Lectures on Kinetic Theory and Magnetohydrodynamics of Plasmas

(Oxford MMathPhys/MSc in Mathematical and Theoretical Physics)

Alexander A. Schekochihin†

The Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3NP, UK
Merton College, Oxford OX1 4JD, UK

(compiled on 6 June 2025)

These are the notes for my lectures on plasma physics, taught since 2014 as part of the MMathPhys/MScMTP programme at Oxford. Part I contains the lectures on plasma kinetics that formed part of a core course on "Kinetic Theory." The plasma lectures were intended as part of an overall introduction to the concepts and methods of the subject, alongside neutral-gas kinetics (taught by Paul Dellar) and kinetics of gravitating particles (taught by James Binney, succeeded by Jean-Baptiste Fouvry and then Chris Hamilton), for each of which its own lecture notes are available. The more advanced sections assembled in Part II cover the material first taught in the "Advanced Topics in Plasma Physics" course during the dreadful Trinity Term of 2020, under the Covid-19 lockdown. Extracts from these notes were also used in (and in part written for) my lectures at successive plasma-physics sessions of École de Physique des Houches in 2017, 2019, and 2023. Part III is an introduction to magnetohydrodynamics, which was part of the course on "Advanced Fluid Dynamics," taught by me in 2015-21 (with Paul Dellar covering the other part of the course, dedicated to complex fluids). These notes evolved from two earlier courses: "Advanced Plasma Theory," taught as a graduate course at Imperial College in 2008, and "Magnetohydrodynamics and Turbulence," taught three times as a Mathematics Part III course at Cambridge in 2005-06. Finally, Part IV is dedicated to the marriage of kinetics and MHD. It originates from the Les Houches lectures of 2013 and 2015 (as well as the unfinished draft of a review of KMHD that Matt Kunz and I once planned to write), and has been taught as part of the MMathPhys course on "Collisionless Plasma Physics" since Plamen Ivanov and I took it over from Felix Parra in 2022. The lectures have accreted a lot of material over the years, not all of which it would obviously be a good idea to teach, or indeed learn, at least at one's first encounter with the subject. I have made a concerted effort to typeset in small font the parts that can be skipped at first reading, although what may feel inessential in the initial exposition will sometimes acquire greater importance, technical and/or conceptual, later on. I will be grateful for any feedback from students, tutors, and sympathisers.

CONTENTS

† E-mail: alex.schekochihin@merton.ox.ac.uk

PART I

Kinetic Theory of Plasmas

1. Kinetic Description of a Plasma

We shall study a gas consisting of charged particles—ions and electrons. In general, there may be many different species of ions, with different masses and charges, and, of course, only one type of electrons.

I will index particle species by α ($\alpha = e$ for electrons, $\alpha = i$ for ions). Each is characterised by its mass m_α and charge $q_\alpha = Z_\alpha e$, where e is the magnitude of the electron charge and Z_α is a positive or negative integer (e.g., $Z_e = -1$).

1.1. *Quasineutrality*

We shall always assume that plasma is neutral overall:

$$\sum_\alpha q_\alpha N_\alpha = eV \sum_\alpha Z_\alpha \bar{n}_\alpha = 0, \tag{1.1}$$

where N_α is the number of the particles of species α, $\bar{n}_\alpha = N_\alpha/V$ is their mean number density and V the volume of the plasma. This condition is known as *quasineutrality*.

1.2. *Weak Interactions*

Interaction between charged particles is governed by the Coulomb potential:

$$U\left(\boldsymbol{r}_i^{(\alpha)}, \boldsymbol{r}_j^{(\alpha')}\right) = -\frac{q_\alpha q_{\alpha'}}{\left|\boldsymbol{r}_i^{(\alpha)} - \boldsymbol{r}_j^{(\alpha')}\right|}, \tag{1.2}$$

where by $\boldsymbol{r}_i^{(\alpha)}$ I mean the position of the i-th particle of species α. It is a safe bet that we will only be able to have a nice closed kinetic description if the gas is approximately ideal, i.e., if particles interact weakly, viz.,

$$k_\mathrm{B} T \gg U \sim \frac{e^2}{\varDelta r} \sim e^2 n^{1/3}, \tag{1.3}$$

where k_B is the Boltzmann constant, which will henceforth be absobed into the temperature T, and $\varDelta r \sim n^{-1/3}$ is the typical interparticle distance. Let us see what this condition means and implies physically.

1.3. *Debye Shielding*

Let us consider a plasma in thermodynamic equilibrium (as one does in statistical mechanics, I will refuse to discuss, for the time being, how exactly it got there). Take one particular particle, of species α. It creates an electric field around itself, $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$; all other particles sit in this field (Fig. 1)—and, indeed, also affect it, as we will see below. In equilibrium, the densities of these particles ought to satisfy Boltzmann's formula:

$$n_{\alpha'}(\boldsymbol{r}) = \bar{n}_{\alpha'}\, e^{-q_{\alpha'}\varphi(\boldsymbol{r})/T} \approx \bar{n}_{\alpha'} \left(1 - \frac{q_{\alpha'}\varphi}{T}\right), \tag{1.4}$$

where $\bar{n}_{\alpha'}$ is the mean density of the particles of species α' and $\varphi(\boldsymbol{r})$ is the electrostatic potential, which depends on the distance $\boldsymbol{r}$ from our "central" particle. As $r \to \infty$, $\varphi \to 0$ and $n_{\alpha'} \to \bar{n}_{\alpha'}$. The exponential can be Taylor-expanded provided the weak-interaction condition (1.3) is satisfied ($e\varphi \ll T$).

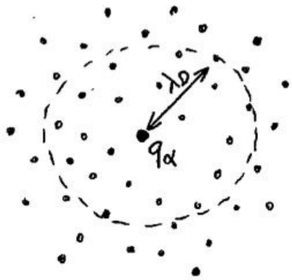


FIGURE 1. A particle amongst particles and its Debye sphere.

By the Gauss–Poisson law, we have

$$\begin{aligned}\boldsymbol{\nabla}\cdot\boldsymbol{E} = -\nabla^2\varphi &= 4\pi q_\alpha\delta(\boldsymbol{r}) + 4\pi\sum_{\alpha'} q_{\alpha'}n_{\alpha'}\\ &\approx 4\pi q_\alpha\delta(\boldsymbol{r}) + \underbrace{4\pi\sum_{\alpha'} q_{\alpha'}\bar{n}_{\alpha'}}_{\substack{=0\text{ by}\\ \text{quasineutral-}\\ \text{ity}}} - \underbrace{\left(\sum_{\alpha'}\frac{4\pi\bar{n}_{\alpha'}q_{\alpha'}^2}{T}\right)}_{\equiv 1/\lambda_{\mathrm{D}}^2}\varphi. \end{aligned} \tag{1.5}$$

In the first line of this equation, the first term on the right-hand side is the charge density associated with the "central" particle and the second term is the charge density of the rest of the particles. In the second line, I used the Taylor-expanded Boltzmann expression (1.4) for the particle densities and then the quasineutrality (1.1) to establish the vanishing of the second term. The combination that has arisen in the last term as a prefactor of φ has dimensions of inverse square length, so we define the *Debye length* to be

$$\boxed{\lambda_{\mathrm{D}} \equiv \left(\sum_\alpha \frac{4\pi\bar{n}_\alpha q_\alpha^2}{T}\right)^{-1/2}}\,. \tag{1.6}$$

Using also the obvious fact that the solution of (1.5) must be spherically symmetric, we recast this equation as follows

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\varphi}{\partial r} - \frac{1}{\lambda_{\mathrm{D}}^2}\,\varphi = -4\pi q_\alpha\delta(\boldsymbol{r}). \tag{1.7}$$

The solution to this that asymptotes to the Coulomb potential $\varphi \to q_\alpha/r$ as $r\to 0$ and to zero as $r\to\infty$ is

$$\varphi = \frac{q_\alpha}{r}\,e^{-r/\lambda_{\mathrm{D}}}. \tag{1.8}$$

Thus, in a quasineutral plasma, charges are shielded on typical distances $\sim \lambda_{\mathrm{D}}$.

Obviously, this statistical calculation only makes sense if the "Debye sphere" has very many particles in it, viz., if

$$\boxed{N_{\mathrm{D}} \equiv n\lambda_{\mathrm{D}}^3 \gg 1}\,. \tag{1.9}$$

Let us check that this is the case:

$$N_{\mathrm{D}} \sim n\left(\frac{T}{ne^2}\right)^{3/2} = \left(\frac{T}{n^{1/3}e^2}\right)^{3/2} \gg 1, \tag{1.10}$$

provided the weak-interaction condition (1.3) is satisfied. The quantity N_D is called the *plasma parameter.*

1.4. *Micro- and Macroscopic Fields*

This calculation tells us something very important about electromagnetic fields in a plasma. Let $\boldsymbol{E}^{(\mathrm{micro})}(\boldsymbol{r},t)$ and $\boldsymbol{B}^{(\mathrm{micro})}(\boldsymbol{r},t)$ be the exact *microscopic fields* at a given location $\boldsymbol{r}$ and time t. These fields are responsible for interactions between particles. On distances $l \ll \lambda_\mathrm{D}$, these will be just the two-particle interactions—binary collisions between particles in vacuo, just like in a neutral gas (except the interparticle potential is the Coulomb potential). In contrast, on distances $l \gtrsim \lambda_\mathrm{D}$, individual particles' fields are shielded and what remains are fields due to collective influence of large numbers of particles—*macroscopic fields:*

$$\boldsymbol{E}^{(\mathrm{micro})} = \underbrace{\langle \boldsymbol{E}^{(\mathrm{micro})}\rangle}_{\equiv \boldsymbol{E}} + \delta\boldsymbol{E}, \quad \boldsymbol{B}^{(\mathrm{micro})} = \underbrace{\langle \boldsymbol{B}^{(\mathrm{micro})}\rangle}_{\equiv \boldsymbol{B}} + \delta\boldsymbol{B}, \tag{1.11}$$

where the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are averages over some intermediate scale l such that

$$\varDelta r \sim n^{-1/3} \ll l \ll \lambda_\mathrm{D}. \tag{1.12}$$

Such averaging (or "coarse-graining") is made possible by the condition (1.9).

Thus, plasma has a new feature compared to neutral gas: because the Coulomb potential is long-range ($\propto 1/r$), the fields decay on a length scale that is long compared to the interparticle distances [$\lambda_\mathrm{D} \gg \varDelta r \sim n^{-1/3}$ according to (1.9)] and so, besides interactions between individual particles, there are also collective effects: interaction of particles with mean macroscopic fields due to all other particles.

Before I use this approach to construct a description of the plasma as a continuum (on scales $\gg l$), let us check that particles travel sufficiently long distances between collisions in order to feel the macroscopic fields, viz., that their mean free path is $\lambda_\mathrm{mfp} \gg \lambda_\mathrm{D}$. The mean free path can be estimated in terms of the collision cross-section σ:

$$\lambda_\mathrm{mfp} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4} \tag{1.13}$$

because $\sigma \sim d^2$ and the effective distance d by which particles have to approach each other in order to have significant Coulomb interaction is inferred by balancing the Coulomb potential energy (1.2) with the particle temperature, $e^2/d \sim T$. Using (1.13) and (1.6), we find

$$\frac{\lambda_\mathrm{mfp}}{\lambda_\mathrm{D}} \sim \frac{T^2}{ne^4}\left(\frac{ne^2}{T}\right)^{1/2} \sim n\lambda_\mathrm{D}^3 = N_\mathrm{D} \gg 1, \quad \text{q.e.d.} \tag{1.14}$$

Thus, it makes sense to talk about a particle travelling long distances experiencing the macroscopic fields exerted by the rest of the plasma collectively before being deflected by a much larger, but also much shorter-range, microscopic field of another individual particle.

1.5. *Maxwell's Equations*

The exact microscopic fields satisfy *Maxwell's equations* and, since Maxwell's equations are linear, so do the macroscopic fields: by direct averaging,

$$\boldsymbol{\nabla}\cdot\langle\boldsymbol{E}^{(\mathrm{micro})}\rangle = 4\pi\langle\sigma^{(\mathrm{micro})}\rangle, \tag{1.15}$$

$$\boldsymbol{\nabla}\cdot\langle\boldsymbol{B}^{(\mathrm{micro})}\rangle = 0, \tag{1.16}$$

$$\boldsymbol{\nabla}\times\langle\boldsymbol{E}^{(\mathrm{micro})}\rangle + \frac{1}{c}\frac{\partial\langle\boldsymbol{B}^{(\mathrm{micro})}\rangle}{\partial t} = 0, \tag{1.17}$$

$$\boldsymbol{\nabla}\times\langle\boldsymbol{B}^{(\mathrm{micro})}\rangle - \frac{1}{c}\frac{\partial\langle\boldsymbol{E}^{(\mathrm{micro})}\rangle}{\partial t} = \frac{4\pi}{c}\langle\boldsymbol{j}^{(\mathrm{micro})}\rangle. \tag{1.18}$$

The new quantities here are the averages of the microscopic charge density $\sigma^{(\mathrm{micro})}$ and the microscopic current density $\boldsymbol{j}^{(\mathrm{micro})}$. How do we calculate them?

Clearly, they depend on where all the particles are at any given time and how fast these particles move. We can assemble all this information in one function:

$$F_\alpha(\boldsymbol{r},\boldsymbol{v},t) = \sum_{i=1}^{N_\alpha}\delta^3\big(\boldsymbol{r}-\boldsymbol{r}_i^{(\alpha)}(t)\big)\delta^3\big(\boldsymbol{v}-\boldsymbol{v}_i^{(\alpha)}(t)\big), \tag{1.19}$$

where $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ are the exact phase-space coordinates of particle i of species α at time t, i.e., these are the solutions of the exact equations of motion for all these particles moving in the microscopic fields $\boldsymbol{E}^{(\mathrm{micro})}(t,\boldsymbol{r})$ and $\boldsymbol{B}^{(\mathrm{micro})}(t,\boldsymbol{r})$. The function F_α is called the *Klimontovich distribution function.* It is a random object (i.e., it fluctuates on scales $\sim n^{-1/3} \ll \lambda_{\mathrm{D}}$) because it depends on the exact particle trajectories, which depend on the exact microscopic fields.[1] In terms of this distribution function,

$$\sigma^{(\mathrm{micro})}(\boldsymbol{r},t) = \sum_\alpha q_\alpha\int d\boldsymbol{v}\, F_\alpha(\boldsymbol{r},\boldsymbol{v},t), \tag{1.20}$$

$$\boldsymbol{j}^{(\mathrm{micro})}(\boldsymbol{r},t) = \sum_\alpha q_\alpha\int d\boldsymbol{v}\, \boldsymbol{v}F_\alpha(\boldsymbol{r},\boldsymbol{v},t). \tag{1.21}$$

We now need to average these quantities for use in (1.15) and (1.18). We shall assume that the average over microscales (1.12) and the ensemble average (i.e., the average over many different initial conditions for individual particles) are the same. The ensemble average of F_α is an object familiar from the kinetic theory of gases (Dellar 2015), the so-called *one-particle distribution function:*

$$\langle F_\alpha\rangle = f_{1\alpha}(\boldsymbol{r},\boldsymbol{v},t) \tag{1.22}$$

(I shall henceforth omit the subscript 1). If we learn how to compute f_α, then we can average (1.20) and (1.21), substitute them into (1.15) and (1.18), and have the following

[1] If you find the idea of a "random distribution function" jarring, you can call F_α "phase-space density," reflecting the fact that kinetic theory is the "fluid dynamics" of a particle density field in a 6D phase space $(\boldsymbol{r},\boldsymbol{v})$. This language will return when we get to talking about statistical mechanics of a collisionless plasma in §10 (and onwards).

set of macroscopic Maxwell's equations:

$$\nabla \cdot \boldsymbol{E} = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, f_\alpha(\boldsymbol{r},\boldsymbol{v},t), \tag{1.23}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.24}$$

$$\nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{1.25}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha(\boldsymbol{r},\boldsymbol{v},t). \tag{1.26}$$

1.6. *Vlasov–Landau Equation*

We now need an evolution equation for $f_\alpha(\boldsymbol{r},\boldsymbol{v},t)$, hopefully in terms of the macroscopic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$, so we can couple it to (1.23–1.26) and thus have a closed system of equations describing our plasma.

The process of deriving it starts with Liouville's theorem and is a direct generalisation of the BBGKY procedure familiar from gas kinetics (e.g., Dellar 2015)[2] to the somewhat more cumbersome case of a plasma:

—many species α;

—presence of forces due to the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{B}$;

—Coulomb potential for interparticle collisions, with some attendant complications to do with its long-range nature: in brief, use Rutherford's cross section and cut off long-range interactions at λ_{D} (this is described in many textbooks and plasma-physics courses: see, e.g., Parra 2019*a*, Kunz 2021, or a shortcut in §11.6 of these Lectures, avoiding cross-section dynamics).

The result of this derivation is

$$\frac{\partial f_\alpha}{\partial t} + \{f_\alpha, H_{1\alpha}\} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}. \tag{1.27}$$

The Poisson bracket contains $H_{1\alpha}$, the Hamiltonian for a single particle of species α moving in the *macroscopic* fields $\boldsymbol{E}$ and $\boldsymbol{B}$—all the microscopic fields $\delta\boldsymbol{E}$ are gone into the collision operator on the right-hand side, of which more will be said shortly (§1.8).[3]

Technically speaking, one ought to be working with canonical variables, but dealing with canonical momenta of charged particles in a magnetic field is an unnecessary complication, so I shall stick to the $(\boldsymbol{r},\boldsymbol{v})$ representation of the phase space. Then (1.27) takes the form of Liouville's equation, but with microscopic fields hidden inside the collision operator [see (1.41)]:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}}\cdot(\dot{\boldsymbol{r}} f_\alpha) + \frac{\partial}{\partial \boldsymbol{v}}\cdot(\dot{\boldsymbol{v}} f_\alpha) = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}, \tag{1.28}$$

where

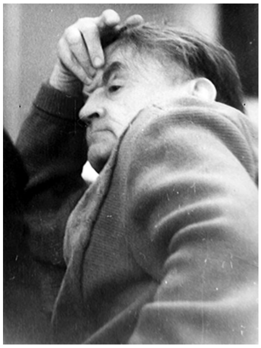
$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} = \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right). \tag{1.29}$$

This gives us the *Vlasov–Landau equation:*

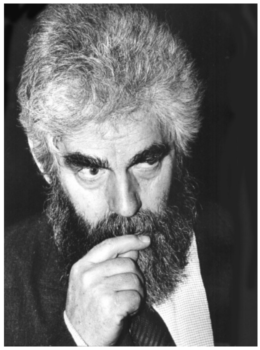
$$\boxed{\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}.} \tag{1.30}$$

[2] In §1.7, I will sketch Klimontovich's version of this procedure (Klimontovich 1967).

[3] $\delta\boldsymbol{B}$ turns out to be irrelevant as long as the particle motion is non-relativistic, $v/c \ll 1$.



(a)



(b)

FIGURE 2. The fathers of kinetic equations: (a) Anatoly Vlasov (1908-1975); photo taken from `http://jinrmag.jinr.ru/2018/8/v18.htm`. (b) Yuri Klimontovich (1924-2002); photo taken from `https://7iskusstv.com/2016/Nomer4/Klimontovich1.php`. Landau will appear in Fig. 4.

Any other macroscopic force that the plasma might be subject to (e.g., gravity) can be added to the Lorentz force in the third term on the left-hand side, as long as its divergence in velocity space is $(\partial/\partial \boldsymbol{v}) \cdot \text{force} = 0$. Equation (1.30) is closed by Maxwell's equations (1.23–1.26).

1.7. *Klimontovich's Version of BBGKY*

By way of a technical digression, let me outline (the beginning of) the derivation of (1.30) due to Klimontovich (1967). Consider the Klimontovich distribution function (1.19) and calculate its time derivative: by the chain rule,

$$\frac{\partial F_\alpha}{\partial t} = -\sum_i \frac{\mathrm{d}\boldsymbol{r}_i^{(\alpha)}(t)}{\mathrm{d}t} \cdot \left[\frac{\partial}{\partial \boldsymbol{r}} \delta^3\big(\boldsymbol{r} - \boldsymbol{r}_i^{(\alpha)}(t)\big)\delta^3\big(\boldsymbol{v} - \boldsymbol{v}_i^{(\alpha)}(t)\big)\right]$$
$$- \sum_i \frac{\mathrm{d}\boldsymbol{v}_i^{(\alpha)}(t)}{\mathrm{d}t} \cdot \left[\frac{\partial}{\partial \boldsymbol{v}} \delta^3\big(\boldsymbol{r} - \boldsymbol{r}_i^{(\alpha)}(t)\big)\delta^3\big(\boldsymbol{v} - \boldsymbol{v}_i^{(\alpha)}(t)\big)\right]. \tag{1.31}$$

First, because $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ obviously do not depend on the phase-space variables $\boldsymbol{r}$ and $\boldsymbol{v}$, the derivatives $\partial/\partial\boldsymbol{r}$ and $\partial/\partial\boldsymbol{v}$ can be pulled outside, so the right-hand side of (1.31) can be written as a divergence in phase space. Secondly, the particle equations of motion give us

$$\frac{\mathrm{d}\boldsymbol{r}_i^{(\alpha)}(t)}{\mathrm{d}t} = \boldsymbol{v}_i^{(\alpha)}(t), \tag{1.32}$$

$$\frac{\mathrm{d}\boldsymbol{v}_i^{(\alpha)}(t)}{\mathrm{d}t} = \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}^{(\mathrm{micro})}\big(\boldsymbol{r}_i^{(\alpha)}(t), t\big) + \frac{\boldsymbol{v}_i^{(\alpha)}(t) \times \boldsymbol{B}^{(\mathrm{micro})}\big(\boldsymbol{r}_i^{(\alpha)}(t), t\big)}{c}\right], \tag{1.33}$$

which are to be substituted into the right-hand side of (1.31)—*after* it is written in the divergence form. Since the time derivatives of $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ inside the divergence multiply delta functions identifying $\boldsymbol{r}_i^{(\alpha)}(t)$ with $\boldsymbol{r}$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ with $\boldsymbol{v}$, $\boldsymbol{r}_i^{(\alpha)}(t)$ may be replaced by $\boldsymbol{r}$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ by $\boldsymbol{v}$ in the right-hand sides of (1.32) and (1.33) when they go into (1.31). This gives (wrapping all the sums of delta functions back into F_α)

$$\frac{\partial F_\alpha}{\partial t} = -\boldsymbol{\nabla} \cdot (\boldsymbol{v} F_\alpha) - \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}^{(\mathrm{micro})}(\boldsymbol{r}, t) + \frac{\boldsymbol{v} \times \boldsymbol{B}^{(\mathrm{micro})}(\boldsymbol{r}, t)}{c}\right) F_\alpha\right]. \tag{1.34}$$

Finally, because $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables and the Lorentz force has zero divergence in $\boldsymbol{v}$ space, F_α satisfies *exactly*

$$\boxed{\frac{\partial F_\alpha}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} F_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E}^{(\mathrm{micro})} + \frac{\boldsymbol{v}\times\boldsymbol{B}^{(\mathrm{micro})}}{c}\right)\cdot\frac{\partial F_\alpha}{\partial \boldsymbol{v}} = 0}\,. \tag{1.35}$$

This is the *Klimontovich equation*. There is no collision integral here because microscopic fields are explicitly present. The equation is closed by the microscopic Maxwell's equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}^{(\mathrm{micro})} = 4\pi\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, F_\alpha(\boldsymbol{r},\boldsymbol{v},t), \tag{1.36}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}^{(\mathrm{micro})} = 0, \tag{1.37}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}^{(\mathrm{micro})} + \frac{1}{c}\frac{\partial \boldsymbol{B}^{(\mathrm{micro})}}{\partial t} = 0, \tag{1.38}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B}^{(\mathrm{micro})} - \frac{1}{c}\frac{\partial \boldsymbol{E}^{(\mathrm{micro})}}{\partial t} = \frac{4\pi}{c}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v} F_\alpha(\boldsymbol{r},\boldsymbol{v},t). \tag{1.39}$$

Now let us separate the microscopic fields into mean (macroscopic) and fluctuating parts according to (1.11); also

$$F_\alpha = \underbrace{\langle F_\alpha\rangle}_{\equiv f_\alpha} + \delta F_\alpha. \tag{1.40}$$

Maxwell's equations are linear, so averaging them gives the same equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of f_α [see (1.23–1.26)] and for $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$ in terms of δF_α. Averaging the Klimontovich equation (1.35) gives the Vlasov–Landau equation:

$$\begin{aligned}\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}&\\ = -\frac{q_\alpha}{m_\alpha}\left\langle\left(\delta\boldsymbol{E} + \frac{\boldsymbol{v}\times\delta\boldsymbol{B}}{c}\right)\cdot\frac{\partial \delta F_\alpha}{\partial \boldsymbol{v}}\right\rangle &\equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}.\end{aligned} \tag{1.41}$$

The macroscopic fields in the left-hand side satisfy the macroscopic Maxwell's equations (1.23–1.26). The microscopic fluctuating fields $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$ inside the average in the right-hand side satisfy microscopic Maxwell's equations with fluctuating charge and current densities expressed in terms of δF_α. Thus, the right-hand side is quadratic in δF_α. In order to close this equation, we need an expression for the correlation function $\langle \delta F_\alpha \delta F_{\alpha'}\rangle$ in terms of f_α and $f_{\alpha'}$. This is basically what the BBGKY procedure plus truncation of velocity integrals based on an expansion in $1/n\lambda_{\mathrm{D}}^3$ achieve. The result is the Landau collision operator (§11.6).

