac{e}{\varphi_k(t)} \langle \varphi_k(t) \delta f_{k-k'}(t, v) \delta f_{k'}^*(t, v') \rangle \\
= D_k \left[ k' \cdot \frac{\partial}{\partial v} \langle \delta f_k(t, v) \delta f_{k'}^*(t, v') \rangle + k' \cdot \frac{\partial}{\partial v'} \langle \delta f_{k-k'}^*(t', v') \delta f_{k-k'}(t', v) \rangle \right] \\
= D_k k' \left[ \frac{\partial C_k(v, v')}{\partial v} + \frac{\partial C_{k-k'}(v, v')}{\partial v'} \right]. \\
(11.25)
\]

This has worked out this way because, when (11.20) was used, \( k'' \) in the first term in (11.24) was set to \(-k'\) and \( k'' \) in the second term to \( k' \). Finally, since \( C_k(v', v) = C_k(v, v') \), the nonlinear term in (11.19) becomes

\[
N_k(v, v') = \sum_{k'} D_k k' k': \left( 2 \frac{\partial^2}{\partial v \partial v'} C_k - \frac{\partial^2}{\partial v \partial v'} C_k + \frac{\partial^2}{\partial v' \partial v'} C_k \right) \\
= D \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial v'} \right)^2 C_k + 2 \sum_{k'} D_k k' k' : \frac{\partial^2}{\partial v \partial v'} (C_{k-k'} - C_k), \\
(11.26)
\]

where \( D = \sum_k k'^2 D_{k'}/3 \) is the QL diffusion coefficient (cf. Q10).

We have paid a heavy price both in dodgy assumptions and in algebra, but the evolution equation (11.19) for the phase-space correlation function \( C_k(v, v') \) is now closed.

**Exercise 11.2. Furutsu–Novikov formula.** At least one element of dodgyness can, in fact, be removed. The average-splitting scheme that I employed above is, more formally, based on the following mathematical formula, due to Furutsu (1963) and Novikov (1965). If \( \varphi(q) \) is a Gaussian random field that depends on variable(s) \( q \) (which can include time, space, wave number, vector
index, etc.) and \( f[\varphi] \) is a functional of \( \varphi \), then their correlation function can be expressed as a convolution of the second-order correlator of \( \varphi \) and the functional derivative of \( F \) with respect to \( \varphi \):

\[
\langle \varphi(q)F[\varphi] \rangle = \int dq' \langle \varphi(q)\varphi(q') \rangle \left\langle \frac{\delta F[\varphi]}{\delta \varphi(q')} \right\rangle .
\] (11.27)

Convince yourself that the results (11.23) and (11.26) can be derived from (11.21) using this formula. If you are unsure about functional derivatives, here are three basic rules:

\[
\frac{\delta \varphi(q)}{\delta \varphi(q')} = \delta(q - q'), \quad \frac{\delta}{\delta \varphi(q)} \int dq' \varphi(q')K(q') = K(q), \quad \frac{\delta}{\delta \varphi} FG = \frac{\delta F}{\delta \varphi} G + F \frac{\delta G}{\delta \varphi}. \] (11.28)

Note also that \( \delta F/\delta \varphi(q') = 0 \) if \( F \) does not depend on the value of \( \varphi \) at \( q' \), e.g., if \( q \) is time, \( F \) is some function that only depends on the past history of \( \varphi \) (as \( \delta f \) does in the calculation above), and we are trying to take a functional derivative of it with respect to \( \varphi \) at a future moment.

**Exercise 11.3. Two-time correlation function.** Consider the two-time correlation function

\[
\mathcal{C}_k(t - t', v, v') = \langle \delta f_k(t, v)\delta f_k^*(t', v') \rangle,
\] (11.29)

where, without loss of generality, \( t > t' \), and, in a statistical steady state, \( \mathcal{C}_k \) depends on the time delay \( \tau = t - t' \), rather than separately on \( t \) and \( t' \). Using the same methods, assumptions and approximations as above, derive the following equation:

\[
\frac{\partial \mathcal{C}_k}{\partial \tau} + ik \cdot v' \mathcal{C}_k = D \frac{\partial^2 \mathcal{C}_k}{\partial v'^2}. \] (11.30)

Solve this equation to show that

\[
\mathcal{C}_k(\tau, v, v') = \int d^3v'' \frac{1}{(4\pi D\tau)^{3/2}} \exp \left( -\frac{|v - v''|^2}{4D\tau} - ik \cdot \frac{v + v''}{2}\tau - \frac{Dk^2\tau^3}{12} \right) C_k(v'', v'),
\] (11.31)

where \( C_k \) is the one-time correlation function (11.17). What is, therefore, the correlation time of \( \delta f_k \)?

11.2.2. Phase Mixing and Unmixing

Let us now discuss what this equation tells us. For a moment, let us ignore \( N_k \) and solve the linear equation

\[
\frac{\partial C_k}{\partial t} + ik \cdot (v - v')C_k = S_k.
\] (11.32)

The solution is

\[
C_k = S_k \frac{1 - e^{-k \cdot (v - v')t}}{i k \cdot (v - v')} \to S_k \pi \delta(k \cdot (v - v')) \quad \text{as} \quad t \to \infty;
\] (11.33)

the delta-function limit was derived in (5.39). This tells us that linear physics wants to make \( C_k(v, v') \) very fine scaled in phase space, with \( \delta f(v) \) taken at very close neighbouring values of \( v \) becoming decorrelated [cf. (10.35)]. This is a manifestation of the familiar phenomenon—phase mixing.

In this context, it seems natural to change variables:

\[
(v, v') \to (u, w), \quad \text{where} \quad u = \frac{v + v'}{2}, \quad w = v - v'.
\] (11.34)

In these new variables, phase mixing will look like growth of \( \partial/\partial w \), whereas the \( u \) dependence of the correlation function will just capture the large-scale, inhomogeneous structure of the velocity space—basically the fact that \( f_0 \) has a scale \( \sim v_{th} \). In fact, there will be a tendency for this scale to increase via what is recognisably QL diffusion: for \( f_0 \),
the version of QLT with an external field was the subject of Q10, while for \( C_k \), the first term in (11.26) is obviously just diffusion in \( u \). Thus, let us assume that \( \partial/\partial w \gg \partial/\partial u \) and write out (11.19) plus (11.23) and (11.26) in this approximation:

\[
\frac{\partial C_k}{\partial t} + i k \cdot w C_k \approx S_k + \frac{D}{4} \frac{\partial^2 C_k}{\partial u^2} - 2 \sum_{k'} D_{k'} \left( k' \cdot \frac{\partial}{\partial w} \right)^2 \left( C_{k-k'} - C_k \right). \tag{11.35}
\]

Since our objective is to characterise the emergence, or otherwise, of the small-scale structure in \( w \), let us make another formal step and Fourier transform in this variable:

\[
C_{k,s}(u) = \int \frac{d^3 w}{(2\pi)^3} C_k(u, w)e^{is \cdot w}. \tag{11.36}
\]

This “mixed-variable” object, containing information both about the large-scale real-space inhomogeneity and Fourier-space representation of small-scale structure, is known in quantum mechanics (and in signal processing) as the Wigner function. It satisfies

\[
\frac{\partial C_{k,s}}{\partial t} + k \cdot \frac{\partial C_{k,s}}{\partial s} \approx S_k(u, u)\delta(s) + \frac{D}{4} \frac{\partial^2 C_{k,s}}{\partial u^2} + 2 \sum_{k'} D_{k'}(k' \cdot s)^2 (C_{k-k',s} - C_{k,s}) \tag{11.37}
\]

A further approximation made above was \( S_k(v, v') \approx S_k(u, u) = 2D_k|k \cdot \partial f_0(u)/\partial u|^2 \), because \( w \) is assumed small while \( S_k \) only contains large-scale velocity dependence, via \( f_0 \). This new equation gives us a rather transparent way to see phase mixing: the second term on the left-hand side is just a propagation term in \( s \) space, at “velocity” \( k \), while the source \((S_k)\) term on the right is a “boundary condition” at small \( s \). Perturbations seeded by this source travel to larger values of \( s \) provided \( k \cdot s > 0 \)—this is phase mixing. In principle, \( k \cdot s < 0 \) would take perturbations from larger to smaller \( s \)—phase unmixing!—but there are no sources at large \( s \) within linear theory. The nonlinear game changer is the last, mode-coupling term on the right-hand side: it can easily couple a phase-mixing perturbation with \((k - k') \cdot s > 0\) to a phase-unmixing one with \( k \cdot s < 0 \) and thus send a pulse back to low \( s \). This is a stochastic version of the echo effect: perturbations chase each other into phase space and can turn around if they couple in the right way.87

Will they couple in the right way? Even though (11.37) is a closed equation, it is not so easy to solve, and it is also unknown how well it describes, even qualitatively, the systems with self-consistent electric field. We do know one solution of it, in 1D and with further simplifying assumptions (Adkins & Schekochihin 2018), which I will show in §11.2.3—on this basis, the tentative answer is yes, phase unmixing can indeed change things in a dramatic way.

**Exercise 11.4.** **Multispecies phase-space turbulence.** Generalise the above developments to the case of multiple species, i.e., derive the equations for

\[
C_{k_{\alpha \alpha'},v_{\alpha \alpha}',t} = \langle \delta f_{k_\alpha}(t,v)\delta f_{k_{\alpha'}}^{\ast}(t',v') \rangle, \tag{11.38}
\]

\[
\langle \delta f_{k_\alpha}(t-v)\delta f_{k_{\alpha'}}^{\ast}(t',v') \rangle = \langle \delta f_{k_\alpha}(t,v)\delta f_{k_{\alpha'}}^{\ast}(t',v') \rangle, \quad t > t'. \tag{11.39}
\]

What is the new version of the \((u, w)\) variables?

---

87In Schekochihin et al. (2016) and Adkins & Schekochihin (2018), you will find the original presentation of this argument in terms of Hermite, rather than Fourier, transforms, which, of course are equivalent at large Hermite order and large \( s \) (see Q7 for a quick tutorial). In the Hermite formalism, there is a neat decomposition of the distribution function into phase-mixing and phase-unmixing parts, with the nonlinear term represented as their coupling, although I now prefer the Fourier language because it does not shackle one to Maxwellian \( f_0 \).

---
The equation that you will have derived for $C_k'(\tau)$ is identical to (11.30) and is solved in the same way. The equation for the one-time function $C_k'$ has not been properly analysed, so it is not known what interesting effects might lurk there, if any. If you investigate, you might find something new.

11.2.3. Adkins’ Solution

To make (11.37) solvable, we make three further approximations:

(i) go to 1D;

(ii) ignore the the diffusion term in $u$ on the grounds that $\partial/\partial u \ll s$;

(iii) assume that the spectrum of the electric perturbations, $D_{k'}$, is concentrated at larger scales than $C_{k,s}$, i.e., $k' \ll k$ (in the context of passive-scalar advection, this is known as the Batchelor 1959 regime).

The last of these allows us to approximate

$$C_{k-k',s} - C_{k,s} \approx -k' \frac{\partial C_{k,s}}{\partial k} + \frac{k'^2}{2} \frac{\partial^2 C_{k,s}}{\partial k^2}.$$  \hfill (11.40)

The first term vanishes under the $k'$ integration in (11.37), and we arrive at the following rather cute equation:

$$\frac{\partial C_{k,s}}{\partial t} + k \frac{\partial C_{k,s}}{\partial s} = S_k \delta(s) + \gamma s^2 \frac{\partial^2 C_{k,s}}{\partial k^2}, \quad \gamma = \sum_{k'} k'^4 D_{k'}.$$ \hfill (11.41)

We shall also drop the time derivative, in pursuit of a steady-state solution.

What this equation does is quite easy to see qualitatively. There are two competing effects: phase (un)mixing, i.e., propagation in $s$ at velocity $k$, and mode coupling, which in the Batchelor regime takes the form of $k$-space diffusion, with diffusivity $\gamma s^2$. At lower $s$, phase (un)mixing dominates, but at higher $s$, the wave-number diffusion takes over. This change of behaviour takes place when the characteristic time scales associated with the two processes become equal:

$$k_s \sim \frac{\gamma s^2}{k^2} \Rightarrow s \sim \frac{k}{\gamma^{1/3}} \equiv s_c.$$ \hfill (11.42)

At $s \ll s_c$, and away from $s = 0$, (11.41) simply reduces to

$$\frac{\partial C_{k,s}}{\partial s} = 0 \Rightarrow C_{k,s} = f(k),$$ \hfill (11.43)

a flat $s$ spectrum with non-obvious $k$ dependence, decided by how it connects to other parts of the solution.

**Exercise 11.5.** This is just the linear phase-mixing spectrum. Show that it is equivalent to the Hermite spectrum (12.40) derived in Q7.

At $s \gg s_c$, (11.41) reduces to

$$\frac{\partial^2 C_{k,s}}{\partial k^2} = 0 \Rightarrow C_{k,s} = g(s),$$ \hfill (11.44)

a flat $k$ spectrum with non-obvious $s$ dependence—the result of wave-number diffusion taking over. This diffusion can take perturbations from positive to negative $k$, or vice versa, and thus allow them to go from phase mixing to phase unmixing.

To be cont’d...
11.2.4. Role of Collisions

11.3. Implications for Collisionless Relaxation

12. Particle Trapping

12.1. Nonlinear Landau Damping

Coming soon. See O’Neil (1965); Mazitov (1965).

12.2. BGK Solutions

Coming soon. See Bernstein et al. (1957).
1. **Industrialised linear theory with the Z function.** Consider a two-species plasma close to Maxwellian equilibrium. Rederive all the results obtained in §§3.4, 3.5, 3.8, 3.9, 3.10 starting from (3.82) and using the asymptotic expansions (3.89) and (3.90) of the plasma dispersion function.

Namely, consider the limits \( \zeta_e \gg 1 \) or \( \zeta_e \ll 1 \) and \( \zeta_i \gg 1 \), find solutions in these limits and establish the conditions on the wave number of the perturbations and on the equilibrium parameters under which these solutions are valid.

In particular, for the case of \( \zeta_e \ll 1 \) and \( \zeta_i \gg 1 \), obtain general expressions for the wave frequency and damping without assuming \( k\lambda_{De} \) to be either small or large. Recover from your solution the cases considered in §§3.8, 3.9 and 3.10.

Find also the ion contribution to the damping of the ion acoustic and Langmuir waves and comment on the circumstances in which it might be important to know what it is.

Convince yourself that you believe the sketch of longitudinal plasma waves in Fig. 14. If you feel computationally inclined, solve the plasma dispersion relation (3.82) numerically [using, e.g., (3.88)] and see if you can reproduce Fig. 14.

You may wish to check your results against some textbook: e.g., Krall & Trivelpiece (1973) and Alexandrov et al. (1984) give very thorough treatments of the linear theory (although in rather different styles than I did).

2. **Transverse plasma waves.** Go back to the Vlasov–Maxwell, rather than Vlasov–Poisson, system and consider electromagnetic perturbations in a Maxwellian unmagnetised plasma (unmagnetised in the sense that in equilibrium, \( B_0 = 0 \)):

\[
\frac{\partial \delta f_\alpha}{\partial t} + i k \cdot v \delta f_\alpha + \frac{q_\alpha}{m_\alpha} \left( E + \frac{v \times B}{c} \right) \cdot \frac{\partial \delta f_\alpha}{\partial v} = 0, \tag{12.1}
\]

where \( E \) and \( B \) satisfy Maxwell’s equations (1.23–1.26) with charge and current densities determined by the perturbed distribution function \( \delta f_\alpha \).

(a) Consider an initial-value problem for such perturbations and show that the equation for the Laplace transform of \( E \) can be written in the form\(^88\)

\[
\hat{\epsilon}(p, k) \cdot \hat{E}(p) = \begin{pmatrix} \text{terms associated with initial} \\
\text{perturbations of } \delta f_\alpha, \ E \text{ and } B \end{pmatrix}, \tag{12.2}
\]

where the dielectric tensor \( \hat{\epsilon}(p, k) \) is, in tensor notation,

\[
\epsilon_{ij}(p, k) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \epsilon_{TT}(p, k) + \frac{k_i k_j}{k^2} \epsilon_{LL}(p, k) \tag{12.3}
\]

and the longitudinal dielectric function \( \epsilon_{LL}(p, k) \) is the familiar electrostatic one, given by (3.82), while the transverse dielectric function is

\[
\epsilon_{TT}(p, k) = 1 + \frac{1}{p^2} \left[ k^2 c^2 - \sum_\alpha \omega_{p\alpha}^2 \zeta_\alpha Z(\zeta_\alpha) \right]. \tag{12.4}
\]

\(^88\)In Q3, dealing with the Weibel instability, you will have to do essentially the same calculation, but with a non-Maxwellian equilibrium. To avoid doing the work twice, you could do that question first and then specialise to a Maxwellian \( f_{0\alpha} \). However, the algebra is a bit hairier for the non-Maxwellian case, so it may be useful to do the simpler case first, to train your hand—and also to have a way to cross-check the later calculation.
(b) Hence solve the transverse dispersion relation, \( \epsilon_{TT}(p, k) = 0 \), and show that, in the high-frequency limit (|\( \zeta_e \)| \( \gg 1 \)), the resulting waves are simply the light waves, which, at long wave lengths, turn into plasma oscillations, viz.,

\[
\omega^2 = k^2 c^2 + \omega^2_{pe}.
\]

(12.5)

What is the wave length above which light can “feel” that it is propagating through plasma?—this is called the **plasma (electron) skin depth**, \( d_e \). Are these waves damped?

(c) In the low-frequency limit (|\( \zeta_e \)| \( \ll 1 \)), show that perturbations are aperiodic (have zero frequency) and damped. Find their damping rate and show that this result is valid for perturbations with wave lengths longer than the plasma skin depth (\( kd_e \ll 1 \)). Explain physically why these perturbations fail to propagate.

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**3. Weibel instability.**

Weibel (1958) realised that transverse plasma perturbations can go unstable if the equilibrium distribution is anisotropic with respect to some special direction \( \hat{n} \), namely if \( f_{0\alpha} = f_{0\alpha}(v_{\perp}, v_{\parallel}) \), where \( v_{\parallel} = v \cdot \hat{n} \), \( v_{\perp} = |v_{\perp}| \) and \( v_{\perp} = v - v_{\parallel} \hat{n} \). The anisotropy can be due to some beam or stream of particles injected into the plasma, it also arises in collisionless shocks or, generically, when plasma is sheared or non-isotropically compressed by some external force. The simplest model for an anisotropic distribution of the required type is a **bi-Maxwellian**:

\[
f_{0\alpha} = \frac{n_{\alpha}}{\pi^{3/2} v_{\theta \perp \alpha} v_{\theta \parallel \alpha}} \exp \left( -\frac{v_{\perp}^2}{v_{\theta \perp \alpha}^2} - \frac{v_{\parallel}^2}{v_{\theta \parallel \alpha}^2} \right),
\]

(12.6)

where, formally, \( v_{\theta \perp \alpha} = \sqrt{2 T_{\perp \alpha}/m_{\alpha}} \) and \( v_{\theta \parallel \alpha} = \sqrt{2 T_{\parallel \alpha}/m_{\alpha}} \) are the two “thermal speeds” in a plasma characterised by two effective temperatures \( T_{\perp \alpha} \) and \( T_{\parallel \alpha} \) (for each species).

(a) Using exactly the same method as in Q2, consider electromagnetic perturbations in a bi-Maxwellian plasma, assuming their wave vectors to be parallel to the direction of anisotropy, \( k \parallel \hat{n} \). Show that the dielectric tensor again has the form (12.3) and the longitudinal dielectric function is again given by (3.82), while the transverse dielectric function is

\[
\epsilon_{TT}(p, k) = 1 + \frac{1}{p^2} \left[ k^2 c^2 + \sum_{\alpha} \omega^2_{p\alpha} \left( 1 - \frac{T_{\perp \alpha}}{T_{\parallel \alpha}} [1 + \zeta_{\alpha} Z(\zeta_{\alpha})] \right) \right].
\]

(12.7)

(b) Show that in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

\[
\gamma = \frac{k v_{\theta \parallel \alpha} T_{\parallel \alpha} e}{\sqrt{\pi} T_{\perp \alpha} \Delta_e} \left( \Delta_e - k^2 d_e^2 \right),
\]

(12.8)

where \( \Delta_e = T_{\perp \alpha}/T_{\parallel \alpha} - 1 \) is the fractional temperature anisotropy, which must be positive in order for the instability to occur. Find the maximum growth rate and the corresponding

\[89\text{In Exercise 4.8, you need the dielectric tensor in terms of a general equilibrium distribution } f_{0\alpha}(v_x, v_y, v_z). \text{ If you are planning to do that exercise, it may save time (at the price of a very slight increase in algebra) to do the derivation with a general } f_{0\alpha} \text{ and then specialise to the bi-Maxwellian (12.6). You can check your algebra by looking up the result in Krall & Trivelpiece (1973) or in Davidson (1983).} \]
wave number. Under what condition(s) is the asymptotic limit in which you worked indeed a valid approximation for this solution?

(c) Are there any other unstable solutions? (cf. Weibel 1958)

(d) What happens if the electrons are isotropic but ions are not?

(e***) If you want a challenge and a test of stamina, work out the case of perturbations whose wave number is not necessarily in the direction of the anisotropy \((k \times \hat{n} \neq 0)\). Are the \(k \parallel \hat{n}\) or some oblique perturbations the fastest growing? This is a lot of algebra, so only do it if you enjoy this sort of thing. The dispersion relation for this general case appears to be in the Appendix of Ruyer et al. (2015), but they only solve it numerically; no one seems to have looked at asymptotic limits. This could be the start of a dissertation.

4. Two-stream instability. Consider one-dimensional, electrostatic perturbations in a two-species (electron-ion) plasma. Let the electron distribution function with respect to velocities in the direction \((z)\) of the spatial variation of perturbations be a “double Lorentzian” consisting of two counterpropagating beams with velocity \(u_b\) and width \(v_b\), viz.,

\[
F_e(v_z) = \frac{n_e v_b}{2\pi} \left[ \frac{1}{(v_z - u_b)^2 + v_b^2} + \frac{1}{(v_z + u_b)^2 + v_b^2} \right]
\]

(12.9)

(see Fig. 12b), while the ions are Maxwellian with thermal speed \(v_{thi} \ll u_b\). Assume also that the phase velocity \((p/k)\) will be of the same order as \(u_b\) and hence that the ion contribution to the dielectric function (3.26) is negligible.

(a) By integrating by parts and then choosing the integration contour judiciously, or otherwise, calculate the dielectric function \(\epsilon(p,k)\) for this plasma and hence show that the dispersion relation is

\[
\sigma^4 + (2u_b^2 + v_p^2)\sigma^2 + u_b^2(u_b^2 - v_p^2) = 0,
\]

(12.10)

where \(\sigma = v_b + p/k\) and \(v_p = \omega_{pe}/k\).

(b) In the long-wavelength limit, viz., \(k \ll \omega_{pe}/u_b\), find the condition for an instability to exist and calculate the growth rate of this instability. Is the nature of this instability kinetic (due to Landau resonance) or hydrodynamic?

(c) Consider the case of cold beams, \(v_b = 0\). Without making any a priori assumptions about \(k\), calculate the maximum growth rate of the instability. Sketch the growth rate as a function of \(k\).

(d) Allowing warm beams, \(v_b > 0\), show that the system is unstable provided

\[
u_b > v_b \quad \text{and} \quad k < \omega_{pe} \frac{\sqrt{u_b^2 - v_p^2}}{u_b^2 + v_b^2}.
\]

(12.11)

What is the effect that a finite beam width has on the stability of the system and on the kind of perturbations that can grow?

In §4.4, you might find it instructive to compare the results that you have just obtained by solving the dispersion relation (12.10) directly with what can be inferred via Penrose’s criterion and Nyquist’s method.

5. Free energy and stability. (a) Starting from the linearised Vlasov–Poisson system

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\(^{90}\)This is based on the 2019 exam question.
and assuming a Maxwellian equilibrium, show by direct calculation from the equations, rather than via expansion of the entropy function and the use of energy conservation (as was done in §5.2), that free energy is conserved:

\[
\frac{d}{dt} \int d^3r \left[ \sum_\alpha \int d^3v \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} + \frac{\nabla \varphi^2}{8\pi} \right] = 0. \quad (12.12)
\]

This is an exercise in integrating by parts.

(b) Now consider the full Vlasov–Maxwell equations and prove, again for a Maxwellian plasma plus small perturbations,

\[
\frac{d}{dt} \int d^3r \left[ \sum_\alpha \int d^3v \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} + \frac{|E|^2 + |B|^2}{8\pi} \right] = 0. \quad (12.13)
\]

(c) Consider the same problem, this time with an equilibrium that is not Maxwellian, but merely isotropic, i.e., \(f_{0\alpha} = f_{0\alpha}(v)\), or, in what will prove to be a more convenient form,

\[
f_{0\alpha} = f_{0\alpha}(\varepsilon_\alpha), \quad (12.14)
\]

where \(\varepsilon_\alpha = m_\alpha v^2/2\) is the particle energy. Find an integral quantity quadratic in perturbed fields and distributions that is conserved by the Vlasov–Maxwell system under these circumstances and that turns into the free energy (12.13) in the case of a Maxwellian equilibrium (if in difficulty, you will find the answer in, e.g., Davidson 1983 or in Kruskal & Oberman 1958, which appears to be the original source). Argue that

\[
\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \quad (12.15)
\]

is a sufficient condition for stability of small (\(\delta f_\alpha \ll f_{0\alpha}\), but not necessarily infinitesimal) perturbations in such a plasma.

6. Fluctuation-dissipation relation for a collisionless plasma. Let us consider a linear kinetic system in which perturbations are stirred up by an external force, which we can think of as an imposed (time-dependent) electric field \(E_{\text{ext}} = -\nabla \chi\). The perturbed distribution function then satisfies

\[
\frac{\partial \delta f_\alpha}{\partial t} + v \cdot \nabla \delta f_\alpha - \frac{q_\alpha}{m_\alpha} (\nabla \varphi_{\text{tot}}) \cdot \frac{\partial f_{0\alpha}}{\partial v} = 0, \quad (12.16)
\]

where \(\varphi_{\text{tot}} = \varphi + \chi\) is the total potential, whose self-consistent part, \(\varphi\), obeys the usual Poisson equation

\[
- \nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int d^3v \delta f_\alpha \quad (12.17)
\]

and the equilibrium \(f_{0\alpha}\) is assumed to be Maxwellian.

(a) By considering an initial-value problem for (12.16) and (12.17) with zero initial perturbation, show that the Laplace transforms of \(\varphi_{\text{tot}}\) and \(\chi\) are related by

\[
\hat{\varphi}_{\text{tot}}(p) = \frac{\hat{\chi}(p)}{\epsilon(p)}, \quad (12.18)
\]

where \(\epsilon(p)\) is the dielectric function given by (3.82).
(b) Consider a time-periodic external force,
\[ \chi(t) = \chi_0 e^{-i\omega_0 t}. \]  
(12.19)

Working out the relevant Laplace transforms and their inverses [see (3.14)], show that, after transients have decayed, the total electric field in the system will oscillate at the same frequency as the external force and be given by
\[ \varphi_{\text{tot}}(t) = \frac{\chi_0 e^{-i\omega_0 t}}{\epsilon(-i\omega_0)}. \]  
(12.20)

(c) Now consider the plasma-kinetic Langevin problem: assume the external force to be a white noise, i.e., a random process with the time-correlation function
\[ \langle \chi(t)\chi^*(t') \rangle = 2D\delta(t-t'). \]  
(12.21)

Show that the resulting steady-state mean-square fluctuation level in the plasma will be
\[ \langle |\varphi_{\text{tot}}(t)|^2 \rangle = D\pi \int_{-\infty}^{+\infty} \frac{d\omega}{\epsilon(-i\omega)^2}. \]  
(12.22)

This is a kinetic fluctuation-dissipation relation: given a certain level of external stirring, parametrised by \( D \), this formula predicts the fluctuation energy in terms of \( D \) and of the internal dissipative properties of the plasma, encoded by its dielectric function.

(d) For a system in which the Landau damping is weak, \( |\gamma| \ll kv_{\text{th}}\alpha \), calculate the integral (12.22) using Plemelj’s formula (3.25) to show that
\[ \langle |\varphi_{\text{tot}}(t)|^2 \rangle = D \sum_i \frac{1}{|\gamma_i|} \left[ \frac{\partial \Re \epsilon(-i\omega)}{\partial \omega} \right]^{-2}_{\omega=\omega_i}, \]  
(12.23)

where \( p_i = -i\omega_i + \gamma_i \) are the weak-damping roots of the dispersion relation.

Here is a reminder of how the standard Langevin problem can be solved using Laplace transforms. The Langevin equation is
\[ \frac{\partial \varphi}{\partial t} + \gamma \varphi = \chi(t), \]  
(12.24)

where \( \varphi \) describes some quantity, e.g., the velocity of a Brownian particle, subject to a damping rate \( \gamma \) and an external force \( \chi \). In the case of a Brownian particle, \( \chi \) is assumed to be a white noise, as per (12.21). Assuming \( \varphi(t=0) = 0 \), the Laplace-trasform solution of (12.24) is
\[ \hat{\varphi}(p) = \frac{\hat{\chi}(p)}{p + \gamma}. \]  
(12.25)

Considering first a non-random oscillatory force (12.19), we have
\[ \hat{\chi}(p) = \int_0^\infty dt e^{-pt} \chi(t) = \frac{\chi_0}{p + i\omega_0} \quad \Rightarrow \quad \hat{\varphi}(p) = \frac{\chi_0}{(p + \gamma)(p + i\omega_0)}. \]  
(12.26)

The inverse Laplace transform of \( \hat{\varphi} \) is calculated by shifting the integration contour to large negative \( \Re p \) while not allowing it to cross the two poles, \( p = -\gamma \) and \( p = -i\omega_0 \), in a manner analogous to that explained in §3.1 (Fig. 5) and shown in Fig. 42. The integral is then dominated by the contributions from the poles:
\[ \varphi(t) = \frac{1}{2\pi i} \int_{i\infty - \sigma}^{i\infty + \sigma} dp e^{pt} \hat{\varphi}(p) = \chi_0 \left( \frac{e^{-i\omega_0 t}}{-\omega_0 + \gamma} + \frac{e^{-\gamma t}}{-\gamma + i\omega_0} \right) \to \chi_0 \frac{e^{-i\omega_0 t}}{-\omega_0 + \gamma} \quad \text{as} \quad t \to \infty, \]  
(12.27)

which is quite obviously the right solution of (12.24) with a periodic force (the second term in the brackets is the decaying transient needed to enforce the zero initial condition).
In the more complicated case of a white-noise force [see (12.21)],

\[
\langle |\varphi(t)|^2 \rangle = \frac{1}{(2\pi)^2} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp \, dp' \, \frac{e^{pt} \, \hat{\chi}(p) \, \hat{\chi}(p')}{p + \gamma} \right)^2
\]

\[
= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dp' \, e^{-i(\omega - \omega') t} \left\{ \hat{\chi}(\omega + \sigma) \hat{\chi'}(\omega' + \sigma) \right\} \frac{e^{i(\omega - \omega') t} \, \hat{\chi}(\omega + \sigma) \hat{\chi'}(\omega' + \sigma)}{(-i\omega + \sigma + \gamma)(i\omega' + \sigma + \gamma)},
\]  

(12.28)

where we have changed variables \( p = -i\omega + \sigma \) and similarly for the second integral. The correlation function of the Laplace-transformed force is, using (12.21),

\[
\langle \hat{\chi}(p) \hat{\chi}(p') \rangle = \int_0^\infty dt \int_0^\infty dt' \, e^{-|p-p'| t} \langle \chi(t) \chi(t') \rangle = 2D \int_0^\infty dt \, e^{-|p-p'| t} = \frac{2D}{p + p'},
\]

provided Re \( p > 0 \) and Re \( p' > 0 \). Then (12.28) becomes

\[
\langle |\varphi(t)|^2 \rangle = \frac{D}{2\pi^2} \int_{-\infty}^{i\infty} dp \int_{-\infty}^{i\infty} dp' \, e^{i(\omega + \sigma) t} \left\{ \frac{e^{i(\omega - \omega') t} \, \hat{\chi}(\omega + \sigma) \hat{\chi'}(\omega' + \sigma)}{(-i\omega + \sigma + \gamma)(i\omega' + \sigma + \gamma)} \right\} \frac{e^{p t}}{\gamma + \omega' + \sigma}
\]

\[
= \frac{D}{\pi} \int_{-\infty}^{i\infty} dp \int_{-\infty}^{i\infty} dp' \, \frac{e^{i(\omega + \sigma) t} \, e^{p t}}{(i\omega' + \sigma + \gamma)} \left\{ \frac{1}{2} + \int_{-\infty}^{i\infty} \frac{dp \, e^{p t} \, (p + i\omega' + \sigma)(p + \gamma)}{(i\omega' + \sigma + \gamma)} \right\},
\]

(12.30)

where we have reverted to the \( p \) variable in one of the integrals and then performed the integration by the same manipulation of the contour as in (12.27). We now note that, since there are no exponentially growing solutions in this system, \( \sigma > 0 \) can be chosen arbitrarily small. Taking \( \sigma \rightarrow +0 \) and neglecting the decaying transient in (12.30), we get, in the limit \( t \rightarrow \infty \),

\[
\langle |\varphi(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{i\infty} \frac{dp \, e^{p t}}{-i\omega' + \sigma} = \frac{D}{\pi} \int_{-\infty}^{i\infty} \frac{dp'}{-i\omega' + \sigma} = \frac{D}{\gamma}.
\]

(12.31)

Note that, while the integral in (12.31) is doable exactly, it can, for the case of weak damping, also be computed via Plemelj’s formula.