Further details are a bit complicated (see Klimontovich 1967 or §11), but my aim here was just to show how the fields are split into macroscopic and microscopic ones, with the former appearing explicitly in the kinetic equation and the latter wrapped up inside the collision operator. The presence of the macroscopic fields and the consequent necessity for coupling the kinetic equation with Maxwell's equations for these fields is the main mathematical difference between the kinetics of neutral gases and the kinetics of plasmas.

1.8. *Collision Operator*

Finally, a few words about the plasma collision operator (or "collision integral"), first derived explicitly by Landau (1936) (the same considerations apply to the the operator due to Lenard 1960 and Balescu 1960, usually viewed as more general). It describes two-particle collisions both within the species α and with other species α' and so depends both on f_α and on all other $f_{\alpha'}$. Its derivation is left to you as an exercise in BBGKY'ing, calculating cross sections and velocity integrals (or in googling; shortcut: see Parra 2019*a*, Kunz 2021, or Swanson 2008).[4] In these Lectures, I shall focus on collisionless aspects of plasma kinetics. Whenever a need arises for invoking the collision operator, the important things about it for us will be its properties:

• *conservation of particles* (within each species α),

$$\int \mathrm{d}\boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0; \tag{1.42}$$

• *conservation of momentum,*

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha \boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0 \tag{1.43}$$

(same-species collisions conserve momentum, whereas different-species collisions conserve it only after summation over species—there is friction of one species against another; e.g., the friction of electrons against the ions is the Ohmic resistivity of the plasma, known as "Spitzer resistivity": see Parra 2019*a*, Kunz 2021, or Helander & Sigmar 2005);

• *conservation of energy,*

$$\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, \frac{m_\alpha v^2}{2} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0; \tag{1.44}$$

• Boltzmann's *H-theorem:* the kinetic entropy

$$S = -\sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, f_\alpha \ln f_\alpha \tag{1.45}$$

cannot decrease, and, as S is conserved by all the collisionless terms in (1.30), the collision operator must have the property that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} \ln f_\alpha \geqslant 0, \tag{1.46}$$

with equality obtained if and only if f_α is a local Maxwellian;

• unlike the Boltzmann operator for neutral gases, the Landau operator expresses the cumulative effect of many glancing (rather than "head-on") collisions (due to the long-range nature of the Coulomb interaction) and so it is a Fokker–Planck operator:[5]

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \sum_{\alpha'} \left(\boldsymbol{A}_{\alpha\alpha'}[f_{\alpha'}] + \mathsf{D}_{\alpha\alpha'}[f_{\alpha'}] \cdot \frac{\partial}{\partial \boldsymbol{v}}\right) f_\alpha, \tag{1.47}$$

where the drag $\boldsymbol{A}_{\alpha\alpha'}$ (vector) and diffusion $\mathsf{D}_{\alpha\alpha'}$ (matrix) coefficients are integral (in $\boldsymbol{v}$ space) functionals of $f_{\alpha'}$. The Fokker–Planck form (1.47) of the Landau operator means that it describes diffusion in velocity space and so will erase sharp gradients in f_α with respect to $\boldsymbol{v}$—a property that we will find very important in §5 (and then again in §12).

[4]A somewhat unorthodox derivation of both the Lenard–Balescu and Landau operators will be given in §§11.5 and 11.6, respectively, as a by-product of a discussion of collisionless (*sic*) relaxation.

[5]The simplest example that I can think of in which the collision operator is a velocity-space diffusion operator of this kind is a gas of Brownian particles [each with velocity described by Langevin's equation (6.65)]. This is treated in detail in §6.9 of Schekochihin (2024). In these Lectures, the general Fokker–Planck form (1.47) emerges in (7.37), and then again, from a different angle, in (11.22); the Landau operator is (11.36).

1.9. *So What's New and What Now?*

Let me summarise the new features that have appeared in the kinetic description of a plasma compared to that of a neutral gas.

• First, particles are charged, so they interact via Coulomb potential. The collision operator is, therefore, different: the cross-section is the Rutherford cross-section, most collisions are glancing (with interaction on distances up to the Debye length), leading to diffusion of the particle distribution function in velocity space. Mathematically, this is manifested in the collision operator in (1.30) having the Fokker–Planck structure (1.47).

One can spin out of the Vlasov–Landau equation (1.30) a theory that is analogous to what is done with Boltzmann's equation in gas kinetics (Dellar 2015): derive *fluid equations*, calculate viscosity, thermal conductivity, Ohmic resistivity, etc., of a collisionally dominated plasma, i.e., of a plasma in which the collision frequency of the particles is much greater than all other relevant time scales. This is done in the same way as in gas kinetics, but now applying the Chapman–Enskog procedure to the Landau collision operator. This is quite a lot of work—and constitutes core textbook material (see Parra 2019*a*; Kunz 2021). In magnetised plasmas especially, the resulting fluid dynamics of the plasma are quite interesting and quite different from neutral fluids—we shall see some of this in Parts III and IV of these Lectures, while the classic treatment of the transport theory can be found in Braginskii (1965); a great textbook on collisional transport is Helander & Sigmar (2005) (see Krommes 2018 for a modernist approach).

• Secondly, Coulomb potential is long-range, so the electric and magnetic fields have a macroscopic (mean) part on scales longer than the Debye length—a particle experiencing these fields is not undergoing a collision in the sense of bouncing off another particle, but is, rather, interacting, via the fields, with the collective of all the other particles. Mathematically, this manifests itself as a Lorentz-force term appearing in the right-hand side of the Vlasov–Landau kinetic equation (1.30). The macroscopic $\boldsymbol{E}$ and $\boldsymbol{B}$ fields that figure in it are determined by the particles via their mean charge and current densities that enter the macroscopic Maxwell's equations (1.23–1.26).

In the case of neutral gas, all the interesting kinetic physics is in the collision operator, hence the focus on transport theory in gas-kinetic literature (see, e.g., the classic monograph by Chapman & Cowling 1991 if you want an overdose of this). In the collisionless limit, the kinetic equation for a neutral gas,

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f = 0, \tag{1.48}$$

simply describes particles with some initial distribution individualistically flying in straight lines along their initial directions of travel. In contrast, for a plasma, even the *collisionless kinetics* (and, indeed, especially the collisionless—or *weakly collisional*—kinetics) is interesting and nontrivial because the particles, via the average properties of their distribution—charge densities and currents,—collectively modify $\boldsymbol{E}$ and $\boldsymbol{B}$, which then act on individual particles and thus modify f_α, etc. This "plasma democracy" is a whole new conceptual world and it is on the effects involving interactions between particles and fields that I shall focus here, in pursuit of maximum novelty.[6]

[6]Similarly interesting things happen when the field tying the particles together is gravity—an even more complicated situation because, while the potential is long-range, like the Coulomb potential, gravity is not shielded and so all particles feel each other's presence at all distances. This gives rise to a very sophisticated theory, within which there is no plasma-style distinction between "collisional" and "collisionless" dynamics (Binney 2018; Fouvry 2021; Hamilton 2022; Hamilton & Fouvry 2024).

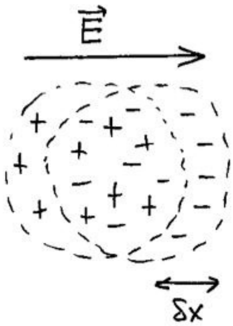


FIGURE 3. A displaced population of electrons will set up a quasineutrality-restoring electric field, leading to plasma oscillations.

I shall also be in pursuit of maximum simplicity (well, to use Einstein's dictum, "as simple as possible, but not simpler"!) and so will mostly restrict my considerations to the *"electrostatic approximation"*:

$$\boldsymbol{B} = 0, \quad \boldsymbol{E} = -\boldsymbol{\nabla}\varphi. \tag{1.49}$$

This, of course, eliminates a huge number of interesting and important phenomena without which plasma physics would not be the voluminous subject that it is, but I cannot do them justice in just a few lectures (a course largely devoted to collisionless magnetised plasmas is offered in Part IV of these Lectures; another version of that plus the magnetised plasma waves is Parra 2019*b*). Some opportunities for generalising electrostatic theory to electromagnetic one will be provided in Q-2 and Q-3.

Thus, we shall henceforth consider a simplified kinetic system, called the *Vlasov–Poisson system*:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}} = 0, \tag{1.50}$$

$$-\nabla^2\varphi = 4\pi\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, f_\alpha. \tag{1.51}$$

Formally, considering a collisionless plasma[7] would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. What are the characteristic time scales (and length scales) in a plasma and what phenomena occur on these scales? These questions bring us to our next theme.

2. Equilibrium and Fluctuations

2.1. *Plasma Frequency*

Consider a plasma in equilibrium, in a happy quasineutral state. Suppose a population of electrons strays from this equilibrium and upsets quasineutrality a bit (Fig. 3). If they

[7]Or, I stress again, a weakly collisional plasma. The collision operator is dropped in (1.50), but let us not forget about it entirely even if the collision frequency is small; it will make a come back in §5.

have shifted by distance δx, the restoring force on each electron will be

$$m_e \delta\ddot{x} = -eE = -4\pi e^2 n_e \delta x \quad \Rightarrow \quad \delta\ddot{x} = -\underbrace{\frac{4\pi e^2 n_e}{m_e}}_{\equiv \omega_{\rm pe}^2} \delta x, \tag{2.1}$$

so there will be oscillations at what is known as the (electron) *plasma frequency*:

$$\boxed{\omega_{\rm pe} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}}\,. \tag{2.2}$$

Thus, we expect *fluctuations* of electric field in a plasma with characteristic frequencies $\omega \sim \omega_{\rm pe}$ (these are Langmuir waves; I will derive their dispersion relation formally in §3.4). These fluctuations are due to *collective* motions of the particles—so they are still *macroscopic* fields in the nomenclature of §1.4.

The time scale associated with $\omega_{\rm pe}$ is the scale of restoration of quasineutrality. The distance an electron can travel over this time scale before the restoring force kicks in, i.e., the distance over which quasineutrality can be violated, is (using the *thermal speed* $v_{\rm the} \sim \sqrt{T/m_e}$ to estimate the electron's velocity)

$$\frac{v_{\rm the}}{\omega_{\rm pe}} \sim \sqrt{\frac{T}{m_e}}\sqrt{\frac{m_e}{e^2 n_e}} = \sqrt{\frac{T}{e^2 n_e}} \sim \lambda_{\rm D}, \tag{2.3}$$

the Debye length (1.6)—not surprising, as this is, indeed, the scale on which microscopic fields are shielded and plasma is quasineutral (§1.3).

Finally, let us check that the plasma oscillations happen on collisionless time scales. The collision frequency of the electrons is, using (2.3) and (1.14),

$$\nu_e \sim \frac{v_{\rm the}}{\lambda_{\rm mfp}} = \frac{v_{\rm the}}{\omega_{\rm pe}}\frac{\omega_{\rm pe}}{\lambda_{\rm mfp}} \sim \frac{\lambda_{\rm D}}{\lambda_{\rm mfp}}\,\omega_{\rm pe} \sim N_{\rm D}^{-1}\omega_{\rm pe} \ll \omega_{\rm pe}, \quad \text{q.e.d.} \tag{2.4}$$

2.2. *Slow vs. Fast*

The plasma frequency $\omega_{\rm pe}$ is only one of the characteristic frequencies (the largest) of the fluctuations that can occur in plasmas. We will think of the scales of all these fluctuations as short and of the associated variation in time and space as *fast*. They occur against the background of some *equilibrium state*,[8] which is either constant or varies *slowly* in time and space. The slow evolution and spatial variation of the equilibrium state can be due to slowly changing, large-scale external conditions that gave rise to this state or, as we will discover soon, it can be due to the average effect of a sea of small fluctuations.

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function:

$$f(\boldsymbol{r},\boldsymbol{v},t) = f_0(\epsilon^a\boldsymbol{r},\boldsymbol{v},\epsilon t) + \delta f(\boldsymbol{r},\boldsymbol{v},t), \tag{2.5}$$

where ϵ is some small parameter characterising the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales, hence ϵ^a). To avoid clutter, I am dropping the species index where this does not lead to ambiguity.

For simplicity, I will abolish the spatial dependence of the equilibrium distribution

[8]Or even just an initial state that is slow to change.

altogether and consider *homogeneous* systems:

$$f_0 = f_0(\boldsymbol{v}, \epsilon t), \tag{2.6}$$

which also means $\boldsymbol{E}_0 = 0$ (there is no equilibrium electric field). Equivalently, all our considerations are restricted to scales much smaller than the characteristic system size. Formally, this equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:[9]

$$f_0(\boldsymbol{v}, t) = \langle f(\boldsymbol{r}, \boldsymbol{v}, t)\rangle \equiv \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} \mathrm{d}t' \int \frac{\mathrm{d}\boldsymbol{r}}{V}\, f(\boldsymbol{r}, \boldsymbol{v}, t'), \tag{2.7}$$

where $\omega^{-1} \ll \Delta t \ll t_{\rm eq}$, where $t_{\rm eq}$ is the equilibrium time scale.

2.3. *Multiscale Dynamics*

It is convenient to work in Fourier space:

$$\varphi(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\varphi_{\boldsymbol{k}}(t), \quad f(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\boldsymbol{v}, t) + \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\delta f_{\boldsymbol{k}}(\boldsymbol{v}, t). \tag{2.8}$$

Then the Poisson equation (1.51) becomes

$$\varphi_{\boldsymbol{k}} = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\, \delta f_{\boldsymbol{k}\alpha} \tag{2.9}$$

and the Vlasov equation (1.50) written for $\boldsymbol{k} = 0$ (i.e., the spatial average of the equation) is

$$\frac{\partial f_0}{\partial t} + \frac{\partial \delta f_{\boldsymbol{k}=0}}{\partial t} = -\frac{q}{m}\sum_{\boldsymbol{k}} \varphi_{-\boldsymbol{k}} i\boldsymbol{k}\cdot\frac{\partial \delta f_{\boldsymbol{k}}}{\partial \boldsymbol{v}}, \tag{2.10}$$

where we can replace $\varphi_{-\boldsymbol{k}} = \varphi_{\boldsymbol{k}}^*$ because $\varphi(\boldsymbol{r}, t)$ is a real field. Averaging over time according to (2.7) eliminates fast variation and gives

$$\boxed{\frac{\partial f_0}{\partial t} = -\frac{q}{m}\sum_{\boldsymbol{k}}\left\langle \varphi_{\boldsymbol{k}}^* i\boldsymbol{k}\cdot\frac{\partial \delta f_{\boldsymbol{k}}}{\partial \boldsymbol{v}}\right\rangle = -\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{q}{m}\sum_{\boldsymbol{k}} i\boldsymbol{k}\,\langle\varphi_{\boldsymbol{k}}^*\delta f_{\boldsymbol{k}}\rangle\right)}\,. \tag{2.11}$$

The right-hand side of (2.11) describes the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations (see §§6 and 11). In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of f_0. This should be obtainable, at least in principle, because the fluctuating fields appearing in the right-hand side of (2.11) themselves depend on f_0: indeed, writing the Vlasov equation (1.50) for the $\boldsymbol{k} \neq 0$ modes, we find the following evolution equation

[9]I use angular brackets to denote this average, but it should be clear that this is not the same thing as the average (1.11) that separated the macroscopic fields from the microscopic ones. The latter average was over sub-Debye, inter-particle scales, whereas the new average (2.7) is over scales that are larger than fluctuation scales but smaller than the system size; both fluctuations and equilibrium are "macroscopic" in the language of §1.4.

for the fluctuations:

$$\boxed{\frac{\partial \delta f_{\boldsymbol{k}}}{\partial t} + \underbrace{i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_{\boldsymbol{k}}}_{\substack{\text{particle}\\ \text{streaming}\\ \text{(phase}\\ \text{mixing)}}} = \underbrace{\frac{q}{m}\varphi_{\boldsymbol{k}} i\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}}_{\substack{\text{wave-particle}\\ \text{interaction}\\ \text{(linear)}}} + \underbrace{\frac{q}{m}\sum_{\boldsymbol{k}'}\varphi_{\boldsymbol{k}'} i\boldsymbol{k}'\cdot\frac{\partial \delta f_{\boldsymbol{k}-\boldsymbol{k}'}}{\partial \boldsymbol{v}}}_{\substack{\text{nonlinear}\\ \text{interactions}}}}\,. \tag{2.12}$$

The three terms that control the evolution of the perturbed distribution function in (2.12) represent the three physical effects that I shall focus on in these Lectures. The second term on the left-hand side describes the free ballistic motion of particles ("streaming"). It gives rise to the phenomenon of phase mixing (§5) and, in its interplay with plasma waves, to Landau damping and kinetic instabilities (§3). The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution (§3). The second term on the right-hand side captures the nonlinear interactions between the fluctuating fields and the perturbed distribution—it is negligible only when fluctuation amplitudes are small enough (which, sadly, they rarely are) and is responsible for *plasma turbulence* (§§7.2, 8, and 12) and other nonlinear phenomena (e.g., §12.1).

The programme for determining the slow evolution of the equilibrium is "simple": solve (2.12) together with (2.9), calculate the *correlation function* of the fluctuations, $\langle\varphi^*_{\boldsymbol{k}}\delta f_{\boldsymbol{k}}\rangle$, as a functional of f_0, and use it to close (2.11); then proceed to solve the latter. Obviously, this is impossible to do in most cases. But it is possible to construct a hierarchy of approximations to the answer and learn much interesting physics in the process.

2.4. *Hierarchy of Approximations*

2.4.1. *Linear Theory*

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, (2.11) turns into $f_0 = \text{const}$ and (2.12) becomes

$$\frac{\partial \delta f_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_{\boldsymbol{k}} = \frac{q}{m}\varphi_{\boldsymbol{k}} i\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}, \tag{2.13}$$

the *linearised kinetic equation*. Solving this together with (2.9) allows one to find oscillating and/or growing/decaying[10] perturbations of a particular equilibrium f_0. The theory for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape (§3).

Physically, the linear solutions will describe what happens over short term, viz., over times t such that

$$\omega^{-1} \ll t \ll t_{\rm eq} \text{ and } t_{\rm nl}, \tag{2.14}$$

where ω is the characteristic frequency of the perturbations, $t_{\rm eq}$ is the time after which the equilibrium starts getting modified by the perturbations via (2.11) (which depends on the amplitude to which they can grow; if perturbations do grow, i.e., the equilibrium is unstable, they can modify the equilibrium by this mechanism so as to render it stable), and $t_{\rm nl}$ is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter [second term on the right-hand side of (2.12); growing perturbations can saturate by this mechanism].

[10]We shall see (§5) that growing/decaying linear solutions imply the equilibrium distribution giving/receiving energy to/from the fluctuations.

2.4.2. *Quasilinear Theory (QLT)*

Suppose

$$t_{\rm eq} \ll t_{\rm nl}, \tag{2.15}$$

i.e., *growing perturbations start modifying the equilibrium before they saturate nonlinearly.* Then the strategy is to solve (2.13) [together with (2.9)] for the perturbations, use the result to calculate their correlation function needed in the right-hand side of (2.11), then work out how the equilibrium therefore evolves and hence how large the perturbations must grow in order for this evolution to turn the unstable equilibrium into a stable one. This is a classic piece of theory, important conceptually—I will describe it in detail and do one example in §6. In reality, however, it happens relatively rarely that unstable perturbations saturate at amplitudes small enough for the nonlinear interactions not to matter [i.e., for (2.15) to hold true].

2.4.3. *Weak-Turbulence (WT) Theory*

Sometimes, one is not lucky enough to get away with QLT (i.e., alas, $t_{\rm nl} \ll t_{\rm eq}$ or $t_{\rm nl} \sim t_{\rm eq}$), but is lucky enough to have perturbations saturating nonlinearly at a *small amplitude* such that[11]

$$t_{\rm nl} \gg \omega^{-1}, \tag{2.16}$$

i.e., *perturbations oscillate linearly faster than they interact nonlinearly* (this can happen, e.g., because propagating wave packets do not stay together long enough to break up completely in one encounter). Because waves are fast compared to nonlinear evolution in this approximation, it is possible to "quantise" them, i.e., to treat a nonlinear turbulent state of the plasma as a cocktail consisting of both "true" particles (ions and electrons) and "quasiparticles" representing electromagnetic excitations (§7).

In this approximation, one can do perturbation theory treating the nonlinear term in (2.12) as small and expanding in the small parameter $(\omega t_{\rm nl})^{-1}$. The resulting *weak (or "wave") turbulence theory* is quite an analytical *tour de force*—but it is a lot of work to do it properly! I will provide an introduction to it in §§7.2 and 8.4 (some pointers to relevant literature are at the end of §7.1, just before the start of §7.2).

Note that because the nonlinear term couples perturbations at different $\boldsymbol{k}$'s (scales), this theory will lead to multi-scale (usually, power-law) fluctuation spectra.

2.4.4. *Strong-Turbulence Theory*

If perturbations manage to grow to a level at which

$$t_{\rm nl} \sim \omega^{-1}, \tag{2.17}$$

we are a facing *strong turbulence*. This is actually what mostly happens (including in WT systems, where turbulence often transitions into the strong regime at small enough, or large enough, scales). Theory of such regimes tends to be of phenomenological/scaling kind, often in the spirit of the classic Kolmogorov (1941) theory of hydrodynamic turbulence. No one really knows how to move very far beyond this sort of approach—and not for lack of trying! I will return to this topic in §§8.6 and 12.2 onwards (see also §16.4).

---

[11] Note that the nonlinear time scale is typically inversely proportional to the amplitude: from (2.12), $t_{\rm nl}^{-1} \sim qk\varphi\delta f/mv_{\rm th}$.

(a)

(b)

FIGURE 4. Lev Landau (1908-1968), great Soviet physicist, quintessential theoretician, author of the Book, cult figure. It is a minor feature of his scientific biography that he wrote the two most important plasma-physics papers of all time (Landau 1936, 1946). He also got a Nobel Prize (1962), but not for plasma physics. (a) Cartoon by A. A. Yuzefovich (from Landau & Lifshitz 1976); the caption says "Dau spake..." (...unto the students, also depicted). (b) Landau's mugshot from NKVD prison (1938), where he ended up for sedition and from whence he was released in 1939 after Peter Kapitsa's personal appeal to Stalin.

3. Linear Theory I: Waves, Landau Damping and Kinetic Instablities

Enough idle chatter, let us calculate! In this section, we are concerned with the linearised Vlasov–Poisson system, (2.13) and (2.9):

$$\frac{\partial \delta f_{\boldsymbol{k}\alpha}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_{\boldsymbol{k}\alpha} = \frac{q_\alpha}{m_\alpha}\varphi_{\boldsymbol{k}} i\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}, \tag{3.1}$$

$$\varphi_{\boldsymbol{k}} = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\delta f_{\boldsymbol{k}\alpha}. \tag{3.2}$$

For compactness of notation, I will drop both the species index α and the wave number $\boldsymbol{k}$ in the subscripts, unless they are necessary for understanding.

We will discover that electrostatic perturbations in a plasma described by (3.1) and (3.2) oscillate, can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to know some complex analysis.

3.1. *Initial-Value Problem*

I shall follow Landau's original paper (Landau 1946) in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not appropriate to think of them over the entire timeline $t \in [-\infty, +\infty]$ (and—NB!—the damped perturbations are *not* pure eignenmodes; see §5.3). So we look for $\delta f(\boldsymbol{v}, t)$ satisfying (3.1) with the initial condition

$$\delta f(\boldsymbol{v}, t = 0) = g(\boldsymbol{v}). \tag{3.3}$$

It is, therefore, appropriate to use the *Laplace transform* to solve (3.1):[12]

$$\boxed{\hat{\delta f}(p) = \int_0^\infty \mathrm{d}t\, e^{-pt}\delta f(t)}\,. \tag{3.4}$$

[12] A handy primer on complex analysis, including Laplace transforms and other concepts used below, is Binney (1999).

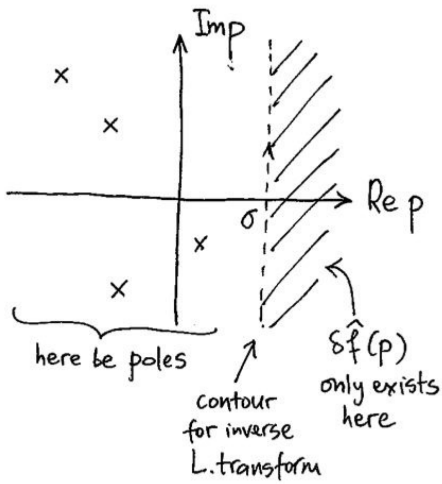


FIGURE 5. Layout of the complex-p plane: $\delta\hat{f}(p)$ is analytic for $\mathrm{Re}\,p \geqslant \sigma$. At $\mathrm{Re}\,p < \sigma$, $\delta\hat{f}(p)$ may have singularities (poles).

It is a mathematical certainty that if there exists a real number $\sigma > 0$ such that

$$|\delta f(t)| < e^{\sigma t} \text{ as } t \to \infty, \tag{3.5}$$

then the integral (3.4) exists (i.e., is finite) for all values of p such that $\mathrm{Re}\,p \geqslant \sigma$. The inverse Laplace transform, giving us back our distribution function as a function of time, is then

$$\delta f(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \mathrm{d}p\, e^{pt}\delta\hat{f}(p), \tag{3.6}$$

where the integral in the complex plane is along a straight line parallel to the imaginary axis and intersecting the real axis at $\mathrm{Re}\,p = \sigma$ (Fig. 5).

Since we expect to be able to recover our desired time-dependent function $\delta f(\boldsymbol{v}, t)$ from its Laplace transform, it is worth knowing the latter. To find it, we Laplace-transform (3.1):

$$\begin{aligned}
\text{l.h.s.} &= \int_0^\infty \mathrm{d}t\, e^{-pt}\frac{\partial \delta f}{\partial t} = \left[e^{-pt}\delta f\right]_0^\infty + p\int_0^\infty \mathrm{d}t\, e^{-pt}\delta f = -g + p\,\delta\hat{f},\\
\text{r.h.s.} &= -i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta\hat{f} + \frac{q}{m}\,\hat{\varphi}\, i\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}.
\end{aligned} \tag{3.7}$$

Equating these two expressions, we find the solution:

$$\boxed{\delta\hat{f}(p) = \frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\left[i\,\frac{q}{m}\,\hat{\varphi}(p)\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + g\right]}\,. \tag{3.8}$$

The Laplace transform of the potential, $\hat{\varphi}(p)$, itself depends on $\delta\hat{f}$ via (3.2):

$$\begin{aligned}
\hat{\varphi}(p) &= \int_0^\infty \mathrm{d}t\, e^{-pt}\,\varphi(t) = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\delta\hat{f}_\alpha(p)\\
&= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\left[i\,\frac{q_\alpha}{m_\alpha}\,\hat{\varphi}(p)\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} + g_\alpha\right].
\end{aligned} \tag{3.9}$$

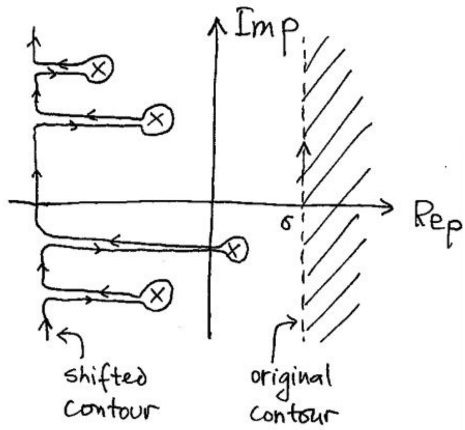


FIGURE 6. New contour for the inverse Laplace transform.

This is an algebraic equation for $\hat{\varphi}(p)$. Collecting terms, we get

$$\underbrace{\left(1-\sum_\alpha \frac{4\pi q_\alpha^2}{k^2 m_\alpha}\, i\int \mathrm{d}\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\right)}_{\equiv\,\epsilon(p,\boldsymbol{k})}\hat{\varphi}(p)=\frac{4\pi}{k^2}\sum_\alpha q_\alpha\int \mathrm{d}\boldsymbol{v}\,\frac{g_\alpha}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.10}$$

The prefactor in the left-hand side, which I denote $\epsilon(p,\boldsymbol{k})$, is called the *dielectric function*, because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, so let us write it out beautifully:

$$\boxed{\epsilon(p,\boldsymbol{k})=1-\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\,\frac{i}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}}\,, \tag{3.11}$$

where the plasma frequency of species α is defined by [cf. (2.2)]

$$\omega_{\mathrm{p}\alpha}^2=\frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}. \tag{3.12}$$

The solution of (3.10) is

$$\hat{\varphi}(p)=\frac{4\pi}{k^2\epsilon(p,\boldsymbol{k})}\sum_\alpha q_\alpha\int \mathrm{d}\boldsymbol{v}\,\frac{g_\alpha}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.13}$$

To calculate $\varphi(t)$, we need to inverse-Laplace-transform $\hat{\varphi}$: similarly to (3.6),

$$\varphi(t)=\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathrm{d}p\,e^{pt}\hat{\varphi}(p). \tag{3.14}$$

How do we do this integral? Recall that $\delta\hat{f}$ and, therefore, $\hat{\varphi}$ only exists (i.e., is finite) for $\mathrm{Re}\,p\geqslant\sigma$, whereas at $\mathrm{Re}\,p<\sigma$, it can have singularities, i.e., poles—let us call them p_i, indexed by i. If we analytically continue $\hat{\varphi}(p)$ everywhere to $\mathrm{Re}\,p<\sigma$ except those poles, the result must have the form

$$\hat{\varphi}(p)=\sum_i \frac{c_i}{p-p_i}+A(p), \tag{3.15}$$

where c_i are some coefficients (residues) and $A(p)$ is the analytic part of the solution. The integration contour in (3.14) can be shifted to $\mathrm{Re}\, p \to -\infty$ but with the proviso that it cannot cross the poles, as shown in Fig. 6 (this is proven by making a closed loop out of the old and the new contours, joining them at $\pm i\infty$, and noting that this loop encloses no poles). Then the contributions to the integral from the vertical segments of the contour are exponentially small,[13] the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy's formula, be expressed in terms of the poles and residues:

$$\boxed{\varphi(t) = \sum_i c_i e^{p_i t}}\,. \tag{3.16}$$

Thus, in the long-time limit, *perturbations of the potential will evolve* $\propto e^{p_i t}$*, where* p_i *are poles of* $\hat{\varphi}(p)$. In general, $p_i = -i\omega_i + \gamma_i$, where ω_i is a real frequency (giving wave-like behaviour of perturbations), $\gamma_i < 0$ represents damping and $\gamma_i > 0$ growth of the perturbations (instability).

Note that we need not be particularly interested in what c_i's are because, if we set up an initial perturbation with a given $\boldsymbol{k}$ and then wait long enough, only the fastest-growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes in comparison. Thus, a typical outcome of the linear theory is $\varphi(t)$ oscillating at some frequency and growing or decaying at some unique rate. Since this is a solution of a linear equation, the prefactor in front of the exponential can be scaled arbitrarily and so does not matter.

Going back to (3.13), we realise that the poles of $\hat{\varphi}(p)$ are zeros of the dielectric function:

$$\boxed{\epsilon(p_i, \boldsymbol{k}) = 0} \quad \Rightarrow \quad p_i = p_i(\boldsymbol{k}) = -i\omega_i(\boldsymbol{k}) + \gamma_i(\boldsymbol{k}). \tag{3.17}$$

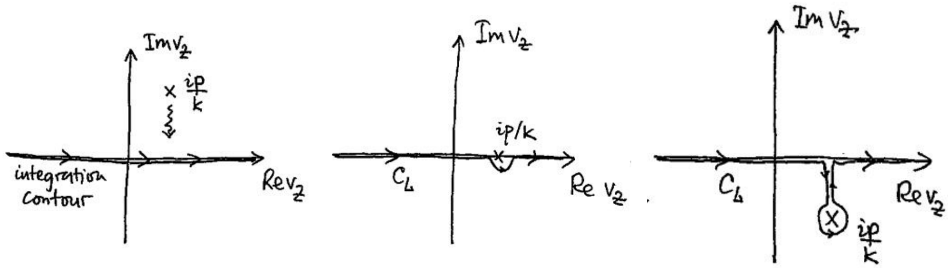
To find the wave frequencies ω_i and the damping/growth rates γ_i, we must solve this equation, which is called the *plasma dispersion relation.*

3.2. *Calculating the Dielectric Function: the "Landau Prescription"*

In order to be able to solve $\epsilon(p, \boldsymbol{k}) = 0$, we must learn how to calculate $\epsilon(p, \boldsymbol{k})$ for any given p and $\boldsymbol{k}$. Before I wrote (3.15), I said that $\hat{\varphi}$, given by (3.13), had to be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ($\mathrm{Re}\, p \geqslant \sigma$), but I did not explain how this was to be done. In order to do it, we must learn how to calculate the velocity integral in (3.11)—if we want $\epsilon(p, \boldsymbol{k})$ and, therefore, its zeros p_i—and also how to calculate the similar integral in (3.13) containing g_α.

First of all, let us turn these integrals into a 1D form. Given $\boldsymbol{k}$, we can always choose

[13] They are exponentially small *in time* as $t \to \infty$ because the integrand of the inverse Laplace transform (3.14) contains a factor of $e^{\mathrm{Re}\, pt}$, which decays faster than any of the "modes" in (3.16). If $\hat{\varphi}(p)$ does not grow too fast at large p, the integral along the vertical part of the contour may also vanish at any finite t, but that is not guaranteed in general: indeed, looking ahead to the explicit expression (3.27) for $\hat{\varphi}(p)$, with the Landau prescription for analytic continuation to $\mathrm{Re}\, p < 0$ analogous to (3.20), we see that $\hat{\varphi}(p)$ will contain a term $\propto G_\alpha(ip/k)$, which can be large at large $\mathrm{Re}\, p$, e.g., if $G_\alpha(v_z)$ is a Maxwellian. Note also that we need the (wildly oscillating in time) integral of $e^{pt}\hat{\varphi}(p)$ over the horizontal segments with $\mathrm{Im}\, p \to \pm\infty$ to vanish. This is fine provided $\hat{\varphi}(\pm i\infty) = 0$, which is usually OK, as is obvious from (3.13).



(a) $\mathrm{Re}\, p > 0$ (b) $\mathrm{Re}\, p = 0$ (c) $\mathrm{Re}\, p < 0$

FIGURE 7. The Landau prescription for the contour of integration in (3.18).

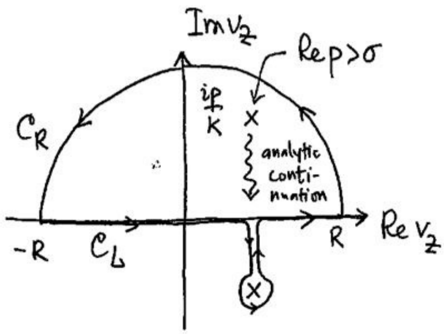


FIGURE 8. Proof of Landau's prescription [see (3.19)].

the z axis to be along $\boldsymbol{k}$.[14] Then

$$\int \mathrm{d}\boldsymbol{v}\, \frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\, \boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = \int \mathrm{d}v_z\, \frac{1}{p + ikv_z}\, k\, \frac{\partial}{\partial v_z} \underbrace{\int \mathrm{d}v_x \int \mathrm{d}v_y\, f_0(v_x, v_y, v_z)}_{\equiv F(v_z)}$$

$$= -i \int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k}. \tag{3.18}$$

Assuming, reasonably, that $F'(v_z)$ is a nice (analytic) function everywhere, we conclude that the integrand in (3.18) has one pole, $v_z = ip/k$. When $\mathrm{Re}\, p \geqslant \sigma > 0$, this pole is harmless because, in the complex plane associated with the v_z variable, it lies above the integration contour, which is the real axis, $v_z \in (-\infty, +\infty)$. We can think of analytically continuing the above integral to $\mathrm{Re}\, p < \sigma$ as moving the pole $v_z = ip/k$ down, towards and below the real axis. As long as $\mathrm{Re}\, p > 0$, this can be done with impunity, in the sense that the pole stays above the integration contour, and so the analytic continuation is simply the same integral (3.18), still along the real axis. However, if the pole moves so far down that $\mathrm{Re}\, p = 0$ or $\mathrm{Re}\, p < 0$, we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. 7. This is called the *Landau prescription* and the contour thus deformed is called the *Landau contour*, C_{L}.

Let me prove that this is indeed an analytic continuation, i.e., that the integral (3.18), adjusted to be along C_{L}, is an analytic function for all values of p. Let us cut the Landau contour at $v_z = \pm R$ and close it in the upper half-plane with a semicircle C_R of radius $R > \sigma/k$ (Fig. 8). Then, with integration running along the truncated C_{L} and

[14] NB: This means that in what follows, $k \geqslant 0$ by definition.

counterclockwise along C_R, we get, by Cauchy's formula,

$$\int_{C_\mathrm{L}} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} + \int_{C_R} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} = 2\pi i\, F'\left(\frac{ip}{k}\right). \tag{3.19}$$

Since analyticity is guaranteed for $\mathrm{Re}\, p \geqslant \sigma$, the integral along C_R is analytic. The right-hand side is also analytic, by assumption. Therefore, the integral along C_L is analytic—this is the integral along the Landau contour if we take $R \to \infty$. Q.e.d.

With the Landau prescription, our integral is calculated as follows:

$$\boxed{\int_{C_\mathrm{L}} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} & \text{if} \quad \mathrm{Re}\, p > 0, \\[2ex] \displaystyle\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} + i\pi F'\left(\frac{ip}{k}\right) & \text{if} \quad \mathrm{Re}\, p = 0, \\[2ex] \displaystyle\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - ip/k} + i2\pi F'\left(\frac{ip}{k}\right) & \text{if} \quad \mathrm{Re}\, p < 0, \end{cases}} \tag{3.20}$$

where the integrals are again over the real axis and the imaginary bits come from the contour making a half (when $\mathrm{Re}\, p = 0$) or a full (when $\mathrm{Re}\, p < 0$) circle around the pole.[15] In the case of $\mathrm{Re}\, p = 0$, or $ip = \omega$, the integral along the real axis is formally divergent and so we take its *principal value*, defined as

$$\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{\omega/k - \varepsilon} + \int_{\omega/k + \varepsilon}^{+\infty}\right] \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k}. \tag{3.21}$$

The difference between (3.21) and the usual Lebesgue definition of an integral is that the latter would be

$$\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k} = \left[\lim_{\varepsilon_1 \to 0} \int_{-\infty}^{\omega/k - \varepsilon_1} + \lim_{\varepsilon_2 \to 0} \int_{\omega/k + \varepsilon_2}^{+\infty}\right] \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k}, \tag{3.22}$$

and this, with, in general, $\varepsilon_1 \neq \varepsilon_2$, diverges logarithmically, whereas in (3.21), the divergences neatly cancel.

The $\mathrm{Re}\, p = 0$ case in (3.20),

$$\int_{C_\mathrm{L}} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k} = \mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{F'(v_z)}{v_z - \omega/k} + i\pi F'\left(\frac{\omega}{k}\right), \tag{3.23}$$

which tends to be of most use in analytical theory, is a particular instance of *Plemelj's formula:* for a real ζ and a well-behaved function f (no poles on or near the real axis),

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{f(x)}{x - \zeta \mp i\varepsilon} = \mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{f(x)}{x - \zeta} \pm i\pi f(\zeta), \tag{3.24}$$

also sometimes written as

$$\lim_{\varepsilon \to +0} \frac{1}{x - \zeta \mp i\varepsilon} = \mathcal{P}\frac{1}{x - \zeta} \pm i\pi\delta(x - \zeta), \tag{3.25}$$

---

[15] Despite what it might look like, the imaginary part of (3.20) is *not* discontinuous at $\mathrm{Re}\, p = 0$. While the second term in the expressions for the $\mathrm{Re}\, p = 0$ and $\mathrm{Re}\, p < 0$ cases jumps by a factor of 2, this is compensated by the imginary part of the first term, which is purely real at $\mathrm{Re}\, p = 0$ but not at $\mathrm{Re}\, p < 0$. You must believe me that this is true as long as you agree that Landau's analytic-continuation scheme is correct.

Finally, armed with Landau's prescription, we are ready to calculate. The dielectric function (3.11) becomes

$$\boxed{\epsilon(p,\boldsymbol{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\int_{C_{\mathrm{L}}} \mathrm{d}v_z\,\frac{F'_\alpha(v_z)}{v_z - ip/k}}\,, \tag{3.26}$$

and, analogously, our Laplace-transformed solution (3.13) becomes

$$\hat{\varphi}(p) = -\frac{4\pi i}{k^3\epsilon(p,\boldsymbol{k})}\sum_\alpha q_\alpha \int_{C_{\mathrm{L}}} \mathrm{d}v_z\,\frac{G_\alpha(v_z)}{v_z - ip/k}, \tag{3.27}$$

where $G_\alpha(v_z) = \int \mathrm{d}v_x \int \mathrm{d}v_y\, g_\alpha(v_x, v_y, v_z)$.

3.3. *Solving the Dispersion Relation: the Limit of Slow Damping/Growth*

A particularly analytically solvable and physically interesting case is one in which, for $p = -i\omega + \gamma$, $\gamma \ll \omega$ and $\gamma \ll kv_{\mathrm{th}\alpha}$, i.e., the case of the damping or growth time of the waves being longer than either their period or the time particles take to cross them. In this limit, the dispersion relation (3.17) is

$$\epsilon(p,\boldsymbol{k}) \approx \epsilon(-i\omega,\boldsymbol{k}) + i\gamma\,\frac{\partial}{\partial\omega}\,\epsilon(-i\omega,\boldsymbol{k}) = 0. \tag{3.28}$$

Setting the real part of (3.28) to zero gives the equation for the real frequency:

$$\boxed{\mathrm{Re}\,\epsilon(-i\omega,\boldsymbol{k}) = 0}\,. \tag{3.29}$$

Setting the imaginary part of (3.28) to zero gives us the damping/growth rate in terms of the real frequency:

$$\boxed{\gamma = -\mathrm{Im}\,\epsilon(-i\omega,\boldsymbol{k})\left[\frac{\partial}{\partial\omega}\,\mathrm{Re}\,\epsilon(-i\omega,\boldsymbol{k})\right]^{-1}}\,. \tag{3.30}$$

Thus, we now only need $\epsilon(p,\boldsymbol{k})$ with $p = -i\omega$. Using (3.23), we get

$$\mathrm{Re}\,\epsilon = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\,\mathcal{P}\!\int \mathrm{d}v_z\,\frac{F'_\alpha(v_z)}{v_z - \omega/k}, \tag{3.31}$$

$$\mathrm{Im}\,\epsilon = -\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{\pi}{n_\alpha}\,F'_\alpha\!\left(\frac{\omega}{k}\right). \tag{3.32}$$

Let us consider a two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- *"High-frequency"* electron waves: $\omega \gg kv_{\mathrm{th}e}$, where $v_{\mathrm{th}e} = \sqrt{2T_e/m_e}$ is the "thermal speed" of the electrons;[16] this limit will give us Langmuir waves (§3.4), slowly damped or growing (§3.5).
- *"Low-frequency"* ion waves: a particularly tractable limit will be that of "hot" electrons and "cold" ions, viz., $kv_{\mathrm{th}e} \gg \omega \gg kv_{\mathrm{th}i}$, where $v_{\mathrm{th}i} = \sqrt{2T_i/m_i}$ is the "thermal speed" of the ions; this limit will give us the sound ("ion-acoustic waves"; §3.8), which also can be damped or growing (§3.9).

[16] This is a standard well-defined quantity for a Maxwellian equilibrium distribution $F_e(v_z) = (n_e/\sqrt{\pi}\,v_{\mathrm{th}e})\exp(-v_z^2/v_{\mathrm{th}e}^2)$, but if we wish to consider a non-Maxwellian F_e, let $v_{\mathrm{th}e}$ be some typical speed characterising the width of the equilibrium distribution, defined by, e.g., (3.36).

3.4. *Langmuir Waves*

Consider the limit

$$\frac{\omega}{k} \gg v_{\rm the}, \tag{3.33}$$

i.e., the phase velocity of the waves is much greater than the typical velocity of a particle from the "thermal bulk" of the distribution. This means that in (3.31), we can expand in $v_z \sim v_{\rm the}$ being small compared to ω/k (higher values of v_z are cut off by the "thermal" fall-off of the equilibrium distribution function). Note that $\omega \gg k v_{\rm the}$ also implies $\omega \gg k v_{{\rm th}i}$ because

$$\frac{v_{{\rm th}i}}{v_{\rm the}} = \sqrt{\frac{T_i}{T_e}\frac{m_e}{m_i}} \ll 1 \tag{3.34}$$

as long as T_i/T_e is not huge.[17] Thus, (3.31) becomes

$$\begin{aligned}
{\rm Re}\,\epsilon &= 1 + \sum_\alpha \frac{\omega_{{\rm p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\frac{k}{\omega}\mathcal{P}\!\!\int {\rm d}v_z\, F_\alpha'(v_z)\left[1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + \left(\frac{kv_z}{\omega}\right)^3 + \dots\right]\\
&= 1 + \sum_\alpha \frac{\omega_{{\rm p}\alpha}^2}{k\omega}\left[\underbrace{\frac{1}{n_\alpha}\int {\rm d}v_z\, F_\alpha'(v_z)}_{=0} - \frac{k}{\omega}\underbrace{\frac{1}{n_\alpha}\int {\rm d}v_z\, F_\alpha(v_z)}_{=1}\right.\\
&\qquad\left. - 2\frac{k^2}{\omega^2}\underbrace{\frac{1}{n_\alpha}\int {\rm d}v_z\, v_z F_\alpha(v_z)}_{=0} - 3\frac{k^3}{\omega^3}\underbrace{\frac{1}{n_\alpha}\int {\rm d}v_z\, v_z^2 F_\alpha(v_z)}_{=v_{{\rm th}\alpha}^2/2} + \dots\right]\\
&= 1 - \sum_\alpha \frac{\omega_{{\rm p}\alpha}^2}{\omega^2}\left[1 + \frac{3}{2}\frac{k^2 v_{{\rm th}\alpha}^2}{\omega^2} + \dots\right],
\end{aligned} \tag{3.35}$$

where I have integrated by parts everywhere, assumed that there are no mean flows, $\langle v_z\rangle = 0$, and, in the last term, used

$$\langle v_z^2\rangle = \frac{v_{{\rm th}\alpha}^2}{2}, \tag{3.36}$$

which is indeed the case for a Maxwellian F_α or, if F_α is not a Maxwellian, can be viewed as the definition of $v_{{\rm th}\alpha}$.

The ion contribution to (3.35) is small because

$$\frac{\omega_{{\rm p}i}^2}{\omega_{\rm pe}^2} = \frac{Zm_e}{m_i} \ll 1, \tag{3.37}$$

so ions do not participate in this dynamics at all. Therefore, to lowest order, the dispersion relation (3.29) becomes

$${\rm Re}\,\epsilon \approx 1 - \frac{\omega_{\rm pe}^2}{\omega^2} = 0 \quad\Rightarrow\quad \boxed{\omega^2 = \omega_{\rm pe}^2 = \frac{4\pi e^2 n_e}{m_e}}\,. \tag{3.38}$$

This is the Tonks & Langmuir (1929) dispersion relation for what is known as *Langmuir, or plasma, oscillations*. This is the formal derivation of the result that we already had in §2.1, on less mathematically rigorous, physical grounds.

---

[17] For hydrogen plasma, $\sqrt{m_i/m_e} \approx 42$, the answer to the Ultimate Question of Life, Universe and Everything (Adams 1979).

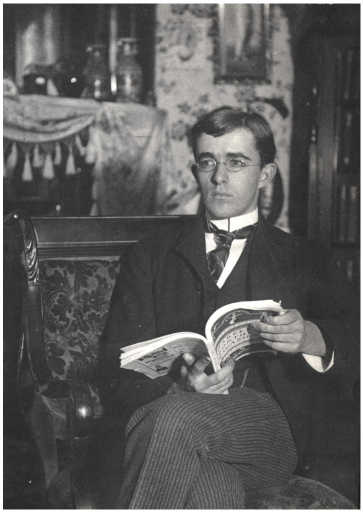


FIGURE 9. Irving Langmuir (1881-1957), the prolific American engineer who first called plasma *plasma*, and discovered its oscillations, as well as many other things. Photo taken from https://www.sciencehistory.org/education/scientific-biographies/irving-langmuir-and-katharine-burr-blodgett/.

We can do a little better if we retain the (small) k-dependent term in (3.35):

$$\mathrm{Re}\,\epsilon \approx 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\left(1 + \underbrace{\frac{3}{2}\frac{k^2 v_{\mathrm{th}e}^2}{\omega^2}}_{\substack{\text{use}\\ \omega^2\approx\omega_{\mathrm{pe}}^2}}\right) = 0 \quad \Rightarrow \quad \boxed{\omega^2 \approx \omega_{\mathrm{pe}}^2(1 + 3k^2\lambda_{\mathrm{D}e}^2)}\,, \tag{3.39}$$

where $\lambda_{\mathrm{D}e} = v_{\mathrm{th}e}/\sqrt{2}\,\omega_{\mathrm{pe}} = \sqrt{T_e/4\pi e^2 n_e}$ is the "electron Debye length" [cf. (1.6)]. Equation (3.39) is the Bohm & Gross (1949*a*) dispersion relation, describing an upgrade of the Langmuir oscillations to dispersive *Langmuir waves*, which have a non-zero group velocity (this effect is due to the electron pressure perturbation joining the electric field in providing the restoring force for the waves: see Exercise 3.1).