Equation (12.31) is the standard Langevin fluctuation-dissipation relation. It can also be obtained without Laplace transforms, either by directly integrating (12.24) and correlating \( \varphi(t) \) with itself or by noticing that

\[
\frac{\partial}{\partial t} \langle \varphi^2 \rangle - \gamma \langle \varphi^2 \rangle = \langle \chi(t) \varphi(t) \rangle = \left\langle \chi(t) \int_0^t dt' \left[ -\gamma \varphi(t') + \chi(t') \right] \right\rangle = D,
\]

(12.32)

where we have used (12.21) and the fact that \( \langle \chi(t) \varphi(t') \rangle = 0 \) for \( t' \leq t \), by causality. Equation (12.31) is the steady-state solution to the above, but (12.32) also teaches us that, if we interpret \( \langle \varphi^2 \rangle /2 \) as energy, \( D \) is the power that is injected into the system by the external force. Thus, fluctuation-dissipation relations such as (12.31) tells us what fluctuation energy will persist in a dissipative system if a certain amount of power is pumped in.
7: Phase-mixing spectrum. Here we study the velocity-space structure of the perturbed distribution function $\delta f$ derived in Q6.

In order to do this, we need to review the Hermite transform:

$$\delta f_m = \frac{1}{n} \int dv_z \frac{H_m(u)\delta f(v_z)}{\sqrt{2^m m!}}, \quad u = \frac{v_z}{v_{th}}, \quad H_m(u) = (-1)^m e^{u^2} \frac{d^m}{du^m} e^{-u^2},$$

(12.33)

where $H_m$ is the Hermite polynomial of (integer) order $m$. We are only concerned with the $v_z$ dependence of $\delta f$ (where $z$, as always, is along the wave number of the force); all $v_x$ and $v_y$ dependence is Maxwellian and can be integrated out. The inverse transform is given by

$$\delta f(v_z) = \sum_{m=0}^{\infty} \frac{H_m(u)F(v_z)}{\sqrt{2^m m!}} \delta f_m, \quad F(v_z) = \frac{n}{\sqrt{\pi v_{th}}} e^{-u^2}.$$  

(12.34)

Because $H_m(u)$ are orthogonal polynomials, viz.,

$$\frac{1}{n} \int dv_z H_m(u)H_{m'}(u)F(v_z) = 2^m m! \delta_{m,m'},$$

(12.35)

they have a Parseval theorem and so the contribution of the perturbed distribution function to the free energy [see (5.18)] can be written as

$$\int d^3v \frac{T|\delta f|^2}{2f_0} = \frac{nT}{2} \sum_m |\delta f_m|^2.$$  

(12.36)

In a plasma where perturbations are constantly stirred up by a force, Landau damping must be operating all the time, removing energy from $\varphi$ to provide “dissipation” of the injected power. The process of phase mixing that accompanies Landau damping must then lead to a certain fluctuation level $\langle|\delta f_m|^2\rangle$ in the Hermite moments of $\delta f$. Lower $m$’s correspond to “fluid” quantities: density ($m = 0$), flow velocity ($m = 1$), temperature ($m = 2$). Higher $m$’s correspond to finer structure in velocity space: indeed, for $m \gg 1$, the Hermite polynomials can be approximated by trigonometric functions,

$$H_m(u) \approx \sqrt{2} \left(\frac{2m}{e}\right)^{m/2} \cos \left(\sqrt{2m} u - \frac{\pi m}{2}\right) e^{u^2/2},$$

(12.37)

and so the Hermite transform is somewhat analogous to a Fourier transform in velocity space with “frequency” $\sqrt{2m}/v_{th}$.

(a) Show that in the kinetic Langevin problem described in Q6(c), the mean square fluctuation level of the $m$-th Hermite moment of the perturbed distribution function is given by

$$\langle|\delta f_m(t)|^2\rangle = \frac{q^2D}{T^2\pi 2^m m!} \int_{-\infty}^{+\infty} d\omega \left|\zeta Z^{(m)}(\zeta)\right|^2, \quad \zeta = \frac{\omega}{kv_{th}},$$

(12.38)

where $Z^{(m)}(\zeta)$ is the $m$-th derivative of the plasma dispersion function [note (3.91)].

(b)** Show that, assuming $m \gg 1$ and $\zeta \ll \sqrt{2m}$,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi} i^{m+1} \left(\frac{2m}{e}\right)^{m/2} e^{i\sqrt{2m-\zeta^2}/2}$$

(12.39)

and, therefore, that

$$\langle|\delta f_m(t)|^2\rangle \approx \frac{\text{const}}{\sqrt{m}}.$$  

(12.40)

Thus, the Hermite spectrum of the free energy is shallow and, in particular, the total free energy diverges—it has to be regularised by collisions. This is a manifestation of a
copious amount of fine-scale structure in velocity space (note also how this shows that 
Landau-damped perturbations involve all Hermite moments, not just the “fluid” ones).

Deriving (12.39) is a (reasonably hard) mathematical exercise: it involves using (3.91) and (12.37) 
and manipulating contours in the complex plane. This is a treat for those who like this sort of 
things. Getting to (12.40) will also require the use of Stirling’s formula.

The Hermite order at which the spectrum (12.40) must be cut off due to collisions can be quickly 
deduced as follows. We saw in § 5.5 that the typical velocity derivative of \( \delta f \) can be estimated 
according to (5.26) and the time it takes for this perturbation to be wiped out by collisions is 
given by (5.31). But, in view of (12.37), the velocity gradients probed by the Hermite moment 
\( m \) are of order \( \sqrt{2m/v} \). The collisional cut off \( m_c \) in Hermite space can then be estimated so:

\[
m_c \sim v_{th}^2 \frac{\partial^2}{\partial v^2} \sim \left( \frac{k v_{th} t_c}{\nu} \right)^{2/3}.
\] (12.41)

Therefore, the total free energy stored in phase space diverges: using (12.36) and (12.40),

\[
\frac{1}{n} \int d^3v \frac{\delta f^2}{2f_0} = \frac{1}{2} \sum_m \langle |\delta f_m|^2 \rangle \sim \int m_c dm \frac{\text{const}}{\sqrt{m}} \propto \nu^{-1/3} \rightarrow \infty \text{ as } \nu \rightarrow +0.
\] (12.42)

In contrast, the total free-energy dissipation rate is finite, however small is the collision frequency: estimating the right-hand side of (5.18), we get

\[
\frac{1}{n} \int d^3v \frac{\delta f}{f_0} \left( \frac{\partial \delta f}{\partial t} \right)_c \sim -\nu \sum_m m \langle |\delta f_m|^2 \rangle \propto \nu \int m_c dm \sqrt{m} \sim kv_{th}.
\] (12.43)

Thus, the kinetic system can collisionally produce entropy at a rate that is entirely independent 
of the collision frequency.

If you find phase-space turbulence and generally life in Hermite space as fascinating as I do, you can learn more from Kanekar et al. (2015) (on fluctuation-dissipation relations and Hermite 
spectra) and from §11.2 (on what happens when nonlinearity strikes).

---

8. **QL theory of Landau damping.** In §6, we discussed the QL theory of an unstable 
system, in which, whatever the size of the initial electric perturbations, they eventually 
grow large enough to affect the equilibrium distribution and modify it so as to suppress 
further growth. In a stable equilibrium, any initial perturbations will be Landau-damped, 
but, if they are sufficiently large to start with, they can also affect \( f_0 \) quasilinearly in a 
way that will slow down this damping.

Consider, in 1D, an initial spectrum \( W(0, k) \) of plasma oscillations (waves) excited 
in the wave-number range \( [k_2, k_1] = [\omega_{pe}/v_2, \omega_{pe}/v_1] \ll \lambda_{De}^{-1} \), with total electric energy \( \mathcal{E}(0) \). Modify the QL theory of §6 to show the following.

(a) A steady state can be achieved in which the distribution develops a plateau in 
the velocity interval \( [v_1, v_2] \) (Fig. 43). Find \( F_{\text{plateau}} \) in terms of \( v_1, v_2 \) and the initial 
distribution \( F(0, v) \). What is the energy of the waves in this steady state? What is the 
lower bound on initial electric energy \( \mathcal{E}(0) \) below which the perturbations would just 
decay without forming a fully-fledged plateau?

(b) Derive the evolution equation for the thermal (nonresonant) bulk of the distribution 
and show that it cools during the QL evolution, with the total thermal energy declining 
by the same amount as the electric energy of the waves:

\[
K_{\text{th}}(t) - K_{\text{th}}(0) = -[\mathcal{E}(0) - \mathcal{E}(t)].
\] (12.44)

Identify where all the energy lost by the thermal particles and the waves goes and thus
Figure 43. Initial stable equilibrium distribution and the final outcome of the QL evolution of a system with Landau-damped electric perturbations.

confirm that the total energy in the system is conserved. Why, physically, do thermal particles lose energy?

(c) Show that we must have

$$E(0) \gg \frac{\gamma_k}{\omega_{pe}} \frac{\delta v}{v}$$

in order for the wave energy to change only by a small fraction before saturating and

$$E(0) \gg \frac{\gamma_k}{\omega_{pe}} \left(\frac{\delta v}{v}\right)^3 \frac{1}{(k\lambda_{De})^2}$$

in order for the QL evolution to be faster than the damping. Here $\delta v = v_2 - v_1$ and $v \sim v_1 \sim v_2$.

This question requires some nuance in handling the calculation of the QL diffusion coefficient. In §6.1, we used the expression (6.6) for $\delta f_k$ in which only the eigenmode-like part was retained, while the phase-mixing terms were dropped on the grounds that we could always just wait long enough for them to be eclipsed by the term containing an exponentially growing factor $e^{i\gamma_k t}$. When we are dealing with damped perturbations, there is no point in waiting because the exponential term is getting smaller, while the phase-mixing terms do not decay (except by collisions, see §§5.3 and 5.5, but we are not prepared to wait for that).

Let us, therefore, bite the bullet and use the full expression (5.25) for the perturbed distribution function, where we single out the slowest-damped mode and assume that all others, if any, will be damped fast enough never to produce significant QL effects:

$$\delta f_k = \frac{q}{m} \frac{1 - e^{-i(k \cdot v - \omega_k) t - i\gamma_k t}}{k \cdot v - \omega_k - i\gamma_k - \partial f_0 / \partial v} + e^{-i\omega_k t} (g_k + \ldots),$$

where “…” stand for any possible undamped, phase-mixing remnants of other modes. When the solution (12.47) is substituted into (6.4), where it is multiplied by $\varphi_k$ and time averaged [according to (2.7)], the second term vanishes because, for resonant particles ($k \cdot v \approx \omega_k$), it contains no resonant denominators and so is smaller than the first term, whereas for the nonresonant particles, it is removed by time averaging (check that this works at least for $|\gamma_k| t \lesssim 1$ and indeed beyond that). Keeping only the first term in the expression (12.47), substituting it into (6.4) and going through a calculation analogous to that given in (6.8), we find that the diffusion matrix is (check this)

$$D(v) = \frac{q^2 m^2}{k^2} \sum_k E_k^2 \Im \left\{ \frac{1 - e^{-i(k \cdot v - \omega_k) t - i\gamma_k t}}{k \cdot v - \omega_k - i\gamma_k} \right\},$$

which is a generalisation of the penultimate line of (6.8). For nonresonant particles, the phase-mixing term is eliminated by time averaging and we end up with the old result: the last line
of (6.8). For resonant particles, assuming $|\gamma_k| \ll |k \cdot v - \omega_k| \ll \omega_k \sim k \cdot v$ and $|\gamma_k| t \ll 1$, we may adopt the approximation (5.39), which we have previously used to analyse the structure of $\delta f$. This gives us

$$D(v) = \frac{q^2}{m^2} \sum_k \frac{kk}{k^2} |E_k|^2 \pi \delta(k \cdot v - \omega_k), \quad (12.49)$$

which is the same result as (6.16)—including, importantly, the sign, which we would have gotten wrong had we just mechanically applied Plemelj’s formula to (6.12) with $\gamma_k < 0$. This is equivalent to saying that the $k$ integral in (6.16) should be taken along the Landau contour, rather than simply along the real line.

Note that the above construction was done assuming $|\gamma_k| t \ll 1$, i.e., all the QL action has to occur before the initial perturbations decay away (which is reasonable). Note also that there is nothing above that would not apply to the case of unstable perturbations ($\gamma_k > 0$) and so we conclude that results of §6, derived formally for $\gamma_k t \gg 1$, in fact also hold on shorter time scales ($\gamma_k t \ll 1$, but, obviously, still $\omega_k t \gg 1$).

9. QL theory of Weibel instability. (a) Starting from the Vlasov equations including magnetic perturbations, show that the slow evolution of the equilibrium distribution function is described by the following diffusion equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \cdot D(v) \cdot \frac{\partial f_0}{\partial v}, \quad (12.50)$$

where the QL diffusion matrix is

$$D(v) = \frac{q^2}{m^2} \sum_k \frac{1}{i(k \cdot v - \omega_k)} + \gamma_k \left( E_k^* + \frac{v \times B_k^*}{c} \right) \left( E_k + \frac{v \times B_k}{c} \right) \quad (12.51)$$

and $\omega_k$ and $\gamma_k$ are the frequency and the growth rate, respectively, of the fastest-growing mode.

(b) Consider the example of the low-frequency electron Weibel instability with wave numbers $k$ parallel to the anisotropy direction [see (12.8)]. Take $k = k \hat{z}$ and $B_k = B_k \hat{y}$ and, denoting $\Omega_k = eB_k/m_e c$ (the Larmor frequency associated with the perturbed magnetic field), show that (12.50) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_x} \left( D_{xx} \frac{\partial f_0}{\partial v_x} + D_{xz} \frac{\partial f_0}{\partial v_z} \right) + \frac{\partial}{\partial v_z} D_{zz} \frac{\partial f_0}{\partial v_z}, \quad (12.52)$$

where the coefficients of the QL diffusion tensor are

$$D_{xx} = \sum_k \frac{\gamma_k}{k^2} |\Omega_k|^2, \quad D_{xz} = -\sum_k \frac{2\gamma_k v_x v_z}{k^2 v_x^2 + \gamma_k^2} |\Omega_k|^2, \quad D_{zz} = \sum_k \frac{\gamma_k v_x^2}{k^2 v_x^2 + \gamma_k^2} |\Omega_k|^2. \quad (12.53)$$

(c) Suppose the electron distribution function $f_0$ is initially the bi-Maxwellian (12.6) with $0 < T_\perp / T_\parallel - 1 \ll 1$ (as should be the case for this instability to work). As QL evolution starts, we may define the temperatures of the evolving distribution according to

$$T_\perp = \frac{1}{n} \int d^3v \frac{m(v_x^2 + v_y^2)}{2} f_0, \quad T_\parallel = \frac{1}{n} \int d^3v \frac{mv_z^2}{2} f_0. \quad (12.54)$$

Show that initially, viz., before $f_0$ has time to change shape significantly so as no longer to be representable as a bi-Maxwellian, the two temperatures will evolve approximately
(using $\gamma_k \ll k v_{th}$) according to

$$\frac{\partial T_\perp}{\partial t} = -\lambda T_\perp, \quad \frac{\partial T_\parallel}{\partial t} = 2\lambda T_\perp,$$

where $\lambda(T_\perp, T_\parallel) = \sum_k 2\gamma_k |\Omega_k|^2 k^2 v_{th}^2$. \hspace{1cm} (12.55)

Thus, QL evolution will lead, at least initially, to the reduction of the temperature anisotropy, thus weakening the instability (these equations should not be used to trace $T_\perp/T_\parallel - 1$ all the way to zero because there is no reason why the QL evolution should preserve the bi-Maxwellian shape of $f_0$).

Note that, even modulo the caveat about the bi-Maxwellian not being a long-term solution, this does not give us a way to estimate (or even guess) what the saturated fluctuation level will be. The standard Weibel lore is that saturation occurs when the approximations that were used to derive the linear theory (Q3) break down, namely, when magnetic field becomes strong enough to magnetise the plasma, rendering the Larmor scale $\rho_e = v_{th}/\Omega_k$ associated with the fluctuations small enough to be comparable to the latter’s wavelengths $\sim k^{-1}$. Using the typical values of $k$ from (12.8), we can write this condition as follows

$$\Omega_k \sim k v_{th} \sim \sqrt{\Delta e} v_{th} \iff \beta_e \equiv B^2/8\pi n_e T_e \sim \Delta_e.$$ \hspace{1cm} (12.56)

Thus, Weibel instability will produce fluctuations the ratio of whose magnetic-energy density to the electron-thermal-energy density (customarily referred to as the inverse of “plasma beta,” $1/\beta_e$) is comparable to the electron pressure anisotropy $\Delta_e$. Because at that point the fluctuations will be relaxing this pressure anisotropy at the same rate as they can grow in the first place [in (12.55), $\lambda \sim \gamma_k$], the QL approach is not valid anymore.

These considerations are, however, usually assumed to be qualitatively sound and lead people to believe that, even in collisionless plasmas, the anisotropy of the electron distribution must be largely self-regulating, with unstable Weibel fluctuations engendered by the anisotropy quickly acting to isotropise the plasma (or at least the electrons).

This is all currently very topical in the part of the plasma-astrophysics world preoccupied with collisionless shocks, origin of the cosmic magnetism, hot weakly collisional environments such as the intergalactic medium (in galaxy clusters) or accretion flows around black holes and many other interesting subjects.

(d) Equations (12.55) say that the total mean kinetic energy,

$$\int d^3 v \frac{m v^2}{2} f_0 = n \left( T_\perp + T_\parallel \right),$$ \hspace{1cm} (12.57)

does not change. But fluctuations are generated and grow at the rate $\gamma_k$! Without much further algebra, can you tell whether you should therefore doubt the result (12.55)?

10. **QL theory of stochastic acceleration.**\(^{91}\) Consider a population of particles of charge $q$ and mass $m$. Assume that collisions are entirely negligible. Assume further that an electrostatic fluctuation field $E = -\nabla \varphi$ (with zero spatial mean) is present and that this field is given and externally determined, i.e., it is unaffected by the particles that are under consideration. This might happen physically if, for example, the particles are a low-density admixture in a plasma consisting of some more numerous species of ions and electrons, which dominate the plasma’s dielectric response.

As usual, we assume that the distribution function can be represented as $f = f_0(t, v) + \delta f(t, r, v)$, where $f_0$ is spatially homogeneous and changes slowly in time compared to the perturbed distribution $\delta f \ll f_0$. Its evolution is described by (2.11), where angle brackets again denote the time average over the fast variation of the fluctuation field.

\(^{91}\)Except for part (d), this is based on the 2018 exam question.
(a) Assume that $\varphi$ is sufficiently small for it to be possible to determine $\delta f$ from the linearised kinetic equation. Let $\delta f = 0$ at $t = 0$. Show that $f_0$ satisfies a QL diffusion equation with the diffusion matrix

$$D(v) = \frac{q^2}{m^2} \sum_k k k 1 \int dp \frac{1}{p + i k \cdot v} \int_{-\infty}^{t} d\tau e^{\nu \tau} C_k(\tau), \quad (12.58)$$

where the $p$ integration is along a contour appropriate for an inverse Laplace transform and $C_k(t - t') = \langle \varphi_k^*(t) \varphi_k(t') \rangle$ is the correlation function of the fluctuation field (which is taken to be statistically stationary, so $C_k$ depends only on the time difference $t - t'$).

(b) Let the correlation function have the form

$$C_k(\tau) = A_k e^{-\gamma_k |\tau|}, \quad (12.59)$$

i.e., $\gamma_k^{-1}$ is the correlation time of the fluctuation field and $A_k$ its spectrum; assume $\gamma_{-k} = \gamma_k$. Do the integrals in (12.58) and show that, at $t \gg \gamma_k^{-1}$,

$$D(v) = \frac{q^2}{m^2} \sum_k k k \frac{\gamma_k A_k}{\gamma_k^2 + (k \cdot v)^2}. \quad (12.60)$$

(c) Restrict consideration to 1D in space and to the limit in which $\gamma_k \gg kv$ for typical wave numbers of the fluctuations and typical particle velocities (i.e., the fluctuation field is short-time correlated). Assuming that $f_0$ at $t = 0$ is a Maxwellian with temperature $T_0$, predict the evolution of $f_0$ with time. Discuss what physically is happening to the particles. Discuss the validity of the short-correlation-time approximation and of the assumption of slow evolution of $f_0$. What is, roughly, the condition on the amplitude and the correlation time of the fluctuation field that makes these assumptions compatible?

(d) When the distribution “heats up” sufficiently, the short-correlation-time approximation will be broken. Staying in 1D, consider the opposite limit, $\gamma_k \ll kv$. Show that the resulting QL equation admits a subdiffusive solution, with

$$f_0(t, v) \propto \frac{e^{-v^4/\alpha t}}{t^{1/4}}, \quad \alpha = 16 \frac{q^2}{m^2} \sum_k \gamma_k A_k. \quad (12.61)$$

In view of this result and of (c), discuss qualitatively how an initially “cold” particle distribution would evolve with time.

The original, classic paper on stochastic acceleration is Sturrock (1966). Note that the velocity dependence of the diffusion matrix (12.60) is determined by the functional form of $C_k(\tau)$, so interesting $\tau$ dependences of the latter can lead to all kinds of interesting distributions $f_0$ of the accelerated particles.

Advanced Plasma Kinetics Problem Set

Option I. Pick any 8 Exercises from §§4, 7–11. This set should include Exercises from at least 3 different sections, but beware of the long independent-study pieces (see Option II).

Option II. Do Exercise 4.8 OR the series of Exercises 8.4, 8.5, 8.7, and 8.9 (at least a stab at the last one is a sine qua non) OR Exercise 8.10.

IUCUNDI ACTI LABORES.
13. MHD Equations

Like hydrodynamics from gas kinetics, MHD can be derived systematically from the Vlasov–Maxwell–Landau equations for a plasma in the limit of large collisionality + a number of additional assumptions (see, e.g., Goedbloed & Poedts 2004; Parra 2019a). Here I will adopt a purely fluid approach—partly to make these lectures self-consistent and partly because there is a certain beauty in it: we need to know relatively little about the properties of the constituent substance in order to spin out a very sophisticated and complete theory about the way in which it flows. This approach is also more generally applicable because the substance that we will be dealing with need not be gaseous, like plasma—you may also think of liquid metals, various conducting solutions, etc.

So, let us declare an interest in the flow of a conducting fluid and attempt to be guided in our description of it by the very basic things: conservation laws of mass, momentum and energy plus Maxwell’s equations for the electric and magnetic fields. This will prove sufficient for most of our purposes. So we shall consider a fluid characterised by the following quantities:

\[ \rho \]—mass density,
\[ u \]—flow velocity,
\[ p \]—pressure,
\[ \sigma \]—charge density,
\[ j \]—current density,
\[ E \]—electric field,
\[ B \]—magnetic field.

Our immediate objective is to find a set of closed equations that would allow us to determine all of these quantities as functions of time and space within the fluid.


This is the most standard of all arguments in fluid dynamics (Fig. 44):

\[
\frac{d}{dt} \int_V d^3 r \rho = - \int_{\partial V} (\rho u) \cdot dS = - \int_V d^3 r \nabla \cdot (\rho u). \tag{13.1}
\]

As this equation holds for any \( V \), however small, it can be converted into a differential relation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{13.2}
\]

This is the continuity equation.
13.2. Conservation of Momentum

A similar approach:

\[
\frac{d}{dt} \int_V d^3r \, \rho \mathbf{u} = - \int_{\partial V} \left( \rho \mathbf{u} \right) \cdot d\mathbf{S} - \int_{\partial V} p \, d\mathbf{S} - \int_{\partial V} \Pi \cdot d\mathbf{S} + \int_V d^3r \, \mathbf{F}.
\]

In differential form, this becomes

\[
\frac{\partial}{\partial t} \rho \mathbf{u} = - \nabla \cdot (\rho \mathbf{u}) - \nabla p - \nabla \cdot \Pi + \mathbf{F},
\]

and so, finally,

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = - \nabla p - \nabla \cdot \Pi + \mathbf{F}.
\]

This is the momentum equation.

One part of this equation does have to be calculated from some knowledge of the microscopic properties of the constituent fluid or gas—the viscous stress. For a gas, it is done in kinetic theory (e.g., Lifshitz & Pitaevskii 1981; Dellar 2015; Schekochihin 2019, §6.8):

\[
\Pi = -\rho \nu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbf{I} \right],
\]

where \( \nu \) is the kinematic (Newtonian) viscosity.

In a magnetised plasma (i.e., such that its collision frequency \( \ll \) Larmor frequency of the gyrating charges), the viscous stress is much more complicated and anisotropic with respect to the direction of the magnetic field: because of their Larmor motion, charged particles diffuse differently across and along the field. This gives rise to the so-called Braginskii (1965) stress (see, e.g., Helander & Sigmar 2005; Parra 2019a).
In what follows, we will never require the explicit form of $\Pi$.

13.3. Electromagnetic Fields and Forces

The fact that the fluid is conducting means that it can have distributed charges ($\sigma$) and currents ($j$) and so the electric ($E$) and magnetic ($B$) fields will exert body forces on the fluid. Indeed, for one particle of charge $q$, the Lorentz force is

$$f_L = q \left( E + \frac{v \times B}{c} \right),$$

(13.7)

and if we sum this over all particles (or, to be precise, average over their distribution and sum over species), we will get

$$F = \sigma E + \frac{j \times B}{c}.$$  

(13.8)

This body force (force density) goes into (13.5) and so we must know $E$, $B$, $\sigma$ and $j$ in order to compute the fluid flow $u$.

Clearly it is a good idea to bring in Maxwell’s equations:

$$\nabla \cdot E = 4\pi \sigma \quad \text{(Gauss)},$$  

(13.9)

$$\nabla \cdot B = 0,$$  

(13.10)

$$\frac{\partial B}{\partial t} = -c \nabla \times E \quad \text{(Faraday)},$$  

(13.11)

$$\nabla \times B = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial E}{\partial t} \quad \text{(Ampère–Maxwell)}.$$  

(13.12)

To these, we append Ohm’s law in its simplest form: The electric field in the frame of a fluid element moving with velocity $u$ is

$$E' = E + \frac{u \times B}{c} = \eta j,$$

(13.13)

where $E$ is the electric field in the laboratory frame and $\eta$ is the Ohmic resistivity.

Normally, the resistivity, like viscosity, has to be computed from kinetic theory (see, e.g., Helander & Sigmar 2005; Parra 2019a) or tabulated by assiduous experimentalists. In a magnetised plasma, the simple form (13.13) of Ohm’s law is only valid at spatial scales longer than the Larmor radii and time scales longer than the Larmor periods of the particles (see, e.g., Goedbloed & Poedts 2004; Parra 2019b).

Equations (13.9–13.13) can be reduced somewhat if we assume (quite reasonably for most applications) that our fluid flow is non-relativistic. Let us stipulate that all fields evolve on time scales $\sim \tau$, have spatial scales $\sim \ell$ and that the flow velocity is

$$u \sim \frac{\ell}{\tau} \ll c.$$  

(13.14)

Then, from Ohm’s law (13.13),

$$E \sim \frac{u}{c} B \ll B,$$

(13.15)

so electric fields are small compared to magnetic fields.
In Ampère–Maxwell’s law (13.12),

\[
\frac{1}{c} \frac{\partial E}{\partial t} \sim \frac{1}{c} \frac{u}{\ell} \frac{B}{c} \sim \frac{u^2}{c^2} \ll 1,
\]

so the displacement current is negligible (note that at this point we have ordered out light waves; see Q2 in Kinetic Theory). This allows us to revert to the pre-Maxwell form of Ampère’s law:

\[
j = \frac{c}{4\pi} \nabla \times B.
\]

(13.17)

Thus, the current is no longer an independent field, there is a one-to-one correspondence \( j \leftrightarrow B \).

Finally, comparing the electric and magnetic parts of the Lorentz force (13.8), and using Gauss’s law (13.9) to estimate \( \sigma \sim E/\ell \), we get

\[
\frac{|\sigma E|}{\frac{1}{c} j \times B} \sim \frac{1}{\ell} \frac{E^2}{c^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{c^2} \ll 1.
\]

(13.18)

Thus, the MHD body force is

\[
F = \frac{j \times B}{c} = \frac{(\nabla \times B) \times B}{4\pi}.
\]

(13.19)

This goes into (13.5) and we note with relief that \( \sigma, j \) and \( E \) have all fallen out of the momentum equation—we only need to know \( B \).

13.4. Maxwell Stress and Magnetic Forces

Let us take a break from formal derivations to consider what (13.19) teaches us about the sort of new dynamics that our fluid will experience as a result of being conducting. To see this, it is useful to play with the expression (13.19) in a few different ways.

By simple vector algebra,

\[
F = \frac{B \cdot \nabla B}{4\pi} - \nabla \left( \frac{B^2}{8\pi} \right) = -\nabla \left( \frac{B^2}{8\pi} I - \frac{BB}{4\pi} \right),
\]

(13.20)

where the last expression was obtained with the aid of \( \nabla \cdot B = 0 \). Thus, the action of the Lorentz force in a conducting fluid amounts to a new form of stress. Mathematically, this “Maxwell stress” is somewhat similar to the kind of stress that would arise from a suspension in the fluid of elongated molecules—e.g., polymer chains, or other kinds of “balls on springs” (see, e.g., Dellar 2017; the analogy can be made rigorous: see Ogilvie & Proctor 2003). Thus, we expect that the magnetic field threading the fluid will impart to it a degree of “elasticity” (you will have an opportunity of a practical engagement with this analogy in Exercise 14.4).

Exactly what this means dynamically becomes obvious if we rewrite the magnetic tension and pressure forces in (13.20) in the following way. Let \( b = B/B \) be the unit
vector in the direction of $B$ (the unit tangent to the field line). Then

$$B \cdot \nabla B = Bb \cdot \nabla(Bb) = B^2 b \cdot \nabla b + bb \cdot \nabla \frac{B^2}{2}$$

(13.21)

and, putting this back into (13.20), we get

$$F = \frac{B^2}{4\pi} b \cdot \nabla b - (1 - bb) \cdot \nabla \frac{B^2}{8\pi} \equiv \nabla \perp \text{magnetic}$$

Thus, we learn that the Lorentz force consists of two distinct parts (Fig. 45):

- curvature force, so called because $b \cdot \nabla b$ is the vector curvature of the magnetic field line—the implication being that field lines, if bent, will want to straighten up;
- magnetic pressure, whose presence implies that field lines will resist compression or rarefaction (the field wants to be uniform in strength).

Note that both forces act perpendicularly to $B$, as they must, since magnetic field never exerts a force along itself on a charged particle [see (13.7)].

So this is the effect of the field on the fluid. What is the effect of the fluid on the field?