Note that all this is only valid for $\omega \gg k v_{\mathrm{th}e}$, which we now see is equivalent to

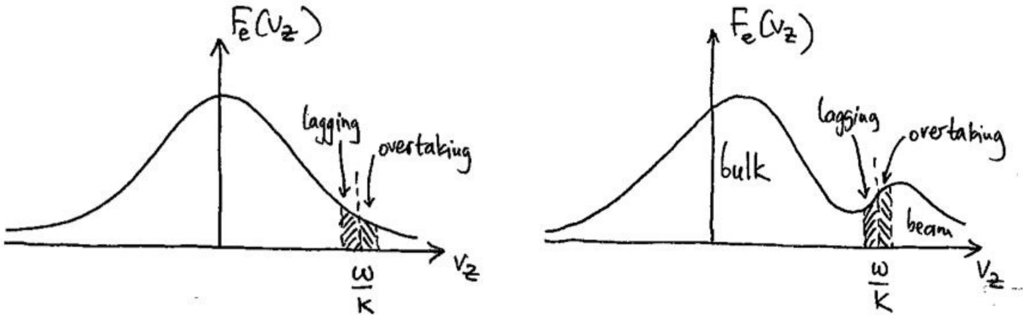
$$k\lambda_{\mathrm{D}e} \ll 1 \tag{3.40}$$

(the wave length of the perturbation is long compared to the Debye length).

Exercise 3.1. Langmuir hydrodynamics.[18] Starting from the linearised kinetic equation for electrons and ignoring perturbations of the ion distribution function completely, work out the fluid equations for electrons (i.e., the evolution equations for the electron density n_e and velocity u_e) and show that you can recover the Langmuir waves (3.39) if you assume that electrons behave as a 1D adiabatic fluid (i.e., have the equation of state $p_e n_e^{-\gamma} = \mathrm{const}$ with $\gamma = 3$). You can prove that they indeed do this by calculating their density and pressure directly from the Landau solution for the perturbed distribution function (see §§5.3 and 5.6), ignoring resonant particles. The "hydrodynamic" description of Langmuir waves will reappear in §8.

---

[18]This is based on the 2017 exam question.



(a) $\omega F'(\omega/k) < 0$: Landau damping (b) $\omega F'(\omega/k) > 0$: instability

FIGURE 10. The Landau resonance (particle velocities equalling phase speed of the wave $v_z = \omega/k$) leads to damping of the wave if more particles lag just behind than overtake the wave and to instability in the opposite case.

3.5. *Landau Damping and Kinetic Instabilities*

Now let us calculate the damping rate of Langmuir waves using (3.30):

$$\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega} \approx \frac{2\omega_{\mathrm{pe}}^2}{\omega^3}, \quad \mathrm{Im}\,\epsilon \approx -\frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{\pi}{n_e}F_e'\left(\frac{\omega}{k}\right) \quad \Rightarrow \quad \boxed{\gamma \approx \frac{\pi}{2}\frac{\omega^3}{k^2}\frac{1}{n_e}F_e'\left(\frac{\omega}{k}\right)}, \tag{3.41}$$

where ω is given by (3.39) (if F_e is Maxwellian, the dispersive correction in ω^2, when substituted into the exponential, will change γ by a factor of order unity, hence it is worth keeping). Provided $\omega F'(\omega/k) < 0$ (as would be the case, e.g., for any distribution monotonically decreasing with $|v_z|$; see Fig. 10a), $\gamma < 0$ and so this is indeed a damping rate, the celebrated *Landau damping* (Landau 1946; it was confirmed experimentally two decades later, by Malmberg & Wharton 1964).

The same theory also describes a class of *kinetic instabilities:* if $\omega F'(\omega/k) > 0$, then $\gamma > 0$, so perturbations grow exponentially with time. An iconic example is the *bump-on-tail instability* (Fig. 10b), which arises when a high-energy ($v_z \gg v_{\mathrm{th}e}$) electron beam is injected into a plasma[19] and whose quasilinear saturation we will study in great detail in §6.

We see that the damping or growth of plasma waves occur via their interaction with the particles whose velocities coincide with the phase velocity of the wave (*"Landau resonance"*). Because such particles are moving in phase with the wave, its electric field is stationary in their reference frame and so can do work on these particles, giving its energy to them (damping) or receiving energy from them (instability). In contrast, other, out-of-phase, particles experience no mean energy change over time because the field that they "see" is oscillating. It turns out (§3.6) that the process works in the spirit of socialist redistribution: the particles slightly lagging behind the wave will, on average, receive energy from it, damping the wave, whereas those overtaking the wave will have some of their energy taken away, amplifying the wave. The condition $\omega F'(\omega/k) < 0$ corresponds to the stragglers being more numerous than the strivers, leading to net

[19]Here we are dealing with the case of a "warm beam" (meaning that it has a finite width in velocity space). It turns out that there exists also another instability, leading to growth of perturbations with ω/k to the right of the bump's peak, due to a different, "fluid" kind of resonance and possible even for "cold beams" (i.e., beams of particles that all have the same velocity): see §3.7.

damping; $\omega F'(\omega/k) > 0$ implies the opposite, leading to an instability (which then leads to flattening of the distribution: see §6).

Let us note again that these results are quantitatively valid only in the limit (3.33), or, equivalently, (3.40). It makes sense that damping should be slow ($\gamma \ll \omega$) in the limit where the waves propagate much faster than the majority of the electrons ($\omega/k \gg v_{\rm the}$) and so can interact only with a small number of particularly fast particles (for a Maxwellian equilibrium distribution, it is an exponentially small number $\sim e^{-\omega^2/k^2 v_{\rm the}^2}$). If, on the other hand, $\omega/k \sim v_{\rm the}$, the waves interact with the majority population and the damping should be strong: *a priori*, we might expect $\gamma \sim k v_{\rm the}$.[20]

Exercise 3.2. Stability of isotropic distributions. Prove that if $f_{0e}(v_x, v_y, v_z) = f_{0e}(v)$, i.e., if it is a 3D-isotropic distribution, monotonic or otherwise, the Langmuir waves at $k\lambda_{\rm De} \ll 1$ are always damped (this is solved in Lifshitz & Pitaevskii 1981; the statement of stability of isotropic distributions is in fact valid much more generally than just for long-wavelength Langmuir waves: see Exercise 4.2).

Landau's method of working out waves and damping in collisionless plasmas, and in particular his prescription for dealing with the singularities in the integrals, has always elicited a degree of dissatisfaction in the minds of some mathematically inclined physicists and motivated them to search for alternatives. Perhaps the earliest and best known such alternative is the formalism due to van Kampen. His objective was more mathematical rigour—but even if this is of limited appeal to you, the book by van Kampen & Felderhof (1967) is still a good read and a good chance to question and re-examine your understanding of how it all works.

A key question that preoccupied van Kampen and many of those who re-examined Landau's solution later on was whether the initial-value problem for the linear evolution of perturbations in a plasma could be solved in the usual way in which such things are done elsewhere in physics: by decomposing the initial perturbation into some convenient set of normal modes, advancing each of them in time, and then reassembling them back into the desired solution. The answer is yes—van Kampen did find a complete set of modes, although they were not eigenfunctions of a Hermitian operator and thus (arguably) were not very user-friendly. In a short paper by Ramos & White (2018), you will find the most recent and the most transparent (in my view) scheme for how to construct normal modes that *are* eigenfunctions of a Hermitian operator.[21] One curious corollary is that it is possible to cook up special initial perturbations that will not decay at the Landau rate and, in fact, can have any time evolution that one cares to specify! If this intrigues or disturbs you, follow the paper trail from Ramos & White (2018) backward in time.[22]

Landau damping became a *cause célèbre* in the hard-core mathematics community, as well as in the wider science world, with the award of the Fields Medal in 2010 to Cédric Villani, who proved (with C. Mouhot) that, basically, Landau's solution of the linearised Vlasov equation survived as a solution of the full nonlinear Vlasov equation for small enough and regular enough initial perturbations: a "popular" account of this is Villani (2014). The regularity restriction is apparently important and the result can break down in interesting ways (Bedrossian 2016). The culprit is plasma echo, of which more will be said in §12 (without any claim to mathematical rigour).

---

[20] This is indeed correct. You can confirm it numerically using (3.90) and (3.96).

[21] This line of thinking subsequently made its way into galactic dynamics: see Lau & Binney (2021*a*,*b*).

[22] Another amusing latter-day exercise is the paper by Heninger & Morrison (2018), where (following up on Morrison 1994, 2000), van Kampen's scheme is recast as a new transform, called "G-transform", to be used instead of the Laplace transform to solve Landau's initial-value problem.

3.6. *Physical Picture of Landau Damping*

The following simple argument (Lifshitz & Pitaevskii 1981) illustrates the physical mechanism of Landau damping.

Consider an electron moving along the z axis, subject to a wave-like electric field:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = v_z, \tag{3.42}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = -\frac{e}{m_e}\,E(z,t) = -\frac{e}{m_e}\,E_0\cos(\omega t - kz)e^{\varepsilon t}. \tag{3.43}$$

I have given the electric field a slow time dependence, $E \propto e^{\varepsilon t}$, but will later take $\varepsilon \to +0$—this describes the field switching on infinitely slowly from $t=-\infty$. Let us assume that the amplitude E_0 of the electric field is so small that it changes the electron's trajectory only a little over several wave periods. Then the equations of motion can be solved perturbatively.

The lowest-order ($E_0 = 0$) solution is

$$v_z(t) = v_0 = \mathrm{const}, \quad z(t) = v_0 t. \tag{3.44}$$

In the next order, let

$$v_z(t) = v_0 + \delta v_z(t), \quad z(t) = v_0 t + \delta z(t). \tag{3.45}$$

Equation (3.43) becomes

$$\frac{\mathrm{d}\delta v_z}{\mathrm{d}t} = -\frac{e}{m_e}\,E(z(t),t) \approx -\frac{e}{m_e}\,E(v_0 t,t) = -\frac{eE_0}{m_e}\,\mathrm{Re}\,e^{[i(\omega-kv_0)+\varepsilon]t}. \tag{3.46}$$

Integrating this gives

$$\begin{aligned}\delta v_z(t) &= -\frac{eE_0}{m_e}\int_0^t \mathrm{d}t'\,\mathrm{Re}\,e^{[i(\omega-kv_0)+\varepsilon]t'}\\ &= -\frac{eE_0}{m_e}\,\mathrm{Re}\,\frac{e^{[i(\omega-kv_0)+\varepsilon]t}-1}{i(\omega-kv_0)+\varepsilon}\\ &= -\frac{eE_0}{m_e}\,\frac{\varepsilon e^{\varepsilon t}\cos[(\omega-kv_0)t] - \varepsilon + (\omega-kv_0)e^{\varepsilon t}\sin[(\omega-kv_0)t]}{(\omega-kv_0)^2+\varepsilon^2}.\end{aligned} \tag{3.47}$$

Integrating again, one gets

$$\begin{aligned}\delta z(t) &= \int_0^t \mathrm{d}t'\,\delta v_z(t')\\ &= -\frac{eE_0}{m_e}\int_0^t \mathrm{d}t'\,\mathrm{Re}\,\frac{e^{[i(\omega-kv_0)+\varepsilon]t'}-1}{i(\omega-kv_0)+\varepsilon}\\ &= -\frac{eE_0}{m_e}\left\{\mathrm{Re}\,\frac{e^{[i(\omega-kv_0)+\varepsilon]t}-1}{[i(\omega-kv_0)+\varepsilon]^2} - \frac{\varepsilon t}{(\omega-kv_0)^2+\varepsilon^2}\right\}\\ &= -\frac{eE_0}{m_e}\left\{\frac{\left[\varepsilon^2-(\omega-kv_0)^2\right]\left\{e^{\varepsilon t}\cos[(\omega-kv_0)t]-1\right\} + 2\varepsilon(\omega-kv_0)e^{\varepsilon t}\sin[(\omega-kv_0)t]}{[(\omega-kv_0)^2+\varepsilon^2]^2}\right.\\ &\qquad\left. - \frac{\varepsilon t}{(\omega-kv_0)^2+\varepsilon^2}\right\}.\end{aligned} \tag{3.48}$$

The work done by the field on the electron per unit time, averaged over time, is the power gained

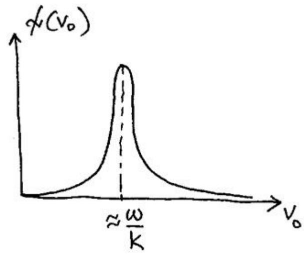


FIGURE 11. The function $\chi(v_0)$ defined in (3.49).

by the electron:

$$
\begin{aligned}
\delta P(v_0) &= -e\left\langle E(z(t),t)v_z(t)\right\rangle \\
&\approx -e\left\langle\left[E(v_0t,t)+\delta z(t)\,\frac{\partial E}{\partial z}(v_0t,t)\right][v_0+\delta v_z(t)]\right\rangle \\
&= -eE_0e^{\varepsilon t}\left\langle \underbrace{v_0\cos[(\omega-kv_0)t]}_{\substack{\text{vanishes}\\ \text{under}\\ \text{averaging}}} + \underbrace{\delta v_z(t)\cos[(\omega-kv_0)t]}_{\substack{\text{only cos term}\\ \text{from (3.47)}\\ \text{survives}\\ \text{averaging}}} + \underbrace{v_0\delta z(t)k\sin[(\omega-kv_0)t]}_{\substack{\text{only sin term}\\ \text{from (3.48)}\\ \text{survives}\\ \text{averaging}}}\right\rangle \\
&= \frac{e^2E_0^2}{2m_e}\,e^{2\varepsilon t}\left\{\frac{\varepsilon}{(\omega-kv_0)^2+\varepsilon^2}+\frac{2kv_0\varepsilon(\omega-kv_0)}{[(\omega-kv_0)^2+\varepsilon^2]^2}\right\} \\
&= \frac{e^2E_0^2}{2m_e}\,e^{2\varepsilon t}\,\frac{\mathrm{d}}{\mathrm{d}v_0}\underbrace{\frac{\varepsilon v_0}{(\omega-kv_0)^2+\varepsilon^2}}_{\equiv\,\chi(v_0)}.
\end{aligned}
\tag{3.49}
$$

We see (Fig. 11) that
—if the electron is lagging behind the wave, $v_0 \lesssim \omega/k$, then $\chi'(v_0) > 0 \;\Rightarrow\; \delta P(v_0) > 0$, so energy goes from the field to the electron (the wave is damped);
—if the electron is overtaking the wave, $v_0 \gtrsim \omega/k$, then $\chi'(v_0) < 0 \;\Rightarrow\; \delta P(v_0) < 0$, so energy goes from the electron to the field (the wave is amplified).

Now remember that we have a whole distribution of these electrons, $F(v_z)$. So the total power per unit volume going into (or out of) them is

$$
\begin{aligned}
P = \int \mathrm{d}v_z\,F(v_z)\delta P(v_z) &= \frac{e^2E_0^2e^{2\varepsilon t}}{2m_e}\int \mathrm{d}v_z\,F(v_z)\chi'(v_z) \\
&= -\frac{e^2E_0^2e^{2\varepsilon t}}{2m_e}\int \mathrm{d}v_z\,F'(v_z)\chi(v_z).
\end{aligned}
\tag{3.50}
$$

Noticing that, by Plemelj's formula (3.25), in the limit $\varepsilon\to+0$,

$$
\chi(v_z) = \frac{\varepsilon v_z}{(\omega-kv_z)^2+\varepsilon^2} = -\frac{iv_z}{2}\left(\frac{1}{kv_z-\omega-i\varepsilon}-\frac{1}{kv_z-\omega+i\varepsilon}\right) \to \pi\,\frac{\omega}{k^2}\,\delta\Big(v_z-\frac{\omega}{k}\Big), \tag{3.51}
$$

we conclude

$$
P = -\frac{e^2E_0^2}{2m_ek^2}\,\pi\omega F'\Big(\frac{\omega}{k}\Big). \tag{3.52}
$$

As in §3.5, we find damping if $\omega F'(\omega/k) < 0$ and instability if $\omega F'(\omega/k) > 0$.

Thus, around the wave-particle resonance $v_z = \omega/k$, the particles just lagging behind the wave receive energy from the wave and those just overtaking it give up energy to it. Therefore, qualitatively, damping occurs if the former particles are more numerous than the latter. We see that Landau's mathematics in §§3.1–3.5 led us to a result that makes physical sense.

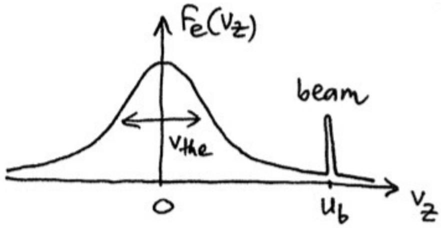


FIGURE 12. Electron distribution with a cold beam; see (3.54).

3.7. *Hot and Cold Beams*

Let us return to the unstable situation, when a high-energy beam produces a bump on the tail of the distribution function and thus electrostatic perturbations can suck energy out of the beam and grow in the region of wave numbers where $v_0 < \omega/k < u_{\rm b}$. Here v_0 is the point of the minimum of the distribution in Fig. 10(b) and $u_{\rm b}$ is the point of the maximum of the bump, which is the velocity of the beam; we are assuming that $u_{\rm b} \gg v_{\rm the}$. In view of (3.41), the instability will have a greater growth rate if the bump's slope is steeper, i.e., if the beam is colder (narrower in v_z space).

Imagine modelling the beam by a little Maxwellian distribution with mean velocity $u_{\rm b}$, tucked onto the bulk distribution:[23]

$$F_e(v_z) = \frac{n_e - n_{\rm b}}{\sqrt{\pi}\, v_{\rm the}} \exp\left(-\frac{v_z^2}{v_{\rm the}^2}\right) + \frac{n_{\rm b}}{\sqrt{\pi}\, v_{\rm b}} \exp\left[-\frac{(v_z - u_{\rm b})^2}{v_{\rm b}^2}\right], \tag{3.53}$$

where $n_{\rm b}$ is the density of the beam, $v_{\rm b}$ is its width, and so $T_{\rm b} = m_e v_{\rm b}^2/2$ is its "temperature", just like $T_e = m_e v_{\rm the}^2/2$ is the temperature of the thermal bulk. A colder beam will have less of a thermal spread around $u_{\rm b}$. It turns out that if we make the width of the beam sufficiently small, while the range of wavenumbers of the bump-on-tail instability shrinks, another instability appears, whose origin is hydrodynamic rather than kinetic. Let us work it out.

Consider a very simple limiting case of the distribution (3.53): let $v_{\rm b} \to 0$ and $n_{\rm b} \ll n_e$. Then (Fig. 12)

$$F_e(v_z) = F_{\rm M}(v_z) + n_{\rm b}\delta(v_z - u_{\rm b}), \tag{3.54}$$

where $F_{\rm M}$ is the bulk Maxwellian from (3.53) (with density $\approx n_e$, neglecting $n_{\rm b}$ in comparison). Let us substitute the distribution (3.54) into the dielectric function (3.26), seek solutions with $p/k \gg v_{\rm the}$, expand the part containing $F_{\rm M}$ in the same way as we did in §3.4,[24] neglect the ion contribution for the same reason as we did there, and deal with $\delta'(v_z - u_{\rm b})$ in the integrand via integration by parts. The resulting dispersion relation is

$$\epsilon \approx 1 + \frac{\omega_{\rm pe}^2}{p^2} - \frac{n_{\rm b}}{n_e}\frac{\omega_{\rm pe}^2}{(ku_{\rm b} - ip)^2} = 0. \tag{3.55}$$

Since $n_{\rm b} \ll n_e$, the last term can only matter for those perturbations that are close to

---

[23]The fact that we are working in 1D means that we are restricting our consideration to perturbations whose wave numbers $\boldsymbol{k}$ are parallel to the beam's velocity. In general, allowing transverse wave numbers brings into play the transverse (electromagnetic) part of the dielectric tensor (see Q-2). However, for non-relativistic beams, the fastest-growing modes will still be the longitudinal, electrostatic ones (see, e.g., Alexandrov *et al.* 1984, §32).

[24]We can treat the Landau contour as simply running along the real axis because we are expecting to find a solution with $\mathrm{Re}\, p > 0$ [see (3.20)], for reasons independent of the Landau resonance.

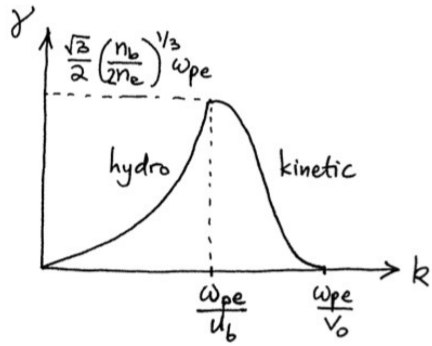


FIGURE 13. Sketch of the growth rate of the hydrodynamic and kinetic beam instabilities: see (3.57) for $k < \omega_{\rm pe}/u_{\rm b}$, (3.59) for $k = \omega_{\rm pe}/u_{\rm b}$, and (3.41) for $\omega_{\rm pe}/u_{\rm b} < k < \omega_{\rm pe}/v_0$, where v_0 is the point of the minimum of the distribution in Fig. 10(b) and $u_{\rm b}$ is the point of the maximum of the bump.

resonance with the beam (this is called the *Cherenkov resonance*):

$$p = -iku_{\rm b} + \gamma, \quad \gamma \ll ku_{\rm b}. \tag{3.56}$$

This turns (3.55) into

$$1 - \frac{\omega_{\rm pe}^2}{k^2u_{\rm b}^2} + \frac{n_{\rm b}}{n_e}\frac{\omega_{\rm pe}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \gamma = \pm\sqrt{\frac{n_{\rm b}}{n_e}}\left(\frac{1}{k^2u_{\rm b}^2} - \frac{1}{\omega_{\rm pe}^2}\right)^{-1/2}. \tag{3.57}$$

The expression under the square root is positive and so there is a growing mode only if $k < \omega_{\rm pe}/u_{\rm b}$. This is in contrast to the case of a hot (or warm) beam in §3.5: there, having a kinetic instability required $\omega F_e'(\omega/k) > 0$, which was only possible at $k > \omega_{\rm pe}/u_{\rm b}$ (the phase speed of the perturbations had to be to the left of the bump's maximum). The new instability that we have found—the *hydrodynamic beam instability*—has the largest growth rate at $ku_{\rm b} = \omega_{\rm pe}$, i.e., when the beam and the plasma oscillations are in resonance, in which case, to resolve the singularity, we need to retain γ in the second term in (3.55). Doing so and expanding in γ, we get

$$\epsilon \approx 1 - \frac{\omega_{\rm pe}^2}{(\omega_{\rm pe} + i\gamma)^2} + \frac{n_{\rm b}}{n_e}\frac{\omega_{\rm pe}^2}{\gamma^2} \approx \frac{2i\gamma}{\omega_{\rm pe}} + \frac{n_{\rm b}}{n_e}\frac{\omega_{\rm pe}^2}{\gamma^2} = 0. \tag{3.58}$$

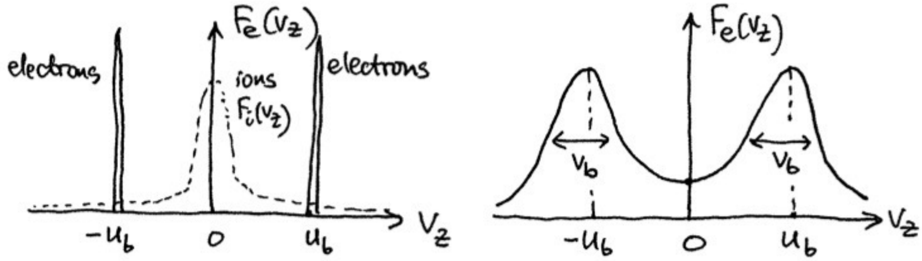
Solution:

$$\boxed{\gamma = \left(\frac{\pm\sqrt{3} + i}{2}, -i\right)\left(\frac{n_{\rm b}}{2n_e}\right)^{1/3}\omega_{\rm pe}}\,. \tag{3.59}$$

The unstable root ($\mathrm{Re}\,\gamma > 0$) is the interesting one. The growth rate of the combined beam instability, hydrodynamic and kinetic, is sketched in Fig. 13.

Exercise 3.3. This instability is called "hydrodynamic" because it can be derived from fluid equations (cf. Exercise 3.1) describing cold electrons ($v_{\rm the} = 0$) and a cold beam ($v_{\rm b} = 0$). Convince yourself that this is the case.

Exercise 3.4. Using the model distribution (3.53), work out the conditions on $v_{\rm b}$ and $n_{\rm b}$ that must be satisfied in order for our derivation of the hydrodynamic beam instability to be valid, i.e., for (3.55) to be a good approximation to the true dispersion relation. What is the effect of finite $v_{\rm b}$ on the hydrodynamic instability? Sketch the growth rate of unstable perturbations as a function of k, taking into account both the hydrodynamic instability and the kinetic one, as



(a) Cold streams: see (3.60) (b) Hot streams: see (4.16) and Q-4

FIGURE 14. Two streams.

well as the Landau damping.

Exercise 3.5. Two-stream instability. This is a popular instability[25] that arises, e.g., in a situation where the plasma consists of two cold counter-streams of electrons propagating against a quasineutrality-enforcing background of effectively immobile ions (Fig. 14a). Model the corresponding electron distribution by

$$F_e(v_z) = \frac{n_e}{2}\left[\delta(v_z - u_b) + \delta(v_z + u_b)\right] \tag{3.60}$$

and solve the resulting dispersion relation (where the ion terms can be neglected for the same reason as in §3.4). Find the wave number at which perturbations grow fastest and the corresponding growth rate. Find also the maximum wave number at which perturbations can grow. If you want to know what happens when the two streams are warm (have a finite thermal spread v_b; Fig. 14b), a nice fully tractable quantitative model of such a situation is the double-Lorentzian distribution (4.16). The dispersion relation for it can be solved exactly: this is done in Q-4. You will again find a hydrodynamic instability, but is there also a kinetic one (due to Landau resonance)? It is an interesting and non-trivial question why not.