### 13.5. Evolution of Magnetic Field

Returning to deriving MHD equations, we use Ohm’s law (13.13) to express $E$ in terms of $u$, $B$ and $j$ in the right-hand side of Faraday’s law (13.11). We then use Ampere’s law (13.17) to express $j$ in terms of $B$. The result is

$$\frac{\partial B}{\partial t} = \nabla \times \left( u \times B - \frac{c^2\eta}{4\pi} \nabla \times B \right).$$

(13.23)

After using also $\nabla \cdot B = 0$ to get $\nabla \times (\nabla \times B) = -\nabla^2 B$ and renaming $c^2\eta/4\pi \rightarrow \eta$, the magnetic diffusivity, we arrive at the magnetic induction equation (due to Hertz):

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta \nabla^2 B.$$  

(13.24)

Note that if $\nabla \cdot B = 0$ is satisfied initially, any solution of (13.24) will remain divergence-free at all times.
13.6. Magnetic Reynolds Number

The relative importance of the diffusion term (it is obvious what this does) and the advection term (to be discussed in the next few sections) in (13.24) is measured by a dimensionless number:

\[
\frac{\nabla \times (u \times B)}{|\eta \nabla^2 B|} \sim \frac{u \ell B}{\eta \ell^2 B} = \frac{u \ell}{\eta} \equiv \text{Rm},
\]

(13.25)
called the magnetic Reynolds number. In nature, it can take a very broad range of values:

- liquid metals in industrial contexts (metallurgy): \( \text{Rm} \sim 10^{-3} \ldots 10^{-1} \),
- planet interiors: \( \text{Rm} \sim 100 \ldots 300 \),
- solar convective zone: \( \text{Rm} \sim 10^6 \ldots 10^9 \),
- interstellar medium (“warm” phase): \( \text{Rm} \sim 10^{18} \),
- intergalactic medium (cores of galaxy clusters): \( \text{Rm} \sim 10^{29} \),
- laboratory “dynamo” experiments: \( \text{Rm} \sim 1 \ldots 10^3 \).

Generally speaking, when flow velocities are large/distances are large/resistivities are low, \( \text{Rm} \gg 1 \) and it makes sense to consider “ideal MHD,” i.e., the limit \( \eta \to 0 \). In fact, \( \eta \) often needs to be brought back in to deal with instances of large \( \nabla B \), which arise naturally from solutions of ideal MHD equations (see §13.14, Q5, §17.2 and Parra 2019a), but let us consider the ideal case for now to understand what the advective part of the induction equation does to \( B \).

13.7. Lundquist Theorem

The ideal (\( \eta = 0 \)) version of the induction equation (13.24),

\[
\frac{\partial B}{\partial t} = \nabla \times (u \times B),
\]

(13.26)
implies that fluid elements that lie on a field line initially will remain on this field line, i.e., “the magnetic field moves with the flow.”

Proof. Unpacking the double vector product in (13.26),

\[
\frac{\partial B}{\partial t} = -u \cdot \nabla B + B \cdot \nabla u - B \nabla \cdot u + u \nabla \times B,
\]

(13.27)
or, using the notation for the “convective derivative” [see (13.5)],

\[
\frac{dB}{dt} \equiv \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) B = B \cdot \nabla u - B \nabla \cdot u.
\]

(13.28)
The continuity equation (13.2) can be rewritten in a somewhat similar-looking form

\[
\frac{d\rho}{dt} \equiv \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \rho = -\rho \nabla \cdot u \quad \Rightarrow \quad \nabla \cdot u = -\frac{1}{\rho} \frac{d\rho}{dt}.
\]

(13.29)
The last expression is now used for \( \nabla \cdot u \) in (13.28):

\[
\frac{dB}{dt} = B \cdot \nabla u + \frac{B}{\rho} \frac{d\rho}{dt}.
\]

(13.30)
Multiplying this equation by \( 1/\rho \) and noting that

\[
\frac{1}{\rho} \frac{dB}{dt} - \frac{B}{\rho^2} \frac{d\rho}{dt} = \frac{d}{dt} \frac{B}{\rho},
\]

(13.31)
we arrive at

\[
\frac{d}{dt} \frac{B}{\rho} = \frac{B}{\rho} \cdot \nabla u. \tag{13.32}
\]

Let us compare the evolution of the vector $B/\rho$ with the evolution of an infinitesimal Lagrangian separation vector in a moving fluid: the convective derivative is the Lagrangian time derivative, so

\[
\frac{d}{dt} \delta r(t) = u(r + \delta r) - u(r) \approx \delta r \cdot \nabla u. \tag{13.33}
\]

Thus, $\delta r$ and $B/\rho$ satisfy the same equation. This means that if two fluid elements are initially on the same field line,

\[
\delta r = \text{const} \frac{B}{\rho}, \tag{13.34}
\]

then they will stay on the same field line, q.e.d.\(^{92}\)

This means that in MHD, the fluid flow will be entraining the magnetic-field lines with it—and, as we saw in §13.4, the field lines will react back on the fluid:

—when the fluid tries to bend the field, the field will want to spring back,
—when the fluid tries to compress or rarefy the field, the field will resist as if it possessed (perpendicular) pressure.

This is the sense in which MHD fluid is “elastic”: it is threaded by magnetic-field lines, which move with it and act as elastic bands.

13.8. Flux Freezing

There is an essentially equivalent formulation of the result of §13.7 that highlights the fact that the ideal induction equation (13.26) is a conservation law—conservation of magnetic flux.

The magnetic flux through a surface $S$ (Fig. 47a) is, by definition,

\[
\Phi = \int_S B \cdot dS \tag{13.35}
\]

($dS \equiv \hat{n} dS$, where $\hat{n}$ is a unit normal pointing out of the surface). The flux $\Phi$ depends on the loop $\partial S$, but not on the choice of the surface spanning it. Indeed, if we consider two

\(^{92}\)Lundquist theorem opens the door to a Lagrangian description of MHD fluid that contains some mathematical and physical delights: I will pick up this thread in §13.12.
surfaces, \( S_1 \) and \( S_2 \), spanning the same loop \( \partial S \) (Fig. 47b) and define \( \Phi_{1,2} = \int_{S_{1,2}} B \cdot dS \), then the flux out of the volume \( V \) enclosed by \( S_1 \cup S_2 = \partial V \) is

\[
\Phi_2 - \Phi_1 = \int_{\partial V} B \cdot dS = \int_V d^3r \nabla \cdot B = 0, \quad \text{q.e.d.} \quad (13.36)
\]

**Alfvén's Theorem.** Flux through any loop moving with the fluid is conserved.

**Proof.** Let \( S(t) \) be a surface spanning the loop at time \( t \). If the loop moves with the fluid (Fig. 48), at the slightly later time \( t+dt \) it is spanned (for example) by the surface

\[
S(t+dt) = S(t) \cup \text{ribbon traced by the loop as it moves over time } dt. \quad (13.37)
\]

Then the flux at time \( t \) is

\[
\Phi(t) = \int_{S(t)} B(t) \cdot dS \quad (13.38)
\]
and at the later time,

\[ \Phi(t + dt) = \int_{S(t+dt)} B(t + dt) \cdot dS \]

\[ = \int_{S(t)} B(t) \cdot dS + dt \int_{S(t)} \frac{\partial B}{\partial t} \cdot dS \]

\[ = \Phi(t) + dt \int_{\text{ribbon}} B(t + dt) \cdot \frac{dS}{dt} = u dt \times dS \]

\[ = dt \int_{\partial S(t)} B(t) \cdot (u \times dS) \]

\[ = -dt \int_{\partial S(t)} (u \times B) \cdot dS \]

\[ = -dt \int_{S(t)} \left[ \nabla \times (u \times B) \right] \cdot dS. \quad (13.39) \]

Therefore,

\[ \frac{d\Phi}{dt} = \frac{\Phi(t + dt) - \Phi(t)}{dt} = \int_{S(t)} \left[ \frac{\partial B}{\partial t} - \nabla \times (u \times B) \right] \cdot dS = 0, \quad \text{q.e.d.} \quad (13.40) \]

This result means that field lines are frozen into the flow. Indeed, consider a flux tube enclosing a field line (Fig. 49). As the tube deforms, the field line stays inside it because fluxes through the ends and sides of the tube cannot change.

Note that Ohmic diffusion breaks flux freezing, as is obvious from (13.40) if in the integrand one uses the induction equation (13.24) keeping the resistive term.

13.9. Amplification of Magnetic Field by Fluid Flow

An interesting physical consequence of these results is that flows of conducting fluid can amplify magnetic fields. For example, consider a flow that stretches an initial cylindrical tube of length \( l_1 \) and cross section \( S_1 \) into a long thin spaghetto of length \( l_2 \) and cross section \( S_2 \) (Fig. 50). By conservation of flux,

\[ B_1 S_1 = B_2 S_2. \quad (13.41) \]

By conservation of mass,

\[ \rho_1 l_1 S_1 = \rho_2 l_2 S_2. \quad (13.42) \]
Therefore,
\[
\frac{B_2}{\rho_2 l_2} = \frac{B_1}{\rho_1 l_1} \quad \Rightarrow \quad \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}.
\] (13.43)

In an incompressible fluid, \(\rho_2 = \rho_1\), and the field is amplified by a factor \(l_2/l_1\). In a compressible fluid, the field can also be amplified by compression.

Going back to the induction equation in the form (13.27),
\[
\frac{\partial B}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{B} \nabla \cdot \mathbf{u},
\] (13.44)

the three terms in it are responsible for, in order, advection of the field by the flow (i.e., the flow carrying the field around with it), “stretching” (amplification) of the field by velocity gradients that make fluid elements longer and, finally, compression or rarefaction of the field by convergent or divergent flows (unless \(\nabla \cdot \mathbf{u} = 0\), as it is in an incompressible fluid).

Hence arises the famous problem of MHD dynamo: are there fluid flows that lead to sustained amplification of the magnetic fields? The answer is yes—but the flow must be 3D (the absence of dynamo action in 2D is a theorem, the simplest version of which is due to Zeldovich 1956; see Q4). Magnetic fields of planets, stars, galaxies, etc. are all believed to owe their origin and persistence to this effect. This topic requires (and merits) a more detailed treatment (see reading suggestions below), but for now let us flag two important aspects:

- resistivity, however small, turns out to be impossible to neglect because large gradients of \(B\) appear as the field is advected by the flow (see §13.14);
- the amplification of the field is checked by the Lorentz force once the field is strong enough that it can act back on the flow, viz., when their energy densities become comparable:
\[
\frac{B^2}{8\pi} \sim \frac{\rho u^2}{2}.
\] (13.45)

If you wish to educate yourself on the topic of MHD dynamos (as everyone should), a classic (and mostly timeless) text is Moffatt (1978). The new classic is the review by Rincon (2019), which I recommend strongly.

If you want to stick with my notes, you will find a (somewhat outdated) review in my handwritten 2007 Les Houches lectures available here: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/leshouches07.pdf. A very short printed review is Schekochihin & Cowley (2007, §3). One of the calculations from these notes is reproduced in §13.14.
13.10. Conservation of Energy

Let us summarise the equations that we have derived so far, namely (13.2), (13.5) and (13.24), expressing conservation of

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{13.46}
\]
momentum

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p - \nabla \cdot \mathbf{\Pi} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi},
\]

\[
= -\nabla \cdot \left[ \left( p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{BB}{4\pi} + \mathbf{\Pi} \right], \tag{13.47}
\]

and flux

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{13.48}
\]

To complete the system, we need an equation for \( p \), which has to come from the one conservation law that we have not yet utilised: conservation of energy.

The total energy density is

\[
\varepsilon = \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{E^2}{8\pi} + \frac{B^2}{8\pi}, \tag{13.49}
\]

where the electric energy can (and, for consistency with §13.3, must) be neglected because \( E^2 / B^2 \sim u^2 / c^2 \ll 1 \). We follow the same logic as we did in §§13.1 and 13.2:

\[
\frac{d}{dt} \int_V d^3 r \varepsilon = -\int_{\partial V} \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} \right) \mathbf{u} \cdot d\mathbf{S} - \int_{\partial V} \left[ (p \mathbf{I} + \mathbf{\Pi}) \cdot \mathbf{u} \right] \cdot d\mathbf{S} - \int_{\partial V} q \cdot d\mathbf{S} - \int_{\partial V} \frac{c}{4\pi} (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{S}. \tag{13.50}
\]

Like the viscous stress \( \mathbf{\Pi} \), the heat flux \( \mathbf{q} \) must be calculated kinetically (in a plasma) or tabulated (in an arbitrary complicated substance). In a gas, \( \mathbf{q} = -\kappa \nabla T \), but it is more complicated in a magnetised plasma (see, e.g., Braginskii 1965; Helander & Sigmar 2005; Parra 2019a).

Note that the magnetic energy and the work done by the Lorentz force are not included in the first two terms on the right-hand side of (13.50) because all of that must already be correctly accounted for by the Poynting flux. Indeed, since \( c\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \nabla \times \mathbf{B} \)
[this is (13.13), with \( \eta \) renamed as in (13.24)], we have

\[
\int_{\partial V} \frac{c}{4\pi} (E \times B) \cdot dS = \int_{\partial V} \frac{B^2}{8\pi} u \cdot dS + \int_{\partial V} \left[ \left( \frac{B^2}{8\pi} \mathbf{l} - \frac{\mathbf{BB}}{4\pi} \right) \cdot u \right] \cdot dS
\]

magnetic energy flow

work done by Maxwell stress

\[
+ \int_{\partial V} \eta \left( \nabla \times B \right) \times B \cdot dS
\]

resistive slippage accounting for field not being precisely frozen into flow

(13.51)

After application of Gauss’s theorem and shrinking of the volume \( V \) to infinitesimality, we get the differential form of (13.50):

\[
\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right) = -\nabla \cdot \left[ \frac{\rho u^2}{2} u + \frac{\gamma}{\gamma - 1} p u + \Pi \cdot u + q \right]
\]

\[
+ \frac{B^2}{4\pi} \mathbf{l} - \frac{\mathbf{BB}}{4\pi} \cdot u + \eta \left( \nabla \times B \right) \times B
\]

(13.52)

It remains to separate the evolution equation for \( p \) by using the fact that we know the equations for \( \rho, u \) and \( B \) and so can deduce the rates of change of the kinetic and magnetic energies.

13.10.1. Kinetic Energy

Using (13.46) and (13.47),

\[
\frac{\partial}{\partial t} \left( \frac{\rho u^2}{2} \right) = \frac{u^2}{2} \frac{\partial \rho}{\partial t} + \rho u \cdot \frac{\partial u}{\partial t}
\]

\[
= -\frac{u^2}{2} \nabla \cdot \left( \rho u \right) - \rho u \cdot \nabla \left( \frac{u^2}{2} \right) - u \cdot \left\{ \nabla \cdot \left[ \left( \frac{B^2}{8\pi} \mathbf{l} - \frac{\mathbf{BB}}{4\pi} \right) \right] \right\}
\]

\[
= -\nabla \cdot \left[ \frac{\rho u^2}{2} u + \rho u + \frac{B^2}{8\pi} \mathbf{l} - \frac{\mathbf{BB}}{4\pi} \cdot u + \Pi \cdot u \right]
\]

\[
+ p \nabla \cdot u + \frac{B^2}{8\pi} \mathbf{l} - \frac{\mathbf{BB}}{4\pi} : \nabla u + \Pi : \nabla u
\]

(13.53)

The flux terms (energy flows and work by stresses on boundaries) that have been crossed out cancel with corresponding terms in (13.52) once (13.53) is subtracted from it.
13.10.2. Magnetic Energy

Using the induction equation (13.48),

$$\frac{\partial B^2}{\partial t} = \frac{B}{8\pi} \left[ -u \cdot \nabla B + B \cdot \nabla u - B \cdot \nabla \times B - B \cdot \nabla \cdot u + \eta \nabla^2 B \right]$$

$$= -\nabla \cdot \left[ \frac{B^2}{8\pi} u + \eta \frac{B \times B}{4\pi} \right] - \frac{B^2}{8\pi} - \frac{BB}{4\pi} : \nabla u - \eta \frac{\nabla \times B}{4\pi}^2.$$

Again, the crossed out flux terms will cancel with corresponding terms in (13.52). The metamorphosis of the resistive term into a flux term and an Ohmic dissipation term is a piece of vector algebra best checked by expanding the divergence of the flux term. Finally, the $u$-to-$B$ energy exchange term (penultimate on the right-hand side) corresponds precisely to the $B$-to-$u$ exchange term in (13.53) and cancels with it if we add (13.53) and (13.54).

13.10.3. Thermal Energy

Subtracting (13.53) and (13.54) from (13.52), consummating the promised cancellations, and mopping up the remaining $\nabla \cdot (pu)$ and $p \nabla \cdot u$ terms, we end up with the desired evolution equation for the thermal (internal) energy:

$$\frac{d}{dt} \frac{p}{\gamma - 1} = -\nabla \cdot q - \frac{\gamma}{\gamma - 1} p \nabla \cdot u - \Pi : \nabla u + \eta \frac{\nabla \times B}{4\pi}^2.$$  

A further rearrangement and the use of the continuity equation (13.46) to express $\nabla \cdot u = -d \ln \rho / dt$ turn (13.55) into

$$\frac{d}{dt} \ln \frac{p}{\rho^\gamma} = \frac{\gamma - 1}{p} \left( -\nabla \cdot q - \Pi : \nabla u + \eta \frac{\nabla \times B}{4\pi}^2 \right).$$

This form of the thermal-energy equation has very clear physical content: the left-hand side represents advection of the entropy of the MHD fluid by the flow—each fluid element behaves adiabatically, except for the sundry non-adiabatic effects on the right-hand side. The latter are the heat flux in/out of the fluid element and the dissipative (viscous and resistive) heating, leading to entropy production. Note that the form of the viscous stress $\Pi$ ensures that the viscous heating is always positive [see, e.g., (13.6)]. In these Lectures, I will, for the most part, focus on ideal MHD and so use the adiabatic version of (13.56), with the right-hand side set to zero.
A. A. Schekochihin

Let me reiterate the equations of ideal MHD, now complete:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{13.57}
\]

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}, \tag{13.58}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{13.59}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \frac{p}{\rho^{\gamma}} = 0. \tag{13.60}
\]

In what follows, we shall study various solutions and asymptotic regimes of these rather nice equations.

**Exercise 13.1. MHD with Self-Gravity.** Consider an MHD system subject to gravity with acceleration \( \mathbf{g} = -\nabla \Phi \), where the gravitational potential \( \Phi \) satisfies Poisson’s equation

\[
\nabla^2 \Phi = 4\pi G \rho, \tag{13.61}
\]

and \( G \) is the gravitational constant. There will then be an additional body force in the momentum equation (13.5), equal to \( \rho \mathbf{g} \). Show that, like the magnetic force (13.20), the gravitational force can be written as a divergence of a stress tensor:

\[
\rho \mathbf{g} = \nabla \cdot \left( \frac{g^2}{8\pi G} \mathbf{I} - \frac{\mathbf{g} \mathbf{g}}{4\pi G} \right). \tag{13.62}
\]

Show also that the total energy of this system is conserved:

\[
\frac{d}{dt} \int d^3 r \left( \frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} + \frac{\rho \Phi}{2} \right) = 0, \tag{13.63}
\]

where the integration is over the entire system, surface integrals over its boundary are assumed to vanish and

\[
\mathcal{E}_G \equiv \int d^3 r \frac{\rho \Phi}{2} = -\int d^3 r \frac{g^2}{8\pi G} < 0 \tag{13.64}
\]

is the gravitational energy of the system.

13.11. **Virial Theorem**

This is a good place to present a rather nice, exact result that follows directly from MHD equations and helps one decide whether an MHD system can “self-confine”, i.e., whether a blob of plasma (or, more generally, a conducting fluid) can exist without blowing itself apart.

Consider an MHD system whose volume is \( V \). Its moment of inertia is

\[
I = \frac{1}{2} \int d^3 r \rho r^2. \tag{13.65}
\]

Let us see how it evolves with time: using the continuity equation (13.2), we get

\[
\frac{dI}{dt} = \frac{1}{2} \int d^3 r r^2 \frac{\partial \rho}{\partial t} = -\frac{1}{2} \int d^3 r r^2 \nabla \cdot (\rho \mathbf{u}) = \int d^3 r \rho \mathbf{u} \cdot \mathbf{r}, \tag{13.66}
\]

after integration by parts and assuming \( \mathbf{u} \perp \partial V \) (no in/outflows). Let us take another time derivative of this, this time using the momentum equation in the form (13.4),

\[
\frac{\partial}{\partial t} \rho \mathbf{u} = -\nabla \cdot \mathbf{T}, \tag{13.67}
\]

where the total stress tensor

\[
\mathbf{T} = \rho \mathbf{uu} + p \mathbf{I} + \Pi + \frac{B^2}{8\pi} \mathbf{I} - \frac{\mathbf{BB}}{4\pi} - \left( \frac{g^2}{8\pi G} \mathbf{I} - \frac{\mathbf{gg}}{4\pi G} \right). \tag{13.68}
\]
includes the Maxwell stress (13.20) and the gravitational stress (13.62). This gives us
\[
\frac{d^2 I}{dt^2} = - \int d^3 r \left( \nabla \cdot T \right) \cdot r = - \int d^3 r \left[ \nabla \cdot \left( T \cdot r \right) - T \cdot \nabla r \right] = - \int dS \cdot T \cdot r + \int d^3 r \text{tr} T. \tag{13.69}
\]
Therefore, in steady state, the stresses on the boundary are
\[
\int dS \cdot T \cdot r = \int d^3 r \text{tr} T = 2\epsilon_{\text{kin}} + 3(\gamma - 1)\epsilon_{\text{th}} + \epsilon_{\text{mag}} + \epsilon_{G}, \tag{13.70}
\]
where \(\epsilon_{\text{kin}}, \epsilon_{\text{th}}, \epsilon_{\text{mag}}\) and \(\epsilon_{G}\) are the total kinetic, thermal, magnetic and gravitational energies of the system—the four terms in (13.63); note that \(\text{tr} \Pi = 0\) (any trace is part of \(p\)). In the absence of gravity, the weighted sum of energies in (13.70) is strictly positive and so it is not possible to have zero stress on the boundary of the system—the system cannot be self-confined.

With gravity, since the gravitational energy (13.64) is negative, the right-hand side of (13.70) can be—and, in a steady, self-confined state, is—zero, telling us gravity can confine MHD systems, e.g., stars. The total energy in this case is, as you might imagine, negative:
\[
\mathcal{E} = \epsilon_{\text{kin}} + \epsilon_{\text{th}} + \epsilon_{\text{mag}} + \epsilon_{G} = -\epsilon_{\text{kin}} - (3\gamma - 4)\epsilon_{\text{th}} < 0. \tag{13.71}
\]
Perhaps an unexpected corollary of this is that if a star radiates and, therefore, loses energy, then \(\epsilon_{\text{th}}\) must increase (assuming \(\epsilon_{\text{kin}} \ll \epsilon_{\text{th}}\), i.e., subsonic motions inside)—i.e., “as the star cools, it heats up”.

13.12. Lagrangian MHD

There is a Lagrangian formulation of ideal MHD, due to Newcomb (1962, a classic and elegant paper, which I recommend for your reading pleasure). This is both mathematically attractive and sheds some physical light.

Let us label each fluid element’s position at \(t = 0\) by the Lagrangian coordinate \(r_0\). Then the Eulerian coordinate \(r\) at any given time \(t\) is the position of the same fluid element at that time:
\[
r(t, r_0) = r_0 + \xi(t, r_0), \tag{13.72}
\]
where \(\xi\) is the displacement. Formally, we shall treat (13.72) as just a coordinate transformation, which can also be inverted: given the Eulerian coordinate \(r\) of a fluid element at time \(t\), the Lagrangian coordinate \(r_0(t, r)\) is where this fluid element was at \(t = 0\). The coordinate transformation is determined by the history of the fluid flow:
\[
\frac{\partial r(t, r_0)}{\partial t} = \frac{\partial \xi(t, r_0)}{\partial t} = u(t, r(t, r_0)) \equiv u_L(t, r_0). \tag{13.73}
\]
I shall use the subscript “L” to designate Lagrangian fields, i.e., MHD fields as functions of the Lagrangian coordinate and time.

Let us learn how to transform derivatives between Eulerian and Lagrangian variables:
\[
\frac{\partial}{\partial r_{0i}} = \frac{\partial r_j}{\partial r_{0i}} \frac{\partial}{\partial r_j} = \left( \delta_{ji} + \frac{\partial \xi_j}{\partial r_{0i}} \right) \frac{\partial}{\partial r_j}, \quad \text{or} \quad \nabla_0 = (\nabla_0 r) \cdot \nabla = (I + \nabla_0 \xi) \cdot \nabla, \tag{13.74}
\]
\[
\frac{\partial}{\partial t_L} \equiv \left( \frac{\partial}{\partial t} \right)_{r_0} = \left( \frac{\partial}{\partial t} \right)_r + \left( \frac{\partial r_i}{\partial t} \right)_{r_0} \frac{\partial}{\partial r_i} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{d}{dt}. \tag{13.75}
\]
Thus, the Lagrangian time derivative is the convective derivative—you knew that! We are now ready to convert the MHD equations (13.57–13.60) to the Lagrangian frame.

13.12.1. Density

The continuity equation is an expression of conservation of mass (§13.1). Let us write that for an infinitesimal volume element moving with the fluid:
\[
\rho_0(r_0)d^3 r_0 = \rho_L(t, r_0)d^3 r, \tag{13.76}
\]
where \(\rho_0\) and \(d^3 r_0\) are the density and the volume of the fluid element at \(t = 0\), and \(\rho_L\) and \(d^3 r\) are its density and volume at time \(t\) (Fig. 51). So we need to know how volumes change under the coordinate transformation (13.72), i.e., we need the Jacobian of the strain matrix \(\partial r_i / \partial r_{0j}\):
\[
d^3 r = J(t, r_0)d^3 r_0, \quad \text{where} \quad J = |\det \nabla_0 r| = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} \frac{\partial r_i}{\partial r_{0l}} \frac{\partial r_j}{\partial r_{0m}} \frac{\partial r_k}{\partial r_{0n}}. \tag{13.77}
\]
The continuity equation in the Lagrangian frame is, therefore,

\[ \rho_L(t, \mathbf{r}_0) = \frac{\rho_0(\mathbf{r}_0)}{J(t, \mathbf{r}_0)} \cdot \frac{\mathbf{J}(t, \mathbf{r}_0)}{\rho_L(t, \mathbf{r}_0)}. \] (13.78)

### 13.12.2. Pressure

This result, together with (13.75), makes it really easy to work out Lagrangian pressure. Indeed, the adiabatic law (13.60) implies that the entropy density of Lagrangian fluid elements never changes:

\[ \frac{\partial}{\partial t} \frac{p_L}{\rho_L} = \frac{d}{dt} \frac{p}{\rho^\gamma} = 0 \Rightarrow \frac{p_L}{\rho_L} = \frac{p_0}{\rho_0}. \] (13.79)

Using (13.78), we get

\[ p_L(t, \mathbf{r}_0) = \frac{p_0(\mathbf{r}_0)}{J^\gamma(t, \mathbf{r}_0)} \cdot \frac{\mathbf{J}(t, \mathbf{r}_0)}{\rho_L(t, \mathbf{r}_0)}. \] (13.80)

### 13.12.3. Magnetic Field

To work out the magnetic field, consider the induction equation in the form (13.32). In Lagrangian variables,

\[ \frac{\partial}{\partial t} \frac{B_L}{\rho_L} = \frac{B_L}{\rho_L} \cdot \nabla \mathbf{u} = \frac{B_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \mathbf{u}_L = \frac{B_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \frac{\partial \mathbf{r}_0}{\partial t}. \] (13.81)

The solution to this that satisfies \( B_L/\rho_L = B_0/\rho_0 \) at \( t = 0 \) is

\[ B_L(t, \mathbf{r}_0) = \frac{B_0}{\rho_0} \cdot \nabla_0 \mathbf{r}_0. \] (13.82)

Indeed:

\[ \frac{\partial}{\partial t} \frac{B_L}{\rho_L} = \frac{B_0}{\rho_0} \cdot \nabla_0 \frac{\partial \mathbf{r}_0}{\partial t} = \frac{B_L}{\rho_L} \cdot (\nabla_0 \mathbf{r})^{-1} \cdot \nabla_0 \frac{\partial \mathbf{r}_0}{\partial t} = \frac{B_L}{\rho_L} \cdot (\nabla \mathbf{r}_0) \cdot \nabla_0 \frac{\partial \mathbf{r}_0}{\partial t}, \] q.e.d. (13.83)

Thus, the magnetic field in the Lagrangian frame satisfies (13.82), or, using (13.78),

\[ B_L(t, \mathbf{r}_0) = \frac{B_0(\mathbf{r}_0) \cdot \nabla_0 \mathbf{r}(t, \mathbf{r}_0)}{J(t, \mathbf{r}_0)}. \] (13.84)

This solution is really just a restatement of the Lundquist theorem: (13.82) says that \( B/\rho \) transforms in the same way as a vector connecting two infinitesimally close material points in the fluid [see (13.33)].
13.12.4. Fluid Flow

Finally, let us deal with the momentum equation (13.58), which in this context is the equation that actually defines the Lagrangian variable transformation (13.72). Using the decomposition (13.20) of the Lorentz force into magnetic pressure and tension, and substituting for \( \rho \), \( p \) and \( B \) from (13.78), (13.80) and (13.84), respectively, we get, noting that \( \nabla_0 r \cdot \nabla = \nabla_0 \),

\[
\frac{\rho_0}{J} \frac{\partial^2 r}{\partial t^2} = - (\nabla_0 r)^{-1} \cdot \nabla_0 \left( p_0 \frac{J}{J^\gamma} + \frac{|B_0 \cdot \nabla_0 r|^2}{8\pi J^2} \right) + \frac{1}{4\pi} \frac{B_0}{J} \cdot \nabla_0 \frac{B_0}{J} \cdot \nabla_0 r.
\]  

(13.85)

Coupled with the initial condition \( r(0) = r_0 \) and the formula (13.77) for \( J \), this equation determines the trajectory \( r(t, r_0) \) of each fluid element and hence its Lagrangian displacement \( \xi = r - r_0 \), its velocity \( u_L = \partial \xi / \partial t \), and the associated density, pressure and magnetic field via (13.78), (13.80) and (13.84). It is not a particularly pretty equation—the price we have paid for being able to integrate explicitly all the other MHD equations.

**Exercise 13.2. Energy in Lagragian MHD.** Show that the total energy of a volume of MHD fluid in Lagrangian variables is

\[
E = \int d^3 r_0 \left[ \frac{1}{2} \rho_0 \left( \frac{\partial r}{\partial t} \right)^2 + \frac{p_0 J^{-(\gamma - 1)}}{\gamma - 1} + \frac{|B_0 \cdot \nabla_0 r|^2}{8\pi J} \right].
\]  

(13.86)

**Exercise 13.3. Action Principle for Lagrangian MHD.** Show that (13.85) can be derived from an action principle, \( \delta S = 0 \), with

\[
S = \int_0^t dt \int d^3 r_0 \mathcal{L}(r, \dot{r}),
\]  

(13.87)

where the Lagrangian density is

\[
\mathcal{L}(r, \dot{r}) = \frac{\rho_0 |\dot{r}|^2}{2} - \frac{p_0 J^{-(\gamma - 1)}}{\gamma - 1} - \frac{|B_0 \cdot \nabla_0 r|^2}{8\pi J}.
\]  

(13.88)


The Lagrangian formalism, besides shedding conceptual light, turns out to give one some useful analytical tools, e.g., for the treatment of explosive MHD instabilities (Plösch & Sudan 1993; Cowley & Artun 1997).

An obvious problem with using this approach to describe the MHD fluid over any significant intervals of time is that it only works for ideal MHD. Even if we restrict ourselves to nice flows that do not have small scales and are thus immune to viscosity, (13.84) tells us that magnetic field can quickly develop small scales and thus access resistivity. An example of how that could happen starting with a fairly generic magnetic configuration is in Q5. An even simpler example is an application of a simple, linear stretching/shearing/compressing transformation (which can be achieved by an arbitrarily large-scale flow) to a curved field line, producing a “folded” field with direction reversals on ever smaller scales (Fig. 52). It is clear that resistivity must kick in, at which point the Lagrangian solution (13.84) can no longer be used. Let me offer two examples of how one might deal with such a situation mathematically and in the process get an idea of what happens physically.

13.13. Eyink’s Stochastic Lundquist Theorem


Let us consider the evolution of magnetic field in a linear flow, i.e.,

\[
u(t, r) = A(t) \cdot r,
\]  

(13.89)

where \( A(t) \) is some matrix that may be a function of time, but not of space. The key simplification that comes with this assumption (admittedly at a heavy price!) is that the strain matrix \( \nabla_0 r \) is independent of position: indeed, it is the solution of

\[
\frac{\partial}{\partial t} \nabla_0 r = \nabla_0 u_L = (\nabla_0 r) \cdot A^T(t), \quad \nabla_0 r(t = 0) = I.
\]  

(13.90)
Figure 52. Effect of stretching/shearing/compression on magnetic field: “folded” field with small-scale direction reversals.