3.8. *Ion-Acoustic Waves*

Let us now see what happens at lower frequencies,

$$v_{\text{the}} \gg \frac{\omega}{k} \gg v_{\text{th}i}, \tag{3.61}$$

i.e., when the waves propagate slower than the bulk of the electron distribution but faster than the bulk of the ion one (Fig. 15). This is another regime in which we might expect to find weakly damped waves: they are out of phase with the majority of the ions, so $F_i'(\omega/k)$ might be small because $F_i(\omega/k)$ is small, while as far as the electrons are concerned, the phase speed of the waves is deep in the core of the distribution, perhaps close to its maximum at $v_z = 0$ (if that is where its maximum is) and so $F_e'(\omega/k)$ might turn out to be small because $F_e(v_z)$ changes slowly in that region.

To make this more specific, let us consider Maxwellian electrons:

$$F_e(v_z) = \frac{n_e}{\sqrt{\pi}\, v_{\text{the}}} \exp\left[-\frac{(v_z - u_e)^2}{v_{\text{the}}^2}\right], \tag{3.62}$$

[25] It was discovered by engineers (Haeff 1949; Pierce & Hebenstreit 1949) and quickly adopted by physicists (Bohm & Gross 1949*b*). Buneman (1958) realised that a case with an electron and an ion stream (i.e., with plasma carrying a current) is unstable in a somewhat analogous way (see Q-5, and also Q-11). The kinetic counterpart to the latter situation is the ion-acoustic instability derived in §3.9 (similarly to the way in which the bump-on-tail instability was the kinetic counterpart to the hydrodynamic beam instability). In §4.4, I will discuss in a more general way the stability of distributions featuring streams.

where we are, in general, allowing the electrons to have a mean flow (current). We will assume that $u_e \ll v_{\rm the}$ but allow $u_e \sim \omega/k$. We can anticipate that this will give us an interesting new effect. Indeed,

$$F_e'(v_z) = -\frac{2(v_z - u_e)}{v_{\rm the}^2}\, F_e(v_z). \tag{3.63}$$

For resonant particles, $v_z = \omega/k$, the prefactor will be small, so we can hope for $\gamma \ll \omega$, as anticipated above, but note that its sign will depend on the relative size of u_e and ω/k and so we might (we will!) get an instability (§3.9).

But let us not get ahead of ourselves: we must first calculate the real frequency $\omega(k)$ of these waves, from (3.29) and (3.31):

$$\mathrm{Re}\,\epsilon = 1 - \frac{\omega_{\rm pe}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\, \frac{F_e'(v_z)}{v_z - \omega/k} - \underbrace{\frac{\omega_{\rm p\it i}^2}{k^2}\frac{1}{n_i}\mathcal{P}\!\int \mathrm{d}v_z\, \frac{F_i'(v_z)}{v_z - \omega/k}}_{\approx \frac{\omega_{\rm p\it i}^2}{\omega^2}\left(1 + 3k^2\lambda_{{\rm D}i}^2\right)} = 0. \tag{3.64}$$

The last (ion) term in this equation can be expanded in $kv_z/\omega \ll 1$ in exactly the same way as it was done in (3.35). The expansion is valid provided

$$k\lambda_{{\rm D}i} \ll 1, \tag{3.65}$$

and I will retain only the lowest-order term, dropping the $k^2\lambda_{{\rm D}i}^2$ correction. The second (electron) term in (3.64) is subject to the opposite limit, $v_z \gg \omega/k$, so, using (3.63),

$$\frac{\omega_{\rm pe}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\, \frac{F_e'(v_z)}{v_z - \omega/k} \approx -\frac{\omega_{\rm pe}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\, \frac{2(v_z - u_e)}{v_{\rm the}^2 v_z}\, F_e(v_z) \approx -\frac{2\omega_{\rm pe}^2}{k^2 v_{\rm the}^2} = -\frac{1}{k^2\lambda_{{\rm D}e}^2}, \tag{3.66}$$

where I have neglected $u_e \ll v_z$ because this integral is over the thermal bulk of the electron distribution.

With all these approximations, (3.64) becomes

$$\mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2\lambda_{{\rm D}e}^2} - \frac{\omega_{\rm p\it i}^2}{\omega^2} = 0. \tag{3.67}$$

The dispersion relation is then

$$\boxed{\omega^2 = \frac{\omega_{\rm p\it i}^2}{1 + 1/k^2\lambda_{{\rm D}e}^2} = \frac{k^2 c_{\rm s}^2}{1 + k^2\lambda_{{\rm D}e}^2}}\,, \tag{3.68}$$

where

$$c_{\rm s} = \omega_{\rm p\it i}\lambda_{{\rm D}e} = \sqrt{\frac{ZT_e}{m_i}} \tag{3.69}$$

is the *sound speed*, called that because, if $k\lambda_{{\rm D}e} \ll 1$, (3.68) describes a wave that is very obviously a sound, or *ion-acoustic, wave:*

$$\boxed{\omega = \pm k c_{\rm s}}\,. \tag{3.70}$$

The phase speed of this wave is the sound speed, $\omega/k = c_{\rm s}$. That the expression (3.69) for $c_{\rm s}$ combines electron temperature and ion mass is a hint as to the underlying physics of sound propagation in plasma: ions provide the inertia, electrons the restoring pressure force (see Exercise 3.6).

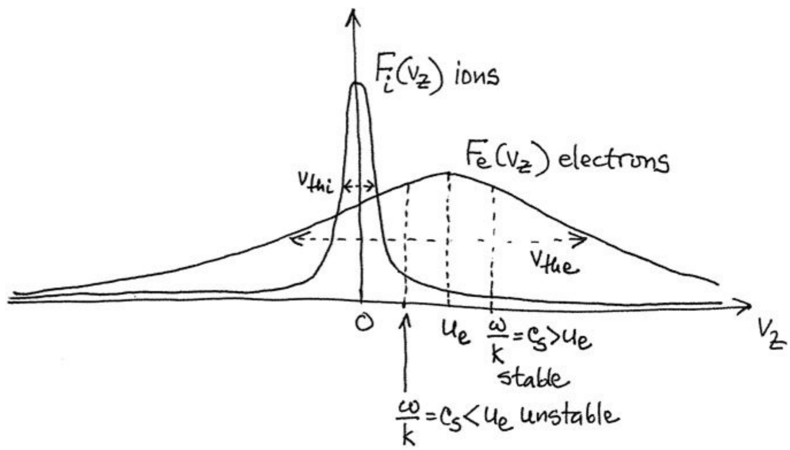


FIGURE 15. Ion-acoustic resonance: damping ($c_s > u_e$) or instability ($c_s < u_e$). Ion Landau damping is weak because $c_s \gg v_{\rm thi}$, so in the tail of $F_i(v_z)$; electron damping/instability is also weak because $u_e, c_s \ll v_{\rm the}$, so close to the peak $F_e(v_z)$.

We can now check under what circumstances the condition (3.61) is indeed satisfied:

$$\frac{c_s}{v_{\rm the}} = \sqrt{\frac{Zm_e}{2m_i}} \ll 1, \qquad \frac{c_s}{v_{\rm thi}} = \sqrt{\frac{ZT_e}{2T_i}} \gg 1, \tag{3.71}$$

with the latter condition requiring that the ions should be colder than the electrons.

Exercise 3.6. Hydrodynamics of sound waves.[26] Starting from the linearised kinetic equations for ions and electrons, work out the fluid equations for the plasma, i.e., the evolution equations for its mass density and mass flow velocity. Assuming $m_e \ll m_i$ (negligible electron inertia) and $T_i \ll T_e$ (cold ions), show that these equations are

$$\frac{\partial \delta n_i}{\partial t} + n_i \boldsymbol{\nabla}\cdot\boldsymbol{u}_i = 0, \qquad m_i n_i \frac{\partial \boldsymbol{u}_i}{\partial t} + \boldsymbol{\nabla}\delta p_e = 0, \tag{3.72}$$

and that the sound waves (3.70) with c_s given by (3.69) are recovered if electrons have the equation of state of an isothermal fluid. Why, and under what assumptions, should they be isothermal physically? Prove mathematically that they indeed are. Why is the equation of state for electrons different in a sound wave than in a Langmuir wave (see Exercise 3.1)? We will revisit ion hydrodynamics in §8.

3.9. *Damping of Ion-Acoustic Waves and Ion-Acoustic Instability*

Are ion acoustic waves damped? Can they grow? We have a standard protocol for answering this question: calculate $\mathrm{Re}\,\epsilon$ and $\mathrm{Im}\,\epsilon$ and substitute into (3.30). Using (3.67) and (3.32), we find

$$\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega} = \frac{2\omega_{\rm pi}^2}{\omega^3}, \qquad \mathrm{Im}\,\epsilon = -\frac{\omega_{\rm pe}^2}{k^2}\frac{\pi}{n_e}F_e'\left(\frac{\omega}{k}\right) - \frac{\omega_{\rm pi}^2}{k^2}\frac{\pi}{n_i}F_i'\left(\frac{\omega}{k}\right). \tag{3.73}$$

The two terms in $\mathrm{Im}\,\epsilon$ represent the interaction between the waves and, respectively, electrons and ions. The ion term is small both on account of $\omega_{\rm pi} \ll \omega_{\rm pe}$ and, assuming Maxwellian ions, of the exponential smallness of $F_i(\omega/k) \propto \exp[-(\omega/kv_{\rm thi})^2]$. We are

[26] The 2020 exam question was based on this.

then left with

$$\gamma = -\frac{\operatorname{Im}\epsilon}{\partial(\operatorname{Re}\epsilon)/\partial\omega} = -\sqrt{\pi}\,\frac{\omega^3}{k^2 v_{\mathrm{the}}^3}\,\frac{m_i}{Zm_e}\left(\frac{\omega}{k} - u_e\right), \tag{3.74}$$

where we have used (3.63). In the long-wavelength limit, $k\lambda_{\mathrm{D}e} \ll 1$, we have $\omega = \pm kc_{\mathrm{s}}$, and so, for the "+" mode,

$$\boxed{\gamma = -\sqrt{\frac{\pi Zm_e}{8m_i}}\,k\left(c_{\mathrm{s}} - u_e\right)}\,. \tag{3.75}$$

If the electron flow is subsonic, $u_e < c_{\mathrm{s}}$, this describes the *Landau damping of ion acoustic waves on hot electrons.* If, on the other hand, the electron flow is supersonic, the sign of γ reverses[27] and we discover the *ion-acoustic instability:* excitation of ion acoustic waves by a fast electron current. The instability belongs to the same general class as, e.g., the bump-on-tail instability (§3.5) in that it involves waves sucking energy from particles, but the new conceptual feature here is that such energy conversion can result from a collaboration between different particle species (electrons supplying the energy, ions carrying the wave).

There is a host of related instabilities involving various combinations of electron and ion beams, currents, streams and counter-streams—excellent treatments of them can be found in the textbooks by Krall & Trivelpiece (1973) and by Alexandrov *et al.* (1984) or in the review by Davidson (1983). I shall return to this topic in §4.4.

Exercise 3.7. Damping of sound waves on ions.[28] Find the ion contribution to the damping of ion-acoustic waves. Under what conditions does it become comparable to, or larger than, the electron contribution?

Exercise 3.8. What happens if $u_e \gg v_{\mathrm{the}}$?

3.10. *Ion Langmuir Waves*

Note that since

$$\frac{\lambda_{\mathrm{D}e}}{\lambda_{\mathrm{D}i}} = \frac{v_{\mathrm{the}}\omega_{\mathrm{pi}}}{v_{\mathrm{th}i}\omega_{\mathrm{pe}}} = \sqrt{\frac{ZT_e}{T_i}}, \tag{3.76}$$

the condition (3.65) need not entail $k\lambda_{\mathrm{D}e} \ll 1$ in the limit of cold ions [see (3.71)]—in this case, the size of the Debye sphere (1.6) is set by the ions, rather than by the electrons, and so we can have perfectly macroscopic (in the language of §1.4) perturbations on scales both larger and smaller than $\lambda_{\mathrm{D}e}$. At larger scales, we have found sound waves (3.70). At smaller scales, $k\lambda_{\mathrm{D}e} \gg 1$, the dispersion relation (3.68) gives us *ion Langmuir oscillations:*

$$\omega^2 = \omega_{\mathrm{pi}}^2 = \frac{4\pi Z^2 e^2 n_i}{m_i}, \tag{3.77}$$

which are analogous to the electron Langmuir oscillations (3.38) and, like them, turn into dispersive *ion Langmuir waves* if the small $k^2\lambda_{\mathrm{D}i}^2$ correction in (3.64) is retained, leading to the Bohm–Gross dispersion relation (3.39), but with ion quantities this time.

Exercise 3.9. Derive the dispersion relation for ion Langmuir waves. Investigate their damping/instability.

---

[27] Recall that $k > 0$ by the choice of the z axis.

[28] The 2016 exam question was loosely based on this.

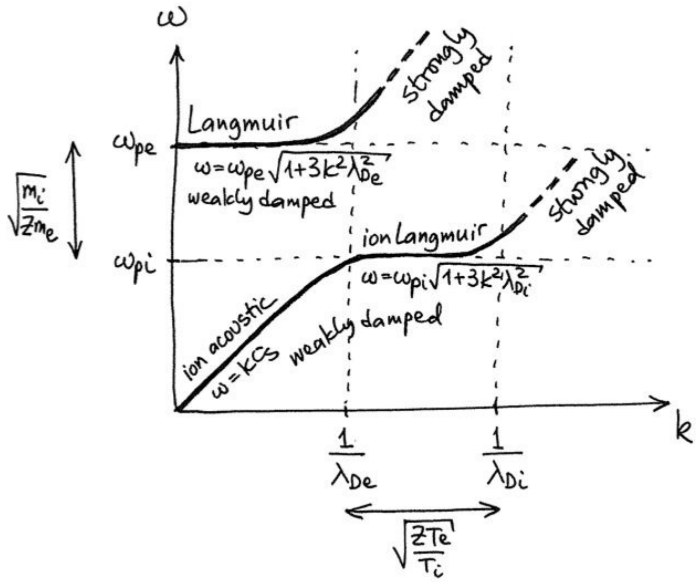


FIGURE 16. Electrostatic (longitudinal) plasma waves.

3.11. *Summary of Electrostatic (Longitudinal) Plasma Waves*

We have achieved what turns out to be a complete characterisation of electrostatic (also known as "longitudinal", in the sense that $\boldsymbol{k} \parallel \boldsymbol{E}$) waves in an unmagnetised plasma. These are summarised in Fig. 16. In the limit of short wavelengths, $k\lambda_{De} \gg 1$ and $k\lambda_{Di} \gg 1$, the electron and ion branches, respectively, becomes dispersive, their damping rates increase and eventually stop being small. This corresponds to waves having phase speeds that are comparable to the speeds of the particles in the thermal bulk of their distributions, so a great number of particles are available to have Landau resonance with the waves and absorb their energy—the damping becomes strong.

Note that if the cold-ion condition $T_i \ll T_e$ is not satisfied, the sound speed is comparable to the ion thermal speed $c_s \sim v_{thi}$, and so the ion-acoustic waves are strongly damped at all wave numbers—it is well-nigh impossible to propagate sound through a collisionless hot plasma (in such an environment, no one will hear you scream)!

Let me digress a little to bring you in contact with a research frontier. In a magnetised plasma, the sound wave looks (and works) essentially the same as (3.70), as long as its wave vector is parallel to the magnetic field—the dispersion relation for such a wave, derived from fluid equations (as in Exercise 3.6) is (16.27). In a *collisionless* such plasma with $T_i \sim T_e$, this wave is again heavily damped. Interestingly, however, at high plasma beta [defined by (16.24)], if a sound wave is allowed to have a finite amplitude (i.e., outside linear theory), it becomes "self-sustaining", by exciting some parasitic micro-instabilities (§§20.8 and 22.1) that in turn render the plasma in its path effectively collisional—a rather beautiful phenomenon that has only been worked out recently: read Kunz *et al.* (2020) if this piques your curiosity.

Exercise 3.10. Longitudinal waves in degenerate plasmas.[29] Sometimes it is meaningful to consider degenerate fermionic plasmas (e.g., electrons in metals or electrons and holes in

[29] This is based on the 2023 exam question.

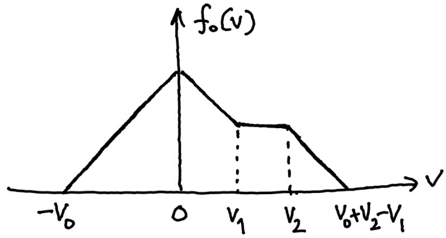


FIGURE 17. The distribution function with a constant slope and a plateau for Exercise 3.11.

superconductors). If they are cold enough, their distribution function is simply

$$f_{0\alpha}(\boldsymbol{v}) = \frac{1}{4}\left(\frac{m_\alpha}{\pi\hbar}\right)^3 H(v_{F\alpha} - v), \tag{3.78}$$

where H is the Heaviside function (= 1 for positive argument, 0 otherwise) and $v_{F\alpha} = (3\pi^2 n_\alpha)^{1/3}\hbar/m_\alpha$ is the "Fermi velocity" assuming spin-1/2 particles (the Fermi energy is $m_\alpha v_{F\alpha}^2/2$). It is easy to check that (3.78) integrates to n_α.

Work out all longitudinal waves for such plasmas. You will find Langmuir waves, ion sound, etc., but there will also be a new feature, the so-called "zero-point sound" at $k\lambda_{De} \gg 1$, where the new Debye length is $\lambda_{De} = \sqrt{3}v_{Fe}/\omega_{pe}$, a wave whose frequency is

$$\omega = \pm k v_{Fe}\left(1 + 2e^{-2-2k^2\lambda_{De}^2/9}\right). \tag{3.79}$$

Work out also whether any of the waves that you have found are Landau damped, and if they are, at what rate. Sketch a version of Fig. 16 for degenerate plasmas. If you find this calculation difficult, you can look up some of it in Lifshitz & Pitaevskii (1981) and all of it in Alexandrov *et al.* (1984).

Exercise 3.11. Electron acoustic waves.[30] Consider a 1D plasma in which the equilibrium electron distribution $f_0(v)$ is as depicted in Fig. 17: $f_0(v) \neq 0$ only for $v \in [-v_0, v_0 + v_2 - v_1]$, with a constant slope $f_0'(v) = -\mathrm{sgn}(v) f_0(0)/v_0$ everywhere inside that interval except at $v \in [v_1, v_2]$, where it has a plateau. We shall seek linear perturbations of this plasma that have phase velocities greatly exceeding the characteristic width of the ion distribution, so the latter's contribution to the dielectric function can be ignored, i.e., (3.26) becomes

$$\epsilon(p,k) = 1 - \frac{\omega_{pe}^2}{k^2}\frac{1}{n_0}\int_{C_L} dv\, \frac{f_0'(v)}{v - ip/k}, \tag{3.80}$$

where, by definition, $n_0 = v_0 f_0(0)$ and $\omega_{pe} = (4\pi e^2 n_0/m_e)^{1/2}$.

(a) At what values of the phase velocity $u = \omega/k$ do you expect *a priori* that completely undamped waves ($p = -i\omega$, where ω is real) might be able to exist?

(b) Show that the frequencies $\omega = ku$ and wavenumbers k of such undamped waves must satisfy the following dispersion relation

$$\boxed{\ln\frac{u^2|v_2 - u|}{|v_0 + u||v_1 - u||v_0 + v_2 - v_1 - u|} = k^2\lambda_{De}^2}, \tag{3.81}$$

where, by definition, $\lambda_{De} = v_0/\omega_{pe}$.

(c) What waves exist in this plasma at short wavelengths, $k\lambda_{De} \gg 1$? Why are they undamped?

(d) Now consider arbitrary wavenumbers $k\lambda_{De}$. Assume that $v_2 - v_1 \ll v_0, v_1$. Show that the

[30] This is based on the 2025 exam question. I am grateful to Gabriele Celebre for getting me interested in these modes.

dispersion relation (3.81) has three solutions:

$$\omega \approx \pm \frac{kv_0}{\sqrt{1 - e^{-k^2\lambda_{\mathrm{De}}^2}}}, \tag{3.82}$$

$$\omega \approx k \left[v_1 + \frac{v_2 - v_1}{1 + e^{k^2\lambda_{\mathrm{De}}^2}(v_0^2 - v_1^2)/v_1^2} \right]. \tag{3.83}$$

Convince yourself that the first two (3.82) are the familiar Langmuir waves at $k\lambda_{\mathrm{De}} \ll 1$. In the same limit, the third (3.83) resembles a sound wave, so could be called the *electron acoustic wave (EAW)* (see discussion below). Sketch ω vs. k for these three branches. Why does the third solution not exist in a Maxwellian plasma?

What I have just called the EAW is, in fact, not quite what is usually called that. Consider for a moment the case with no plateau, i.e., still the distribution function depicted in Fig. 17 but with $v_1 = v_2$. The equation $\mathrm{Re}\,\epsilon(-i\omega, k) = 0$ is then simply

$$\ln \frac{u^2}{|v_0^2 - u^2|} = k^2\lambda_{\mathrm{De}}^2. \tag{3.84}$$

Its two solutions such that $u^2 > v_0^2$ are still the two undamped Langmuir waves (3.82), but it also has two solutions such that $u^2 < v_0^2$:

$$u = \pm \frac{v_0}{\sqrt{1 + e^{-k^2\lambda_{\mathrm{De}}^2}}}. \tag{3.85}$$

These, however, are not legitimate solutions of the dispersion relation because for them $\mathrm{Im}\,\epsilon(-i\omega, k) \neq 0$. Formally (if mathematically somewhat dodgily), this can be fixed by placing an infinitesimal micro-plateau at $v \in [u - \delta, u + \delta]$, $\delta \to 0$, in which case the dispersion relation (3.84) is the same as (3.81). It is these modes that are known as EAWs in the literature, originating from Holloway & Dorning (1991), who derived them for a Maxwellian plasma—the version of the curve (3.84) for a Maxwellian electron distribution is known as the "thumb curve", for a reason that will become obvious to you if you plot it using the technology explained in §3.12. Note that this curve is not, technically, a dispersion curve corresponding to any one electron distribution, because the micro-plateau has to be put in a different place for each k in order for (3.85) to be a solution of the dispersion relation.

The calculation that I offered you in Exercise 3.11 is a version of the calculation done by Valentini *et al.* (2012) for Maxwellian electrons—they called their version of the modes (3.83) "corner modes",[31] rather than EAWs. These modes *are* legitimate solutions, as they sit in a real plateau. They too have a sound-like dispersion relation at $k\lambda_{\mathrm{De}} \ll 1$, which is why I would like to call them EAWs.

3.12. *Plasma Dispersion Function: Putting Linear Theory on Industrial Basis*

Clearly, we have entered the realm of practical calculation—it is now easy to imagine an industry of solving the plasma dispersion relation

$$\epsilon(p, \boldsymbol{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \int_{C_{\mathrm{L}}} \mathrm{d}v_z \, \frac{F_\alpha'(v_z)}{v_z - ip/k} = 0 \tag{3.86}$$

and similar dispersion relations arising from, e.g., considering electromagnetic perturbations (see Q-2), magnetised plasmas (see Parra 2019*b*), different equilibria F_α (see Q-3 and Q-4), etc.

A Maxwellian equilibrium is obviously an extremely important special case because that is, after all, the distribution towards which plasma is pushed by collisions on long time scales:

$$f_{0\alpha}(\boldsymbol{v}) = \frac{n_\alpha}{(\pi v_{\mathrm{th}\alpha}^2)^{3/2}} \, e^{-v^2/v_{\mathrm{th}\alpha}^2} \quad \Rightarrow \quad F_\alpha(v_z) = \frac{n_\alpha}{\sqrt{\pi} v_{\mathrm{th}\alpha}} \, e^{-v_z^2/v_{\mathrm{th}\alpha}^2}. \tag{3.87}$$

---

[31] The reason they called them that is that at $k \to \infty$, $u \approx v_1$, whereas at $k \to 0$, $u \approx v_2$ if one places the plateau very close to the edge of the distribution, viz., if $v_2 - v_1 \sim v_0 - v_1 \ll v_0$. Thus, in the short- and long-wavelength limits, the waves propagate at speeds corresponding to the "corners" of the plateau.

For this case, we would like to introduce a new "special function" that would incorporate the Landau prescription for calculating the velocity integral in (3.86) and that we could in some sense "tabulate" once and for all.