Let us further assume, for simplicity, that the flow is incompressible ($\nabla \cdot u = \text{tr } A = 0$). Then $\rho = \rho_0 = \text{const}$ and so $J = 1$.


The (Lagrangian) magnetic field satisfies the induction equation with resistivity:

$$\frac{\partial B_L}{\partial t} = \frac{B_L \cdot \nabla u}{A \cdot B_L} + \eta \nabla^2 B_L,$$

(13.91)

where the gradients in the right-hand side are still Eulerian. Now let us seek a solution of this equation in the form

$$B_L(t, r_0) = \hat{B}(t) e^{i k(t) \cdot r(t, r_0)}.$$  

(13.92)

At $t = 0$, this is simply a single-Fourier-mode initial field: $B_0(r_0) = \hat{B}_0 e^{i k_0 \cdot r_0}$, where $k_0 = k(0)$ and $\hat{B}_0 = B(0)$. Let us see if such functions $\hat{B}(t)$ and $k(t)$ can be found that (13.92) works. Substituting (13.92) into (13.91), one gets

$$\frac{\partial \hat{B}}{\partial t} + i \left( \frac{\partial k}{\partial t} \cdot r + k \cdot \frac{\partial r}{\partial t} \right) \hat{B} = A \cdot \hat{B} - \eta k^2(t) \hat{B}.$$  

(13.93)

Since $\dot{r} = u = A \cdot r$, the second term on the left-hand side is annihilated if

$$\frac{\partial k}{\partial t} = -k \cdot A \quad \Rightarrow \quad k(t) = (\nabla r_0)(t) \cdot k_0.$$  

(13.94)

The last expression follows from (13.90) if one works out the time derivative of $\nabla r_0$ by differentiating $(\nabla r_0) \cdot (\nabla_0 r) = 1$. We are left with

$$\frac{\partial \hat{B}}{\partial t} = A \cdot \dot{B} - \eta k^2(t) \hat{B} \quad \Rightarrow \quad \hat{B}(t) = \hat{B}_0 \cdot (\nabla_0 r)(t) \exp \left[-\eta \int_0^t dt' k^2(t') \right].$$  

(13.95)

The solution has been obtained by eliminating the resistive term via an integrating factor and taking care of the rest by using (13.90). The Lagrangian solution (13.84) has re-emerged as a prefactor, now attenuated by resistive decay.

Since the induction equation is linear in $B$, any linear combination of solutions of the form (13.92) is also a solution. Therefore, we can accommodate any initial field $B_0$: it will evolve according to

$$B_L(t, r_0) = \sum_{k_0} \hat{B}(t, k_0) e^{i k(t, k_0) \cdot r(t, r_0)}$$

$$= \sum_{k_0} \hat{B}_0(k_0) \cdot (\nabla_0 r)(t) \exp \left[i k(t, k_0) \cdot r(t, r_0) - \eta \int_0^t dt' k^2(t', k_0) \right],$$  

(13.96)
where $\hat{B}_0(k_0)$ is the Fourier coefficient of the initial field $B_0(r_0) = \sum_{k_0} \hat{B}(k_0)e^{ik_0 \cdot r_0}$ and $k(t, k_0)$ satisfies (13.94).

13.14.2. Is There Dynamo?

An interesting question now is whether the energy of the solution (13.96) grows with time—is the velocity field (13.89) a dynamo? It is not hard to prove a version of the Parseval theorem: the volume average of the magnetic energy is

$$\langle B^2 \rangle(t) = \int \frac{d^3 r}{V} |\hat{B}(t, r)|^2 = \sum_{k_0} |\hat{B}(t, k_0)|^2. \quad (13.97)$$


Using (13.95),

$$|\hat{B}(t, k_0)|^2 = \hat{B}_0(k_0) \cdot g(t) \cdot \hat{B}_0^*(k_0) \exp \left[ -2\eta \int_0^t dt' k_0 \cdot g^{-1}(t') \cdot k_0 \right], \quad (13.98)$$

where $g$ is the (covariant) metric tensor associated with the Largangian transformation of variables:

$$g_{ij} = \frac{\partial r_i}{\partial r_i'} \frac{\partial r_j}{\partial r_j'}, \quad (g^{-1})_{ij} = \frac{\partial r_i}{\partial r_i'} \frac{\partial r_j}{\partial r_j'}. \quad (13.99)$$

These can (in principle) be calculated via (13.90) for any given $A(t)$.

It is possible to do this with a degree of generality, but a lot of formalism is needed along the way. I will instead opt for an extremely simple case:

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow g = \begin{pmatrix} e^{2\lambda_1 t} & 0 & 0 \\ 0 & e^{2\lambda_2 t} & 0 \\ 0 & 0 & e^{2\lambda_3 t} \end{pmatrix}, \quad (13.100)$$

where $\lambda_1 > \lambda_2 > 0 > \lambda_3$ are constants and $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (incompressibility). These three rates are rates of stretching and compression by the flow. Ignoring exponentially small terms, we get

$$|\hat{B}(t, k_0)|^2 \approx |\hat{B}_{01}|^2 \exp \left[ 2\lambda_1 t - \eta \left( \frac{k_{01}^2}{\lambda_1} + \frac{k_{02}^2}{\lambda_2} + \frac{k_{03}^2}{|\lambda_3|} e^{2|\lambda_3| t} \right) \right]. \quad (13.101)$$

This says that for most $k_0$, the corresponding modes will decay superexponentially (after initial transient growth at the “ideal” rate $2\lambda_1$). At any given time $t \gg \lambda_1^{-1}$, the domain in the $k_0$ space that will dominate the integral (13.97) is one containing modes for which the resistivity has not yet managed to cut the initial growth rate:

$$\eta \left( \frac{k_{01}^2}{\lambda_1} + \frac{k_{02}^2}{\lambda_2} + \frac{k_{03}^2}{|\lambda_3|} e^{2|\lambda_3| t} \right) \lesssim \text{const.} \quad (13.102)$$

This is the interior of an ellipsoid whose volume is $\propto e^{-|\lambda_3| t}$. Within this volume, $|\hat{B}(t, k_0)|^2 \sim |\hat{B}_{01}|^2 e^{2\lambda_1 t}$. Therefore, the integral (13.97) is, roughly,

$$\langle B^2 \rangle(t) \propto e^{(2\lambda_1 - |\lambda_3|) t} = e^{(\lambda_1 - \lambda_2) t}, \quad (13.103)$$

because $\lambda_3 = -\lambda_1 - \lambda_2$. Since $\lambda_1 > \lambda_2$, the conclusion is that magnetic energy will grow, albeit thanks to an ever shrinking subset of initial modes.

This calculation is due to Zeldovich et al. (1984) (but cleverly anticipated 20 years previously in a one-page note by Moffatt & Saffman 1964). The physical interpretation of it was some time in coming—let me explain it the way I understand it (Schekochihin et al. 2004).

13.14.3. Folded Fields

The three rates $\lambda_1$, $\lambda_2$ and $\lambda_3$ represent the flow’s action along the three (Lyapunov) directions locally associated with it: stretching at the rate $\lambda_1$ along the first direction $⟨e_1⟩$, compression
at the rate $|\lambda_3|$ along the third ($\hat{e}_3$) and something along the second, “null” direction ($\hat{e}_2$)—it can indeed be null ($\lambda_2 = 0$), but it can also be stretching or compression at a smaller rate than the other two. It follows from (13.95) and (13.94) that the magnetic field will align with the stretching direction whereas is wave vector will align with the compression direction, both exponentially fast:

$$B \sim B_0 \hat{e}_1 e^{\lambda_1 t}, \quad k \sim k_0 \hat{e}_3 e^{\lambda_3 t}.$$  

(13.104)

The latter alignment is what makes most modes decay superexponentially—in physical terms, this is the tendency, illustrated by Fig. 52, for the fields to become folded and reverse direction at ever smaller scales. The only modes that survive are those for which the initial wave vector $k_0$ was nearly perpendicular to $\hat{e}_3$, with its permitted angular deviation from $90^\circ$ decaying $\sim e^{-|\lambda_3| t}$. Since $\nabla \cdot B = 0, k_0 \cdot \dot{B}_0 = 0$ must be satisfied for the initial field. Thus, the modes that get amplified most are ones for which $\dot{B}_0 \parallel \hat{e}_1$ and $k_0 \parallel \hat{e}_2$. Such a “winning” fold is sketched in Fig. 53.

**Exercise 13.5.** What happens, mathematically and physically, if (a) $\lambda_2 = 0$, (b) $\lambda_2 < 0$?

**Exercise 13.6.** Work through an analogous argument in 2D and show that magnetic field always decays (no 2D dynamo; cf. Q4). Does this make sense physically?

### 13.14.4. Further Reading

Considering that the above calculation was done for a preposterously simplistic flow, is it at all useful in understanding real dynamos? It turns out to be surprisingly so. One does see folded fields in generic fluctuation dynamos when they are driven by smooth velocity fields, even random ones (Schekochihin et al. 2004; Rincon 2019). Locally such fields can be viewed as linear (13.89), but perhaps not constant time. This is OK as the above construction can be generalised to time-dependent fields. Indeed, since the matrix $g(t)$ is symmetric, it can at any time be diagonalised by an appropriate rotation:

$$g(t) = R^T(t) \cdot \begin{pmatrix} e^{A_1(t)} & 0 \\ 0 & e^{A_2(t)} \end{pmatrix} \cdot R(t),$$  

(13.105)

where the quantities $\Lambda_i(t)/2t$ are known as the finite-time Lyapunov exponents (FTLEs). It is possible to prove (Goldhirsch et al. 1987) that $R(t)$ converges to a constant matrix ($\hat{e}_1 \hat{e}_2 \hat{e}_3$) (Lyapunov basis) exponentially fast in time and that (more slowly) $\Lambda_i(t)/2t \to \lambda_i$ (Lyapunov exponents).

When the flow is random, the FTLEs are random functions and then one can prove that
(13.103) survives in the form

\[ \langle B^2 \rangle(t) \propto e^{(A_1(t) - A_2(t))/2}, \]  

(13.106)

where the overline means averaging over the distribution of the \( A_i \)’s. This distribution is usually quite hard to calculate for any real flow, so it has only been done for a few very special examples.

If you are intrigued by this line of inquiry, you might find further enlightenment in, e.g., Ott (1998) and Chertkov et al. (1999).

14. MHD in a Straight Magnetic Field

Equations (13.57–13.60) have a very simple static, uniform equilibrium solution:

\[ \rho_0 = \text{const}, \quad p_0 = \text{const}, \quad u_0 = 0, \quad B_0 = B_0 \hat{z} = \text{const}. \]  

(14.1)

We will turn to more nontrivial equilibria in due course, but first we shall study this one carefully—because it is very generic in the sense that many other, more complicated, equilibria locally look just like this.

14.1. MHD Waves

If you have an equilibrium solution of any set of equations, your first reflex ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly interesting nonlinear behaviour of small perturbations (e.g., §§6–8).

So we now seek solutions to the MHD equations (13.57–13.60) in the form

\[ \rho = \rho_0 + \delta \rho, \quad p = p_0 + \delta p, \quad u = \frac{\partial \xi}{\partial t}, \quad B = B_0 \hat{z} + \delta B, \]  

(14.2)

where we have introduced the fluid displacement field \( \xi \) (cf. §13.12).\(^{93}\) To start with, we consider all perturbations to be infinitesimal and so linearise the MHD equations

---

\(^{93}\) Thinking in terms of displacements makes sense in MHD but not so much in (homogeneous) hydrodynamics because in the latter case, just displacing a fluid element produces no back reaction, whereas in MHD, because magnetic fields are frozen into the fluid and are elastic, displacing fluid elements causes magnetic restoring forces to switch on. In other words, an (ideal) MHD fluid “remembers” the state from which it has been displaced, whereas neutral (Newtonian) fluids only “know” about velocities at which they flow.
Figure 54. Magnetic perturbations.

\( \delta b = \delta B_\perp / B_0 \)

\( \delta B_\parallel = \delta B \)

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \rho &= -\rho \nabla \cdot u \quad \Rightarrow \quad \frac{\partial \delta \rho}{\partial t} = -\rho_0 \nabla \cdot \frac{\partial \xi}{\partial t} \\
&\Rightarrow \quad \frac{\delta \rho}{\rho_0} = -\nabla \cdot \xi, \quad (14.3) \\
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \frac{p}{\rho^{\gamma}} &= 0 \quad \Rightarrow \quad \frac{\partial}{\partial t} \frac{\delta p}{p_0^{\gamma}} = \gamma \frac{\partial}{\partial t} \frac{\delta \rho}{\rho_0} \\
&\Rightarrow \quad \frac{\delta p}{p_0^{\gamma}} = -\gamma \nabla \cdot \xi, \quad (14.4) \\
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) B &= B \cdot \nabla u - B \nabla \cdot u \quad \Rightarrow \quad \frac{\partial \delta B}{\partial t} = B_0 \nabla_\parallel \frac{\partial \xi}{\partial t} - \hat{z} B_0 \nabla \cdot \frac{\partial \xi}{\partial t} \\
&\Rightarrow \quad \frac{\delta B}{B_0} = \nabla_\parallel \xi_\perp - \hat{z} \nabla_\perp \cdot \xi_\perp - \frac{\delta B_\parallel}{B_0} = -\nabla_\perp \cdot \xi_\perp, \quad (14.5)
\end{align*}
\]

where \( \parallel \) and \( \perp \) denote projections onto the direction \((z)\) of \( B_0 \) and onto the plane \((x, y)\) perpendicular to it, respectively. Equations (14.5) tell us that parallel displacements produce no perturbation of the magnetic field—obviously not, because the magnetic field is carried with the fluid flow and nothing will happen if you displace a straight uniform field parallel to itself.

The physics of magnetic-field perturbations becomes clearer if we observe that

\[
\frac{\delta B}{B_0} = \delta (Bb) / B_0 = \delta b + \hat{z} \frac{\delta B}{B_0}. \quad (14.6)
\]

The perturbed field-direction vector \( \delta b \) must be perpendicular to \( \hat{z} \) (otherwise the field direction is unperturbed; formally this is shown by perturbing the equation \( \mathbf{b}^2 = 1 \)). Therefore, the perpendicular and parallel perturbations of the magnetic field are the perturbations of its direction and strength, respectively (Fig. 54):

\[
\frac{\delta B_\perp}{B_0} = \delta b, \quad \frac{\delta B_\parallel}{B_0} = \frac{\delta B}{B_0}. \quad (14.7)
\]
Finally, linearising (13.58) gives us

\[ \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p = -\nabla \delta p = \gamma p_0 \nabla \cdot \xi, \]

from (14.4)

\[ = -\frac{\nabla B^2}{8\pi} + \frac{B \cdot \nabla B}{4\pi}, \]

(14.8)

Assembling all this, we get

\[ \frac{\partial^2 \xi}{\partial t^2} = c_s^2 \nabla \cdot \xi + v_A^2 \left( \nabla_\perp \nabla_\perp \cdot \xi_\perp + \nabla_\parallel^2 \xi_\perp \right), \]

where two special velocities have emerged:

\[ c_s = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad v_A = \frac{B_0}{\sqrt{4\pi \rho_0}}, \]

(14.10)

the sound speed and the Alfvén speed, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

**Exercise 14.1.** Derive (14.3–14.5) and (14.9) directly from Lagrangian MHD equations (13.78), (13.80), (13.84) and (13.85). A useful starting point is that, for an infinitesimal displacement, \( J = 1 + \nabla_0 \cdot \xi \) [follows from (13.77)].

Let us seek wave-like solutions of (14.9), \( \xi \propto \exp(-i\omega t + ik \cdot r) \). For such perturbations,

\[ \omega^2 \xi = c_s^2 k k \cdot \xi + v_A^2 \left( k_\perp k_\perp \cdot \xi_\perp + k_\parallel^2 \xi_\perp \right). \]

(14.11)

Without loss of generality, let \( k = (k_\perp, 0, k_\parallel) \) (i.e., by definition, \( x \) is the direction of \( k_\perp \);
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Figure 56. Hannes Olof Gösta Alfvén (1908-1995), Swedish electrical engineer and plasma physicist. He was the father of MHD, distrusted religion, computers and Big Bang theory, and got a Nobel Prize “for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics” (1970). In this picture, he is receiving it from King Gustaf VI Adolf of Sweden.

see Fig. 55). Then (14.11) becomes

\[ \omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_\parallel) + v_A^2 k_\perp^2 \xi_x, \]  
(14.12)

\[ \omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y, \]  
(14.13)

\[ \omega^2 \xi_\parallel = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel). \]  
(14.14)

The perturbations of the rest of the fields are

\[ \frac{\delta \rho}{\rho_0} = -i k_\cdot \xi = -i (k_\perp \xi_x + k_\parallel \xi_\parallel), \]  
(14.15)

\[ \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}, \]  
(14.16)

\[ \delta b = i k_\parallel \xi_\perp = i k_\parallel \begin{pmatrix} \xi_x \\ \xi_y \\ 0 \end{pmatrix}, \]  
(14.17)

\[ \frac{\delta B}{B_0} = -i k_\perp \xi_x. \]  
(14.18)

14.1.1. Alfvén Waves

We start by spotting, instantly, that (14.13) decouples from the rest of the system. Therefore, \( \xi = (0, \xi_y, 0) \) is an eigenvector, with two associated eigenvalues

\[ \omega = \pm k_\parallel v_A, \]  
(14.19)

representing Alfvén waves propagating parallel and antiparallel to \( B_0 \). An Alfvénic perturbation is (Fig. 57a)

\[ \xi = \xi_y \hat{y}, \quad \delta p = 0, \quad \delta B = 0, \quad \delta b = i k_\parallel \xi_y \hat{y}, \]  
(14.20)

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force. Note
that these waves can have \( k_\perp \neq 0 \) even though their dispersion relation (14.19) does not depend on \( k_\perp \) (Fig. 57b).

14.1.2. Magnetosonic Waves

Equations (14.12) and (14.14) form a closed 2D system:

\[
\omega^2 \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix} = \begin{pmatrix} c_s^2 k_\perp^2 + v_A^2 k_\parallel^2 & c_s^2 k_\parallel k_\perp \\ c_s^2 k_\parallel k_\perp & c_s^2 k_\perp^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix}. \tag{14.21}
\]

The resulting dispersion relation is

\[
\omega^4 - k^2 (c_s^2 + v_A^2) \omega^2 + c_s^2 v_A^2 k^2 k_\parallel^2 = 0. \tag{14.22}
\]

This has four solutions:

\[
\omega^2 = \frac{1}{2} k^2 \left[ c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta} \right], \quad \cos^2 \theta = \frac{k_\parallel^2}{k^2}. \tag{14.23}
\]

The two “+” solutions are the “fast magnetosonic waves” and the two “−” ones are the “slow magnetosonic waves”.

Since both sound and Alfvén speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of
the thermal to magnetic energies in the MHD medium, known as the plasma beta:

\[
\beta = \frac{p_0}{B_0^2/8\pi} = \frac{2}{\gamma} \frac{c_s^2}{v_A^2}.
\]

(14.24)

The magnetosonic waves can be conveniently summarised by the so-called Friedricks diagram, a graph of (14.23) in polar coordinates where the radius is the phase speed \(\omega/k\) and the angle is \(\theta\), the direction of propagation with respect to \(B_0\) (Fig. 58).

Clearly, magnetosonic waves contain perturbations of both the magnetic field and of the “hydrodynamic” quantities \(\rho\), \(p\), \(u\), but working them all out for the case of general oblique propagation (\(\theta \sim 1\)) is a bit messy. The physics of what is going on is best understood via a few particular cases.

14.1.3. Parallel Propagation

Consider \(k_\perp = 0\) (\(\theta = 0\)). Then \((\xi_x, 0, 0)\) and \((0, 0, \xi_\parallel)\) are eigenvectors of the matrix in (14.21) and the two corresponding waves are

- another Alfvén wave, this time with perturbation in the \(x\) direction (which, however, is not physically different from the \(y\) direction when \(k_\perp = 0\)):

\[
\omega^2 \xi_x = k_\parallel^2 c_s^2 \xi_x \Rightarrow \omega = \pm k_\parallel c_s,
\]

\[
\xi = \xi_x \hat{x}, \quad \delta \rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta b = ik_\parallel \xi_x \hat{x}
\]

(at high \(\beta\), this is the slow wave, at low \(\beta\), this is the fast wave);

- the parallel-propagating sound wave (Fig. 59a):

\[
\omega^2 \xi_\parallel = k_\parallel^2 c_s^2 \xi_\parallel \Rightarrow \omega = \pm k_\parallel c_s,
\]

\[
\xi = \xi_\parallel \hat{z}, \quad \frac{\delta \rho}{\rho_0} = -ik_\parallel \xi_\parallel, \quad \frac{\delta p}{\rho_0} = \gamma \frac{\delta \rho}{\rho_0}, \quad \delta B = 0, \quad \delta b = 0
\]

(at high \(\beta\), this is the fast wave, at low \(\beta\), this is the slow wave); the magnetic field does not participate here at all.
14.1.4. Perpendicular Propagation

Now consider \( \mathbf{k} \parallel = 0 \) \((\theta = 90^\circ)\). Then \((\xi_x, 0, 0)\) is again an eigenvector of the matrix in (14.21).\(^\text{94}\) The resulting fast magnetosonic wave is again a sound wave, but because it is perpendicular-propagating, both thermal and magnetic pressures get involved, the perturbations are compressions/rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfvén speeds (with the latter now representing the magnetic pressure response):

\[
\omega^2 \xi_x = k^2_\perp (c_s^2 + v_A^2) \xi_x \quad \Rightarrow \quad \omega = \pm k_\perp \sqrt{c_s^2 + v_A^2}, \quad (14.29)
\]

\[
\xi = \xi_x \hat{x}, \quad \frac{\delta \rho}{\rho_0} = -ik_\perp \xi_x, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -i k_\perp \xi_x, \quad \delta b = 0. \quad (14.30)
\]

Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field (Fig. 59b).

14.1.5. Anisotropic Perturbations: \( k_\parallel \ll k_\perp \)

Taking \( k_\parallel = 0 \) in §14.1.4 was perhaps a little radical as we lost all waves apart from the fast one. As we are about to see, a lot of babies were thrown out with this particular bathwater.

So let us consider MHD waves in the limit \( k_\parallel \ll k_\perp \). This turns out to be an extremely relevant regime, because, in a strong magnetic field, realistically excitable perturbations, both linear and nonlinear, tend to be highly elongated in the direction of the field. Going back to (14.23) and enforcing this limit, we get

\[
\omega^2 = \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[ 1 \pm \sqrt{1 - \frac{4c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_\parallel^2}{k^2}} \right]
\]

\[
\approx \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[ 1 \pm 1 \mp \frac{2c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_\parallel^2}{k^2} \right]. \quad (14.31)
\]

The upper sign gives the familiar fast wave

\[
\omega = \pm k \sqrt{c_s^2 + v_A^2}. \quad (14.32)
\]

This is just the magnetically enhanced sound wave that was considered in §14.1.4. The small corrections to it due to \( k_\parallel / k \) are not particularly interesting.

The lower sign in (14.31) gives the slow wave

\[
\omega = \pm k_\parallel \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}}, \quad (14.33)
\]

which is more interesting. Let us find the corresponding eigenvector: from (14.14),

\[
(\omega^2 - k_\parallel^2 c_s^2) \xi_\parallel = k_\parallel k_\perp^2 c_s^2 \xi_x. \quad (14.34)
\]

\[
= -k_\parallel^2 \frac{c_s^4}{c_s^2 + v_A^2}, \quad \text{from (14.33)}
\]

\(^\text{94}\)As is \((0, 0, \xi_\parallel)\), but with \( \omega = 0 \); we will deal with this mode in §14.3.4.
Therefore, the displacements are mostly parallel:

\[
\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel c_s^2}{k_\perp c_s^2 + v_A^2} < 1.
\]

(14.35)

Using this equation together with (14.15–14.18), we find that the perturbations in the remaining fields are

\[
\frac{\delta \rho}{\rho_0} = -i(k_\perp \xi_x + k_\parallel \xi_\parallel) = -i \frac{v_A^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel, \quad (14.36)
\]

\[
\frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}, \quad (14.37)
\]

\[
\delta b = ik_\parallel \xi_x \hat{x} = -i \frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel \hat{x} \to 0, \quad (14.38)
\]

\[
\frac{\delta B}{B_0} = -ik_\perp \xi_x = i \frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel. \quad (14.39)
\]

Thus, to lowest order in \(k_\parallel/k_\perp\), this wave involves no bending of the magnetic field, but has a pressure/density perturbation and a magnetic-field-strength perturbation—the latter in counter-phase to the former (Fig. 60). To be precise, the slow-wave perturbations are pressure balanced:

\[
\delta \left( p + \frac{B^2}{8\pi} \right) = \rho_0 \frac{\delta \rho}{\rho_0} + \frac{B_0^2}{4\pi} \frac{\delta B}{B_0} = \rho_0 \left( c_s^2 \frac{\delta \rho}{\rho_0} + v_A^2 \frac{\delta B}{B_0} \right) = 0. \quad (14.40)
\]

The same is, of course, already obvious from the momentum equation (14.8), where, in the limit \(k_\parallel \ll k_\perp\) and \(\omega \ll kc_s\) (“incompressible” perturbations; see §14.2), the dominant balance is

\[
\nabla_\perp \left( p + \frac{B^2}{8\pi} \right) = 0. \quad (14.41)
\]

Finally, the Alfvén waves in the limit of anisotropic propagation are just the same as ever (§14.1.1)—they are unaffected by \(k_\perp\), while being perfectly capable of having perpendicular variation (Fig. 57b).
14.1.6. **High-β Limit:** $c_s \gg v_A$

Another limit in which high-frequency acoustic response (fast waves) and low-frequency, pressure-balanced Alfvénic response (slow and Alfvén waves) are separated is $\beta \gg 1 \iff c_s \gg v_A$.\(^{95}\) In this limit, the approximate expression (14.31) for the magnetosonic frequencies is still valid, but because $v_A/c_s$, rather than $k_\parallel/k$, is small. The rest of the calculations in §14.1.5 are also valid, with the following simplifications arising from $v_A$ being negligible compared to $c_s$.

The upper sign in (14.31) again gives us the *fast wave*, which, this time, is a pure sound wave:

$$\omega = \pm kc_s. \quad (14.42)$$

This is natural because, at high $\beta$, the magnetic pressure is negligible compared to thermal pressure and sound can propagate oblivious of the magnetic field.

The lower sign in (14.31) yields the *slow wave*: (14.33) is still valid and becomes, for $v_A \ll c_s$,

$$\omega = \pm k_\parallel v_A. \quad (14.43)$$

Because the slow wave’s dispersion relation in this limit looks exactly like the dispersion relation (14.19) of an Alfvén wave, it is called the *pseudo-Alfvén wave*. The similarity is deceptive as the nature of the perturbation (the eigenvector) is completely different. Substituting $\omega^2 = k^2 v_A^2$ into (14.14), we find

$$k_\perp \xi_x + k_\parallel \xi_\parallel = \frac{v_A^2}{c_s^2} k_\parallel \xi_\parallel \ll k_\parallel \xi_\parallel. \quad (14.44)$$

This just says that, to lowest order in $1/\beta$, $\nabla \cdot \xi = 0$, i.e., the perturbations are incompressible. In contrast to the anisotropic case (14.35), the perpendicular and parallel displacements are now comparable (assuming, in general, $k_\parallel \sim k_\perp$):

$$\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel}{k_\perp}. \quad (14.45)$$

Also in contrast to the anisotropic case, the density and pressure perturbations are now

\(^{95}\)This limit is astrophysically very interesting because magnetic fields locally produced by plasma motions in various astrophysical environments (e.g., interstellar and intergalactic media) can only be as strong energetically as the motions that make them [see (13.45)] and so, the latter being subsonic, $v_A \sim u \ll c_s$. 

---

**Figure 61.** Slow wave in the high-β limit: pressure balanced, $\xi_x \sim \xi_\parallel$. 

\[\text{Diagram showing slow wave characteristics.}\]
vanishingly small, but the field can be bent as well as compressed:

\[
\frac{\delta \rho}{\rho_0} = -i(k_\perp \xi_x + k_\parallel \xi_\parallel) = -i \frac{v_A^2}{c_s^2} k_\parallel \xi_\parallel \to 0, \tag{14.46}
\]

\[
\frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0} \to 0, \tag{14.47}
\]

\[
\delta b = i k_\parallel \xi_x \hat{x} = -i \frac{k_\parallel}{k_\perp} k_\parallel \xi_\parallel \hat{x}, \tag{14.48}
\]

\[
\frac{\delta B}{B_0} = -i k_\perp \xi_x = ik_\parallel \xi_\parallel. \tag{14.49}
\]

The $\delta B$ and $\delta b$ perturbations are in counter-phase, as are $\xi_\parallel$ and $\xi_x$ (Fig. 61). It is easy to check that pressure balance (14.40) is again maintained by these perturbations.

In the more general case of oblique propagation ($k_\parallel \sim k_\perp$) and finite beta ($\beta \sim 1$), the fast and slow magnetosonic waves generally have comparable frequencies and contain perturbations of all relevant fields, with the fast waves tending to have the perturbations of the thermal and magnetic pressure in phase and slow waves in counter-phase (Fig. 62).

### 14.2. Subsonic Ordering

Enough linear theory! We shall now occupy ourselves with the behaviour of finite (although still small) perturbations of a straight-field equilibrium. While we abandon linearisation (i.e., the neglect of nonlinear terms), much of what the linear theory has taught us about the basic responses of an MHD fluid remains true and useful. In particular, the linear relations between the perturbation amplitudes of various fields provide us with a guidance as to the relative size of finite perturbations of these fields. This makes sense if, while allowing the nonlinearities back in, we do not assume the linear physics to be completely negligible, i.e., if we allow the linear and nonlinear time scales to compete (§14.2.3). We shall see that solutions for which this is the case satisfy self-consistent equations, so can be expected to be realisable (and, as we know from experimental, observational and numerical evidence, are realised).

I shall start by constructing nonlinear equations that describe the incompressible limit, i.e., fields and motions that are subsonic: both their phase speeds and flow velocities will
be assumed small compared to the speed of sound:

\[
\frac{\omega/k}{\sqrt{c^2_s + v_A^2}} \ll 1, \quad \text{Ma} \equiv \frac{u}{\sqrt{c^2_s + v_A^2}} \ll 1. \tag{14.50}
\]

In this limit, we expect all fast-wave-like perturbations to disappear (in a similar way to the sound waves disappearing in the incompressible Navier–Stokes hydrodynamics) and for the MHD dynamics to contain only Alfvenic and slow-wave-like perturbations. We saw in §§14.1.5 and 14.1.6 that, linearly, fast and slow waves are well separated either in the limit of \( k_\parallel/k_\perp \ll 1 \) or in the limit of \( \beta \gg 1 \). Indeed, comparing the Alfvén frequency (14.19) and slow-wave frequency (14.33) to the sound (fast-wave) frequency (14.32), we get

\[
\frac{\omega_{\text{Alfvén}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel v_A}{k \sqrt{c^2_s + v_A^2}} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1 + \beta}}, \quad \frac{\omega_{\text{slow}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel c_s v_A}{k (c_s^2 + v_A^2)} \sim \frac{k_\parallel}{k} \frac{\sqrt{\beta}}{1 + \beta},
\]

both of which are small in either of the two limits, satisfying the first of the conditions (14.50).

The second condition (14.50) involves the “magnetic Mach number” Ma (generalised to compare the flow velocity to the speed of sound in a magnetised fluid), which measures the size of the perturbations themselves—in the linear theory, this was arbitrarily small, but now we will need to relate it to our other small parameter(s), \( k_\parallel/k \) or \( 1/\beta \). This means that we would like to construct an asymptotic ordering in which there will be some prescription as to how small, or otherwise, various (fractional) perturbations and small parameters are—not by themselves, i.e., compared to 1, but compared to each other (compared to 1, the small parameters can all formally be taken to be as small as we desire).