Taking F_α to be (3.87) and letting $u = v_z/v_{\mathrm{th}\alpha}$ and $\zeta_\alpha = ip/kv_{\mathrm{th}\alpha}$, we can rewrite the velocity integral in (3.86) as follows

$$\frac{1}{n_\alpha}\int_{C_\mathrm{L}} \mathrm{d}v_z\,\frac{F'_\alpha(v_z)}{v_z - ip/k} = -\frac{2}{\sqrt{\pi}\,v_{\mathrm{th}\alpha}^2}\int_{C_\mathrm{L}} \mathrm{d}u\,\frac{u\,e^{-u^2}}{u-\zeta_\alpha} = -\frac{2}{v_{\mathrm{th}\alpha}^2}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right], \tag{3.88}$$

where the *plasma dispersion function* is defined to be

$$\boxed{Z(\zeta) = \frac{1}{\sqrt{\pi}}\int \mathrm{d}u\,\frac{e^{-u^2}}{u-\zeta}}\,. \tag{3.89}$$

In these terms, the plasma dispersion relation (3.86) becomes

$$\boxed{\epsilon = 1 + \sum_\alpha \frac{1+\zeta_\alpha Z(\zeta_\alpha)}{k^2\lambda_{\mathrm{D}\alpha}^2} = 0}\,. \tag{3.90}$$

Note that if the Maxwellian distribution (3.87) has a mean flow, as it did, e.g., in (3.62), this amounts to a shift by some mean velocity u_α and all one needs to do to adjust the above results is to shift the argument of Z accordingly:

$$\zeta_\alpha \to \zeta_\alpha - \frac{u_\alpha}{v_{\mathrm{th}\alpha}}. \tag{3.91}$$

3.12.1. *Some Properties of $Z(\zeta)$*

It is not hard to see that

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}}\int \mathrm{d}u\,e^{-u^2}\frac{\partial}{\partial u}\frac{1}{u-\zeta} = -\frac{2}{\sqrt{\pi}}\int \mathrm{d}u\,\frac{u\,e^{-u^2}}{u-\zeta} = -2\left[1+\zeta Z(\zeta)\right]. \tag{3.92}$$

Let us treat this identity as a differential equation: the integrating factor is e^{ζ^2}, so

$$e^{\zeta^2}Z(\zeta) = -2\int_0^\zeta \mathrm{d}t\,e^{t^2} + Z(0). \tag{3.93}$$

We know the boundary condition at $\zeta = 0$ from (3.23): for real ζ,

$$\frac{1}{\sqrt{\pi}}\int \mathrm{d}u\,\frac{e^{-u^2}}{u-\zeta} = \frac{1}{\sqrt{\pi}}\underbrace{\mathcal{P}\!\int_{-\infty}^{+\infty}\mathrm{d}u\,\frac{e^{-u^2}}{u-\zeta}}_{\substack{=0\text{ for }\zeta=0\\ \text{because}\\ \text{integrand is}\\ \text{odd}}} + i\sqrt{\pi}\,e^{-\zeta^2} \quad\Rightarrow\quad Z(0) = i\sqrt{\pi}. \tag{3.94}$$

Using this in (3.93) and changing the integration variable $t = -ix$, we find

$$Z(\zeta) = e^{-\zeta^2}\left(i\sqrt{\pi} + 2i\int_0^{i\zeta}\mathrm{d}x\,e^{-x^2}\right) = 2i\,e^{-\zeta^2}\int_{-\infty}^{i\zeta}\mathrm{d}x\,e^{-x^2}. \tag{3.95}$$

FIGURE 18. Vera Faddeeva (1906-1983), the mother of the plasma dispersion function.

This turns out to be a uniformly valid expression for $Z(\zeta)$: our function is simply a complex erf![32] Here is a MATHEMATICA script for calculating it:

$$\texttt{Z[zeta\_]} := \texttt{I Sqrt[Pi] Exp[}-\texttt{zeta}^2\texttt{](1 + I Erfi[zeta])}. \tag{3.96}$$

You can use this to code up (3.90) and explore, e.g., the strongly damped solutions ($\zeta \sim 1$, $\gamma \sim \omega$).

3.12.2. *Asymptotics of $Z(\zeta)$*

If you are a devotee of the ancient art of asymptotic theory, you will find most useful (as, effectively, we did in §§3.4–3.9) various limiting forms of $Z(\zeta)$. At small argument $|\zeta| \ll 1$, the Taylor series is

$$\boxed{Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2} - 2\zeta\left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - \frac{8\zeta^6}{105} + \dots\right)}\,. \tag{3.97}$$

At large argument, $|\zeta| \gg 1$, $|\mathrm{Re}\,\zeta| \gg |\mathrm{Im}\,\zeta|$, the asymptotic series is

$$\boxed{Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \dots\right)}\,. \tag{3.98}$$

All the results (for a Maxwellian equilibrium) that I derived in §§3.4–3.10 can be readily obtained from (3.90) by using the above limiting cases (see Q-1). It is, indeed, a general practical strategy for studying this and similar plasma dispersion relations to look for solutions in the limits $\zeta_\alpha \to 0$ or $\zeta_\alpha \to \infty$, then check under what physical conditions the solutions thus obtained are valid (i.e., that they indeed satisfy $|\zeta_\alpha| \ll 1$ or $|\zeta_\alpha| \gg 1$, $|\mathrm{Re}\,\zeta| \gg |\mathrm{Im}\,\zeta|$), and then fill in the non-asymptotic blanks in the same way that an experienced hunter espying antlers sticking out above the shrubbery can reconstruct, in contour outline, the rest of the hiding deer.

Exercise 3.12. Work out the Taylor series (3.97). A useful step might be to prove this interesting formula (which also turns out to be handy in other calculations; see, e.g., Q-8):

$$\frac{\mathrm{d}^m Z}{\mathrm{d}\zeta^m} = \frac{(-1)^m}{\sqrt{\pi}} \int_{C_\mathrm{L}} \mathrm{d}u\, \frac{H_m(u)\, e^{-u^2}}{u-\zeta}, \tag{3.99}$$

where $H_m(u)$ are Hermite polynomials [defined in (6.74)].

[32] When written in this form, and with $i\sqrt{\pi}$ factored out, $w(\zeta) = Z(\zeta)/i\sqrt{\pi} \equiv \mathrm{erfcx}(-i\zeta)$ is called the Faddeeva function, after the Soviet mathematician Vera Faddeeva (Fig. 18), who published the first book of tables of it (Faddeeva & Terent'ev 1954), before those American upstarts Fried & Conte (1961). People meant business in those days. In the 21st century, we have an easier life: just teach a computer to compute our function and make an app. The Wikipedia article `https://en.wikipedia.org/wiki/Faddeeva_function` offers some useful links to numerical algorithms for doing it, more sophisticated than (3.96).

Exercise 3.13. Work out the asymptotic series (3.98) using the Landau prescription (3.20) and expanding the principal-value integral similarly to the way it was done in (3.35). Work out also (or look up; e.g., Fried & Conte 1961) other asymptotic forms of $Z(\zeta)$, relaxing the condition $|\mathrm{Re}\,\zeta| \gg |\mathrm{Im}\,\zeta|$.

---

4. Linear Theory II: General Stability Theory

In §3, we learned how to perturb some given equilibrium distribution $f_{0\alpha}$ infinitesimally and work out whether this perturbation will decay, grow, oscillate, and how quickly. Let me now pose the question in a more general way. In a collisionless plasma, there can be infinitely many possible equilibria, including quite complicated ones. If we set one up, will it persist, i.e., is it stable? If it is not stable, what modification do we expect it to undergo in order to become stable? Other than solving the dispersion relation (3.17) to answer the first question and developing various types of nonlinear theories to answer the second (along the lines advertised in §2.4 and developed in §6 and subsequent sections), both of which can be quite complicated and often intractable technical challenges, do we have at our disposal any general principles that allow us to pronounce on stability? Is there a general insight that we can cultivate as to what sort of distributions are likely to be stable or unstable and to what sorts of perturbations?

We have had glimpses of such general principles already. For example, in §3.5, it was ascertained, by an explicit calculation, that one could encounter a situation with a (small) growth rate if the equilibrium distribution had a positive derivative somewhere along the direction (z) of the wave number of the perturbation, viz., $v_z F_e'(v_z) > 0$. I developed this further in §3.7, finding that not only hot but also cold beams and streams triggered instabities. In Exercise 3.2, I dropped a hint that general statements could perhaps be made about certain general classes of distributions: 3D-isotropic equilibria could be proven stable (we shall prove this again, by a different method, in Exercise 4.2). How general are such statements? Are they sufficient or also necessary criteria? Is there a universal stability litmus test? Let us attack the problem of kinetic stability with an aspiration to generality—although still, for now, for electrostatic perturbations only. We shall also, for now, limit our ambition to determining *linear* stability of generic equilibria, i.e., their stability against infinitesimal perturbations. *Nonlinear* stability will have to wait till §9.

4.1. *Nyquist's Method*

The problem of linear stability comes down to the question of whether the dispersion relation (3.17) has any unstable solutions: roots with growth rates $\gamma_i(\boldsymbol{k}) > 0$.

It is going to be useful to write the dielectric function (3.26) as follows

$$\epsilon(p,\boldsymbol{k}) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2}\int_{C_{\mathrm{L}}} \mathrm{d}v_z\,\frac{\bar{F}'(v_z)}{v_z - ip/k}, \tag{4.1}$$

$$\bar{F} = \frac{1}{n_e}\sum_\alpha Z_\alpha^2\frac{m_e}{m_\alpha}F_\alpha = \frac{F_e}{n_e} + \frac{Zm_e}{m_i}\frac{F_i}{n_i}, \tag{4.2}$$

where the last expression in (4.2) is for the case of a two-species plasma. Thus, the distribution functions of different species come into the linear problem additively, weighted by their species' charges and (inverse) masses.

Let us develop a method (due to Nyquist 1932) for counting zeros of $\epsilon(p)$ (I will henceforth suppress $\boldsymbol{k}$ in the argument) in the half-plane $\mathrm{Re}\,p > 0$ (the unstable roots of the dispersion relation). Observe that $\epsilon(p)$ is analytic (by virtue of our efforts in §3.2 to make it so) and that if $p = p_i$ is its zero of order N_i, then in its vicinity,

$$\epsilon(p) = \mathrm{const}\,(p-p_i)^{N_i} + \dots \quad \Rightarrow \quad \frac{\partial \ln\epsilon(p)}{\partial p} = \frac{N_i}{p-p_i} + \dots, \tag{4.3}$$

so zeros of $\epsilon(p)$ are poles of $\partial\ln\epsilon(p)/\partial p$; the latter function has no other poles because $\epsilon(p)$ is analytic. If we now integrate this function over a closed contour C_R running along the imaginary axis (and just to the right of it: $p = -i\omega + 0$) in the complex p plane from iR to $-iR$ and then along a semicircle of radius R back to iR (Fig. 19), we will, in the limit $R \to \infty$, capture all

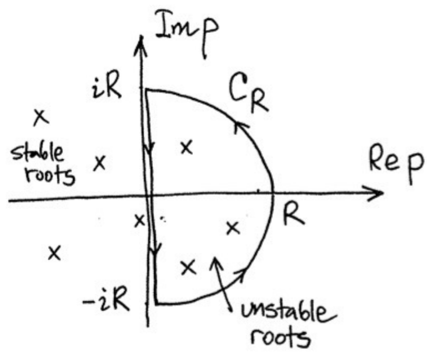


FIGURE 19. Integration contour for (4.4).

these poles:

$$\lim_{R\to\infty}\int_{C_R} \mathrm{d}p\, \frac{\partial \ln \epsilon(p)}{\partial p} = 2\pi i \sum_i N_i = 2\pi i N, \tag{4.4}$$

where N is the total number of zeros of $\epsilon(p)$ in the half-plane $\mathrm{Re}\, p > 0$. It turns out (as I shall prove in a moment) that the contribution to the integral over C_R from the semicircle vanishes at $R \to \infty$ and so we need only integrate along the imaginary axis:

$$\boxed{2\pi i N = \int_{+i\infty+0}^{-i\infty+0} \mathrm{d}p\, \frac{\partial \ln \epsilon(p)}{\partial p} = \ln \frac{\epsilon(-i\infty)}{\epsilon(+i\infty)}}\,. \tag{4.5}$$

Proof. All we need to show is that

$$|p|\frac{\partial \ln \epsilon(p)}{\partial p} \to 0 \quad \text{as} \quad |p| \to \infty,\ \mathrm{Re}\, p > 0. \tag{4.6}$$

Indeed, using (4.1) and the Landau integration rule (3.20), we have in this limit:

$$\epsilon(p) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \bar{F}'(v_z)\, \frac{ik}{p}\left(1 - \frac{ikv_z}{p} + \dots\right) \approx 1 + \frac{1}{p^2}\sum_\alpha \omega_{\mathrm{p}\alpha}^2, \tag{4.7}$$

where I have integrated by parts and used $\int \mathrm{d}v_z\, F_\alpha = n_\alpha$. Manifestly, the condition (4.6) is satisfied.

Note that, along the imaginary axis $p = -i\omega$, by the same expansion and using also the Plemelj formula (3.23), we have

$$\epsilon(-i\omega) \approx 1 - \frac{1}{\omega^2}\sum_\alpha \omega_{\mathrm{p}\alpha}^2 - i\pi \frac{\omega_{\mathrm{pe}}^2}{k^2}\bar{F}'\left(\frac{\omega}{k}\right) \to 1 \mp i0 \quad \text{as} \quad \omega \to \mp\infty. \tag{4.8}$$

This is going to be useful shortly.

In view of (4.8) and of our newly proven formula (4.5), as the function $\epsilon(-i\omega)$ runs along the real line in ω, it changes from

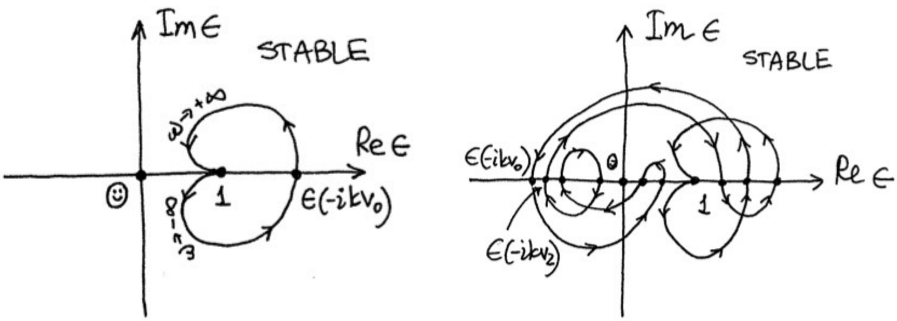
$$\epsilon(i\infty) = 1 - i0 \quad \text{at} \quad \omega = -\infty, \tag{4.9}$$

where I have arbitrarily fixed its phase, to

$$\epsilon(-i\infty) = e^{2\pi i N} + i0 \quad \text{at} \quad \omega = +\infty, \tag{4.10}$$

where N is the number of times the function

$$\epsilon(-i\omega) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2}\left[\mathcal{P}\!\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{\bar{F}'(v_z)}{v_z - \omega/k} + i\pi \bar{F}'\left(\frac{\omega}{k}\right)\right] \tag{4.11}$$



(a) Single-maximum, stable equilibrium (b) Strange but stable equilibrium

FIGURE 20. Two examples of Nyquist diagrams showing stability (because failing to circle zero): (a) the case of a monotonically decreasing distribution (§4.2, Fig. 21a); (b) another stable case, even though very complicated (it also illustrates the argument in §4.3).

circles around the origin in the complex ϵ plane. Since N is also the number of unstable roots of the dispersion relation, this gives us a way to count these roots by sketching $\epsilon(-i\omega)$—this sketch is called the *Nyquist diagram*. Two examples of Nyquist diagrams implying stability are given in Fig. 20: the curve $\epsilon(-i\omega)$ departs from $1 - i0$ and comes back to $1 + i0$ via a path that, however complicated, never makes a full circle around zero. Two examples of unstable situations appear in Fig. 22(b,d): in these cases, zero is circumnavigated, implying that the equilibrium distribution $\bar{F}$ is unstable (at a given value of k).

In order to work out whether the Nyquist curve circles zero (and how many times), all one needs to do is find $\mathrm{Re}\,\epsilon(-i\omega)$ at all points ω where $\mathrm{Im}\,\epsilon(-i\omega) = 0$, i.e., where the curve intersects the real line, and hence sketch the Nyquist diagram. We shall see in a moment, with the aid of some important examples, how this is done, but let us do a little bit of preparatory work first.

It follows immediately from (4.11) that these crossings happen whenever $\omega/k = v_*$ is a velocity at which $\bar{F}(v_z)$ has an extremum, $\bar{F}'(v_*) = 0$. At these points, the dielectric function (4.11) is real and can be expressed so:

$$\boxed{\epsilon(-ikv_*) = 1 + \frac{\omega_{\mathrm{pe}}^2}{k^2} P(v_*)}\,. \tag{4.12}$$

Here $P(v_*)$ is (minus) the principal-value integral in (4.11), which can be manipulated as follows:

$$\begin{aligned} P(v_*) &= -\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{\bar{F}'(v_z)}{v_z - v_*} = -\mathcal{P}\int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{1}{v_z - v_*}\frac{\partial}{\partial v_z}\left[\bar{F}(v_z) - \bar{F}(v_*)\right] \\ &= \int_{-\infty}^{+\infty} \mathrm{d}v_z\, \frac{\bar{F}(v_*) - \bar{F}(v_z)}{(v_z - v_*)^2}, \end{aligned} \tag{4.13}$$

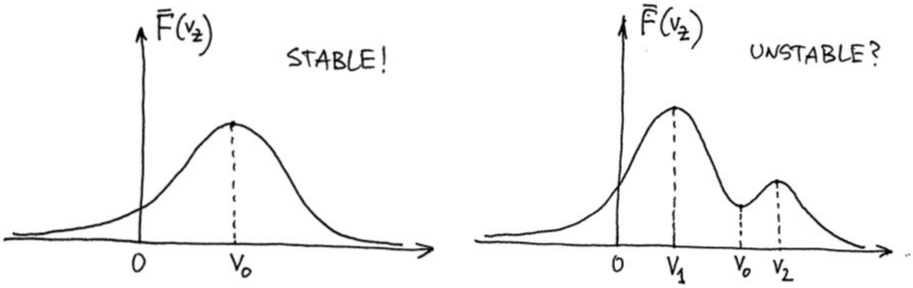
where I have integrated by parts; the additional term $\bar{F}(v_*)$ was inserted under the derivative in order to eliminate the boundary terms arising in this integration by parts around the pole $v_z = v_*$.[33]

Now we are ready to analyse particular (and, as we shall see, also generic) equilibrium distributions $\bar{F}(v_z)$.

4.2. *Stability of Monotonically Decreasing Distributions*

Consider first a distribution function that has a single maximum at $v_z = v_0$ and monotonically decays in both directions away from it (Fig. 21a): $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) < 0$. This means that,

[33] Note that in the final expression in (4.13), there is no longer a need for principal-value integration because, v_* being a point of extremum of $\bar{F}$, the numerator of the integrand is quadratic in $v_z - v_*$ in the vicinity of v_*.



(a) Single-maximum distribution (§4.2) (b) Single-minimum distribution (§4.3)

FIGURE 21. Two examples of equilibrium distributions.

besides at $\omega = \mp\infty$, $\mathrm{Im}\,\epsilon(-i\omega) \propto \bar{F}'(\omega/k)$ also vanishes at $\omega = kv_0$. It is then clear that

$$\epsilon(-ikv_0) = 1 + \frac{\omega_{\mathrm{pe}}^2}{k^2}P(v_0) > 1 \tag{4.14}$$

because $\bar{F}(v_0) > \bar{F}(v)$ for all v_z and so $P(v_0) > 0$. Thus, the Nyquist curve departs from $1 - i0$ at $\omega = -\infty$, intersects the real line once at $\omega = kv_0$ and then comes back to $1 + i0$ without circling zero; the corresponding Nyquist digram is sketched in Fig. 20(a). Conclusion:

Monotonically decreasing distributions are stable against electrostatic perturbations.

We do not in fact need all this mathematical machinery just to prove the stability of monotonically decreasing distributions (in §9.5, we shall see that this is a very robust result)—but it will come handy when dealing with less simple cases. Parenthetically, let us work out some direct proofs of stability.

Exercise 4.1. Direct proof of linear stability of monotonically decreasing distributions. (a) Consider the dielectric function (4.1) with $p = -i\omega + \gamma$ and assume $\gamma > 0$ (so the Landau contour is just the real axis). Work out the real and imaginary parts of the dispersion relation $\epsilon(p) = 0$ and show that it can never be satisfied if $v_z\bar{F}'(v_z) \leqslant 0$, i.e., that any equilibrium distribution that has a maximum at zero and decreases monotonically on both sides of it is stable against electrostatic perturbations.[34]

(b) What if the maximum is at $v_z = v_0 \neq 0$?

Exercise 4.2. Direct proof of linear stability of isotropic distributions. (a) Recall Exercise 3.2 and show that all homogeneous, 3D-isotropic (in velocity) equilibria are stable against electrostatic perturbations (with no need to assume long wave lengths).

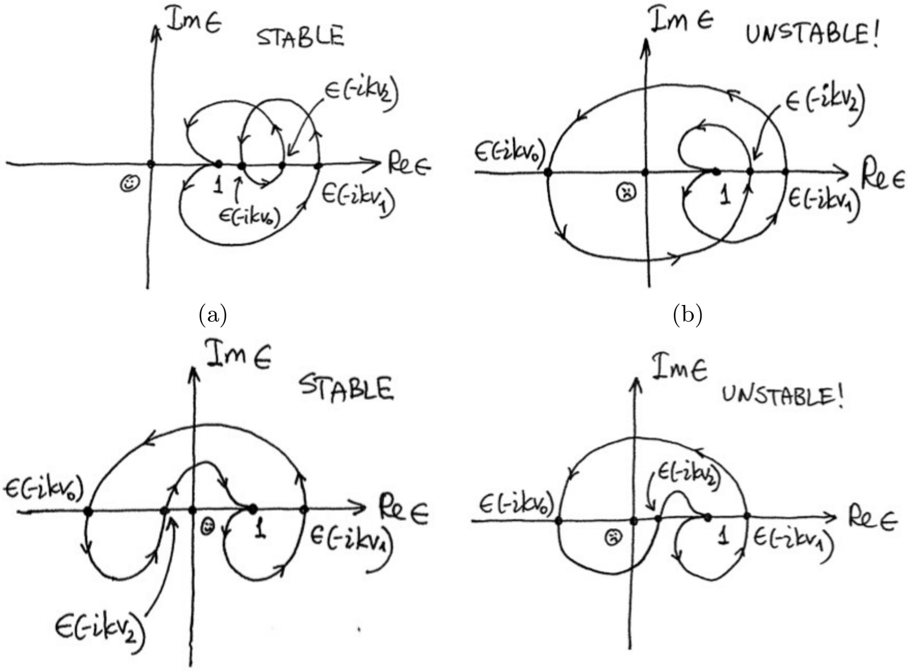
(b) Prove, in the same way, that isotropic equilibria are also stable against *electromagnetic* perturbations. You will need to derive the transverse dielectric function in the same way as in Q-2 or Q-3, but for a general equilibrium distribution $f_{0\alpha}(v_x, v_y, v_z)$; failing that, you can look it up in a book, e.g., Krall & Trivelpiece (1973) or Davidson (1983).

4.3. *Penrose's Instability Criterion*

It would be good to learn how to test for stability generic distributions that have multiple minima and maxima: the simplest of them is shown in Fig. 21b, evoking the bump-on-tail situation discussed in §3.5 and thus posing a risk (but, as we are about to see, not a certainty!) of being unstable.

The Nyquist curve $\epsilon(-i\omega)$ departs from $1 - i0$ at $\omega = -\infty$, then crosses the real line for the

[34]This kind of argument can also be useful in stability considerations applying to more complicated situations, e.g., magnetised plasmas (Bernstein 1958).



(a) (b) (c) (d)

FIGURE 22. Various possible forms of the Nyquist diagram for a single-minimum distribution sketched in Fig. 21b: (a) $\epsilon(-ikv_0) > 1$, stable; (b) $\epsilon(-ikv_0) < 0$, $\epsilon(-ikv_2) > 1$, unstable; (c) $\epsilon(-ikv_0) < \epsilon(-ikv_2) < 0$, stable; (d) $\epsilon(-ikv_0) < 0 < \epsilon(-ikv_2) < 1$, unstable.

first time at $\omega = kv_1$, corresponding to the leftmost maximum of $\bar{F}$.[35] This crossing is upwards, from the lower to the upper half-plane, and it is not hard to see that a maximum will always correspond to such an upward crossing and a minimum to a downward one, from the upper to the lower half-plane: this follows directly from the change of sign of $\mathrm{Im}\,\epsilon$ in Eq. (4.11) because $\bar{F}'(\omega/k)$ goes from positive to negative at any point of maximum and vice versa at any minimum. After a few crossings back and forth, corresponding to local minima and maxima (if any), the Nyquist curve will come to the the downward crossing corresponding to the *global* minimum (other than at $v_z = \pm\infty$) of the distribution function at, say, $\omega = kv_0$. If at this point $P(v_0) > 0$, then $\epsilon(-ikv_0) > 1$ and the same is true at all other crossing points v_* because v_0 is the global minimum of $\bar{F}$ and so $P(v_*) > P(v_0) > 0$ for all other extrema. In this situation, illustrated in Fig. 22(a), the Nyquist curve never circumnavigates zero and, therefore, $P(v_0) > 0$ is a sufficient condition of stability. It is also the necessary one, which is proved in the following way.