The general strategy for ordering perturbations with respect to each other will be to use the linear relations obtained in the two incompressible limits (\( k_\parallel/k \ll 1 \) or \( \beta \gg 1 \)). If we do not specifically expect one perturbation to be larger or smaller than another on some physical grounds (like the properties of the linear response), we must order them the same; this does not stop us later from constructing subsidiary expansions in which they might be different. For example, MHD equations themselves were an expansion in a number of small parameters, in particular \( u/c \) [see (13.14)]. However, at the time of deriving them, I did not want to rule out sonic or supersonic motions and so, effectively, I ordered \( \text{Ma} \sim 1, k_\parallel/k \sim 1 \) and \( \beta \sim 1 \), as far as the \( u/c \) expansion was concerned, i.e., \( \text{Ma}, k_\parallel/k, 1/\beta \gg u/c \). Now we are constructing a subsidiary expansion in these other parameters, keeping in mind that they are allowed to be small but not as small as the small parameter already used in the derivation of the MHD equations.\footnote{In principle, you should always feel a little paranoid about the question of whether such “nested” asymptotic expansions commute, i.e., whether it matters in which order they are done. They usually do commute, but this is not guaranteed and you ought to check if you want to be sure. Another formally justified mathematical worry is whether asymptotic solutions of exact equations are the same as exact solutions of asymptotic equations. This will lead you on a journey to the world of proofs of existence and uniqueness—where I wish you an enjoyable stay.}

14.2.1. Ordering of Alfvenic Perturbations

Since the Alfvenic perturbations decouple completely from the rest (§14.1.1), linear theory does not give us a way to relate \( u_y \) to \( u_\parallel \), so we shall exercise the no-prejudice principle stated above and assume

\[
u_y \sim u_\parallel, \tag{14.52}\]
i.e., the Mach numbers for the Alfvénic and slow-wave-like motions are comparable. We can, however, relate \( u_y \) to \( \delta b \), via the curvature-force response (14.20):

\[
|\delta b| \sim k_\parallel \xi_y \sim \frac{k_\parallel u_y}{\omega} \sim \frac{u_y}{v_A} \sim \text{Ma} \sqrt{1 + \beta}.
\]  

(14.53)

### 14.2.2. Ordering of Slow-Wave-Like Perturbations

For slow-wave-like perturbations, in either the anisotropic or the high-\( \beta \) limit, from (14.14) and (14.33),

\[
\nabla \cdot \mathbf{u} \sim \omega (k_\perp \xi_x + k_\parallel \xi_\parallel) \sim \frac{\omega^2}{k_\parallel c_s^2} \omega_\parallel \sim \frac{v_A^2}{c_s^2 + v_A^2} k_\parallel u_\parallel \sim \frac{k_\parallel u_\parallel}{1 + \beta}.
\]  

(14.54)

Thus, the divergence of the flow velocity is small (the dynamics are incompressible) in all three of our (potentially) small parameters:

\[
\frac{\nabla \cdot \mathbf{u}}{k \sqrt{c_s^2 + v_A^2}} \sim \frac{k_\parallel}{k} \frac{1}{1 + \beta} \text{Ma}.
\]  

(14.55)

From this, we can immediately obtain an ordering for the density and pressure perturbations: using (14.3), (14.4), (14.33) and (14.54) [cf. (14.36) and (14.46)],

\[
\frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \nabla \cdot \xi \sim \frac{\nabla \cdot \mathbf{u}}{\omega} \sim \text{Ma} \frac{1}{\sqrt{\beta}}.
\]  

(14.56)

The magnetic-field-strength (magnetic-pressure) perturbation is, using (14.39) and (14.33) [cf. (14.49)],

\[
\frac{\delta B}{B_0} \sim k_\perp \xi_x \sim \frac{c_s^2}{c_s^2 + v_A^2} \frac{k_\parallel u_\parallel}{\omega} \sim \sqrt{\beta} \text{Ma},
\]  

(14.57)

or, perhaps more straightforwardly, from pressure balance (14.40) and using (14.56),

\[
\frac{\delta B}{B_0} = -\frac{\beta}{2} \frac{\delta p}{p_0} \sim \sqrt{\beta} \text{Ma}.
\]  

(14.58)

Finally, in a similar fashion, using (14.17) and (14.57) [cf. (14.38) and (14.48)], we find

\[
|\delta b| \sim k_\parallel \xi_x \sim \frac{k_\parallel}{k_\perp} \sqrt{\beta} \text{Ma}
\]  

(14.59)

for slow-wave-like perturbations. Note that in all interesting limits this is superceded by the Alfvénic ordering (14.53).

### 14.2.3. Ordering of Time Scales

Let us recall that our motivation for using linear relations between perturbations to determine their relative sizes in a nonlinear regime was that linear response will lose its exclusive sway but remain non-negligible. In formal terms, this means that we must order the linear and nonlinear time scales to be comparable.\(^{97}\) The nonlinearity in MHD equations are advective, i.e., they are of the form \( \mathbf{u} \cdot \nabla (\text{stuff}) \) and similar, so the rate of nonlinear interaction is \( \sim ku \) (in the case of anisotropic perturbations, \( \sim k_\perp u_\perp \)). Ordering this to be comparable to the frequencies of the Alfvén and slow waves [see the theory of MHD turbulence, this principle, applied at each scale, is known as the critical balance (see §14.4).]
(14.51)] gives us
\[ \omega_{\text{Alfvén}} \sim ku \Rightarrow Ma \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1 + \beta}}, \] (14.60)
\[ \omega_{\text{slow}} \sim ku \Rightarrow Ma \sim \frac{k_\parallel}{k} \frac{\sqrt{\beta}}{1 + \beta}. \] (14.61)

Note that the first of these relations supersedes the second in all interesting limits.

14.2.4. Summary of Subsonic Ordering

Thus, the ordering of the time scales determines the size of the perturbations via (14.60). Using this restriction on Ma, we may summarise our subsonic ordering as follows:

\[ Ma \equiv \frac{u}{\sqrt{c_s^2 + v_\perp^2}} \sim \frac{\delta b}{\sqrt{1 + \beta}} \sim \frac{1}{\sqrt{\beta}} \frac{\delta B}{B_0} \sim \frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1 + \beta}} \ll 1 \] (14.62)

and \( \omega \sim ku \). The ordering can be achieved either in the limit of \( k_\parallel/k \ll 1 \) or \( 1/\beta \ll 1 \), or both. Note that if one of these parameters is small, the other can be order unity or even large (as long as it is not larger than the inverse of the small one).

The case of anisotropic perturbations and arbitrary \( \beta \) applies in a broad range of plasmas, from magnetically confined fusion ones (tokamaks, stellarators) to space (e.g., the solar corona or the solar wind). We shall consider the implications of this ordering in §14.3.

The case of high \( \beta \) applies, e.g., to high-energy galactic and extragalactic plasmas. It is the direct generalisation to MHD of incompressible Navier–Stokes hydrodynamics, i.e., in this case, all one needs to do is solve MHD equations assuming \( \rho = \text{const} \) and \( \nabla \cdot \mathbf{u} = 0 \). We shall consider this case now.

14.2.5. Incompressible MHD Equations

Assuming \( \beta \gg 1 \), our ordering becomes
\[ \frac{u}{c_s} \sim \frac{\omega}{k c_s} \sim \frac{1}{\sqrt{\beta}} \sim Ma, \quad \frac{|\delta b|}{\sqrt{1 + \beta}} \sim \frac{\delta B}{B_0} \sim \sqrt{\beta} Ma \sim 1, \quad \frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{Ma}{\sqrt{\beta}} \sim \frac{Ma^2}{1}. \] (14.63)

Thus, the density and pressure perturbations are minuscule, while magnetic perturbations are order unity—magnetic fields are relatively easy to bend (i.e., subsonic motions can tangle the field substantially in this regime). Because of this, it will not make sense to split \( \mathbf{B} \) into \( B_0 \) and \( \delta \mathbf{B} \) explicitly, we will treat the magnetic field as a single field, with no need for a strong mean component.

Let us examine the MHD equations (13.57–13.60) under the ordering (14.63).

Since \( \omega \sim ku \), the convective derivative \( \frac{d}{dt} = \partial / \partial t + \mathbf{u} \cdot \nabla \) survives intact in all equations, allowing the advective nonlinearity to enter.

The continuity equation (13.57) simply reiterates our earlier statement that the velocity field is divergenceless to lowest order:
\[ \nabla \cdot \mathbf{u} = -\frac{1}{\rho} \frac{d\rho}{dt} \sim \omega \frac{\delta \rho}{\rho_0} \sim Ma^3 k c_s \to 0. \] (14.64)

\(^{98}\)Note that it is not absolutely necessary to work out the detailed linear theory of a set of equations in order to be able to construct such orderings; it is often enough to know roughly where you are going and simply balance terms representing the physics that you wish to keep (or expect to have to keep). An example of this approach is given in §14.2.8.
The momentum equation (13.58) becomes

\[
\left( 1 + \frac{\delta \rho}{\rho_0} \right) \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla \left( \frac{c_s^2 \delta p}{\gamma \rho_0} + \frac{B^2}{8 \pi \rho_0} \right) + \frac{B \cdot \nabla B}{4 \pi \rho_0}. \tag{14.65}
\]

The density perturbation in the left-hand side is \( \sim \Ma^2 \) and so negligible compared to unity. The remaining terms in this equation are all the same order (\( \sim \Ma^2 k c_s^2 \)) and so they must all be kept. The total “pressure” \( \tilde{p} \) is determined by enforcing \( \nabla \cdot u = 0 \) [see (14.64)]. Namely, our equations are

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla \tilde{p} + B \cdot \nabla B, \tag{14.66}
\]

where

\[
\nabla^2 \tilde{p} = -\nabla \nabla : (uu - BB) \tag{14.67}
\]

and we have rescaled the magnetic field to velocity units, \( B/\sqrt{4\pi \rho_0} \rightarrow B \).

In the induction equation, best written in the form (13.27), all terms are the same order \( \sim k u B \sim \Ma k c_s B \) except the one containing \( \nabla \cdot u \), which is \( \sim \Ma^3 k c_s B \) and so must be neglected. We are left with

\[
\frac{\partial B}{\partial t} + u \cdot \nabla B = B \cdot \nabla u. \tag{14.68}
\]

Finally, the internal-energy equation (13.60), which, keeping only the lowest-order terms, becomes

\[
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \left( \frac{\delta p}{p_0} - \gamma \frac{\delta \rho}{\rho_0} \right) = 0, \tag{14.69}
\]

can be used to find \( \delta \rho/\rho_0 \), once \( \delta p/p_0 = \gamma (\tilde{p} - B^2/2)/c_s^2 \) is calculated from the solution of (14.66–14.68). Note that \( \delta \rho/\rho_0 \) is merely a spectator quantity, not required to solve (14.66–14.68), which form a closed set.

Equations (14.66–14.68) are the equations of incompressible MHD (let us call it \( iMHD \)). Note that while they have been obtained in the limit of \( \beta \gg 1 \), all \( \beta \) dependence has disappeared from them—basically, they describe subsonic dynamics on top of an infinite heat bath. This is how it should be: formally, in any good asymptotic theory, it must be possible to make the small parameter arbitrarily small without changing anything in the equations.

**Exercise 14.2.** Show that \( iMHD \) conserves the sum of kinetic and magnetic energies,

\[
\frac{dt}{dt} \int d^3r \left( \frac{u^2}{2} + \frac{B^2}{2} \right) = 0. \tag{14.70}
\]

**Exercise 14.3.** Check that you can obtain the right waves, viz., Alfvén (§14.1.1) and pseudo-Alfvén (§14.1.6), directly from \( iMHD \).

**Exercise 14.4.** Magnetoelastic waves.\(^99\) Show that the \( iMHD \) equations can be rewritten as the following closed set describing the evolution of the velocity field \( u \) and the Maxwell

\(^{99}\) This is based on the 2019 exam question.
tensor $M_{ij} = B_i B_j$:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = - \frac{\partial \tilde{p}}{\partial r_i} + \frac{\partial M_{ij}}{\partial r_j},$$  \hspace{1cm} (14.71)

$$\frac{\partial M_{ij}}{\partial t} + u_n \frac{\partial M_{ij}}{\partial r_n} = M_{nj} \frac{\partial u_i}{\partial r_n} + M_{in} \frac{\partial u_j}{\partial r_n}$$  \hspace{1cm} (14.72)

(summation over repeated indices is implied).

Imagine that there is no mean magnetic field, the MHD medium is static, and it is filled with chaotically tangled magnetic fields that are constant in time. Denote their Maxwell tensor $M_{ij}^{(0)}$. Assume that these fields have a characteristic scale that is no larger than $\ell$ and are statistically isotropic, so if we introduce an average (denoted by angle brackets) over scales of order $\ell$, then

$$\langle M_{ij}^{(0)} \rangle = v_A^2 \delta_{ij},$$  \hspace{1cm} (14.73)

where (obviously) $v_A^2 = \langle B^2 \rangle / 3 = \text{const.}$ This is clearly a static ($u_i = 0$) equilibrium solution of (14.71) and (14.72). Consider infinitesimal perturbations $\delta u_i$ and $\delta M_{ij}$ around this equilibrium and assume that they vary in space on scales much longer than $\ell$, viz.,

$$\langle u_i \rangle = 0 + \delta u_i \ll v_A, \quad \langle M_{ij} \rangle = \langle M_{ij}^{(0)} \rangle + \delta M_{ij}, \quad \delta M_{ij} \ll v_A^2.$$  \hspace{1cm} (14.74)

Ignore any possible perturbations of $u_i$ and $M_{ij}$ on scales $\ell$ or smaller. Show that $\delta u_i$ and $\delta M_{ij}$ will describe propagating waves, derive their dispersion relation and also the relationship between $\delta M_{ij}$ and the displacement vector $\xi_i$ associated with $\delta u_i = \partial \xi_i / \partial t$. These are called magnetoelastic waves. Think about their physical nature, their similarities with, or differences from, Alfvén waves.

As I already intimated in §13.4, the iMHD equations written in the form of (14.71) and (14.72) are mathematically similar to the equations describing certain kinds of polymer-laden fluids. The intrinsic “elasticity” of the Maxwell stress leads to a kind of isotropic Alfvénic response that gives rise to the magnetoelastic waves. Note, however, that a significant difference between polymer chains and magnetic fields is that the latter have a sign, so there is a distinction between parallel and antiparallel fields, while polymers do not have that. Consequently, the analogy between MHD and polymer fluids becomes very imperfect if dissipation of $\delta M_{ij}$ is included: for polymers, there is a relaxation term in (14.72) of the form $- (M_{ij} - v_A^2 \delta_{ij}) / \tau$, describing the polymers’ desire to curl up due to entropic forces; whereas in MHD, the resistive term $\eta (B_i \nabla^2 B_j + B_j \nabla^2 B_i)$ (§13.5) cannot be converted into anything that depends only on $M_{ij}$—indeed, it would, e.g., heavily damp antiparallel fields that reverse direction on small scales, an effect invisible to $M_{ij}$, where the field’s sign cancels out.

14.2.6. Elsasser MHD

The iMHD equations possess a remarkable symmetry. Let us introduce Elsasser (1950) fields

$$Z^\pm = u \pm B$$  \hspace{1cm} (14.75)

and rewrite (14.66) and (14.68) as evolution equations for $Z^\pm$: after trivial algebra,

$$\frac{\partial Z^\pm}{\partial t} + Z^\mp \cdot \nabla Z^\pm = - \nabla \tilde{p}$$  \hspace{1cm} (14.76)

and, since $\nabla \cdot Z^\pm = 0$,

$$\nabla^2 \tilde{p} = - \nabla \nabla : Z^+ Z^-.$$  \hspace{1cm} (14.77)

Thus, one can think of iMHD as representing the evolution of two incompressible “velocity fields” advecting each other.

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100 As you might imagine, this assumption is quite dodgy, although, as it turns out, not always: see Hosking & Schekochihin (2020).
Let us restore the separation of the magnetic field into its mean and perturbed parts, \( B = B_0 + \delta B = v_A \hat{z} + \delta B \) (recall that \( B \) is in velocity units). Then

\[
Z^\pm = \pm v_A \hat{z} + \delta Z^\pm
\]  

and (14.76) becomes

\[
\frac{\partial \delta Z^\pm}{\partial t} + v_A \nabla_\parallel \delta Z^\pm + \delta Z^\mp \cdot \nabla \delta Z^\pm = - \nabla \tilde{p} .
\]  

Thus, \( \delta Z^\pm \) are finite, counter-propagating (at the Alfvén speed \( v_A \)) perturbations—and they interact nonlinearly only with each other, not with themselves. If we let, say, \( \delta Z^- = 0 \iff u = \delta B \), then \( \delta Z^+ \) satisfies

\[
\frac{\partial \delta Z^+}{\partial t} - v_A \nabla_\parallel \delta Z^+ = 0,
\]  

and similarly for \( \delta Z^- \) (propagating at \( -v_A \)) if \( \delta Z^+ = 0 \). Therefore,

\[
\delta Z^\pm = f(r \pm v_A t \hat{z}), \quad \delta Z^\mp = 0,
\]  

where \( f \) is an arbitrary function, are exact nonlinear solutions of iMHD. They are called Elsasser states. Physically, they are isolated Alfvén-wave packets that propagate along the guide field and never interact (because they all travel at the same speed and so can never catch up with or overtake one another). In order to have any interesting nonlinear dynamics, the system must have counter-propagating Alfvén-wave packets (see §14.4).

14.2.7. Cross-Helicity

Equations (14.76) manifestly support two conservation laws:

\[
\frac{d}{dt} \int d^3 r \frac{|Z^\pm|^2}{2} = 0,
\]  

i.e., the energy of each Elsasser field is individually conserved. This can be reformulated as conservation of the total energy,

\[
\frac{d}{dt} \int d^3 r \frac{1}{2} \left( \frac{|Z^+|^2}{2} + \frac{|Z^-|^2}{2} \right) = \frac{d}{dt} \int d^3 r \left( \frac{u^2}{2} + \frac{B^2}{2} \right) = 0,
\]  

and of a new quantity, known as the cross-helicity:

\[
\frac{d}{dt} \int d^3 r \frac{1}{2} \left( \frac{|Z^+|^2}{2} - \frac{|Z^-|^2}{2} \right) = \frac{d}{dt} \int d^3 r u \cdot B = 0.
\]

In the Elsasser formulation, the cross-helicity is a measure of energy imbalace between the two Elsasser fields\(^{101} \)—this is observed, for example, in the solar wind, where there is significantly more energy in the Alfvénic fluctuations propagating away from the Sun than towards it (see, e.g., Wicks et al. 2013).

**Exercise 14.5.** To see why we needed incompressibility to get this new conservation law, work out the time evolution equation for \( \int d^3 r u \cdot B \) from the general (compressible) MHD equations and hence the condition under which the cross-helicity is conserved.

\(^{101}\)Cross-helicity can also be interpreted as a topological invariant, counting the linkages between flux tubes and vortex tubes analogously to what magnetic helicity does for the flux tubes alone (see §15.2).
14.2.8. Stratified MHD

It is quite instructive to consider a very simple example of non-uniform MHD equilibrium: the case of a stratified atmosphere. Let us introduce gravity into MHD equations, viz., the momentum equation (13.58) becomes

\[ \frac{\rho}{\rho_0} \frac{d\mathbf{u}}{dt} = -\nabla \left( p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \rho g \hat{z} \]  
(14.85)

(uniform gravitational acceleration pointing downward, against the \( z \) direction). We wish to consider a static equilibrium inhomogeneous in the \( z \) direction and threaded by a uniform magnetic field (which may be zero):

\[ \rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \mathbf{u}_0 = 0, \quad \mathbf{B}_0 = B_0 \mathbf{b}_0 = \text{const}, \]  
(14.86)

where \( \mathbf{b}_0 \) is at some general angle to \( \hat{z} \) and \( p_0(z) \) and \( \rho_0(z) \) are constrained by the vertical force balance:

\[ \frac{dp_0}{dz} = -\rho_0 g \] \( \Rightarrow \) \( g = -\frac{\rho_0}{\rho_0} \frac{d \ln p_0}{dz} = \frac{c_s^2}{\gamma} \frac{1}{H_p}, \)  
(14.87)

where it has been opportune to define the pressure scale height \( H_p \). We shall now seek time-dependent solutions of the MHD equations for which

\[ \rho = \rho_0(z) + \delta \rho, \quad \frac{\delta \rho}{\rho_0} \ll 1, \quad p = p_0(z) + \delta p, \quad \frac{\delta p}{p_0} \ll 1, \]  
(14.88)

and the spatial variation of all perturbations occurs on scales that are small compared to the pressure scale height \( H_p \) or the analogously defined density scale height \( H_\rho = -(d \ln \rho_0/dz)^{-1} \) (for ordering purposes, we denote them both \( H \)):

\[ kH \gg 1. \]  
(14.89)

After the equilibrium pressure balance is subtracted from (14.85), this equation becomes, under any ordering in which \( \delta \rho \ll \rho_0 \),

\[ \rho_0 \frac{d\mathbf{u}}{dt} = -\nabla \left( \delta p + \frac{B^2}{8\pi} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi} - \delta \rho g \hat{z}. \]  
(14.90)

The last term is the buoyancy (Archimedes) force. In order for this new feature to give rise to any nontrivial new physics, it must be ordered comparable to all the other terms in the equation: using (14.87) to express \( g \sim p_0/\rho_0 H \), we find

\[ \delta \rho g \sim k \delta p \quad \Rightarrow \quad \frac{\delta \rho}{\rho_0} \sim kH \frac{\delta p}{p_0} \gg \frac{\delta p}{p_0}, \]  
(14.91)

\[ \delta \rho g \sim \frac{kB^2}{4\pi} \quad \Rightarrow \quad \frac{\delta \rho}{\rho_0} \sim \frac{kH}{\beta} \ll 1 \quad \Rightarrow \quad \beta \gg kH \gg 1. \]  
(14.92)

So we learn that the density perturbations must now be much larger than the pressure perturbations, but, in order for the former to remain small and for the magnetic field to be in the game, \( \beta \) must be high (it is in anticipation of this that we did not split \( \mathbf{B} \) into \( \mathbf{B}_0 \) and \( \delta \mathbf{B} \), expecting them to be of the same order).

Let us now expand the internal-energy equation (13.60) in small density and pressure perturbations. Denoting \( s = p/\rho^n = s_0(z) + \delta s \) (entropy density) and introducing the entropy scale height

\[ \frac{1}{H_s} = \frac{d \ln s_0}{dz} = -\frac{1}{H_p} + \frac{\gamma}{H_\rho}, \]  
(14.93)

(assumed positive), we find\(^{102}\)

\[ \frac{d}{dt} \frac{\delta s}{s_0} = -\frac{d \mathbf{u}}{H_s}, \quad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \frac{\gamma}{\rho_0} \approx -\frac{\gamma}{\rho_0}. \]  
(14.94)

\(^{102}\)We are able to take equilibrium quantities in and out of spatial derivatives because \( kH \gg 1 \) and the perturbations are small.
The last, approximate, expression follows from the smallness of pressure perturbations [see (14.91)]. This then gives us

\[
\frac{d}{dt} \frac{\delta \rho}{\rho_0} = \frac{u_z}{\gamma H_s}.
\] (14.95)

But, on the other hand, the continuity equation (13.57) is

\[
\frac{d}{dt} \frac{\delta \rho}{\rho_0} = -\nabla \cdot u + \frac{u_z}{H_s} \Rightarrow \nabla \cdot u = u_z \left( \frac{1}{H_p} - \frac{1}{\gamma H_s} \right) = \frac{u_z}{\gamma H_p} \Rightarrow \nabla \cdot u \sim \frac{1}{k H} \ll 1.
\] (14.96)

Thus, the dynamics are incompressible again and the role of the continuity equation is to tell us that we must find \(\delta \rho\) from the momentum equation (14.90) by enforcing \(\nabla \cdot u = 0\) to lowest order. The difference with iMHD (§14.2.5) is that \(\delta \rho/\rho\) now participates in the dynamics via the buoyancy force and must be found self-consistently from (14.95).

Finally, we rewrite our newly found simplified system of equations for a stratified, high-\(\beta\) atmosphere, in the following neat way:

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla \tilde{p} + B \cdot \nabla B + a \hat{z},
\] (14.97)

\[
\nabla^2 \tilde{p} = -\nabla \nabla : (uu - BB) + \frac{\partial a}{\partial z},
\] (14.98)

\[
\frac{\partial a}{\partial t} + u \cdot \nabla a = -N^2 u_z, \quad N = \frac{c_s}{\gamma \sqrt{H_s H_p}},
\] (14.99)

\[
\frac{\partial B}{\partial t} + u \cdot \nabla B = B \cdot \nabla u,
\] (14.100)

where we have rescaled \(B/\sqrt{4\pi \rho_0} \rightarrow B\) and denoted the Archimedes acceleration

\[
a = -\frac{\delta \rho}{\rho_0} g = -\frac{\delta \rho}{\rho_0} \frac{c_s^2}{\gamma H_p},
\] (14.101)

a quantity also known as the buoyancy of the fluid. We shall call (14.97–14.100) the equations of stratified MHD (SMHD).

A new frequency \(N\), known as the Brunt–Väisälä frequency, has appeared in our equations. A new frequency \(N\), known as the Brunt–Väisälä frequency, has appeared in our equations. In order for all the linear and nonlinear time scales that are present in our equations to coexist legitimately within our ordering, we must demand that the Alfvén, Brunt–Väisälä and nonlinear time scales all be comparable:

\[
k v_A \sim N \sim k u \Rightarrow \frac{1}{\sqrt{\beta}} \sim \frac{1}{k H} \sim Ma.
\] (14.102)

This gives us a relative ordering between all the small parameters that have appeared so far, including the new one, \(1/k H\). Using (14.95) and recalling (14.91), let us summarise the ordering of the perturbations:

\[
u/c_s \sim \frac{\delta \rho}{\rho_0} \sim Ma, \quad \frac{\delta \rho}{\rho_0} \sim Ma^2, \quad |\delta b| \sim \frac{\delta B}{B_0} \sim 1.
\] (14.103)

The difference with the iMHD high-\(\beta\) ordering (14.63) is that the density perturbations have now been promoted to dynamical relevance, thankfully without jeopardising incompressibility (i.e., still ordering out the sonic perturbations). The ordering (14.103) can be thought of as a generalisation to MHD of the Boussinesq approximation in hydrodynamics.

Further investigations of the SMHD equations are undertaken in Q6.

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\(^{103}\) \(N\) is real because we assumed \(H_s > 0\) (a “stably stratified atmosphere”), otherwise the atmosphere becomes convectively unstable—this happens when the equilibrium entropy decreases upwards (cf. §16.3, Q10 and Q6c).
14.3. Reduced MHD

We now turn to the anisotropic ordering, \( k_\parallel/k \ll 1 \) (while \( \beta \sim 1 \), in general), for which we studied the linear theory in §14.1.5. Specialising to this case from our general ordering (14.62), we have

\[
\text{Ma} \sim \frac{u_\perp}{c_s} \sim \frac{u_\parallel}{c_s} \sim |\delta b| \sim \frac{\delta B}{B_0} \sim \frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_\perp c_s} \sim \frac{k_\parallel}{k_\perp} \ll 1.
\]  

(14.104)

Starting again with the continuity equation (13.57), dividing through by \( \rho_0 \) and ordering all terms, we get

\[
\left( \frac{\partial}{\partial t} + u_\perp \cdot \nabla_\perp + u_\parallel \cdot \nabla_\parallel \right) \frac{\delta \rho}{\rho_0} = - \left( 1 + \frac{\delta \rho}{\rho_0} \right) \left( \nabla_\perp \cdot u_\perp + \nabla_\parallel u_\parallel \right).
\]  

(14.105)

Thus, to lowest order, the perpendicular velocity field is 2D-incompressible:

\[
\mathcal{O}(\text{Ma}) : \quad \nabla_\perp \cdot u_\perp = 0.
\]  

(14.106)

In the next order (which we will need in §14.3.2),

\[
\mathcal{O}(\text{Ma}^2) : \quad (\nabla \cdot u)_2 = - \left( \frac{\partial}{\partial t} + u_\perp \cdot \nabla_\perp \right) \frac{\delta \rho}{\rho_0} = - \frac{d}{dt} \frac{\delta \rho}{\rho_0},
\]  

(14.107)

where, to leading order, the convective derivative now involves only perpendicular advection.

Equation (14.106) implies that \( u_\perp \) can be written in terms of a stream function:

\[
uu_\perp = \hat{z} \times \nabla_\perp \Phi.
\]  

(14.108)

Similarly, for the magnetic field, we have

\[
0 = \nabla \cdot B = \nabla_\perp \cdot \delta B_\perp + \nabla_\parallel \cdot \delta B_\parallel \approx \nabla_\perp \cdot \delta B_\perp,
\]  

(14.109)

so \( \delta B_\perp \) is also 2D-solenoidal and can be written in terms of a flux function:

\[
\frac{\delta B_\perp}{\sqrt{4\pi \rho_0}} = \hat{z} \times \nabla_\perp \Psi.
\]  

(14.110)

Note that \( \Psi = -A_\parallel/\sqrt{4\pi \rho_0} \), the parallel component of the vector potential.

Thus, Alfvénically polarised perturbations, \( u_\perp \) and \( \delta B_\perp \) (see §14.1.1), can be described by two scalar functions, \( \Phi \) and \( \Psi \). Let us work out the evolution equations for them.

14.3.1. Alfvénic Perturbations

We start with the induction equation, again most useful in the form (13.27). Dividing through by \( B_0 \), we have

\[
\frac{d}{dt} \frac{\delta B}{B_0} = b \cdot \nabla u - b \nabla \cdot u.
\]  

(14.111)

Throwing out the obviously subdominant \( \delta b \) contribution in the last term on the right-hand side (i.e., approximating \( b \approx \hat{z} \) in that term), then taking the perpendicular part
of the remaining equation, we get

$$\frac{d}{dt} \frac{\delta B_\perp}{B_0} = b \cdot \nabla u_\perp. \quad (14.112)$$

As we saw above, the convective derivative is with respect to the perpendicular velocity only and, in view of the stream-function representation \((14.108)\) of the latter, for any function \(f\), we have, to leading order,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{u}_\perp \cdot \nabla f = \frac{\partial f}{\partial t} + \frac{1}{v_A} \hat{z} \cdot (\nabla \perp \Psi \times \nabla \perp f) = \frac{\partial f}{\partial t} + \{\Phi, f\}, \quad (14.113)$$

where the “Poisson bracket” is

$$\{\Phi, f\} = \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial x}. \quad (14.114)$$

Similarly, to leading order,

$$b \cdot \nabla f = \frac{\partial f}{\partial z} + \frac{1}{v_A} \hat{z} \cdot (\nabla \perp \Psi \times \nabla \perp f) = \frac{\partial f}{\partial z} + \frac{1}{v_A} \{\Psi, f\}. \quad (14.115)$$

Finally, using \((14.113)\) and \((14.115)\) in \((14.112)\) and expressing \(\delta B_\perp\) in terms of \(\Psi\) [see \((14.110)\)] and \(\mathbf{u}_\perp\) in terms of \(\Phi\) [see \((14.108)\)], it is a straightforward exercise to show, after “uncurling” \((14.112)\), that

$$\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_A \frac{\partial \Phi}{\partial z}. \quad (14.116)$$

Turning now to the momentum equation \((13.58)\), taking its perpendicular part and dividing through by \(\rho \approx \rho_0\), we get

$$\frac{d\mathbf{u}_\perp}{dt} = \frac{1}{\rho_0} \left[ -\nabla \perp \left( p + \frac{B^2}{8\pi} \right) + \frac{B \cdot \nabla \delta B_\perp}{4\pi} \right] = -\nabla \perp \left( \frac{c_s^2 \delta p}{\gamma p_0} + \frac{v_A^2 \delta B}{B_0} \right) + \frac{v_A^2 b \cdot \nabla \delta B_\perp}{B_0}. \quad \text{Ma}^2$$

To lowest order,

$$\mathcal{O}(\text{Ma}) : \quad \nabla \perp \left( \frac{c_s^2 \delta p}{\gamma p_0} + \frac{v_A^2 \delta B}{B_0} \right) = 0 \quad \Rightarrow \quad \frac{\delta p}{p_0} = -\gamma \frac{v_A^2 \delta B}{c_s^2 B_0}. \quad (14.118)$$

This is a statement of pressure balance, which is physically what has been expected [see \((14.41)\)] and which will be useful in \(\S 14.3.2\). In the next order, \((14.117)\) contains the perpendicular gradient of the second-order contribution to the total pressure. To avoid having to calculate it, we take the curl of \((14.117)\) and thus obtain

$$\mathcal{O}(\text{Ma}^2) : \quad \nabla \perp \times \frac{d\mathbf{u}_\perp}{dt} = v_A^2 \nabla \perp \times \left( b \cdot \nabla \frac{\delta B_\perp}{B_0} \right). \quad (14.119)$$

Finally, using again \((14.108)\), \((14.110)\), \((14.113)\) and \((14.115)\) in \((14.119)\), some slightly tedious algebra leads us to

$$\frac{\partial}{\partial t} \nabla \perp ^2 \Phi + \{\Phi, \nabla \perp ^2 \Phi\} = v_A \frac{\partial}{\partial z} \nabla \perp ^2 \Psi + \{\Psi, \nabla \perp ^2 \Psi\}. \quad (14.120)$$

\(^{104}\) Another easy route to this equation is to start from the induction equation in the form \((13.59)\), let \(\mathbf{B} = \nabla \times \mathbf{A}\), “uncurl” \((13.59)\) and take the \(z\) component of the resulting evolution equation for \(\mathbf{A}\).
Note that $\nabla^2_\perp \Phi$ is the vorticity of the flow $u_\perp$ and so the above equation is the MHD generalisation of the 2D Euler equation.

To summarise the equations (14.120) and (14.116) in their most compact form, we have

$$\frac{d}{dt} \nabla^2_\perp \Phi = v_A b \cdot \nabla \nabla^2_\perp \Psi, \quad (14.121)$$
$$\frac{d\Psi}{dt} = v_A \frac{\partial \Phi}{\partial z}, \quad (14.122)$$

where the convective time derivative $d/dt$ and the parallel spatial derivative $b \cdot \nabla$ are given by (14.113) and (14.115), respectively. Beautifully, these nonlinear equations describing Alfvénic perturbations have decoupled completely from everything else: we do not need to know $\delta \rho$, $\delta p$, $u_\parallel$ or $\delta B$ in order to solve for $u_\perp$ and $\delta B_\perp$. Alfvénic dynamics are self-contained.

Equations (14.121) and (14.122) are called the Equations of Reduced MHD (RMHD). They were originally derived in the context of tokamak plasmas (Kadomtsev & Pogutse 1974; Strauss 1976) and are extremely popular as a simple paradigm for MHD in a strong guide field—not just in tokamaks, but also in space.\(^{105}\)

### 14.3.2. Compressive Perturbations

What about the rest of our fields—in the linear language, the slow-wave-like perturbations (§14.1.5)? While we do not need them to compute the Alfvénic perturbations, we might still wish to know them for their own sake.

Returning to the induction equation (14.111) and taking its $z$ component, we get

$$\frac{d}{dt} \frac{\delta B^\parallel}{B_0} = b \cdot \nabla u^\parallel - \nabla \cdot u \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\delta B}{B_0} - \frac{\delta p}{\rho_0} \right) = b \cdot \nabla u^\parallel, \quad (14.123)$$

where all terms are $O(Ma^2)$, $\delta B^\parallel \approx \delta B$ to leading order and we used (14.107) to express $\nabla \cdot u$. The derivatives $d/dt$ and $b \cdot \nabla$ contain the nonlinearities involving $\Phi$ and $\Psi$, which we already know from (14.121) and (14.122).

To find an equation for $u^\parallel$, we take the $z$ component of the momentum equation (13.58):

$$\frac{du^\parallel}{dt} = \frac{1}{\rho_0} \left[ \frac{\partial}{\partial z} \left( p + \frac{B^2}{8\pi} \right) + \frac{B \cdot \nabla \delta B}{4\pi} \right] \Rightarrow \frac{du^\parallel}{dt} = v_A^2 b \cdot \nabla \frac{\delta B}{B_0}. \quad (14.124)$$

The parallel pressure gradient is $O(Ma^3)$ because there is pressure balance (14.118) to lowest order.

Finally, let us bring in the energy equation (13.60), as yet unused. To leading order, it is

$$\frac{d}{dt} \frac{\delta s}{s_0} = \frac{d}{dt} \left( \frac{\delta p}{\rho_0} - \gamma \frac{\delta \rho}{\rho_0} \right) = 0 \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\delta p}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right) = 0, \quad (14.125)$$

where, to obtain the final version of the equation, we substituted (14.118) for $\delta p/\rho_0$.

\(^{105}\)In the latter context, they are used most prominently as a description of Alfvénic turbulence at small scales (see §14.4), for which the RMHD equations can be shown to be the correct description even if the plasma is collisionless and in general requires kinetic treatment (Schekochihin et al. 2009; Kunz et al. 2015, 2018).
Equations (14.123–14.125) are a complete set of equations for $\delta B$, $u_\parallel$ and $\delta \rho$, given $\Phi$ and $\Psi$. These equations are linear in the Lagrangian frame associated with the Alfvénic perturbations, provided the parallel distances are measured along perturbed field lines. Physically, they tell us that slow waves propagate along perturbed field lines and are passively (i.e., without acting back) advected by the perpendicular Alfvénic flows.

In what follows, when we refer to RMHD, we will mean all five equations (14.121–14.122) and (14.123–14.125).

**Exercise 14.6.** Check that the linear relationships between various perturbations in a slow wave derived in §14.1.5 are manifest in (14.123–14.125).

**Exercise 14.7.** Show that RMHD equations possess the following exact symmetry: $\forall \epsilon$ and $a$, one can simultaneously scale all perturbation amplitudes by $\epsilon$, perpendicular distances by $a$, parallel distances and times by $a/\epsilon$. This means that parallel and perpendicular distances in RMHD are effectively measured in different units. It also means that the small parameter $Ma$ in RMHD can be made arbitrarily small, without any change in the form of the equations, so RMHD is a *bona fide* asymptotic theory (see remark at the end of §14.2.5).

### 14.3.3. Elsasser Fields and the Energetics of RMHD

The Elsasser approach (§14.2.6) can be adapted to the RMHD system. Defining Elsasser potentials

$$\zeta^\pm = \Phi \pm \Psi \quad \Leftrightarrow \quad \delta Z^\pm_\perp = u_\perp \pm \frac{\delta B_\perp}{\sqrt{4\pi \rho_0}} = \hat{z} \times \nabla_\perp \zeta^\pm,$$

(14.126)

it is a straightforward exercise to show that the “vorticities” of the the two Elsasser fields,

$$\omega^\pm = \hat{z} \cdot (\nabla_\perp \times \delta Z^\pm_\perp) = \nabla^2_\perp \zeta^\pm$$

(14.127)

(fluid vorticities ± electric currents), satisfy the following evolution equation

$$\frac{\partial \omega^\pm}{\partial t} \mp v_A \frac{\partial \omega^\pm}{\partial z} + \{\zeta^\mp, \omega^\pm\} = \{\partial_j \zeta^\pm, \partial_j \zeta^\mp\},$$

(14.128)

where summation over the repeated index $j$ is implied. The main corollary of this equation is the same as in §14.2.6, although here it applies to perpendicular perturbations only: *only counter-propagating Alfvénic perturbations can interact and any finite-amplitude perturbation composed of just one Elsasser field is a nonlinear solution.*

Some light is perhaps shed on the nature of the interaction between Elsasser fields if we notice that the left-hand side of (14.128) tells us that the Elsasser vorticity $\omega^\pm$ is propagated along the mean field at the speed $v_A$ and advected across the field by the Elsasser field $\delta Z^\pm_\perp$. The right-hand side of (14.128) is a kind of vortex-stretching term, implying a tendency for vortices and current layers to be produced in the $(x,y)$ plane. There is a preference for current layers, as it turns out. The term in the right-hand side of (14.128) has opposite signs for the two Elsasser fields. Therefore, arguably, nonlinear dynamics favour $\omega^+ \omega^- < 0$, i.e., $|\nabla^2_\perp \Psi|^2 > |\nabla^2_\perp \Phi|^2$ (larger currents than vorticities). This is, indeed, what is seen in numerical simulations of MHD turbulence (see Zhdankin et al. 2016 and §14.4).

The energies of the two Elsasser fields are individually conserved (cf. §14.2.7),

$$\frac{d}{dt} \int d^3r |\nabla_\perp \zeta^\pm|^2 = \frac{d}{dt} \int d^3r |\delta Z^\pm_\perp|^2 = 0,$$

(14.129)

i.e., when the two fields do interact, they scatter each other nonlinearly, but do not exchange energy.
There is an Elsasser-like formulation for the slow waves as well:

\[ \delta Z_{\parallel}^\pm = u_{\parallel} \pm \frac{\delta B}{\sqrt{4\pi \rho_0}} \sqrt{1 + \frac{v_A^2}{c_s^2}}. \]  

(14.130)

Then, from (14.123–14.125), one gets, after more algebra,

\[ \frac{\partial \delta Z_{\parallel}^\pm}{\partial t} \mp \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}} \frac{\partial \delta Z_{\parallel}^\pm}{\partial z} = -\frac{1}{2} \left( \left( 1 \mp \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{ \zeta^+, \delta Z_{\parallel}^+ \} + \left( 1 \mp \frac{1}{\sqrt{1 + v_A^2/c_s^2}} \right) \{ \zeta^-, \delta Z_{\parallel}^- \} \right). \]  

(14.131)

Note the (expected) appearance of the slow-wave phase speed [cf. (14.33)] in the left-hand side. Thus, slow waves interact only with Alfvénic perturbations—when \( v_A \ll c_s \), only with the counterpropagating ones, but at finite \( \beta \), because the slow waves are slower, a co-propagating Alfvénic perturbation can catch up with a slow one, have its way with it in passing and speed on (it’s a tough world).

There is no energy exchange in these interactions: the “+” and “−” slow-wave energies are individually conserved:

\[ \frac{d}{dt} \int d^3 r |\delta Z_{\parallel}^\pm|^2 = 0. \]  

(14.132)

14.3.4. Entropy Mode

There are only two equations in (14.131), whereas we had three equations (14.123–14.125) for our three compressive fields \( \delta B, u_{\parallel} \) and \( \delta \phi \). The third equation, (14.125), was in fact for the entropy perturbation:

\[ \frac{d \tilde{s}}{dt} = 0, \quad \frac{\delta s}{s_0} = -\gamma \left( \frac{\delta \phi}{\rho_0} + \frac{v_A^2}{c_s^2} \frac{\delta B}{B_0} \right). \]  

(14.133)

We see that \( \delta s \) is a decoupled variable, independent from \( \zeta^\pm \) or \( \delta Z_{\parallel}^\pm \) (because it is the only one that involves \( \delta \phi/\rho_0 \)). Equation (14.133) says that \( \delta s \) is a passive scalar field, simply carried around by the Alfvénic velocity \( u_\perp \) (via \( d/dt \)). At high \( \beta \), this is just a density perturbation.

The associated linear mode is not a wave: its dispersion relation is

\[ \omega = 0. \]  

(14.134)

This is the (famously often forgotten) 7th MHD mode, known as the entropy mode (there are 7 equations in MHD, so there must be 7 linear modes: two fast waves, two Alfvén waves, two slow waves and one entropy mode).

Exercise 14.8. Go back to §14.1 and find where this mode was overlooked.

Since the entropy mode is decoupled, its “energy” (variance) is individually conserved:

\[ \frac{d}{dt} \int d^3 r |\delta s|^2 = 0. \]  

(14.135)

---

\(^{106}\)At high \( \beta \), \( v_A \ll c_s \), so we recover from (14.130) and (14.126) the Elsasser fields as defined for iMHD in (14.75).
Thus, in RMHD, the (nonlinear) evolution of all perturbations is constrained by 5 separate conservation laws: $\int d^3r |\delta Z_\parallel|^2$, $\int d^3r |\delta Z_\perp|^2$ and $\int d^3r |\delta s|^2$ are all invariants.

14.3.5. Discussion

Such are the simplifications allowed by anisotropy. Besides greater mathematical simplicity, what is the moral of this story, physically? Let me leave you with two observations.

- In a strong magnetic field, linear propagation is a parallel effect, whilst nonlinearity is a perpendicular effect (advection by $u_\perp$, adjustment of propagation direction by $\delta B_\perp$). RMHD equations express the idea that linear and nonlinear physics play equally important role—this becomes the fundamental guiding principle in the theory of MHD turbulence (§14.4). The idea is that complicated nonlinear dynamics that emerge in the perpendicular plane get teased out along the field because propagating waves enforce a degree of parallel spatial coherence. The distances over which this happens are determined by equating linear and nonlinear time scales, $k_\parallel v_A \sim k_\perp u_\perp$. Dynamics cannot stay coherent over distances longer than $\sim k_\parallel^{-1}$ determined by this balance because of causality: points separated by longer parallel distances cannot exchange information quickly enough to catch up with perpendicular nonlinearities acting locally at each of these points. This principle is called critical balance.

- Restricting the size of perturbations to be small made the RMHD system, in a certain sense, “less nonlinear” than the full MHD (or than iMHD, where $\delta B/B_0 \sim 1$ was allowed). This led to the system’s dynamics being constrained by more invariants: the MHD energy invariant got split into 5 individually conserved quadratic quantities.

**Exercise 14.9.** You might find it an interesting excercise to think about properties of the RMHD system in 2D, in the light of the two observations above. How many invariants are there? In what kind of physical circumstances can we use 2D RMHD without necessarily expecting parallel coherence of the system to break down by the causality argument?

14.4. MHD Turbulence

RMHD is a good starting point for developing the theory of MHD turbulence—a phenomenon observed with great precision in the solar wind and believed ubiquitous in the Universe. I am writing a tutorial review of this topic—a reasonably advanced draft can be found here: [http://www-thphys.physics.ox.ac.uk/research/plasma/JPP/papers17/schekochihin2a.pdf](http://www-thphys.physics.ox.ac.uk/research/plasma/JPP/papers17/schekochihin2a.pdf).

15. MHD Relaxation

So far, we have only considered MHD in a straight field against the background of constant density and pressure (except in §14.2.8, where this was generalised slightly). As any more complicated (static) equilibrium will locally look like this, what we have done has considerable universal significance. Now we shall occupy ourselves with a somewhat less universal (i.e., dependent on the circumstances of a particular problem) and more “large-scale” (compared to the dynamics of wavy perturbations) question: what kind of (static) equilibrium states are there and into which of those states will an MHD fluid normally relax?
15.1. Static MHD Equilibria

Let us go back to the MHD equations (13.57–13.60) and seek static equilibria, i.e., set $u = 0$ and $\partial/\partial t = 0$. The remaining equations are

$$\begin{align*}
-\nabla p + \frac{j \times B}{c} &= 0, \\
j &= \frac{c}{4\pi} \nabla \times B, \\
\nabla \cdot B &= 0
\end{align*}$$

(15.1)

(the force balance, Ampère’s law and the solenoidality-of-$B$ constraint). These are 7 equations for 7 unknowns ($p$, $B$, $j$), so a complete set. Density is irrelevant because nothing moves and so inertia does not matter.

The force-balance equation has two immediate general consequences:

$$B \cdot \nabla p = 0,$$

(15.2)

so magnetic surfaces are surfaces of constant pressure, and

$$j \cdot \nabla p = 0,$$

(15.3)

so currents flow along those surfaces.

Equation (15.2) implies that if magnetic field lines are stochastic and fill the volume of the system, then $p =$ const across the system and so the force balance becomes

$$j \times B = 0.$$

(15.4)

Such equilibria are called force-free and turn out to be very interesting, as we shall discover soon (from §15.1.2 onwards).

15.1.1. MHD Equilibria in Cylindrical Geometry

As the simplest example of an inhomogeneous equilibrium, let us consider the case of cylindrical and axial symmetry:

$$\frac{\partial}{\partial \theta} = 0, \quad \frac{\partial}{\partial z} = 0.$$

(15.5)

Solenoidality of the magnetic field then rules out it having a radial component:

$$\nabla \cdot B = \frac{1}{r} \frac{\partial}{\partial r} rB_r = 0 \quad \Rightarrow \quad rB_r = \text{const} \quad \Rightarrow \quad B_r = 0.$$

(15.6)

Ampère’s law tells us that currents do not flow radially either:

$$j = \frac{c}{4\pi} \nabla \times B \quad \Rightarrow \quad \begin{cases} j_r = 0, \\
j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}, \\
j_z = \frac{c}{4\pi r} \frac{1}{r} \frac{\partial}{\partial r} rB_\theta. \end{cases}$$

(15.7)

Finally, the radial pressure balance gives us

$$\frac{\partial p}{\partial r} = \frac{(j \times B)_r}{c} = \frac{j_\theta B_z - j_z B_\theta}{c} = \frac{1}{4\pi} \left( -B_z \frac{\partial B_z}{\partial r} - \frac{B_\theta}{r} \frac{\partial}{\partial r} rB_\theta \right)$$

$$= -\frac{\partial}{\partial r} \frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r} \frac{B_\theta^2}{8\pi} \quad \Rightarrow \quad \frac{\partial}{\partial r} \left( p + \frac{B^2}{8\pi} \right) = -\frac{B_\theta^2}{4\pi r}.$$

(15.8)

This simply says that the total pressure gradient is balanced by the tension force. A general equilibrium for which this is satisfied is called a screw pinch.
One simple particular case of this is the $z$ pinch (Fig. 63a). This is achieved by letting a current flow along the $z$ axis, giving rise to an azimuthal field:

$$j_\theta = 0, \quad j_z = -\frac{c}{4\pi r} \frac{\partial}{\partial r} r B_\theta \quad \Rightarrow \quad B_\theta = \frac{4\pi}{c} \frac{1}{r} \int_0^r dr' r' j_z(r'), \quad B_z = 0. \quad (15.9)$$

Equation (15.8) becomes

$$\frac{\partial p}{\partial r} = -\frac{1}{c} j_z B_\theta. \quad (15.10)$$

The “pinch” comes from magnetic loops and is due to the curvature force: the loops want to contract inwards, the pressure gradient opposes this and so plasma is confined (Fig. 63b). This configuration will, however, prove to be very badly unstable (§16.4)—which does not stop it from being a popular laboratory set up for short-term confinement experiments (see, e.g., review by Haines 2011).

Another simple particular case is the $\theta$ pinch (Fig. 64a). This is achieved by imposing a straight but radially non-uniform magnetic field in the $z$ direction and, therefore, azimuthal currents:

$$B_\theta = 0, \quad j_z = 0, \quad j_\theta = -\frac{c}{4\pi} \frac{\partial B_z}{\partial r}. \quad (15.11)$$

Equation (15.8) is then just a pressure balance, pure and simple:

$$\frac{\partial}{\partial r} \left( p + \frac{B_z^2}{8\pi} \right) = 0. \quad (15.12)$$

In this configuration, we can confine the plasma (Fig. 64c) or the magnetic flux (Fig. 64d). The latter is what happens, for example, in flux tubes that link sunspots (Fig. 64b). The $\theta$ pinch is a stable configuration (Q11).

The more general case of a screw pinch (15.8) is a superposition of $z$ and $\theta$ pinches, with both magnetic fields and currents wrapping themselves around cylindrical flux surfaces.

The next step in complexity is to assume axial, but not cylindrical symmetry ($\partial/\partial \theta = 0, \partial/\partial z \neq 0$). This is explored in Q9.

For a much more thorough treatment of MHD equilibria, the classic textbook is Freidberg (2014).

15.1.2. Force-Free Equilibria

Another interesting and elegant class of equilibria arises if we consider situations in which $\nabla p$ is negligible and can be completely omitted from the force balance. This can happen in two possible sets of circumstances:
—pressure is the same across the system, e.g., because the field lines are stochastic [a previously mentioned consequence of (15.2)];
—\( \beta = p/(B^2/8\pi) \ll 1 \), so thermal energy is negligible compared to magnetic energy and so \( p \) is irrelevant.

A good example of the latter situation is the solar corona, where \( \beta \sim 1 - 10^{-6} \) (assuming \( n \sim 10^9 \) cm\(^{-3} \), \( T \sim 10^2 \) eV and \( B \sim 1 - 10^3 \) G, the lower value applying in the photosphere, the upper one in the coronal loops; see Fig. 64b)

In such situations, the equilibrium is purely magnetic, i.e., the magnetic field is “force-free,” which implies that the current must be parallel to the magnetic field:

\[
\mathbf{j} \times \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{j} \parallel \mathbf{B} \quad \Rightarrow \quad \frac{4\pi}{c} \mathbf{j} = \nabla \times \mathbf{B} = \alpha(r) \mathbf{B},
\]

where \( \alpha(r) \) is an arbitrary scalar function. Taking the divergence of the last equation tells us that

\[
\mathbf{B} \cdot \nabla \alpha = 0,
\]

so the function \( \alpha(r) \) is constant on magnetic surfaces. If \( \mathbf{B} \) is chaotic and volume-filling, then \( \alpha = \text{const} \) across the system.

The case of \( \alpha = \text{const} \) is called the linear force-free field. In this case, taking the curl of (15.13) and then iterating it once gives us

\[
-\nabla^2 \mathbf{B} = \alpha \nabla \times \mathbf{B} = \alpha^2 \mathbf{B} \quad \Rightarrow \quad \left( \nabla^2 + \alpha^2 \right) \mathbf{B} = 0,
\]

so the magnetic field satisfies a Helmholtz equation (to solve which, one must, of course, specify some boundary conditions).

Thus, there is, potentially, a large zoo of MHD equilibria. Some of them are stable, some are not, and, therefore, some are more interesting and/or more relevant than others. How does one tell? A good question to ask is as follows. Suppose we set up some initial
configuration of magnetic field (by, say, switching on some current-carrying coils, driving currents inside plasma, etc.)—*to what (stable) equilibrium will this system eventually relax?*

In general, any initially arranged magnetic configuration will exert forces on the plasma, these will drive flows, which in turn will move the magnetic fields around; eventually, everything will settle into some static equilibrium. We expect that, normally, some amount of the energy contained in the initial field will be lost in such a relaxation process because the flows will be dissipating, the fields diffusing and/or reconnecting, etc.—the losses occur due to the resistive and viscous terms in the non-ideal MHD equations derived in §13. Thus, one expects that *the final relaxed static state will be a minimum-energy state* and so we must be able to find it by minimising magnetic energy:

\[
\int \frac{d^3r}{8\pi} B^2 \to \min.
\] (15.16)

Clearly, if the relaxation occurred without any constraints, the solution would just be \( B = 0 \). In fact, there are constraints. These constraints are *topological*: if you think of magnetic field lines as a tangled mess, you will realise that, while you can change this tangle by moving field lines around, you cannot easily undo linkages, knots, etc.—anything that, to be undone, would require the field lines to have “ends”. This intuition can be turned into a quantitative theory once we discover that the induction equation (13.59) has an invariant that involves the magnetic field only and is, in a certain sense, “better conserved” than energy.

### 15.2. Helicity

*Magnetic helicity* in a volume \( V \) is defined as

\[
H = \int_V d^3r \, A \cdot B,
\] (15.17)

where \( A \) is the vector potential, \( \nabla \times A = B \).

#### 15.2.1. Helicity Is Well Defined

This is not obvious because \( A \) is not unique: a gauge transformation

\[
A \to A + \nabla \chi,
\] (15.18)

with \( \chi \) an arbitrary scalar function, leaves \( B \) unchanged and so does not affect physics. Under this transformation, helicity stays invariant:

\[
H \to H + \int_V d^3r \, B \cdot \nabla \chi = H + \int_{\partial V} \, dS \cdot B \chi = H,
\] (15.19)

provided \( B \) at the boundary is parallel to the boundary, i.e., provided the volume \( V \) encloses the field (nothing sticks out).

#### 15.2.2. Helicity Is Conserved

Let us go back to the induction equation (13.23) (in which we retain resistivity to keep track of non-ideal effects, i.e., of the breaking of flux conservation):

\[
\frac{\partial B}{\partial t} = \nabla \times (u \times B - \eta \nabla \times B).
\] (15.20)

“Uncurling” this equation, we get

\[
\frac{\partial A}{\partial t} = u \times B - \eta \nabla \times B + \nabla \chi.
\] (15.21)
Using (15.20) and (15.21), we have

\[
\frac{\partial}{\partial t} A \cdot B = B \cdot (u \times B - \eta \nabla \times B + \nabla \chi) + A \cdot [\nabla \times (u \times B - \eta \nabla \times B)] \\
= -\eta B \cdot (\nabla \times B) + \nabla \cdot (B \chi) \\
- \nabla \cdot [A \times (u \times B - \eta \nabla \times B)] + (u \times B - \eta \nabla \times B) \cdot (\nabla \times A) \\
= \nabla \cdot [B \chi - uA \cdot B + BA \cdot u + \eta A \times (\nabla \times B)] - 2\eta B \cdot (\nabla \times B). \quad (15.22)
\]

Integrating this and using Gauss’s theorem, we get

\[
\frac{\partial}{\partial t} \int_V d^3r A \cdot B = \int_{\partial V} dS \cdot [B \chi - uA \cdot B + BA \cdot u + \eta A \times (\nabla \times B)] \\
- 2\eta \int_V d^3r B \cdot (\nabla \times B). \quad (15.23)
\]

The surface integral vanishes provided both \( u \) and \( B \) are parallel to the boundary (no fields stick out and no flows cross). The resistive term in the surface integral can also be ignored either by arranging \( V \) appropriately or simply by taking it large enough so \( B \to 0 \) on \( \partial V \), or, indeed, by taking \( \eta \to +0 \). Thus,

\[
\frac{dH}{dt} = -2\eta \int d^3r B \cdot (\nabla \times B), \quad (15.24)
\]

**magnetic helicity is conserved in ideal MHD.**

Furthermore, it turns out that even in resistive MHD, helicity is “better conserved” than energy, in the following sense. As we saw in §13.10.2, the magnetic energy evolves according to

\[
\frac{d}{dt} \int d^3r \frac{B^2}{8\pi} = \left(\text{energy exchange terms and fluxes}\right) - 2\eta \int d^3r |\nabla \times B|^2. \quad (15.25)
\]

The first term on the right-hand side contains various fluxes and energy exchanges with the velocity field [see (13.54)], all of which eventually decay as the system relaxes (flows decay by viscosity). The second term represents Ohmic heating. If \( \eta \) is small but the Ohmic heating is finite, it is finite because magnetic field develops fine-scale gradients: \( \nabla \sim \eta^{-1/2} \), so

\[
-2\eta \int d^3r |\nabla \times B|^2 \to \text{const} \quad \text{as} \quad \eta \to +0. \quad (15.26)
\]

But then the right-hand side of (15.24) is

\[
-2\eta \int d^3r B \cdot (\nabla \times B) = \mathcal{O}(\eta^{1/2}) \to 0 \quad \text{as} \quad \eta \to +0. \quad (15.27)
\]

Thus, as an initial magnetic configuration relaxes, while its energy can change quickly (on dynamical times), its helicity changes only very slowly in the limit of small \( \eta \). The constancy of \( H \) (as \( \eta \to +0 \)) provides us with the constraint subject to which the energy will need to be minimised.

Before we use this idea, let us discuss what the conservation of helicity means physically, or, rather, topologically.

\footnote{The resistive term in the right-hand side of (15.24) is \( \propto \int d^3r B \cdot j \), a quantity known as the \textit{current helicity}.}
15.2.3. Helicity Is a Topological Invariant

Consider two linked flux tubes, $T_1$ and $T_2$ (Fig. 65). The helicity of $T_1$ is the product of the fluxes through $T_1$ and $T_2$:

$$H_1 = \oint_{T_1} A \cdot B = \oint_{T_1} \frac{dl}{bdS} dS A \cdot dS b$$
$$= \oint_{T_1} A \cdot bdl Bb \cdot bdS = \oint_{T_1} A \cdot dS B \cdot dS = \Phi_1 \oint_{T_1} A \cdot dl = \Phi_1 \Phi_2.$$  (15.28)

By the same token, in general, in a system of many linked tubes, the helicity of tube $i$ is

$$H_i = \Phi_i \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij},$$  (15.29)

where $N_{ij}$ is the number of times tube $j$ passes through the hole in tube $i$. The total helicity of this entire assemblage of flux tubes is then

$$H = \sum_{ij} \Phi_i \Phi_j N_{ij}.$$  (15.30)

Thus, $H$ is the number of linkages of the flux tubes weighted by the field strength in them. It is in this sense that helicity is a topological invariant.

Note that the cross-helicity $\int d^3 r \, \mathbf{u} \cdot \mathbf{B}$ (§14.2.7) can similarly be interpreted as counting the linkages between flux tubes ($\mathbf{B}$) and vortex tubes ($\omega = \nabla \times \mathbf{u}$). The current helicity $\int d^3 r \, \mathbf{B} \cdot \mathbf{j}$ [appearing in the right-hand side of (15.24)] counts the number of linkages between current loops. The latter is not an MHD invariant though.

15.3. J. B. Taylor Relaxation

Let us now work out the equilibrium to which an MHD system will relax by minimising magnetic energy subject to constant helicity:

$$\delta \int_V d^3 r (B^2 - \alpha \mathbf{A} \cdot \mathbf{B}) = 0,$$  (15.31)
where \( \alpha \) is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Let us work out the two terms:

\[
\delta \int_V d^3r \, B^2 = 2 \int_V d^3r \, B \cdot \delta B = 2 \int_V d^3r \, (\nabla \times \delta A)
\]

\[
= 2 \int_V d^3r \, [\nabla \cdot (B \times \delta A) + (\nabla \times B) \cdot \delta A]
\]

\[
= -2 \int_{\partial V} dS \cdot (B \times \delta A) + 2 \int_V d^3r \, (\nabla \times B) \cdot \delta A,
\] (15.32)

\[
\delta H = \delta \int_V d^3r \, A \cdot B = \int_V d^3r \, (B \cdot \delta A + A \cdot \delta B) = \int_V d^3r \, [B \cdot \delta A + A \cdot (\nabla \times \delta A)]
\]

\[
= \int_V d^3r [B \cdot \delta A - \nabla \cdot (A \times \delta A) + (\nabla \times A) \cdot \delta A]
\]

\[
= -\int_{\partial V} dS \cdot (A \times \delta A) + 2 \int_V d^3r \, B \cdot \delta A.
\] (15.33)

Now, since

\[
\frac{\partial \delta B}{\partial t} = \nabla \times (u \times B) = \nabla \times \left( \frac{\partial \xi}{\partial t} \times B \right)
\]

(15.34)

for small displacements, we have \( \delta A = \xi \times B \), whence

\[
B \times \delta A = B^2 \xi - B \cdot \xi B,
\]

(15.35)

\[
A \times \delta A = A \cdot B \xi - A \cdot \xi B.
\]

(15.36)

Therefore, the surface terms in (15.32) and (15.33) vanish if \( B \) and \( \xi \) are parallel to the boundary \( \partial V \), i.e., if the volume \( V \) encloses both \( B \) and the plasma—there are no displacements through the boundary.

This leaves us with

\[
\delta \int_V d^3r \, (B^2 - \alpha A \cdot B) = 2 \int_V d^3r \, (\nabla \times B - \alpha B) \cdot \delta A = 0,
\]

(15.37)

which instantly implies that \( B \) is a linear force-free field:

\[
\nabla \times B = \alpha B \quad \Rightarrow \quad \nabla^2 B = -\alpha^2 B.
\]

(15.38)

Thus, our system will relax to a linear force-free state determined by (15.38) and system-specific boundary conditions. Here \( \alpha = \alpha(H) \) depends on the (fixed by initial conditions) value of \( H \) via the equation

\[
H(\alpha) = \int d^3r \, A \cdot B = \frac{1}{\alpha} \int d^3r \, B^2,
\]

(15.39)

where \( B \) is the solution of (15.38) (since \( \nabla \times B = \alpha B = \alpha \nabla \times A \), we have \( B = \alpha A + \nabla \chi \) and the \( \chi \) term vanishes under volume integration).

Thus, the prescription for finding force-free equilibria is

—solve (15.38), get \( B = B(\alpha) \), parametrically dependent on \( \alpha \),
—calculate \( H(\alpha) \) according to (15.39),
—set \( H(\alpha) = H_0 \), where \( H_0 \) is the initial value of helicity, hence calculate \( \alpha = \alpha(H_0) \) and complete the solution by using this \( \alpha \) in \( B = B(\alpha) \).

Note that it is possible for this procedure to return multiple solutions. In that case, the
solution with the smallest energy must be the right one (if a system relaxed to a local minimum, one can always imagine it being knocked out of it by some perturbation and falling to a lower energy).