Suppose $P(v_0) < 0$. Then, in (4.12), we can always find a range of k that are small enough that $\epsilon(-ikv_0) < 0$, so the downward crossing at v_0 happens on the negative side of zero in the ϵ plane. After this downward crossing, the Nyquist curve will make more crossings, until it finally comes to rest at $1 + i0$ as $\omega = +\infty$. Let us denote by $v_2 > v_0$ the point of extremum for which the corresponding crossing occurs at a point on the $\mathrm{Re}\,\epsilon$ axis that is closest to (but always will be to the right of) $\epsilon(-ikv_0) < 0$. If $\epsilon(-ikv_2) > 0$, then there is no way back, zero has been fully circumnavigated and so there must be at least one unstable root (see Fig. 22b,d). If $\epsilon(-ikv_2) < 0$, there is in principle some wiggle room for the Nyquist curve to avoid circling zero (see Fig. 22c for a single-minimum distribution of Fig. 21b—or Fig. 20b for some serious wiggles). However, since $P(v_2) > P(v_0)$ for any v_2 (because v_0 is the global minimum of $\bar{F}$), we can always increase k in (4.12) just enough so $\epsilon(-ikv_2) > 0$ even though $\epsilon(-ikv_0) < 0$ still (this corresponds to turning Fig. 22c into Fig. 22d). Thus, if $P(v_0) < 0$, there will always be some range of k inside which there is an instability.

[35]For the distribution sketched in Fig. 21(b), this maximum is global, so $P(v_1) > 0$ and, therefore, $\epsilon(-ikv_1) > 1$. This is the rightmost such crossing when v_1 is the global maximum.

We have obtained a sufficient and necessary condition of instability of an equilibrium $\bar{F}(v_z)$ against electrostatic perturbations: if v_0 is the point of global minimum of $\bar{F}$,[36]

$$\boxed{P(v_0) = \int_{-\infty}^{+\infty} \mathrm{d}v_z \, \frac{\bar{F}(v_0) - \bar{F}(v_z)}{(v_z - v_0)^2} < 0 \quad \Leftrightarrow \quad \bar{F} \text{ is unstable}}\,. \tag{4.15}$$

This is the famous *Penrose's instability criterion* (the famous criterion, not the famous Penrose; it was proved by *Oliver* Penrose 1960, in a stylistically somewhat different way than I did it here). Note that considerations of the kind presented above can be used to work out the wave-number intervals, corresponding to various troughs in $\bar{F}$, in which instabilities exist.

Intuitively, the criterion (4.15) says that, in order for a distribution to be unstable, it needs to have a trough and this trough must be deep enough. Thus, if $F(v_0) = 0$, i.e., if the distribution has a "hole", it is always unstable (an extreme example of this is the two-stream instability; see Exercise 3.5). Another corollary is that you cannot stablise a distribution by just adding some particles in a narrow interval around v_0, as this would create two minima nearby, which, the filled interval being narrow, are still going to render the system unstable. To change that, you must fill the trough substantially with particles—hence the tendency to flatten bumps into plateaux, which we will discover in §6 (this answers, albeit in very broad strokes, the question posed at the beginning of §4 about the types of stable distributions towards which the unstable ones will be pushed as the instabilities saturate).

Exercise 4.3. Consider a single-minimum distribution like the one in Fig. 21(b), but with the global maximum on the right and the lesser maximum on the left of the minimum. Draw various possible Nyquist diagrams and convince yourself that Penrose's criterion works. If you enjoy this, think of a distribution that would give rise to the Nyquist diagram in Fig. 20(b).

Exercise 4.4. What happens if the distribution function $\bar{F}$ has an inflection point, i.e., $\bar{F}(v_0) = 0$, $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) = 0$?

Exercise 4.5. What happens if the distribution function has a trough with a flat bottom (i.e., a flat minimum over some interval of velocities)?

4.4. *Bumps, Beams, Streams and Flows*

An elementary example of the use of Penrose's criterion is the two-stream instability, first introduced in Exercise 3.5. The case of two cold streams, represented by (3.60) and Fig. 14(a), is obviously unstable because there is a gaping hole in this distribution. What if we now give these streams some thermal width? This can be modeled by the double-Lorentzian distribution (Fig. 14b)

$$F_e(v_z) = \frac{n_e v_\mathrm{b}}{2\pi} \left[\frac{1}{(v_z - u_\mathrm{b})^2 + v_\mathrm{b}^2} + \frac{1}{(v_z + u_\mathrm{b})^2 + v_\mathrm{b}^2} \right], \tag{4.16}$$

which is particularly easy to handle analytically. For the moment, we will consider the ions to be infinitely heavy, so $\bar{F} = F_e$.

Since the distribution (4.16) is symmetric, it can only have its minimum at $v_0 = 0$. Asking that it should indeed be a minimum, rather than a maximum, i.e., $\bar{F}'(0) > 0$, one finds that the condition for this is

$$u_\mathrm{b} > \frac{v_\mathrm{b}}{\sqrt{3}}. \tag{4.17}$$

Otherwise, the two streams are too wide (in velocity space) and the distribution is monotonically decreasing, so, according to §4.2, it is stable.

If the condition (4.17) is satisfied, the distribution has two bumps, but is this enough to make it unstable? Substituting this distribution into Penrose's criterion (4.15) and doing the integral

---

[36]Another way of putting this is: *a distribution $\bar{F}$ is unstable iff it has a minimum at some v_0 for which $P(v_0) < 0$.* Obviously, if $P(v_0) < 0$ at some minimum, it is also negative at the global minimum.

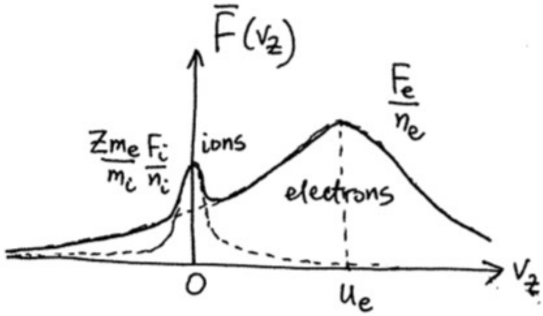


FIGURE 23. Combined distribution (4.2) for cold ions and hot electrons (cf. Fig. 15).

exactly,[37] we get the necessary and sufficient instability condition:

$$P(0) = -\frac{u_\mathrm{b}^2 - v_\mathrm{b}^2}{(u_\mathrm{b}^2 + v_\mathrm{b}^2)^2} < 0 \quad \Leftrightarrow \quad \boxed{u_\mathrm{b} > v_\mathrm{b}}\,. \tag{4.18}$$

Thus, if the streams are sufficiently fast and/or their thermal spread is sufficiently narrow, an instability will occur, but it is not quite enough just to have a little trough. Note, by the way, that Penrose's criterion does not differentiate between hydrodynamic (cold) and kinetic (hot) instability mechanisms (§3.7).

Exercise 4.6. Use Nyquist's method to work out the range of wave numbers at which perturbations will grow for the two-stream instability (you will find the answer in Jackson 1960—yes, *that* Jackson). Convince yourself that this is all in accord with the explicit solution of the dispersion relation obtained in Q-4.

It is obvious how these considerations can be generalised to more complicated situations, e.g., to cases where the streams have different velocities, where one of them is, in fact, the thermal bulk of the distribution and the other is a little bump on its tail (§3.7), where there are more than two streams, etc. The streams also need not be composed of the particles of the same species: indeed, as we saw in (4.1), in the linear theory, the distributions of all species are additively combined into $\bar{F}$ with weights that are inversely proportional to their masses [see (4.2)]. Thus, the ion-acoustic instability (§3.9) is also just a kind of of two-stream—or, if you like, bump-on-tail—instability, with the entire hot and mighty electron distribution making up a magnificent bump on the tail of the cold, m_e/m_i-weighted ion one (Fig. 23).[38] When the streams/beams have thermal spreads, they are more commonly thought of as mean flows—or currents, if the electron flows are not compensated by the ion ones.

Exercise 4.7. Construct an equilibrium distribution to model your favorite plasma system with flows and/or beams and investigate its stability: find the growth rate as a function of wave number, instability conditions, etc.

4.5. *Anisotropies*

So we have found that various holes, bumps, streams, beams, flows, currents and other such nonmonotonic features in the (combined, multispecies) equilibrium distribution present an instability risk, unless they are sufficiently small, shallow, wide and/or close enough to the thermal bulk. All of these are, of course, *anisotropic* features—indeed, as we saw in Exercise 4.2,

[37] The easiest way to do it is to turn the integration path along the real axis into a loop by completing it with a semicircle at positive or negative complex infinity, where the integrand vanishes, and use Cauchy's formula.

[38] In fact, when the two species' temperatures are the same, there is still an instability, whose criterion can again be obtained by the Nyquist-Penrose method: see Jackson (1960).

3D-isotropic distributions are harmless, instability-wise. It turns out that anisotropies of the distribution function in velocity space are dangerous even when the distribution decays monotonically in all directions. However, the instabilities that occur in such situations are *electromagnetic*, rather than electrostatic, and so require an investigation into the properties of the transverse dielectric function of the kind derived in Q-2 or Q-3, but for a general equilibrium. A nice treatment of anisotropy-driven instabilities can be found in Krall & Trivelpiece (1973) and an even more thorough one in Davidson (1983) and Bott *et al.* (2024); three examples of such instabilities will appear in Q-3 (where you get to derive one yourself), §20.8, and §22.1 (in §22.2, I will explain how anisotropic distributions sometimes arise). In §§9.2.1 and 9.5.3, I will show in quite a simple way that, at least in principle, there is always energy available to be extracted from anisotropic distributions.

Exercise 4.8. Criterion of instability of anisotropic distributions. This is an independent-study topic. Consider linear stability of general distribution functions to *electromagnetic* perturbations and work out the stability criterion in the spirit of §4. You should discover that anisotropic distributions such as, e.g., the bi-Maxwellian (6.45), tend to be unstable. Krall & Trivelpiece (1973, §9.10) would be a good place to read about it, but do range beyond.

---

5. Energy, Entropy, Heating, Irreversibility, and Phase Mixing

While we are done with the "calculational" part of linear theory (calculating at what rates the field perturbations oscillate, decay or grow), we are not yet done with the "conceptual" part: what exactly is going on, mathematically and physically? The plan of addressing this question in this section is as follows.

- I will show that Landau damping of perturbations of a plasma in thermal equilibrium leads to the heating of this equilibrium—basically, that energy is conserved. This is not a surprise, but it is useful to see explicitly how this works (§5.1).

- I will then ask how it is possible to have heating (an irreversible process) in a plasma that was assumed collisionless and must conserve entropy. In other words, why, physically, is Landau damping a damping? This will lead us to consider entropy evolution in our system and to introduce an important concept of free energy (§5.2).

- In the above context, we will examine (§§5.3 and 5.6) the Laplace-transform solution (3.8) for the perturbed distribution function and establish the phenomenon of *phase mixing*—emergence of fine structure in velocity (phase) space. This will allow collisions and, therefore, irreversiblity back in (§5.5). We will also see that the Landau-damped solutions are *not* eigenmodes (while growing solutions can be), and so conclude that it made sense to insist on using an initial-value-problem formalism.

5.1. *Energy Conservation and Heating*

Let us go back to the full, nonlinear Vlasov–Poisson system, with the collision term restored:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}, \tag{5.1}$$

$$-\nabla^2\varphi = 4\pi\sum_\alpha q_\alpha\int \mathrm{d}\boldsymbol{v}\, f_\alpha. \tag{5.2}$$

Let us calculate the rate of change of the electric energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int \mathrm{d}\boldsymbol{r}\,\frac{E^2}{8\pi}}_{\equiv\,\mathscr{U}} = \underbrace{\int \mathrm{d}\boldsymbol{r}\,\frac{\boldsymbol{\nabla}\varphi}{4\pi}\cdot\frac{\partial(\boldsymbol{\nabla}\varphi)}{\partial t}}_{\text{by parts}} = -\underbrace{\int \mathrm{d}\boldsymbol{r}\,\frac{\varphi}{4\pi}\frac{\partial}{\partial t}\nabla^2\varphi}_{\text{use (5.2)}} = \sum_\alpha q_\alpha \underbrace{\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\varphi\,\frac{\partial f_\alpha}{\partial t}}_{\text{use (5.1)}}$$
$$= \sum_\alpha q_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\varphi\Bigg[\underbrace{-\boldsymbol{v}\cdot\boldsymbol{\nabla}f_\alpha}_{\text{by parts}} + \underbrace{\frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}}_{\substack{\text{vanishes}\\ \text{because}\\ f(\pm\infty)=0}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}}_{\substack{\text{vanishes}\\ \text{because}\\ \text{number of}\\ \text{particles is}\\ \text{conserved}}}\Bigg]$$
$$= \sum_\alpha q_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,f_\alpha\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi = -\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}, \tag{5.3}$$

where $\boldsymbol{j}$ is the current density. So the rate of change of the electric field is minus the rate at which electric field does work on the charges, a.k.a. Joule heating—not a surprising result. The energy goes into accelerating particles, of course: the rate of change of their kinetic energy is

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = \sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\underbrace{\frac{m_\alpha v^2}{2}\,\frac{\partial f_\alpha}{\partial t}}_{\text{use (5.1)}}$$
$$= \sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2}\Bigg[\underbrace{-\boldsymbol{v}\cdot\boldsymbol{\nabla}f_\alpha}_{\substack{\text{vanishes}\\ \text{because}\\ \text{full}\\ \text{divergence}}} + \underbrace{\frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{v}}}_{\text{by parts in } \boldsymbol{v}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}}_{\substack{\text{vanishes}\\ \text{because}\\ \text{energy is}\\ \text{conserved}}}\Bigg]$$
$$= -\sum_\alpha q_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,f_\alpha\boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi = \int \mathrm{d}\boldsymbol{r}\,\boldsymbol{E}\cdot\boldsymbol{j}. \tag{5.4}$$

Combining (5.3) and (5.4) gives us the law of energy conservation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{K}+\mathscr{U}) = 0}\,. \tag{5.5}$$

Exercise 5.1. Demonstrate energy conservation for the more general case in which magnetic-field perturbations are also allowed (e.g., for the Vlasov–Maxwell system).

Thus, if the perturbations are damped, the energy of the particles must increase—this is usually called *heating*. Strictly speaking, however, heating is a slow, irreversible increase in the mean temperature of the thermal equilibrium. Let us make this statement quantitative. Consider a Maxwellian plasma, homogeneous in space but possibly with some slow dependence on time (cf. §2):

$$f_{0\alpha} = \frac{n_\alpha}{(\pi v_{\mathrm{th}\alpha}^2)^{3/2}}\,e^{-v^2/v_{\mathrm{th}\alpha}^2} = n_\alpha\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} e^{-m_\alpha v^2/2T_\alpha}. \tag{5.6}$$

In a homogeneous system with a fixed volume, the density n_α is constant in time because the number of particles is constant:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,Vn_\alpha = 0. \tag{5.7}$$

The temperature, however, is allowed to change: $T_\alpha = T_\alpha(t)$. The total kinetic energy of the particles is

$$\mathscr{K} = V\sum_\alpha \underbrace{\int \mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2}\,f_{0\alpha}}_{=\frac{3}{2}n_\alpha T_\alpha} + \sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2}\,\delta f_\alpha. \tag{5.8}$$

Let us average this over time, as per (2.7): the perturbed part vanishes and we have

$$\langle\mathscr{K}\rangle = V\sum_\alpha \frac{3}{2}\,n_\alpha T_\alpha. \tag{5.9}$$

Averaging also (5.5) and using (5.9), we get

$$\sum_\alpha \frac{3}{2}\,n_\alpha\,\frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{V}\int\mathrm{d}\boldsymbol{r}\,\frac{\langle E^2\rangle}{8\pi}, \tag{5.10}$$

so the heating rate of the equilibrium equals the rate of decrease of the mean energy of the perturbations.

We saw that the perturbations evolve according to (3.16). If we wait for a while, only the slowest-damped mode will matter, with all others exponentially small in comparison. Let us call its frequency and its damping rate $\omega_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}} < 0$, respectively, so $\boldsymbol{E}_{\boldsymbol{k}} \propto e^{-i\omega_{\boldsymbol{k}}t+\gamma_{\boldsymbol{k}}t}$. If $|\gamma_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}}$, the time average (2.7) can be defined in such a way that $\omega_{\boldsymbol{k}}^{-1} \ll \varDelta t \ll |\gamma_{\boldsymbol{k}}|^{-1}$. Then (5.10) becomes

$$\sum_\alpha \frac{3}{2}\,n_\alpha\,\frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = -\sum_{\boldsymbol{k}} 2\gamma_{\boldsymbol{k}}\,\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} > 0. \tag{5.11}$$

The Landau damping rate of the electric-field perturbations is the heating rate of the equilibrium.[39]

This result, while at first glance utterly obvious, might, on reflection, appear to be paradoxical: surely, the heating of the equilibrium implies increasing entropy—but the damping that is leading to the heating is collisionless and, in a collisionless system, in view of the H-theorem, how can entropy change?

5.2. *Entropy and Free Energy*

The kinetic entropy for each species of particles is defined to be

$$S_\alpha = -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,f_\alpha \ln f_\alpha. \tag{5.12}$$

This quantity [or, indeed, the full-phase-space integral of any quantity that is a function only of f_α; see (9.8)] can only be changed by collisions. Furthermore, the plasma-physics version of Boltzmann's H-theorem says that

$$\frac{\mathrm{d}S_\alpha}{\mathrm{d}t} = -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} \ln f_\alpha \geqslant 0, \tag{5.13}$$

[39] Obviously, the damping of waves on particles of species α increases only the temperature of that species.

where equality is achieved iff all f_α are Maxwellian with the same temperature $T_\alpha = T$.

Thus, if collisions are ignored, the total entropy stays constant and everything that happens is, in principle, reversible. So how can there be net damping of waves and, worse still, net heating of the equilibrium particle distribution?! Presumably, any damping solution can be turned into a growing solution by reversing all particle trajectories—so should the overall perturbation level rather not stay constant?

As I already noted in §5.1, strictly speaking, heating is the increase of the equilibrium temperature—and, therefore, of the equilibrium entropy. Indeed, for each species, the equilibrium entropy is

$$\begin{aligned} S_0 = -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, f_0 \ln f_0 &= -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, f_0 \left\{ \ln\left[n\left(\frac{m}{2\pi}\right)^{3/2}\right] - \frac{3}{2}\ln T - \frac{mv^2}{2T}\right\} \\ &= V\left\{ -n\ln\left[n\left(\frac{m}{2\pi}\right)^{3/2}\right] + \frac{3}{2}\, n\ln T + \frac{3}{2}\, n\right\}, \end{aligned} \tag{5.14}$$

where I have used $\int \mathrm{d}\boldsymbol{v}\,(mv^2/2)f_0 = (3/2)nT$. Since $n = \mathrm{const}$,

$$T\,\frac{\mathrm{d}S_0}{\mathrm{d}t} = V\,\frac{3}{2}\,n\,\frac{\mathrm{d}T}{\mathrm{d}t}, \tag{5.15}$$

so heating is indeed associated with an increase of S_0.

Since, according to (5.11), this can be successfully accomplished by collisionless damping and since entropy overall can only increase due to collisions, we must search for the "missing entropy" (or, rather, for the missing *decrease* of entropy) in the perturbed part of the distribution. The mean entropy associated with the perturbed distribution is[40]

$$\begin{aligned} \langle \delta S\rangle = \langle S - S_0\rangle &= -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\ \langle (f_0+\delta f)\ln(f_0+\delta f) - f_0\ln f_0\rangle \\ &= -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\ \left\langle (f_0+\delta f)\left[\ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2f_0^2} + \dots\right] - f_0\ln f_0\right\rangle \\ &\approx -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{\langle\delta f^2\rangle}{2f_0}, \end{aligned} \tag{5.16}$$

after expanding to second order in small $\delta f/f_0$ and using $\langle\delta f\rangle = 0$. The total mean entropy of each species, $\langle S\rangle = S_0 + \langle\delta S\rangle$, can only by changed by collisions, so, if collisions are ignored, any heating of a given species, i.e., any increase in its S_0 [see (5.15)] must be compensated by a *decrease* in its $\langle\delta S\rangle$. The latter can only be achieved by *increasing* $\langle\delta f^2\rangle$: indeed, using (5.15) and (5.16), we find[41]

$$\begin{aligned} T\left(\frac{\mathrm{d}S_0}{\mathrm{d}t} + \frac{\mathrm{d}\langle\delta S\rangle}{\mathrm{d}t}\right) &= V\,\frac{3}{2}\,n\,\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{T\langle\delta f^2\rangle}{2f_0} \\ &= -\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, T\left\langle \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \ln f\right\rangle, \end{aligned} \tag{5.17}$$

---

[40] As an aside, note that this piece of the calculation is entirely idependent of what f_0 is. It simply demonstrates that the entropy of an averaged distribution f_0 is always larger than that of the exact distribution f, as long as $\delta f \ll f_0$. If the average is reinterpreted as a coarse graining over phase space, this argument is sometimes viewed as a kind of "proof" (or illustration) of the second law of thermodynamics. Indeed, take f_0 and f to be the same at some initial time t. Then $S_0(t) = \langle S(t)\rangle$. Now advance to time $t + \delta t$. Some small δf arises, but coarse graining "deletes" the information contained in it and (5.16) shows that $S_0(t+\delta t) > \langle S(t+\delta t)\rangle$ (cf. the general statistical-mechanical argument to the same effect: Schekochihin 2024, §§12.4 and 13.4).

[41] In the second term, T can be brought inside the time derivative because $\langle\delta f^2\rangle/f_0 \ll f_0$.

the last equality due to Boltzmann's H-theorem (5.13). If collisions are ignored, T can only increase if $\langle \delta f^2 \rangle$ increases too.

It is useful to work out the collision term in (5.17) in terms of f_0 and δf: using the fact that $\langle \delta f \rangle = 0$ by definition and that the number of particles is conserved by the collision operator, we get

$$\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, T \left\langle \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \ln f \right\rangle \approx \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} \left[T \left(\frac{\partial f_0}{\partial t}\right)_{\mathrm{c}} \ln f_0 + \left\langle \frac{T\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t}\right)_{\mathrm{c}} \right\rangle \right]$$
$$= V \int \mathrm{d}\boldsymbol{v}\, \frac{mv^2}{2} \left(\frac{\partial f_0}{\partial t}\right)_{\mathrm{c}} + \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} \left\langle \frac{T\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t}\right)_{\mathrm{c}} \right\rangle. \tag{5.18}$$

The second term is the collisional damping of δf, of which more will be said soon. The first term is the collisional energy exchange between the equilibrium distributions of different species (intra-species collisions conserve energy, but inter-species ones do not, because there is friction between species). If the species under consideration is α, this energy exchange can be represented as $\sum_{\alpha'} \nu_{\alpha\alpha'}(T_\alpha - T_{\alpha'})$ (see, e.g., Helander & Sigmar 2005) and will act to equilibrate temperatures between species as the system strives toward thermal equilibrium. If the collision frequencies $\nu_{\alpha\alpha'}$ are small, this is a slow effect. Due to overall energy conservation, the energy-exchange terms vanish exactly if (5.18) is summed over species.

Finally, let us sum (5.17) over species and use (5.10) to relate the total heating to the rate of change of the electric-perturbation energy:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\left[\sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, \frac{T_\alpha \langle \delta f_\alpha^2 \rangle}{2 f_{0\alpha}} + \int \mathrm{d}\boldsymbol{r}\, \frac{\langle E^2 \rangle}{8\pi} \right]}_{\equiv \mathscr{F}} = \sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} \left\langle \frac{T_\alpha \delta f_\alpha}{f_{0\alpha}} \left(\frac{\partial \delta f_\alpha}{\partial t}\right)_{\mathrm{c}} \right\rangle \leqslant 0}, \tag{5.19}$$

where I used (5.18) in the right-hand side (with the total equilibrium collisional energy-exchange terms vanishing upon summation over species). The right-hand side must be non-positive-definite because collisions cannot decrease entropy [see (5.13)].

Equation (5.19) is a way to express the idea that, except for the effect of collisions, the change in the electric-perturbation energy ($= -$heating) must be compensated by the change in $\langle \delta f^2 \rangle$, in terms of a *conservation law* of a quadratic positive-definite quantity, $\mathscr{F}$, that measures the total amount of perturbation in the system (a quadratic norm of the perturbed solution).[42] It is not hard to realise that this quantity is the *free energy* of the perturbed state, comprising the entropy of the perturbed distribution and the energy of the electric field:

$$\boxed{\mathscr{F} = \mathscr{U} - \sum_\alpha T_\alpha \langle \delta S_\alpha \rangle} \tag{5.20}$$

(to be precise, it is the difference—always positive!—between the free energy of the perturbed state, $\mathscr{U} - \sum_\alpha T_\alpha \langle S_\alpha \rangle$, and the free energy of the equilibrium, $-\sum_\alpha T_\alpha S_{0\alpha}$). You may recall from thermodynamics that free energy is the quantity that systems on their way to equilibrium want to minimise at fixed volume and temperature (Schekochihin 2024, §15.3).

---

[42]Note that the existence of such a quantity implies that the Maxwellian equilibrium is stable: if a quadratic norm of the perturbed solution cannot grow, clearly there cannot be any exponentially growing solutions. This is known as Newcomb's theorem, first communicated to the world in the paper by Bernstein (1958, Appendix I). A generalisation of this principle to general isotropic distributions is the subject of Q-6(c) and of §9.4, where the conserved quantity $\mathscr{F}$ will reemerge in a different way, confirming its status as a Platonic entity that cannot be avoided.