Exercise 15.1. Force-free fields in 2D. Show that for incompressible MHD confined to the 2D plane \((x, y)\), the quantity \(\int d^2r A_z^2\) is conserved, except for resistive dissipation (this 2D invariant is sometimes called “anastrophy”). Work out the 2D version of J. B. Taylor relaxation and show that the resulting equilibrium field is a linear force-free field.

15.4. Relaxed Force-Free State of a Cylindrical Pinch

Let us illustrate how the procedure derived in \(\S\)15.3 works by considering again the case of cylindrical and axial symmetry [see (15.5)]. The \(z\) component of (15.38) gives us the following equation for \(B_z(r)\):

\[
B_z'' + \frac{1}{r} B_z' + \alpha^2 B_z = 0. \tag{15.40}
\]

This is a Bessel equation, whose solution, subject to \(B_z(0) = B_0\) and \(B_z(\infty) = 0\), is

\[
B_z(r) = B_0 J_0(\alpha r). \tag{15.41}
\]

We can now calculate the azimuthal field as follows

\[
\alpha B_\theta = (\nabla \times \mathbf{B})_\theta = -B_z' \implies B_\theta(r) = B_0 J_1(\alpha r). \tag{15.42}
\]

This gives us an interesting twisted field (Fig. 67), able to maintain itself in equilibrium without help from pressure gradients.

Finally, we calculate its helicity according to (15.39): assuming that the length of the
cylinder is $L$, its radius $R$ and so its volume $V = \pi R^2 L$, we have

$$H = \frac{1}{\alpha} \int d^3 r \, B^2 = \frac{2\pi L B_0^2}{\alpha} \int_0^R drr \left[ J_0^2(\alpha r) + J_1^2(\alpha r) \right]$$

$$= \frac{B_0^2 V}{\alpha^2} \left[ J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R} J_1(\alpha R)J_2(\alpha R) \right]. \quad (15.43)$$

If we solve this for $\alpha = \alpha(H)$, our solution is complete.

**Exercise 15.2.** Work out what happens in the general case of $\partial/\partial \theta \neq 0$ and $\partial/\partial z \neq 0$ and whether the simple symmetric solution obtained above is the correct relaxed, minimum-energy state (not always, it turns out). This is not a trivial exercise. The solution is in Taylor & Newton (2015, §9), where you will also find much more on the subject of J. B. Taylor relaxation, relaxed states and much besides—all from the original source.

There are other useful variational principles—other in the sense that the constraints that are imposed are different from helicity conservation. The need for them arises when one considers magnetic equilibria in domains that do not completely enclose the field lines, i.e., when $dS \cdot B \neq 0$ at the boundary. One example of such a variational principle, also yielding a force-free field (although not necessarily a linear one), is given in Q1(e). A specific example of such a field arises in Q9(f).

15.5. Parker’s Problem and Topological MHD

Coming soon... On topology in MHD, a very mathematically minded student might enjoy the book by Arnold & Khesin (1999).

16. MHD Stability and Instabilities

We now wish to take a more general view of the MHD stability problem: given some static\(^{108}\) equilibrium (some $\rho_0$, $p_0$, $B_0$ and $u_0 = 0$), will this equilibrium be stable to small perturbations of it, i.e., will these perturbations grow or decay?

There are two ways to answer this question:

1) Carry out the normal-mode analysis, i.e., linearise the MHD equations around the given equilibrium, just as we did when we studied MHD waves in §14.1, and see if any of the frequencies (solutions of the dispersion relation) turn out to be complex, with

---

\(^{108}\)A treatment of the more general case of a dynamic equilibrium, $u_0 \neq 0$, can be found in the excellent textbook by Davidson (2016).
positive imaginary parts (growth rates). This approach has the advantage of being direct and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do and one might be willing to settle for less: just being able to prove that some configuration is stable or that certain types of perturbations might grow. Hence the the second approach:

2) Check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system increasing. If so, then the equilibrium is stable—this is called the energy principle and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy decreasing, that equilibrium is unstable. The advantage of this second approach is that we do not need to solve the (linearised) MHD equations in order to pronounce on stability, just to examine the properties of the perturbed energy functional.

It should be already quite clear how to do the normal-mode analysis, at least conceptually, so I shall focus on the second approach.

### 16.1. Energy Principle

Recall what the total energy in MHD is (§13.10)

\[ E = \int d^3r \left( \frac{\rho u^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right) \equiv \int d^3r \frac{\rho u^2}{2} + W. \]  

(16.1)

As we saw in §14.1, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements \( \xi \)—we will work this out shortly for a general equilibrium, but for now, let us accept that this will be true.\(^{109}\) As \( u = \frac{\partial \xi}{\partial t} \) by definition of \( \xi \), we have

\[ E = \int d^3r \frac{1}{2} \rho_0 \left| \frac{\partial \xi}{\partial t} \right|^2 + W_0 + \delta W_1[\xi] + \delta W_2[\xi, \xi] + \ldots, \]  

(16.2)

where we have kept terms up to second order in \( \xi \) and so \( W_0 \) is the equilibrium part of \( W \) (i.e., its value for \( \xi = 0 \)), \( \delta W_1[\xi] \) is linear in \( \xi \), \( \delta W_2[\xi, \xi] \) is bilinear (quadratic), etc. Energy must be conserved to all orders, so

\[ \frac{dE}{dt} = \int d^3r \rho_0 \frac{\partial^2 \xi}{\partial t^2} \cdot \frac{\partial \xi}{\partial t} + \delta W_1 \left[ \frac{\partial \xi}{\partial t} \right] + \delta W_2 \left[ \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right] + \delta W_2 \left[ \xi, \frac{\partial \xi}{\partial t} \right] + \cdots = 0. \]  

(16.3)

This must be true at all times, including at \( t = 0 \), when \( \xi \) and \( \frac{\partial \xi}{\partial t} \) can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions \( \xi \) and \( \eta \),

\[ \int d^3r \eta \cdot F[\xi] + \delta W_1[\eta] + \delta W_2[\eta, \xi] + \delta W_2[\xi, \eta] + \cdots = 0. \]  

(16.4)

In the first order, this tells us that

\[ \delta W_1[\eta] = 0, \]  

(16.5)

\(^{109}\)In fact, also the fully nonlinear dynamics can be completely expressed in terms of displacements if the MHD equations are written in Lagrangian coordinates (see §13.12).
which is good to know because it means that $\delta W_1$ disappears from (16.2) (there are no first-order energy perturbations). In the second order, we get

$$
\int d^3 r \, \eta \cdot F[\xi] = -\delta W_2[\eta, \xi] - \delta W_2[\xi, \eta].
$$

Let $\eta = \xi$. Then (16.6) implies

$$
\delta W_2[\xi, \xi] = -\frac{1}{2} \int d^3 r \, \xi \cdot F[\xi].
$$

This is the part of the perturbed energy in (16.2) that can be both positive and negative. The Energy Principle is

$$
\delta W_2[\xi, \xi] > 0 \text{ for any } \xi \iff \text{equilibrium is stable}
$$

(Bernstein et al. 1958). Before we are in a position to prove this, we must do some preparatory work.

16.1.1. Properties of the Force Operator $F[\xi]$

Since the right-hand side of (16.6) is symmetric with respect to swapping $\xi \leftrightarrow \eta$, so must be the left-hand side:

$$
\int d^3 r \, \eta \cdot F[\xi] = \int d^3 r \, \xi \cdot F[\eta].
$$

Therefore, operator $F[\xi]$ is self-adjoint. Since, by definition,

$$
F[\xi] = \rho_0 \frac{\partial^2 \xi}{\partial t^2},
$$

the eigenmodes of this operator satisfy

$$
\xi(t, r) = \xi_n(r) e^{-i\omega_n t} \implies F[\xi_n] = -\rho_0 \omega_n^2 \xi_n.
$$

As always for self-adjoint operators, we can prove a number of useful statements.

1) The eigenvalues $\{\omega_n^2\}$ are real.

**Proof.** If (16.11) holds, so must

$$
F[\xi_n^*] = -\rho_0 (\omega_n^2)^* \xi_n^*,
$$

provided $F$ has no complex coefficients (we shall confirm this explicitly in §16.2.1). Taking the full scalar products (including integrating over space) of (16.11) with $\xi_n^*$ and of (16.12) with $\xi_n$ and subtracting one from the other, we get

$$
- [\omega_n^2 - (\omega_n^2)^*] \int d^3 r \, \rho_0 |\xi_n|^2 = \int d^3 r \, \xi_n^* \cdot F[\xi_n] - \int d^3 r \, \xi_n \cdot F[\xi_n^*] = 0
$$

$$
\implies \omega_n^2 = (\omega_n^2)^*, \quad \text{q.e.d.}
$$

This result implies that, if any MHD equilibrium is unstable, at least one of the eigenvalues must be $\omega_n^2 < 0$ and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 68a), rather than growing oscillations (also known as “overstabilities”; see Fig. 68b).

2) The eigenmodes $\{\xi_n\}$ are orthogonal.
Proof. Taking the full scalar products of (16.11) with $\xi_m$ (assuming $m \neq n$ and non-degeneracy of $\omega_{m,n}^2$), and of the analogous equation

$$F[\xi_m] = -\rho_0 \omega_m^2 \xi_m$$

(16.14)

with $\xi_n$ and subtracting them, we get

$$-(\omega_n^2 - \omega_m^2) \int d^3r \rho_0 \xi_n \cdot \xi_m = \int d^3r \xi_n \cdot F[\xi_n] - \int d^3r \xi_m \cdot F[\xi_m] = 0$$

$$\Rightarrow \int d^3r \rho_0 \xi_n \cdot \xi_m = \delta_{nm} \int d^3r \rho_0 |\xi_n|^2,$$

q.e.d.

(16.15)

16.1.2. Proof of the Energy Principle (16.8)

Let us assume completeness of the set of eigenmodes $\{\xi_n\}$ (not, in fact, an indispensable assumption, but we shall not worry about this nuance here; see Kulsrud 2005, §7.2). Then any displacement at any given time $t$ can be decomposed as

$$\xi(t, r) = \sum_n a_n(t) \xi_n(r).$$

(16.16)

The energy perturbation (16.7) is

$$\delta W_2[\xi, \xi] = -\frac{1}{2} \int d^3r \xi \cdot F[\xi] = -\frac{1}{2} \sum_{nm} a_n a_m \int d^3r \xi_n \cdot F[\xi_m]$$

$$= \frac{1}{2} \sum_{nm} a_n a_m \omega_m^2 \int d^3r \rho_0 \xi_n \cdot \xi_m = \frac{1}{2} \sum_n a_n^2 \omega_n^2 \int d^3r \rho_0 |\xi_n|^2.$$

(16.17)

By the same token,

$$K[\xi, \xi] = \frac{1}{2} \int d^3r \rho_0 |\xi|^2 = \frac{1}{2} \sum_n a_n^2 \int d^3r \rho_0 |\xi_n|^2.$$

(16.18)

Then, if we arrange $\omega_1^2 \leq \omega_2^2 \leq \ldots$, the smallest eigenvalue is

$$\omega_1^2 = \min_\xi \frac{\delta W_2[\xi, \xi]}{K[\xi, \xi]}.$$

(16.19)

Therefore,

\[\text{Figure 68. MHD instabilities.}\]
• condition (16.8) is sufficient for stability because, if \( \delta W[\xi, \xi] > 0 \) for all possible \( \xi \), then the smallest eigenvalue \( \omega_1^2 > 0 \), and so all eigenvalues are positive, \( \omega_n^2 \geq \omega_1^2 > 0 \);

• condition (16.8) is necessary for stability because, if the equilibrium is stable, then all eigenvalues are positive, \( \omega_n^2 > 0 \), whence \( \delta W[\xi, \xi] > 0 \) in view of (16.17), q.e.d.

### 16.2. Explicit Calculation of \( \delta W_2 \)

Now that we know that we need the sign of \( \delta W_2 \) to ascertain stability (or otherwise), it is worth working out \( \delta W_2 \) as an explicit function of \( \xi \). It is a second-order quantity, but (16.7) tells us that all we need to calculate is \( F[\xi] \) to first order in \( \xi \), i.e., we just need to linearise the MHD equations around an arbitrary static equilibrium. The procedure is the same as in §14.1, but without assuming \( \rho_0, p_0 \) and \( B_0 \) to be spatially homogeneous.

#### 16.2.1. Linearised MHD Equations

Thus, generalising somewhat the procedure adopted in (14.3–14.5), we have

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \Rightarrow \frac{\partial \delta \rho}{\partial t} = -\nabla \cdot \left( \rho_0 \frac{\partial \xi}{\partial t} \right)
\]

\[
\Rightarrow \delta \rho = -\nabla \cdot (\rho_0 \xi), \quad (16.20)
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) p = -\gamma p \nabla \cdot \mathbf{u} \Rightarrow \frac{\partial \delta p}{\partial t} = -\nabla \cdot \left( \rho_0 \xi \right) \mathbf{v}_p - \gamma \mathbf{p}_0 \cdot \nabla \cdot \frac{\partial \xi}{\partial t}
\]

\[
\Rightarrow \delta p = -\mathbf{\xi} \cdot \nabla \mathbf{v}_p - \gamma \mathbf{p}_0 \nabla \cdot \frac{\partial \xi}{\partial t}, \quad (16.21)
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \Rightarrow \frac{\partial \delta \mathbf{B}}{\partial t} = \nabla \times \left( \frac{\partial \xi}{\partial t} \times \mathbf{B}_0 \right)
\]

\[
\Rightarrow \delta \mathbf{B} = \nabla \times (\mathbf{\xi} \times \mathbf{B}_0), \quad (16.22)
\]

Note that again \( \delta \rho, \delta p \) and \( \delta \mathbf{B} \) are all expressed as linear operators on \( \xi \)—and so \( \delta W = \delta \int d^3r \left[ B^2/8\pi + p/(\gamma - 1) \right] \) must also be some operator involving \( \xi \) and its gradients but not \( \partial \xi/\partial t \) (as we assumed in §16.1).

Finally, we deal with the momentum equation (to which we add gravity as this will give some interesting instabilities):

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \mathbf{p} + \frac{\nabla \times \mathbf{B}_0 \times \mathbf{B}}{4\pi} + \rho \mathbf{g}. \quad (16.23)
\]

This gives us

\[
F[\xi] = \rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla \delta \rho + \frac{(\nabla \times \mathbf{B}_0) \times \delta \mathbf{B}}{4\pi} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi} + \delta \rho \mathbf{g}
\]

\[
= \nabla \left( \mathbf{\xi} \cdot \nabla \mathbf{v}_p + \gamma \mathbf{p}_0 \nabla \cdot \mathbf{\xi} \right) - \gamma \nabla \cdot (\rho_0 \mathbf{\xi}) + \mathbf{j}_0 \times \frac{\delta \mathbf{B}}{c} + \frac{(\nabla \times \delta \mathbf{B}) \times \mathbf{B}_0}{4\pi}, \quad (16.24)
\]

where \( \mathbf{j}_0 = c(\nabla \times \mathbf{B}_0)/4\pi \), we have used (16.20) and (16.21) for \( \delta \rho \) and \( \delta p \), respectively, and \( \delta \mathbf{B} \) is given by (16.22).
16.2.2. Energy Perturbation

Now we can use (16.24) in (16.7) to calculate explicitly

\[ \delta W_2 = \frac{1}{2} \int d^3r \left[ -\xi \cdot \nabla (\xi \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \xi) + (g \cdot \xi) \nabla \cdot (\rho_0 \xi) \right] \]

\[ = (\xi \cdot \nabla p_0) \nabla \cdot \xi + \gamma p_0 (\nabla \cdot \xi)^2 \]

after integration by parts

\[ = \frac{(j_0 \times \delta B) \cdot \xi}{c} = \frac{(\nabla \times \delta B) \times B_0}{4\pi} \cdot \xi. \]

(16.25)

Thus, we have arrived at a standard textbook (e.g., Kulsrud 2005) expression for the energy perturbation (this expression is non-unique because one can do various integrations by parts):

\[ \delta W_2 = \frac{1}{2} \int d^3r \left[ (\xi \cdot \nabla p_0) \nabla \cdot \xi + \gamma p_0 (\nabla \cdot \xi)^2 + (g \cdot \xi) \nabla \cdot (\rho_0 \xi) \right. \]

\[ + \frac{j_0 \cdot (\xi \times \delta B)}{c} + \frac{|\delta B|^2}{4\pi}, \]

(16.26)

where \( \delta B = \nabla \times (\xi \times B_0) \). Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilising. The terms that are not sign-definite and so potentially destabilising involve equilibrium gradients of pressure, density and magnetic field (currents). It is perhaps not a surprise to learn that Nature, with its fundamental yearning for thermal equilibrium, might dislike gradients—while it is of course not a rule that all such inhomogeneities render the system unstable, we will see that they often do, usually when gradients exceed certain critical thresholds.

All we need to do now is calculate \( \delta W_2 \) according to (16.26) for any equilibrium that interests us and see if it can be negative for any class of perturbations (or show that it is positive for all perturbations).

16.3. Interchange Instabilities

As the first and simplest example of how one does stability calculations using the Energy Principle, we will (perhaps disappointingly) consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic stratified atmosphere:

\[ \rho_0 = \rho_0(z) \text{ and } p_0 = p_0(z) \text{ satisfying } \frac{dp_0}{dz} = -\rho_0 g \]

(16.27)

(gravity acts downward, against the \( z \) direction, \( g = -g \hat{z} \)).
16.3.1. **Formal Derivation of the Schwarzschild Criterion**

With $B_0 = 0$ and the hydrostatic equilibrium (16.27), (16.26) becomes

$$
\delta W_2 = \frac{1}{2} \int d^3 r \left[ \xi_z p_0' \nabla \cdot \xi + \gamma p_0 (\nabla \cdot \xi)^2 - g \xi_z (p_0' \xi_z + p_0 \nabla \cdot \xi) \right]
$$

$$
= \frac{1}{2} \int d^3 r \left[ 2 p_0' \xi_z \nabla \cdot \xi + \gamma p_0 (\nabla \cdot \xi)^2 - \rho_0' g \xi_z^2 \right],
$$

(16.28)

where we have used $\rho_0 g = -p_0'$. We see that $\delta W_2$ depends on $\xi_z$ and $\nabla \cdot \xi$. Let us treat them as independent variables and minimise $\delta W_2$ with respect to them (i.e., seek the most unstable possible situation):

$$
\frac{\partial}{\partial (\nabla \cdot \xi)} \left[ \text{integrand of (16.28)} \right] = 2 p_0' \xi_z + 2 \gamma p_0 (\nabla \cdot \xi) = 0 \Rightarrow \nabla \cdot \xi = -\frac{p_0'}{\gamma p_0} \xi_z.
$$

(16.29)

Substituting this back into (16.28), we get

$$
\delta W_2 = \frac{1}{2} \int d^3 r \left( -\frac{p_0'^2}{\gamma p_0} - \rho_0' g \right) \xi_z^2 = \frac{1}{2} \int d^3 r \frac{\rho_0 g}{\gamma} \left( \frac{p_0'}{p_0} - \frac{\rho_0'}{\rho_0} \right) \xi_z^2.
$$

(16.30)

By the Energy Principle, the system is stable iff

$$
\delta W_2 > 0 \Leftrightarrow \frac{d \ln s_0}{dz} > 0,
$$

(16.31)

where $s_0 = p_0/\rho_0^\gamma$ is the entropy function. The inequality (16.31) is the **Schwarzschild criterion** for convective stability.\textsuperscript{111} If this criterion is broken, there will be an instability, called the **interchange instability**.

This calculation illustrates both the power and the weakness of the method:

— on the one hand, we have obtained a stability criterion quite quickly and without having to solve the underlying equations,

— on the other hand, while we have established the condition for instability, we have as yet absolutely no idea what is going on physically.

16.3.2. **Physical Picture**

We can remedy the latter problem by examining what type of displacements give rise to $\delta W_2 < 0$ when the Schwarzschild criterion is broken. Recalling (16.20) and (16.21) and specialising to the displacements given by (16.29) (as they are the ones that minimise $\delta W_2$), we get

$$
\frac{\delta p}{p_0} = -\frac{\xi \cdot \nabla p_0}{p_0} - \gamma \nabla \cdot \xi = -\frac{p_0'}{p_0} \xi_z - \gamma \nabla \cdot \xi = 0,
$$

(16.32)

$$
\frac{\delta \rho}{\rho_0} = -\frac{\nabla \cdot (\rho_0 \xi)}{\rho_0} = -\frac{\rho_0'}{\rho_0} \xi_z - \nabla \cdot \xi = \frac{1}{\gamma} \left( -\frac{p_0'}{p_0} + \frac{p_0'}{p_0} \right) = \frac{1}{\gamma} \frac{d \ln s_0}{dz} \xi_z.
$$

(16.33)

\textsuperscript{111}We studied perturbations of a stably stratified atmosphere in §14.2.8 and Q6, where we saw that these perturbations indeed did not grow provided the entropy scale length $1/H_s = d \ln s_0/dz$ was positive.
Thus, the offending perturbations maintain themselves in pressure balance (i.e., they are not sound waves) and locally increase or decrease density for blobs of fluid that fall ($\xi_z < 0$) or rise ($\xi_z > 0$), respectively.

This gives us some handle on the situation: if we imagine a blob of fluid slowly rising (slowly, so $\delta p = 0$) from the denser nether regions of the atmosphere to the less dense upper ones, then we can ask whether staying in pressure balance with its surroundings will require the blob to expand ($\delta \rho < 0$) or contract ($\delta \rho > 0$). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly) and the system will be unstable. The direction of the entropy gradient determines which of these two scenarios is realised.

16.3.3. Intuitive Rederivation of the Schwarzschild Criterion

We can use this physical intuition to derive the Schwarzschild criterion directly. Consider two blobs, at two different vertical locations, lower (1) and upper (2), where the equilibrium densities and pressures are $\rho_{01}, p_{01}$ and $\rho_{02}, p_{02}$. Now interchange these two blobs (Fig. 69). Inside the blobs, the new densities and pressures are $\rho_1, p_1$ and $\rho_2, p_2$.

Requiring the blobs to stay in pressure balance with their local surroundings gives

$$p_1 = p_{02}, \quad p_2 = p_{01}. \quad (16.34)$$

Requiring the blobs to rise or fall adiabatically, i.e., to satisfy $p/\rho^\gamma = \text{const}$, and then using pressure balance (16.34) gives

$$\frac{\rho_{01}}{\rho_1} = \frac{p_1}{p_{01}} = \frac{\rho_{02}}{\rho_1} \implies \frac{\rho_1}{\rho_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma}. \quad (16.35)$$

Requiring that the buoyancy of the rising blob overcome gravity, i.e., that the weight of the displaced fluid be larger than the weight of the blob,

$$\rho_{02}g > \rho_1g, \quad (16.36)$$

gives the condition for instability:

$$\rho_1 < \rho_{02} \iff \frac{\rho_1}{\rho_{02}} = \frac{\rho_{01}}{\rho_{02}} \left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma} < 1 \iff \frac{p_{02}}{p_{01}} < \frac{\rho_{01}}{\rho_{02}}. \quad (16.37)$$

This is exactly the same as the Schwarzschild condition (16.31) for the interchange instability (and this is why the instability is called that).

Note that, while this is of course a much simpler and more intuitive argument than the application of the Energy Principle, it only gives us a particular example of the
kind of perturbation that would be \textit{unstable} under particular conditions, not any general criterion of what equilibria might be guaranteed to be stable.

In Q10, we will explore how the above considerations can be generalised to an equilibrium that also features a non-zero magnetic field.

16.4. Instabilities of a Pinch

As our second (also classic) example, we consider the stability of a $z$-pinch equilibrium (§15.1.1, Fig. 63):

$$
B_0 = B_0(r) \hat{\theta}, \quad j_0 = j_0(r) \hat{z} = \frac{c}{4\pi r} (rB_0)' \hat{z}, \quad p_0'(r) = -\frac{1}{c} j_0B_0 = -\frac{B_0(rB_0)'}{4\pi r}. \quad (16.38)
$$

Since we are going to have to work in cylindrical coordinates, we must first write all the terms in (16.26) in these coordinates and with the equilibrium (16.38):

$$
(\xi \cdot \nabla p_0)(\nabla \cdot \xi) = \xi_r p_0' \left( \frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)
$$

$$
= p_0' \frac{\xi_r^2}{r} + p_0' \xi_r \left( \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z} \right), \quad (16.39)
$$

$$
\gamma p_0(\nabla \cdot \xi)^2 = \gamma p_0 \left( \frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2, \quad (16.40)
$$

$$
\delta B = \nabla \times (\xi \times B_0) = \hat{r} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \xi_r B_0 \right) + \hat{\theta} \left( -\frac{\partial}{\partial z} \xi_z B_0 - \frac{\partial}{\partial r} \xi_r B_0 \right) + \hat{z} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \xi_z B_0 \right), \quad (16.41)
$$

$$
\frac{j_0 \cdot (\xi \times \delta B)}{c} = \frac{j_0}{c} \left( \xi_r \delta B_\theta - \xi_\theta \delta B_r \right) = p_0' \left[ \xi_r \left( \frac{\partial \xi_r}{\partial z} + \frac{\partial \xi_r}{\partial \theta} + \xi_r \frac{B_0'}{B_0} \right) + \xi_\theta \left( \frac{1}{r} \frac{\partial \xi_r}{\partial \theta} \right) \right],
$$

$$
= \frac{p_0' \xi_r}{B_0} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial \theta} \right) + \frac{p_0' B_0'}{B_0} \xi_r^2, \quad (16.42)
$$

$$
\frac{|\delta B|^2}{4\pi} = \frac{B_0^2}{4\pi r^2} \left[ \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \frac{B_0^2}{4\pi} \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial \theta} + \xi_r \frac{B_0'}{B_0} \right)^2 \quad (16.43)
$$

The terms that are crossed out have been dropped because they combine into a full derivative with respect to $\theta$ and so, upon substitution into (16.26), vanish under integra-
tion. Assembling all this together, we have

\[
\delta W_2 = \frac{1}{2} \int d^3 r \left\{ \left( p'_0 + \frac{p'_0 r B'_0}{B_0} + \frac{r B'^2_0}{4\pi} \right) \frac{\xi^2_r}{r} + 2 \left( p'_0 + \frac{B_0 B'_0}{4\pi} \right) \xi_r \left( \frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) \right\}
\]

\[
= 2p'_0 + \frac{B^2_0}{4\pi r} + \left( \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right) + \gamma p_0 \left( \frac{\partial \xi_r}{\partial \theta} + \frac{\partial \xi_z}{\partial \theta} \right)^2
\]

\[
= \frac{1}{2} \int d^3 r \left\{ \left( p'_0 \frac{\xi^2_r}{r} + \frac{B^2_0}{4\pi} \left( \frac{\partial \xi_z}{\partial \theta} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \right)^2 \right) + \gamma p_0 \left( \frac{\partial \xi_r}{\partial \theta} + \frac{\partial \xi_z}{\partial \theta} \right)^2 \right\},
\]

where, in simplifying the first two terms in the integrand, we used the equilibrium equation (16.38):

\[
p'_0 = -\frac{B^2_0}{4\pi r} - \frac{B_0 B'_0}{4\pi} \Rightarrow \frac{r B'^2_0}{4\pi} = -\frac{p'_0 r B'_0}{B_0} - \frac{B_0 B'_0}{4\pi} = -\frac{p'_0 r B'_0}{B_0} + p'_0 + \frac{B^2_0}{4\pi r}. \quad (16.45)
\]

Finally, after a little further tiding up,

\[
\delta W_2 = \frac{1}{2} \int d^3 r \left\{ \left( 2p'_0 \frac{\xi^2_r}{r} + \frac{B^2_0}{4\pi} \left( \frac{\partial \xi_z}{\partial \theta} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \right)^2 \right) + \gamma p_0 \left( \frac{\partial \xi_r}{\partial \theta} + \frac{\partial \xi_z}{\partial \theta} \right)^2 \right\} + \frac{B^2_0}{4\pi r^2} \left( \left( \frac{\partial \xi_r}{\partial \theta} \right)^2 + \left( \frac{\partial \xi_z}{\partial \theta} \right)^2 \right), \quad (16.46)
\]

16.4.1. Sausage Instability

Let us first consider axisymmetric perturbations: \( \partial / \partial \theta = 0 \). Then \( \delta W_2 \) depends on two variables only:

\[
\xi_r \quad \text{and} \quad \eta \equiv \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}. \quad (16.47)
\]

Indeed, unpacking all the \( r \) derivatives in (16.46), we get

\[
\delta W_2 = \frac{1}{2} \int d^3 r \left[ 2p'_0 \frac{\xi^2_r}{r} + \frac{B^2_0}{4\pi} \left( \eta - \frac{\xi_z}{r} \right)^2 + \gamma p_0 \left( \eta + \frac{\xi_r}{r} \right)^2 \right]. \quad (16.48)
\]

We shall treat \( \xi_r \) and \( \eta \) as independent variables and minimise \( \delta W_2 \) with respect to \( \eta \):

\[
\frac{\partial}{\partial \eta} \left[ \text{integrand of (16.48)} \right] = 2 \frac{B^2_0}{4\pi} \left( \eta - \frac{\xi_z}{r} \right) + 2\gamma p_0 \left( \eta + \frac{\xi_r}{r} \right) = 0 \Rightarrow \eta = \frac{1 - \gamma \beta / 2 \xi_r}{1 + \gamma \beta / 2 \xi_r}, \quad (16.49)
\]

where, as usual, \( \beta = 8\pi p_0 / B^2_0 \). Putting this back into (16.48), we get

\[
\delta W_2 = \int d^3 r p_0 \left[ \frac{r p'_0}{p_0} + \frac{1}{\beta} \left( \frac{\gamma \beta}{1 + \gamma \beta / 2} \right)^2 + \frac{1}{2} \left( \frac{2}{1 + \gamma \beta / 2} \right)^2 \right] \frac{\xi^2_r}{r^2}
\]

\[
= \int d^3 r p_0 \left( r \frac{\text{d} \ln p_0}{\text{d} r} + \frac{2\gamma}{1 + \gamma \beta / 2} \right) \frac{\xi^2_r}{r^2}. \quad (16.50)
\]
There will be an instability \( \delta W_2 < 0 \) if (but not only if, because we are considering the restricted set of axisymmetric displacements)

\[
-r \frac{d \ln p_0}{dr} > \frac{2\gamma}{1 + \gamma\beta/2},
\]

i.e., when the pressure gradient is too steep, the equilibrium is unstable.

What sort of instability is this? Recall that the perturbations that we have identified as making \( \delta W_2 < 0 \) are axisymmetric, have some radial and axial displacements and are compressible: from (16.49),

\[
\nabla \cdot \xi = \eta + \frac{\xi_r}{r} = \frac{2}{1 + \gamma\beta/2} \frac{\xi_r}{r}.
\]

They are illustrated in Fig. 70. The mechanism of this aptly named sausage instability is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further.

**Exercise 16.1.** Convince yourself that the displacements that have been identified cause magnetic perturbations that are consistent with the cartoon in Fig. 70.

### 16.4.2. Kink Instability

Now consider non-axisymmetric perturbations \( (\partial/\partial \theta \neq 0) \) to see what other instabilities might be there. First of all, since we now have \( \theta \) variation, \( \delta W_2 \) depends on \( \xi_\theta \). However, in (16.46), \( \xi_\theta \) only appears in the third term, where it is part of \( \nabla \cdot \xi \), which enters quadratically and with a positive coefficient \( \gamma p_0 \). We can treat \( \nabla \cdot \xi \) as an independent variable, alongside \( \xi_r \) and \( \xi_z \), and minimise \( \delta W_2 \) with respect to it. Obviously, the energy perturbation is minimal when

\[
\nabla \cdot \xi = 0,
\]

i.e., the most dangerous non-axisymmetric perturbations are incompressible (unlike for the case of the axisymmetric sausage mode in §16.4.1: there we could not—and did not—have such incompressible perturbations because we did not have \( \xi_\theta \) at our disposal, to be chosen in such a way as to enforce incompressibility).

To carry out further minimisation of \( \delta W_2 \), it is convenient to Fourier transform our displacements in the \( \theta \) and \( z \) directions—both are directions of symmetry (i.e., the
equilibrium profiles do not vary in these directions), so this can be done with impunity:

\[ \xi = \sum_{m,k} \xi_{mk}(r) e^{i(m\theta + kz)}. \]  

(16.54)

Then (16.46) (with \( \nabla \cdot \xi = 0 \)) becomes, by Parseval’s theorem (the operator \( F[\xi] \) being self-adjoint; see §16.1.1),

\[
\delta W_2 = \frac{1}{2} \sum_{m,k} 2\pi L_z \int_0^\infty dr \, \left\{ \frac{2p_0}{r} \frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi} \left[ \frac{r}{r} \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right]^2 + \frac{m^2}{r^2} \left( |\xi_r|^2 + |\xi_z|^2 \right) \right\}. 
\]

(16.55)

As \( \xi_z \) and \( \xi_z^* \) only appear algebraically in (16.55) (no \( r \) derivatives), it is easy to minimise \( \delta W_2 \) with respect to them: setting the derivative of the integrand with respect to either \( \xi_z \) or \( \xi_z^* \) to zero, we get

\[- ik \left( r \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right) + \frac{m^2}{r^2} \xi_z = 0 \implies \xi_z = \frac{ikr^3}{m^2 + k^2r^2} \frac{\partial}{\partial r} \frac{\xi_r}{r}. \]

(16.56)

Putting this back into (16.55) and assembling terms, we get

\[
\delta W_2 = \sum_{m,k} \pi L_z \int_0^\infty dr \, \left\{ \frac{2p_0}{r} \left( \frac{r}{p_0} + \frac{B_0^2}{8\pi} \right) \frac{|\xi_r|^2}{r^2} \right. \\
+ \left. \frac{B_0^2}{4\pi} \left[ \left( \frac{1 - k^2r^2}{m^2 + k^2r^2} \right)^2 + \frac{m^2k^2r^2}{(m^2 + k^2r^2)^2} \right] \left| r \frac{\partial}{\partial r} \frac{\xi_r}{r} \right|^2 \right\}. 
\]

(16.57)

The second term here is always stabilising. The most unstable modes will be ones with \( k \to \infty \), for which the stabilising term is as small as possible. The remaining term will allow \( \delta W_2 < 0 \) and, therefore, an instability, if

\[- r \frac{\partial}{\partial r} \frac{\ln p_0}{d r} > \frac{m^2}{\beta} \].

(16.58)

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest \( m \), viz., \( m = 1 \).

Note that another way of writing the instability condition (16.58) is

\[- rp_0' = \frac{B_0^2}{4\pi} + \frac{r B_0 B_0'}{4\pi} > m^2 \frac{B_0^2}{8\pi} \implies \left. \frac{\partial}{\partial r} \frac{\ln B_0}{d r} > \frac{m^2}{2} - 1 \right\].

(16.59)

What does this instability look like? The unstable perturbations are incompressible:

\[ \nabla \cdot \xi = 0 \implies \frac{1}{r} \frac{\partial}{\partial r} \frac{\xi_r}{r} + \frac{im}{r} \frac{\xi_r}{r} + ik \xi_z = 0. \]

(16.60)
Setting $m = 1$ and using (16.56), we find
\[i\xi_\theta = -\frac{\partial}{\partial r} r\xi_r + \frac{k^2 r^4}{m^2 + k^2 r^2} \frac{\partial}{\partial r} \frac{\xi_r}{r} \approx -2\xi_r \quad \text{and} \quad \xi_z \ll \xi_r.\] (16.61)

The basic cartoon (Fig. 71) is as follows: the flux surfaces are bent, with a twist (to remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher $B$ to those with lower $B$, thermal pressure in the latter (convex) areas becomes uncompensated, the field lines open up further, etc. This is called the kink instability.

Similar methodology can be used to show that, unlike the $z$ pinch, the $\theta$ pinch (§15.1.1, Fig. 64) is always stable: see Q11.

17. Further Reading

What follows is not a literature survey, but rather just a few pointers for the keen and the curious.

17.1. MHD Instabilities

There are very many of these, easily a whole course’s worth. They are an interesting topic. A founding text is the old, classic, super-meticulous monograph by Chandrasekhar (2003). In the context of toroidal (fusion) plasmas, you want to learn the so-called ballooning theory, a tour de force of theoretical plasma physics, which, like the relaxation theory, is associated with J. B. Taylor’s name (so his lectures, Taylor & Newton 2015, are a good starting point; the original paper on the subject is Connor et al. 1979). In the unlikely event that you have an appetite for more energy-principle calculations in the style of §16.4, the book by Freidberg (2014) will teach you more than you ever wanted to know. In astrophysics, MHD instabilities have been a hot topic since the early 1990s, not least due the realisation by Balbus & Hawley (1991) that the magnetorotational instability (MRI) is responsible for triggering turbulence and, therefore, maintaining momentum transport in accretion flows—so the lecture notes by Balbus (2015) are an excellent place to start learning about this subject (this is also an opportunity to learn
how to handle equilibria that are not static, e.g., most interestingly, featuring rotating and shear flows.\footnote{Another excellent set of lecture notes on astrophysical fluid dynamics is Ogilvie (2016), this one originating from Cambridge Part III.}

As with everything in physics, the frontier in this subject is nonlinear phenomena. One very attractive theoretical topic has been the theory of explosive instabilities and erupting flux tubes by S. C. Cowley and his co-workers: the founding (quite pedagogically written) paper was Cowley & Artun (1997), the key recent one is Cowley \textit{et al.} (2015); follow the paper trail from there for various refinements and applications (from space to tokamaks).

### 17.2. Resistive MHD

Most of our discussion revolved around properties of ideal MHD equations. It is, in fact, quite essential to study resistive effects, even when resistivity is very small, because many ideal solutions have a natural tendency to develop ever smaller spatial gradients, which can only be regularised by resistivity (we touched on this, e.g., in §15.2.2). The key linear result here is the \textit{tearing mode}, a resistive instability associated with the propensity of magnetic-field lines to reconnect—change their topology in such a way as to release some of their energy. This is covered in the lectures by Parra (2019a); other good places to read about it are Taylor & Newton (2015) again, the original paper by Furth \textit{et al.} (1963), or standard textbooks (e.g., Sturrock 1994, §17).

Here again the frontier is nonlinear: the theory of \textit{magnetic reconnection}: tearing modes, in their nonlinear stage, tend to lead to formation of \textit{current sheets} (which is, in fact, a general tendency of X-point solutions in MHD), and how reconnection happens after that has been a subject of active research since mid-20th century. Magnetic reconnection is believed to be a key player in a host of plasma phenomena, from solar flares to the so-called “sawtooth crash” in tokamaks, to MHD turbulence. Kulsrud (2005, §14) has a good introduction to the history and the basics of the subject from a live witness and key contributor. There has been much going on in it in the last decade, many of the advances occurring on the collisionless reconnection front requiring kinetic theory (some key names to search for in the extensive recent literature are W. Daughton, J. Drake, J. Egedal), but even within MHD, the discovery of the \textit{plasmoid instability} (amounting to the realisation that current sheets are tearing unstable; see Loureiro \textit{et al.} 2007) has led to a new theory of resistive MHD reconnection (Uzdensky \textit{et al.} 2010), a development that I (obviously) find important.

Even more recently, magnetic reconnection became intimately intertwined with the theory of MHD turbulence (§14.4)—you will find an account of this in my (hopefully pedagogical) review, a draft of which is here: \url{http://www-thphys.physics.ox.ac.uk/research/plasma/JPP/papers17/schekochihin2a.pdf}. Appendix C of this document also contains a “reconnection primer” covering tearing modes, current sheets and related topics in the most straightforward non-rigorous way that I could manage.

### 17.3. Dynamo Theory and MHD Turbulence

These are topics of active research, which one can have full access to with the education provided by these notes, and indeed it is to an extent with these topics in mind (or, at any rate, in my mind) that some of these notes were written. In §§13.9, 13.14 and 14.4, further pointers are provided.
17.4. Hall MHD, Electron MHD, Braginskii MHD

These and other “two-fluid” approximations of plasma dynamics have to do with
with (i) what happens at scales where different species (ions and electrons) cannot be
considered to move together (Hall/Electron MHD; see Q7) and (ii) how momentum
transport (viscosity) and energy transport (heat conduction) operate in a magnetised
plasma, i.e., a plasma where the Larmor motion of particles dominates over their Coulomb
collisions, even though the latter might be faster than the fluid motions (Braginskii
1965 MHD). In general, this is a kinetic subject, although certain limits can be treated
by fluid approximations. An introduction to these topics is given in Parra (2019b) and
Parra (2019a) (see also Goedbloed & Poedts 2004, §3 and the excellent monograph by
Helander & Sigmar 2005).

17.5. Double-Adiabatic MHD and Onwards to Kinetics

A conceptually interesting and important paradigm is the so-called double-adiabatic
MHD (or CGL equations, after the orginal authors Chew et al. 1956; see also Kulsrud
1983). This deals with a situation in a magnetised plasma (in the sense defined in §17.4)
when pressure becomes anisotropic, with pressures perpendicular and parallel to the local
direction of the magnetic field evolving each according to its own, separate equation,
replacing the adiabatic law (13.60) and based on the conservation of the adiabatic
invariants of the Larmor-gyrating particles. The dynamics of pressure-anisotropic
plasma, based on CGL equations or, which is usually more correct physically, on the
full kinetic description (and its reduced versions, e.g., Kinetic MHD; see Parra 2019b,
also Kulsrud 1983), are another current frontier, with applications to weakly collisional
astrophysical plasmas (from interplanetary to intergalactic). A key feature that makes
this topic both interesting and difficult is that pressure anisotropies in high-\(\beta\) plasmas
trigger small-scale instabilities (in particular, the Alfvén wave becomes unstable—the so-
called firehose instability), which break the fluid approximation and leave us without a
good mean-field theory for the description of macroscopic motions in such environments
(for a short introduction to these issues, see Schekochihin et al. 2010, although this subject
is developing so fast that anything written 10 years ago is at least partially obsolete; you
can read Squire et al. 2017 for a taste of how hairy things become in what concerns even
such staples as Alfvén waves).
1. Clebsch Coordinates. As $\nabla \cdot \mathbf{B} = 0$, it is always possible to find two scalar functions $\alpha(r)$ and $\beta(r)$ such that
\[
\mathbf{B} = \nabla \alpha \times \nabla \beta.
\] (17.1)

(a) Argue that any magnetic field line can be described by the equations
\[
\alpha = \text{const}, \quad \beta = \text{const}.
\] (17.2)
This means that $(\alpha, \beta, \ell)$, where $\ell$ is the distance (arc length) along the field line, are a good set of curvilinear coordinates, known as the Clebsch coordinates.

(b) Show that the magnetic flux through any area $S$ in the $(x, y)$ plane is
\[
\Phi = \int_{\tilde{S}} d\alpha \, d\beta,
\] (17.3)
where $\tilde{S}$ is the area $S$ in new coordinates after transforming $(x, y) \rightarrow (\alpha(x, y, 0), \beta(x, y, 0))$.

(c) Show that if (17.1) holds at time $t = 0$ and $\alpha$ and $\beta$ are evolved in time according to
\[
\frac{d\alpha}{dt} = 0, \quad \frac{d\beta}{dt} = 0,
\] (17.4)
where $d/dt$ is the convective derivative, then (17.1) correctly describes the magnetic field at all $t > 0$.

(d) Argue from the above that magnetic flux is frozen into the flow and magnetic field lines move with the flow.

(e*) Show that the field that minimises the magnetic energy within some domain subject to the constraint that the values of $\alpha$ and $\beta$ are fixed at the boundary of this domain (i.e., that the “footpoints” of the field lines are fixed) is a force-free field.\[113\]

A prototypical example of the kind of fields that arise from the variational principle in (e) is the “arcade” fields describing magnetic loops sticking out of the Sun’s surface, with footpoints anchored at the surface. One such field will be considered in Q9(f) and more can be found in Sturrock (1994, §13).

2. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with number density $n_{\text{gal}} = 1 \text{ cm}^{-3}$ undergoing a gravitational collapse to a spherical star with number density $n_{\text{star}} = 10^{26} \text{ cm}^{-3}$. The magnetic field in the galactic plasma is $B_{\text{gal}} \sim 3 \times 10^{-6} \text{ G}$. Assuming that flux is frozen, estimate the magnetic field in a star. Find out if this is a good estimate. If not, how, in your view, could we account for the discrepancy?

3. Flux Concentration. Consider a simple 2D model of incompressible convective motion (Fig. 72):
\[
\mathbf{u} = U \left( -\sin \frac{\pi x}{L} \cos \frac{\pi z}{L}, \ 0, \ \cos \frac{\pi x}{L} \sin \frac{\pi z}{L} \right).
\] (17.5)

(a) In the neighbourhood of the stagnation point $(0, 0, 0)$, linearise the flow, assume vertical magnetic field, $\mathbf{B} = (0, 0, B(t, x))$ and derive an evolution equation for $B(t, x)$,
including both advection by the flow and Ohmic diffusion. Suppose the field is initially uniform, \( B(t = 0, x) = B_0 = \text{const} \). It should be clear to you from your equation that magnetic field is being swept towards \( x = 0 \). What is the time scale of this sweeping? Given the magnetic Reynolds number \( R_m = \frac{UL}{\eta} \gg 1 \), show that flux conservation holds on this time scale.

(b) Find the steady-state solution of your equation. Assume \( B(x) = B(-x) \) and use flux conservation to determine the constants of integration (in terms of \( B_0 \) and \( R_m \)). What is the width of the region around \( x = 0 \) where the flux is concentrated? What is the magnitude of the field there?

(c*) Obtain the time-dependent solution of your equation for \( B \) and confirm that it indeed converges to your steady-state solution. Find the time scale on which this happens.

\[ \xi = \sqrt{\pi R_m x/L}, \quad \tau = \pi Ut/L, \quad X = \xi e^\tau, \quad s = (e^{2\tau} - 1)/2. \]

(d) Can you think of a quick heuristic argument based on the induction equation that would tell you that all these answers were to be expected?

4. Zeldovich's Antidynamo Theorem. Consider an arbitrary 2D velocity field: \( u = (u_x, u_y, 0) \). Assume incompressibility. Show that, in a finite system (i.e., in a system that can be enclosed within some volume outside which there are no fields or flows), this velocity field cannot be a dynamo, i.e., any initial magnetic field will always eventually decay.

\[ \text{Hint. Consider separately the evolution equations for } B_z \text{ and for the magnetic field in the } (x, y)-\text{plane. Show that } B_z \text{ decays by working out the time evolution of the volume integral of } B_z^2. \text{ Then write } B_x, B_y \text{ in terms of one scalar function (which must be possible because } \partial B_x/\partial x + \partial B_y/\partial y = 0) \text{ and show that it decays as well.} \]

5*: X-Point Collapse. Consider the following initial magnetic-field configuration:

\[ \mathbf{B}_0(r_0) = B_0 \hat{z} + \hat{z} \times \nabla_0 \Psi(x_0, y_0), \quad \text{(17.6)} \]

where \( r_0 = (x_0, y_0, z_0), \ B_0 = \text{const, and} \)

\[ \Psi(x_0, y_0) = \frac{x_0^2 - y_0^2}{2}. \quad \text{(17.7)} \]

This is called an \textit{X-point} (Fig. 73a).

(a) Use the Lagrangin MHD equation (13.85), where \( \mathbf{r} = (x, y, z) \), and seek a solution
in the form
\[ x = \xi(t)x_0, \quad y = \eta(t)y_0, \quad z = z_0. \] (17.8)
Show that \( \xi \) and \( \eta \) satisfy the following equations
\[ \ddot{\xi} = \eta \left( \frac{1}{\eta^2} - \frac{1}{\xi^2} \right), \quad \ddot{\eta} = \xi \left( \frac{1}{\xi^2} - \frac{1}{\eta^2} \right). \] (17.9)

(b) Consider the possibility that, as time goes on, \( \eta(t) \) becomes ever smaller, \( \eta \to 0 \), while \( \xi(t) \) tends to a constant, \( \xi \to \xi_c \). Show that the solution that has this property is
\[ \xi(t) \approx \xi_c + \frac{9}{4} \left( \frac{2}{9\xi_c} \right)^{1/3} (t_c - t)^{4/3}, \quad \eta(t) \approx \left( \frac{9\xi_c}{2} \right)^{1/3} (t_c - t)^{2/3} \] (17.10)
as \( t \to t_c \), where \( t_c \) is some finite time. This is called the *Syrovatskiı́ (1971) solution*.

(c) Calculate the magnetic field as a function of time and convince yourself that the Syrovatskii solution describes the initial \( X \)-point configuration collapsing explosively to a sheet along the \( x \) axis. What happens after \( t \) reaches \( t_c \)?

(d) Do a similar calculation, but for incompressible Lagrangian MHD, i.e., assuming \( J = 1 = \text{const} \) (which is now the equation that determines the total pressure; cf. §14.2.5). Show that the solution in this case is
\[ \xi(t) = A(t), \quad \eta(t) = \frac{1}{A(t)}, \] (17.11)
where \( A(t) \) is an arbitrary function of time. Take \( A(t) = e^{\lambda t} \) and show that this solution corresponds to an exponentially collapsing \( X \)-point. This is called the *Chapman & Kendall (1963) solution*. Can this evolution continue forever?

(e) Show that the fluid flow associated with a collapsing solution consists of an inflow (into the “sheet”) and an outflow (from the “sheet”). Going back to the general solution (17.11), assume that the outflow velocity \( u_x \) at a given fixed Lagrangian position \( x_{out} \) is equal to some known constant \( u_{out} \) (i.e., as the “sheet” collapses and gets longer, the outflow from its ends is always the same). Find \( \xi(t) \) and \( \eta(t) \) in this case. This solution is due to *Uzdensky & Loureiro (2016)* (read their paper to find out what the use of it is).

6. MHD Waves in a Stratified Atmosphere. The generalisation of iMHD to the case of a stratified atmosphere is explained in §14.2.8. Convince yourself that you understand how the SMHD equations and the SMHD ordering arise and then study them as follows.

(a) Work out all SMHD waves (both their frequencies and the corresponding eigenvec-
It is convenient to choose the coordinate system in such a way that \( k = (k_x, 0, k_z) \), where \( z \) is the vertical direction (the direction of gravity). The mean magnetic field \( B_0 = B_0 b_0 \) is assumed to be straight and uniform, at a general angle to \( z \). We continue referring to the projection of the wave number onto the magnetic-field direction as \( k_\parallel = k \cdot b_0 = k_x b_0 x + k_z b_0 z \). Note that in the case of \( B_0 = 0 \), you are dealing with stratified hydrodynamics, not MHD—the waves that you obtain in this case are the well known gravity waves, or “\( g \)-modes”.

(b) Explain the physical nature of the perturbations (what makes the fluid oscillate) in the special cases (i) \( k_z = 0 \) and \( b_0 = \hat{z} \), (ii) \( k_z = 0 \) and \( b_0 = \hat{x} \), (iii) \( k_x = 0 \), (iv) \( k_z \neq 0, k_x \neq 0 \) and \( b_0 = \hat{z} \).

(c) Under what conditions are the perturbations you have found unstable? What is the physical mechanism for the instability? What role does the magnetic field play (stabilising or destabilising) and why? Cross-check your answers with §16.3 and Q10.

(d) Find the conserved energy (a quadratic quantity whose integral over space stays constant) for the full nonlinear SMHD equations (14.97–14.100). Give a physical interpretation of the quantity that you have obtained—why should it be conserved?

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7. Electron MHD. In certain physical regimes (roughly realised, for example, in the solar-wind and other kinds of astrophysical turbulence at scales smaller than the ion Larmor radius; see Schekochihin et al. 2009 or Boldyrev et al. 2013), plasma turbulence can be described by an approximation in which the magnetic field is frozen into the electron flow \( u_e \), while ions are considered motionless, \( u_i = 0 \). In this approximation, Ohm’s law becomes\(^{114}\)

\[
E = -\frac{u_e \times B}{c}.
\]  

(17.12)

Here \( u_e \) can be expressed directly in terms of \( B \) because the current density in a plasma consisting of motionless hydrogen ions \( (n_i = n_e) \) and moving electrons is

\[
j = en_e (u_i - u_e) = -en_e u_e,
\]  

(17.13)

but, on the other hand, \( j \) is known via Ampère’s law. Here \( n_e \) is the electron number density and \( e \) the electron charge.

(a) Using this and Faraday’s law, show that the evolution equation for the magnetic field in this approximation is

\[
\frac{\partial B}{\partial t} = -d_i \nabla \times [(\nabla \times B) \times B],
\]  

(17.14)

where the magnetic field has been rescaled to Alfvénic velocity units, \( B/\sqrt{4\pi m_i n_i} \rightarrow B \), and \( d_i = c/\omega_{pi} \) is the ion inertial scale (“ion skin depth”), \( \omega_{pi} = \sqrt{4\pi e^2 n_i/m_i} \). Equation (17.14) is the equation of Electron MHD (EMHD), completely self-consistent for \( B \).

(b) Show that magnetic energy is conserved by (17.14). Is magnetic helicity conserved?

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\(^{114}\)Strictly speaking, the generalised Ohm’s law in this approximation also contains an electron-pressure gradient (see, e.g., Goedbloed & Poedts 2004), but that vanishes upon substitution of \( E \) into Faraday’s law.
Does J. B. Taylor relaxation work and what kind of field will be featured in the relaxed state? Is it obvious that this field is a good steady-state solution of (17.14)?

(c) Consider infinitesimal perturbations of a straight-field equilibrium, \( B = B_0 \hat{z} + \delta B \), and show that they are helical waves with the dispersion relation

\[
\omega = \pm k_{\parallel} v_A k_d \, .
\]  

(17.15)

These are called \textit{Kinetic Alfvén Waves (KAW)}.

(d) Now consider finite perturbations and argue that the appropriate ordering in which linear and nonlinear physics can coexist while perturbations remain small is

\[
|\delta b| \sim \frac{\delta B}{B} \sim \frac{k_{\parallel}}{k} \ll 1 .
\]  

(17.16)

Under this ordering, show that the magnetic field can be represented as

\[
\frac{\delta B}{B_0} = \frac{1}{v_A} \hat{z} \times \nabla \perp \Psi + \hat{z} \frac{\delta B}{B}
\]  

(17.17)

and the evolution equations for \( \Psi \) and \( \delta B/B_0 \) are

\[
\frac{\partial \Psi}{\partial t} = v_A^2 d_i b \cdot \nabla \frac{\delta B}{B_0} , \quad \frac{\partial}{\partial t} \frac{\delta B}{B_0} = -d_i b \cdot \nabla \nabla^2 \perp \Psi ,
\]  

(17.18)

where \( b \cdot \nabla \) is given by (14.115). These are the equations of \textit{Reduced Electron MHD}.

(e) Check that the conservation of magnetic energy and the KAW dispersion relation (17.15) are recovered from (17.18). Is there any other conservation law?

8. Hydrodynamics of Rotating Fluid.\textsuperscript{115} Most of this question is not on MHD, but deals with equations describing a somewhat analogous system: also embedded into an external field and supporting anisotropic wave-like perturbations. It is an incompressible fluid rotating at angular velocity \( \Omega = \Omega \hat{z} \), where \( \hat{z} \) is the unit vector in the direction of the z axis. The velocity field \( u \) in such a fluid satisfies the following equation

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + 2u \times \Omega ,
\]  

(17.19)

where pressure \( p \) is found from the incompressibility condition \( \nabla \cdot u = 0 \), the last term on the right-hand side is the Coriolis force, the centrifugal force has been absorbed into \( p \), and viscosity has been ignored.

(a) Consider infinitesimal perturbations of a static (\( u_0 = 0 \)), homogeneous equilibrium of (17.19). Show that the system supports waves with the dispersion relation

\[
\omega = \pm 2\Omega \frac{k_{\parallel}}{k} \, .
\]  

(17.20)

These are called \textit{inertial waves}. Here \( k = (k_\perp, 0, k_{\parallel}) \) (without loss of generality); the subscripts refer to directions perpendicular and parallel to the axis of rotation.

(b) In the case \( k_{\parallel} \ll k_\perp \), determine the direction of propagation of the inertial waves. Determine also the relationship between the components of the velocity vector \( u \) associated with the wave. Comment on the polarisation of the wave.

\textsuperscript{115}This is based on the 2018 exam question.
(c) When rotation is strong, i.e., when $\Omega \gg ku$, perturbations in a rotating system are anisotropic with $\epsilon = k_{\parallel}/k_{\perp} \ll 1$. Order the linear and nonlinear time scales to be similar to each other and work out the ordering of all relevant quantities, namely, $u_{\perp}$ (horizontal velocity), $u_{\parallel}$ (vertical velocity), $\delta p$ (perturbed pressure), $\omega$, $\Omega$, $k_{\parallel}$, $k_{\perp}$ with respect to each other and to $\epsilon$. Using this ordering, show that the motions of a rotating fluid satisfy the following reduced equations

$$\frac{\partial}{\partial t} \nabla^2_{\perp} \Phi + \{\Phi, \nabla^2_{\perp} \Phi\} = 2\Omega \frac{\partial u_{\parallel}}{\partial z}, \quad \frac{\partial u_{\parallel}}{\partial t} + \{\Phi, u_{\parallel}\} = -2\Omega \frac{\partial \Phi}{\partial z},$$

where the “Poisson bracket” is defined by (14.114) and $\Phi$ is the stream function of the perpendicular velocity, i.e., to the lowest order in $\epsilon$, $u_{\perp}^{(0)} = \hat{z} \times \nabla_{\perp} \Phi$. Note that, in order to obtain the above equations, you will need to work out $\nabla_{\perp} \cdot u_{\perp}^{(0)}$ to both the lowest and next order in $\epsilon$, i.e., both $\nabla_{\perp} \cdot u_{\perp}^{(0)}$ and $\nabla_{\perp} \cdot u_{\perp}^{(1)}$.

(d) Show that any purely horizontal flows in a strongly rotating fluid must be exactly two-dimensional (i.e., constant along the axis of rotation).

(e) For a strongly rotating, incompressible, highly electrically conducting fluid embedded in a strong uniform magnetic field $B_0$ parallel to the axis of rotation, discuss qualitatively under what conditions you would expect anisotropic ($k_{\parallel} \ll k_{\perp}$) Alfvénic and slow-wave-like (pseudo-Alfvénic) perturbations to be decoupled from each other? There are certain interesting similarities between MHD turbulence and turbulence in rotating fluid systems described by (17.21) and, indeed, also turbulence in stratified environments that we dealt with in §14.2.8 and Q6. If you would like to know more, see Nazarenko & Schekochihin (2011) and follow the paper trail from there.

9. Grad–Shafranov Equation. Consider static MHD equilibria (15.1) in cylindrical coordinates $(r, \theta, z)$ and assume axisymmetry, $\partial/\partial \theta = 0$.

(a) Using the solenoidality of the magnetic field, show that any axisymmetric such field can be expressed in the form

$$B = I \nabla \psi \times \nabla \theta,$$

where $I$ and $\psi$ are functions of $r$ and $z$ and $\nabla \theta = \hat{\theta}/r$ ($\hat{\theta}$ is the unit basis vector in the $\theta$ direction). Show that magnetic surfaces are surfaces of $\psi = \text{const}$.

(b) Using the force balance, show that $\nabla I \times \nabla \psi = 0$ and $\nabla p \times \nabla \psi = 0$ and hence argue that

$$I = I(\psi) \quad \text{and} \quad p = p(\psi)$$

are functions of $\psi$ only (i.e., they are constant on magnetic surfaces).

(c) Again from the force balance, show that $\psi(r, z)$ satisfies the Grad–Shafranov equation

$$-\left(\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}\right) = 4\pi r^2 \frac{dp}{d\psi} + I \frac{dI}{d\psi}.$$

This defines the shape of an axisymmetric equilibrium, given the profiles $p(\psi)$ and $I(\psi)$.

(d) Show that in cylindrical symmetry ($\partial/\partial \theta = 0$, $\partial/\partial z = 0$), (17.24) reduces to (15.8).

(e*) Assume $I(\psi) = \text{const}$ (so the azimuthal field $B_\theta = I/r$ is similar to the magnetic field from a central current) and $p(\psi) = a\psi$, where $a$ is some constant. Find a solution
of (17.24) that gives rise to magnetic surfaces that resemble nested tori, but with “D-shaped” cross section (Fig. 74; this looks a bit like the modern tokamaks). If you stipulate that $p$ must vanish at $r = 0$ and at $r = R$ along the $z = 0$ axis and also at $z = \pm L$ along the $r = 0$ axis and that the maximum pressure at $r < R$ is $p_0$, show that the corresponding magnetic surfaces are described by

$$
\psi = 2\sqrt{\frac{2\pi p_0}{1 + R^2/4L^2}} r^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{L^2}\right)^{1/2}.
$$

(17.25)

Where is the (azimuthal) magnetic axis of these surfaces? What is the value of $a$?

(f) Seek solutions to (17.24) that are linear force-free fields. Show that in this case, (17.24) reduces to the Bessel equation (a substitution $\psi = rf(r, z)$ will prove useful). Set $B_z(0,0) = B_0$. Find solutions of two kinds: (i) ones in a semi-infinite domain $z \geq 0$, with the field vanishing exponentially at $z \to \infty$; (ii) ones periodic in $z$. If you also impose the boundary condition $B_r = 0$ at $r = R$, how can this be achieved? Can either of these solutions be the result of J. B. Taylor relaxation of an MHD system? If so, how would one decide whether it is more or less likely to be the correct relaxed state than the solution derived in §15.4?

You will find the solution of the type (i) in Sturrock (1994, §13) (who also shows how to construct many other force-free fields, useful in various physical and astrophysical contexts). Think of this solution in the context of Q1(e). The solution of type (ii) is a particular case of the general ($\partial/\partial \theta \neq 0$) equilibrium solution derived and discussed in Taylor & Newton (2015, §9). However, the axisymmetric solution is not very useful because, as they show, depending on the values of helicity and of $R$, the true relaxed state is either the cylindrically and axially symmetric solution derived in §15.4 or one which also has variation in the $\theta$ direction.

10. Magnetised Interchange Instability. Consider the same set up as in §16.3, but now the stratified atmosphere is threaded by straight horizontal magnetic field (Fig. 75):

$$
\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad B_0 = B_0(z) \hat{x}, \quad \frac{d}{dz} \left( p_0 + \frac{B_0^2}{8\pi} \right) = -\rho_0 g.
$$

(17.26)

We shall be concerned with the stability of this equilibrium.

(a) For simplicity, assume $\partial \xi/\partial x = 0$. This rules out any perturbations of the magnetic-field direction, $\delta b = 0$, so there will be no field-line bending, no restoring curvature
forces. For this restricted set of perturbations, work out $\delta W_2$ and observe that, like in the unmagnetised case considered in §16.3, it depends only on $\nabla \cdot \xi$ and $\xi_z$. Minimise $\delta W_2$ with respect to $\nabla \cdot \xi$ and show that

$$\frac{d}{dz} \ln \frac{p_0}{\rho_0} + \frac{2}{\beta} \frac{d}{dz} \ln \frac{B_0}{\rho_0} < 0$$  \hspace{1cm} (17.27)$$

is a sufficient condition for instability (the *magnetised interchange instability*). Would you be justified in expecting stability if the condition (17.27) were not satisfied?

(b) Explain how this instability operates and rederive the condition for instability by considering interchanging blobs (or, rather, flux tubes), in the spirit of §16.3.3. If field-line bending is allowed ($\partial \xi / \partial x \neq 0$), another instability emerges, the *Parker (1966) instability*. Do investigate.

11. **Stability of the $\theta$ Pinch.** Consider the following cylindrically and axially symmetric equilibrium:

$$B_0 = B_0(r) \hat{z}, \quad j_0 = j_0(r) \hat{\theta} = -\frac{c}{4\pi} B'_0(r) \hat{\theta}, \quad \frac{d}{dr} \left( p_0 + \frac{B_0^2}{8\pi} \right) = 0$$  \hspace{1cm} (17.28)$$

(a $\theta$ pinch; see §15.1.1, Fig. 64). Consider general displacements of the form

$$\xi = \xi_{mk}(r)e^{im\theta+ikz}. \hspace{1cm} (17.29)$$

Show that the $\theta$ pinch is always stable. Specifically, you should be able to show that

$$\delta W_2 = \pi L_z \int_0^\infty dr \left\{ \gamma p_0 |\nabla \cdot \xi|^2 + \frac{B_0^2}{4\pi} \left[ k^2 (|\xi_r|^2 + |\xi_\theta|^2) + \left| \frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r} \right|^2 \right] \right\} > 0,$$

where $L_z$ is the length of the cylinder.
PART III
Kinetic MHD
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