It is quite a typical situation in non-equilibrium systems that there is an energy-like (quadratic in the relevant fields and positive definite) quantity that is conserved except for dissipation. For example, in hydrodynamics, the motion of a fluid is governed by the Navier–Stokes equation:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)=-\boldsymbol{\nabla}p+\mu\Delta\boldsymbol{u}, \tag{5.21}$$

where $\boldsymbol{u}$ is velocity, ρ mass density ($\rho=\mathrm{const}$ for an incompressible fluid), p pressure and μ the dynamical viscosity of the fluid. The conservation law is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{r}\,\frac{\rho u^2}{2}=-\mu\int\mathrm{d}\boldsymbol{r}\,|\boldsymbol{\nabla}\boldsymbol{u}|^2\leqslant 0. \tag{5.22}$$

The conserved quadratic quantity is kinetic energy and the negative-definite dissipation (leading to net entropy production) is viscous heating.

You will find a similar conservation law for incompressible MHD if, in §14.10, you work out the time evolution of $\int\mathrm{d}\boldsymbol{r}\,(\rho u^2/2+B^2/8\pi)$ assuming $\rho=\mathrm{const}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$; cf. (16.70).

Exercise 5.2. Free energy and kinetic energy of mean plasma flow. Suppose the perturbation δf contains a mean flow of particles, with velocity $\boldsymbol{u}$. Show that it is then always formally possible to decompose

$$\delta f=\frac{2\boldsymbol{u}\cdot\boldsymbol{v}}{v_{\mathrm{th}}^2}\,f_0+h, \tag{5.23}$$

where $\int\mathrm{d}\boldsymbol{v}\,\boldsymbol{v}h=0$. Hence show that

$$\iint\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{T\langle\delta f^2\rangle}{2f_0}=\int\mathrm{d}\boldsymbol{r}\,\frac{mn\langle u^2\rangle}{2}+\iint\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{T\langle h^2\rangle}{2f_0}, \tag{5.24}$$

i.e., that the entropic part of the free energy is equal to the kinetic energy of the mean plasma flow plus the free energy associated with the part of the perturbed distribution that has no mean velocity. This means that "fluid" energy budgets such as (5.22) and (16.70) (with dissipation included) are not just analogs, but particular cases, of the free-energy budget (5.19). In §5.7, I will show how these ideas play out for Langmuir waves.

A corollary of all this is that, as the electric perturbations decay via Landau damping, the perturbed distribution function must grow. This calls for going back to our solution for it (§3.1) and analysing carefully the behaviour of δf.

5.3. *Structure of Perturbed Distribution Function*

Start with our solution (3.8) for $\delta f(p)$ and substitute into it the solution (3.15) for $\varphi(p)$:

$$\delta\hat{f}(p)=\underbrace{\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}}_{\substack{\text{“kinetic”}\\ \text{(ballistic)}\\ \text{pole}}}\left\{i\,\frac{q}{m}\left[\underbrace{\sum_i\frac{c_i}{p-p_i}}_{\substack{\text{poles}\\ \text{representing}\\ \text{linear}\\ \text{modes, see}\\ \text{(3.17)}}}+A(p)\right]\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}+g\right\}. \tag{5.25}$$

To compute the inverse Laplace transform (3.6), we adopt the same method as in §3.1 (Fig. 6), viz., shift the integration contour to large negative $\mathrm{Re}\,p$ as shown in Fig. 24 and

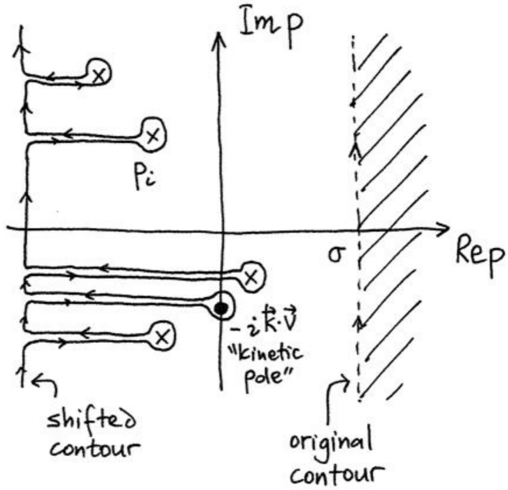


FIGURE 24. Shifting the integration contour in (5.26). This is analogous to Fig. 6 but note the additional "kinetic" pole.

use Cauchy's formula:[43]

$$\delta f(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \mathrm{d}p\, e^{pt}\delta\hat{f}(p) = \underbrace{i\,\frac{q}{m}\sum_i \frac{c_i e^{p_i t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}}_{\substack{\text{eigenmode-like}\\ \text{solution, comes from}\\ \text{the poles of } \varphi(p)}}$$

$$+ \underbrace{e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}\left\{g - i\,\frac{q}{m}\left[\sum_i \frac{c_i}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}} + A(-i\boldsymbol{k}\cdot\boldsymbol{v})\right]\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right\}}_{\substack{\text{ballistic response,}\\ \text{comes from}\\ p=-i\boldsymbol{k}\cdot\boldsymbol{v}}}. \tag{5.26}$$

The solution (5.26) teaches us two important things.

First, *the Landau-damped solution is not an eigenmode.* Even though the evolution of the potential, given by (3.16), does look like a sum of damped eigenmodes of the form $\varphi \propto e^{p_i t}$, $\mathrm{Re}\, p_i < 0$, the full solution of the Vlasov–Poisson system does *not* decay: there is a part of $\delta f(t)$, the "ballistic response" $\propto e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$, that oscillates without decaying—in fact, we shall see in §5.6 that δf even has a growing part! It is this part that is responsible for keeping free energy conserved, as per (5.19) without collisions (§5.7). Thus, you may think of Landau damping as a process of transferring (free) energy from the electric-field perturbations to the perturbations of the distribution function.

In contrast to the case of damping, a growing solution ($\mathrm{Re}\, p_i > 0$) can be viewed as an eigenmode

[43] A perceptive reader has spotted that this formula does not seem to satisfy $\delta f(t=0) = g$ unless $A(-i\boldsymbol{k}\cdot\boldsymbol{v}) = 0$. This is because, as explained in footnote 13, the method for calculating the inverse Laplace transform that involves discarding the integral along the vertical part of the shifted contour in Fig. 24 only works in the limit of long times. It is an instructive exercise in complex analysis to show that, in the (overly restrictive) case of $\hat{\varphi}(p)$ decaying quickly at $\mathrm{Re}\, p \to -\infty$, the solution (5.26) is also valid at finite t and, accordingly, $A(-i\boldsymbol{k}\cdot\boldsymbol{v}) = 0$, i.e., $A(p)$ vanishes for any purely imaginary p.

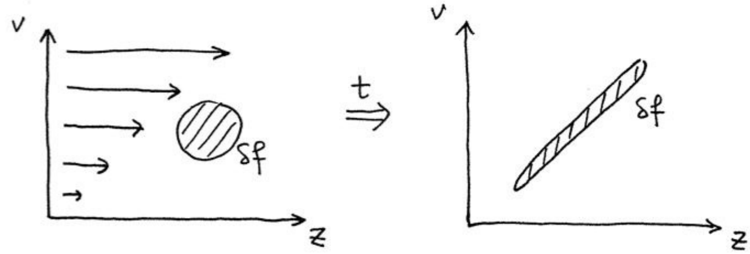


FIGURE 25. Shearing of δf in phase space.

because, after a few growth times, the first term in (5.26) will be exponentially larger than the ballistic term. This will allow us to ignore the latter in our treatment of QLT (§6.1)—a handy, although not necessary (see Q-9), simplification. Note that reversibility is not an issue for the growing solutions: so, there may be (and often are) damped solutions as well, so what? We only care about the growing modes because they will be all that is left if we wait long enough.

Secondly, the δf perturbations have *fine structure in velocity (phase) space.* This structure gets finer with time: roughly speaking, if $\delta f \propto e^{-ikvt}$, then

$$\frac{1}{\delta f}\frac{\partial \delta f}{\partial v} \sim ikt \to \infty \quad \text{as} \quad t \to \infty. \tag{5.27}$$

This phenomenon is called *phase mixing.* You can think of the basic mechanism responsible for it as a shearing in phase space: the homogeneous part of the linearised kinetic equation,

$$\frac{\partial \delta f}{\partial t} + v\,\frac{\partial \delta f}{\partial z} = \dots, \tag{5.28}$$

describes advection of δf by a linear shear flow in the the (z, v) plane. This turns any δf structure in this plane into long thin filaments, with large gradients in v (Fig. 25).

5.4. *Landau Damping Is Enabled by Phase Mixing*

Phase mixing helps us make sense of the notion that, even though φ is the velocity integral of δf, the former can be decaying while the latter is not:

$$\varphi = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \underbrace{\int \mathrm{d}\boldsymbol{v}\,\delta f_\alpha}_{\substack{\text{fine}\\ \text{structure}\\ \text{cancels}}} \propto e^{-\gamma t} \to 0. \tag{5.29}$$

The velocity integral over the fine structure increasingly cancels as time goes on—a perturbation initially "visible" as φ phase-mixes away, disappearing into the negative entropy associated with the fine velocity dependence of δf [see (5.16)].

One can similarly, and more generally, argue that the refinement of velocity dependence of δf causes lower velocity moments of δf (density, flow velocity, pressure, heat flux, and so on) to decrease with time, transferring free energy to higher moments (ever higher as time goes on). One way to formalise this statement neatly is in terms of Hermite moments: since Hermite polynomials are orthogonal, the free energy of the perturbed distribution can be written as a sum of "energies" of the Hermite moments [see (6.77)]. It is then possible to represent the Landau-damped perturbations as having a broad spectrum in Hermite space, with the majority of the free energy residing in high-order moments—infinitely high in the formal limit of zero collisionality and infinite time (see Q-8 and Kanekar *et al.* 2015).

Since the mth-order Hermite moment can, for $m \gg 1$, be asymptotically represented as a cosine function in v space oscillating with the "frequency" $\sqrt{2m}/v_{\rm th}$ [see (6.78)], (5.27) implies that the typical order of the moment in which the free energy resides grows with time as $m \sim (kv_{\rm th}t)^2$. An approach involving a Fourier, rather than Hermite, transform in velocity space will be presented in §12.2.2.

Taking Hermite (or other kind of) moments of the kinetic equation is essentially the procedure for deriving "fluid" equations for the plasma—or, rather, plasma becomes a fluid if this procedure can be stopped after a few moments (e.g., in the limit of strong collisionality, this happens at the third moment; see Dellar 2015, Parra 2019*a*, and Kunz 2021). Since Landau damping is a long-time effect of this phase-mixing process, it cannot be captured by any fluid approximation to the kinetic system involving a truncation of the hierarchy of moment equations at some finite-order moment—it is an essentially kinetic effect "beyond all orders".

One useful way to see this is by examining the structure of Langmuir hydrodynamics, which was the subject of Exercise 3.1. The moment hierarchy can be truncated by assuming $kv_{\rm the}/\omega \gg 1$, but one can never capture Landau damping however many moments one keeps: indeed, the Landau damping rate (3.41) for, say, a Maxwellian plasma, will be $\gamma \propto \exp(-\omega^2/k^2v_{\rm the}^2)$, all coefficients in the Taylor expansion of which in powers of $kv_{\rm the}/\omega$ are zero.

Note that, in a pure phase-mixing process, i.e., one involving particle streaming only, and no field-particle interaction, moments of the distribution function decay faster than exponentially in time, so, in a certain sense, Landau damping is actually a slowing down of this process (§§5.6 and 5.7 might shed some light on why that is). As further reading on this topic, I recommend the paper by Hammett *et al.* (1992), which was in certain respects foundational to the modern understanding of phase mixing and its connexion with Landau damping and with various "fluid" descriptions of plasmas.

Exercise 5.3. Consider a particle system described by the kinetic equation with no electric field (1.48) and with an initial Maxwellian distribution whose density is $n = n_0 + a\cos(kz)$, where $n_0 = \text{const}$. Show that the density perturbation decays in time as

$$\delta n(t) = \delta n(0) e^{-(kv_{\rm th}t)^2/4}. \tag{5.30}$$

5.5. *Role of Collisions*

As ever larger velocity-space gradients emerge, it becomes inevitable that at some point they will become so large that *collisions can no longer be ignored.* This is because the Landau collision operator is a Fokker–Planck (diffusion) operator in velocity space [see (1.47)] and so it will eventually wipe out the fine structure in $\boldsymbol{v}$, however small is the collision frequency ν. Let us estimate how long this takes.

The size of the velocity-space gradients of δf due to ballistic response is given by (5.27). Then the collision term is

$$\left(\frac{\partial \delta f}{\partial t}\right)_{\rm c} \sim \nu v_{\rm th}^2 \frac{\partial^2 \delta f}{\partial v^2} \sim -\nu v_{\rm th}^2 k^2 t^2 \delta f. \tag{5.31}$$

Solving for the time evolution of the perturbed distribution function due to collisions,

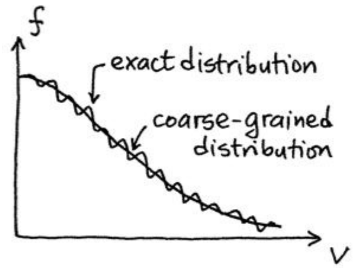


FIGURE 26. Coarse graining of the distribution function.

we get[44]

$$\frac{\partial\delta f}{\partial t} \sim -\nu(kv_{\text{th}}t)^2\delta f \quad \Rightarrow \quad \delta f \sim \exp\left(-\frac{1}{3}\,\nu k^2 v_{\text{th}}^2 t^3\right) \equiv e^{-(t/t_{\text{c}})^3}. \tag{5.32}$$

Therefore, the characteristic collisional decay time is[45]

$$\boxed{t_{\text{c}} \sim \frac{1}{\nu^{1/3}(kv_{\text{th}})^{2/3}}}\,. \tag{5.33}$$

Note that $t_{\text{c}} \ll \nu^{-1}$ provided $\nu \ll kv_{\text{th}}$, i.e., t_{c} is within the range of times over which our "collisionless" theory is valid. After time t_{c}, "collisionless" damping becomes irreversible because the part of δf that is fast-varying in velocity space is lost (entropy has grown) and so it is no longer possible, even in principle, to invert all particle trajectories, have the system retrace back its steps, "phase-unmix" and thus "undamp" the damped perturbation.

In a sufficiently collisionless system, phase unmixing is, in fact, possible if nonlinearity is allowed—giving rise to the beautiful phenomenon of *plasma echo*, in which perturbations can first appear to be damped away but then come back from phase space (§12.1). This effect is a source of much preoccupation to pure mathematicians (Villani 2014; Bedrossian 2016): indeed the validity of the linearised Vlasov equation (3.1) as a sensible approximation to the full nonlinear one (2.12) is in question if the velocity derivative $\partial\delta f/\partial\boldsymbol{v}$ in the last term of the latter starts growing uncontrollably. Phase unmixing has also recently turned out to have interesting consequences for the role of Landau damping in plasma turbulence (see §12.2 onwards).

Some rather purist theoreticians sometimes choose to replace collisional estimates of the type discussed above by a stipulation that $\delta f(\boldsymbol{v})$ must be *"coarse-grained"* beyond some suitably chosen scale in $\boldsymbol{v}$ (Fig. 26)—this is equivalent to saying that the formation of the fine-structured phase-space part of δf constitutes a loss of information and so leads to growth of entropy (i.e., loss of negative entropy associated with $\langle\delta f^2\rangle$). Somewhat non-rigorously, this means that we can just consider the ballistic term in (5.26) to have been wiped out and use the coarse-grained

---

[44] A slightly more rigorous calculation, involving both collisions and phase mixing, can be done along the same lines as in Exercise 12.3.

[45] As far as I can tell, this result, like many other textbook results in early plasma physics, belongs to a Soviet (Karpman 1967) and some Americans in hot pursuit (Su & Oberman 1968).

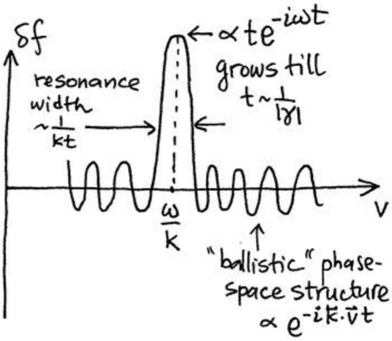


FIGURE 27. Emergence of the Case–van Kampen mode.

(i.e., velocity-space-averaged) version of δf:[46]

$$\overline{\delta f} = i\,\frac{q}{m}\sum_i \frac{c_i e^{p_i t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{5.34}$$

We can check that the correct solution (3.16) for the potential can be recovered from this:

$$\begin{aligned}\varphi &= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{v}\,\overline{\delta f}_\alpha \\ &= \sum_i c_i e^{p_i t}\Bigg[\underbrace{\sum_\alpha i\frac{4\pi q_\alpha^2}{m_\alpha k^2}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} - 1}_{=\,-\epsilon(p_i,\boldsymbol{k})\,=\,0 \text{ by definition of } p_i} + 1\Bigg] = \sum_i c_i e^{p_i t}.\end{aligned} \tag{5.35}$$

If you are wondering how this works without the coarse-graining kludge, read on.

5.6. *Further Analysis of δf*

Having given a rather qualitative analysis of the structure and consequences of the solution (5.26), I anticipate a degree of dissatisfaction from a perceptive reader. Yes, there is a non-decaying piece of δf. But conservation of free energy in a collisionless system in the face of Landau damping in fact requires $\langle \delta f^2\rangle$ to grow, not just to fail to decay [see (5.19)]. How do we see that this does indeed happen? The analysis that follows addresses this question. These considerations are not really necessary for most practical plasma-physics calculations (see, however, Q-9), but it may be necessary for your peace of mind and greater comfort with this whole conceptual framework.

Let us rearrange the solution (5.26) as follows:

$$\boxed{\delta f(t) = i\,\frac{q}{m}\sum_i c_i\,\frac{e^{p_i t} - e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + (g + \dots)e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}\,. \tag{5.36}$$

The second term is the ballistic evolution of perturbations (particles flying apart in straight lines at different velocities)—a homogeneous solution of the kinetic equation (3.1). This develops a lot of fine-scale velocity-space structure, but obviously does not grow. The first term, a particular solution arising from the (linear) wave-particle interaction, is more interesting, especially around the resonances $\mathrm{Re}\,p_i + \boldsymbol{k}\cdot\boldsymbol{v} = 0$.

Consider one of the modes, $p_i = -i\omega + \gamma$, and assume $\gamma \ll \boldsymbol{k}\cdot\boldsymbol{v} \sim \omega$. This allows one to

[46]With the understanding that any integral involving the resonant denominator must be taken along the Landau contour (see Q-9). If you adopt this shorthand, you can, nonrigorously but often expeditiously, use Fourier transforms to map time into the (complex) frequency space, rather than Laplace transforms.

introduce "intermediate" times:

$$\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}} \ll t \ll \frac{1}{\gamma}. \tag{5.37}$$

This means that the wave has had time to oscillate, phase mixing has got underway, but the perturbation has not yet been damped away significantly. We have then, for the relevant piece of the perturbed distribution (5.36),

$$\delta f \propto \frac{e^{p_i t} - e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}} = -ie^{-i\omega t}\frac{e^{\gamma t} - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega - i\gamma} \approx -ie^{-i\omega t}\frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega}, \tag{5.38}$$

with the last, approximate, expression valid at the intermediate times (5.37), assuming also that, even though we might be close to the resonance, we shall not come closer than γ, viz., $|\boldsymbol{k}\cdot\boldsymbol{v}-\omega| \gg \gamma$. Respecting this ordering, but taking $|\boldsymbol{k}\cdot\boldsymbol{v}-\omega| \ll 1/t$, we find

$$\delta f \propto t\, e^{-i\omega t}. \tag{5.39}$$

Thus, δf has a peak that *grows with time*, emerging from the sea of fine-scale but constant-amplitude structures (Fig. 27). The width of this peak is obviously $|\boldsymbol{k}\cdot\boldsymbol{v}-\omega| \sim 1/t$ and so δf around the resonance develops a sharp structure, which, in the formal limit $t\to\infty$ (but respecting $\gamma t \ll 1$, i.e., with infinitesimal damping), tends to a delta function:

$$\boxed{\delta f \propto -ie^{-i\omega t}\frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega} \to e^{-i\omega t}\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)} \quad \text{as} \quad t\to\infty. \tag{5.40}$$

Here is a "proof":

$$\frac{1-e^{-ixt}}{x} = \underbrace{\frac{1-\cos xt}{x}}_{\substack{\text{finite as}\\ t\to\infty,\\ \text{even at}\\ x=0}} + \underbrace{i\frac{\sin xt}{x}}_{\substack{=it \text{ at}\\ x=0,\\ \text{so dom-}\\ \text{inant}}} \approx \frac{e^{ixt}-e^{-ixt}}{2x} = \frac{i}{2}\int_{-t}^{t}\mathrm{d}t' e^{ixt'} \to i\pi\delta(x) \quad \text{as} \quad t\to\infty. \tag{5.41}$$

Note that writing the solution in the vicinity of the resonance in the form of a δ-function, as in (5.40), is tantamount to stipulating that any integral taken with respect to $\boldsymbol{v}$ (or $\boldsymbol{k}$) and involving δf must always be done along the Landau contour, circumventing the pole from below [cf. (3.23)]. You will find the representation (5.40) of δf useful in working out the quasilinear theory of Landau damping in Q-9.

If finite damping is restored, all this goes on until $t \sim 1/\gamma$, with the delta function reaching the height $\propto 1/\gamma$ and width $\propto \gamma$. In the limit $t \gg 1/\gamma$, the damped part of the solution decays, $e^{\gamma t}\to 0$, and we are left with just the ballistic part, the second term in (5.26).

5.7. *Free-Energy Conservation for Landau-Damped Langmuir Waves*

Finally, let us convince ourselves that, if we ignore collisions, we can recover (5.19) with a zero right-hand side from the full collisionless Landau-damped solution given by (3.16) and (5.36). For simplicity, let us consider the case of electron Langmuir waves and prove that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}|^2}{2f_0} = -2\gamma_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}}\,. \tag{5.42}$$

In (5.36), let the relevant root of the dispersion relation be $p_i = -i\omega_{\mathrm{pe}} + \gamma_{\boldsymbol{k}}$, where $\gamma_{\boldsymbol{k}}$ is given by (3.41), and assume a Maxwellian f_0. Based on the discussion §5.6, we should expect $\delta f_{\boldsymbol{k}}$ to develop a growing δ-like peak around the resonance $\boldsymbol{k}\cdot\boldsymbol{v}\approx\omega_{\mathrm{pe}}$. In this region of velocity space, the distribution function (5.36) for electrons ($q=-e$) is

$$\delta f_{\boldsymbol{k}}^{(\mathrm{res})} \approx \frac{e}{m_e}\underbrace{c_i e^{p_i t}}_{=\varphi_{\boldsymbol{k}}}\frac{1-e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-ip_i)t}}{\boldsymbol{k}\cdot\boldsymbol{v}-ip_i}\frac{2\boldsymbol{k}\cdot\boldsymbol{v}}{v_{\mathrm{th}e}^2}f_0 \approx \frac{e\varphi_{\boldsymbol{k}}}{T}\,\boldsymbol{k}\cdot\boldsymbol{v}\underbrace{\frac{1-e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\mathrm{pe}})t}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\mathrm{pe}}}}_{\approx i\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\mathrm{pe}})}f_0. \tag{5.43}$$

We are going to have to compute $|\delta f_{\boldsymbol{k}}|^2$ and squaring delta functions is a dangerous game belonging to the class of games that one must play *veeery carefully.*[47] Here is how:

$$\frac{\partial}{\partial t}\left|\frac{1-e^{-ixt}}{x}\right|^2 = \frac{\partial}{\partial t}\frac{4}{x^2}\sin^2\frac{xt}{2} = \frac{2\sin xt}{x} \xrightarrow[t\to\infty]{} 2\pi\delta(x) \quad\Rightarrow\quad \left|\frac{1-e^{-ixt}}{x}\right|^2 \xrightarrow[t\to\infty]{} 2\pi t\delta(x). \tag{5.44}$$

Using this prescription,

$$\begin{aligned}\int \mathrm{d}\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{res})}|^2}{2f_0} &= \int\mathrm{d}\boldsymbol{v}\,\frac{e^2|\varphi_{\boldsymbol{k}}|^2}{2T}(\boldsymbol{k}\cdot\boldsymbol{v})^2 2\pi t\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\mathrm{pe}})f_0\\ &= t|\varphi_{\boldsymbol{k}}|^2\frac{2\pi e^2\omega_{\mathrm{pe}}^2}{m_e v_{\mathrm{th}e}^2 k}F\left(\frac{\omega_{\mathrm{pe}}}{k}\right)\\ &= 4t\,\frac{k^2|\varphi_{\boldsymbol{k}}|^2}{8\pi}\underbrace{\frac{\omega_{\mathrm{pe}}^4}{k^3}\frac{\pi}{n_e v_{\mathrm{th}e}^2}F\left(\frac{\omega_{\mathrm{pe}}}{k}\right)}_{=-\gamma_{\boldsymbol{k}}>0;\text{ see }(3.41)}\\ &= -4\gamma_{\boldsymbol{k}}t\,\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}.\end{aligned} \tag{5.45}$$

Thus, the entropic part of the free energy grows secularly with time (assuming still $\gamma_{\boldsymbol{k}}t \ll 1$). Its time derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{res})}|^2}{2f_0} \approx -4\gamma_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -2\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}. \tag{5.46}$$

Despite what it looks like, the extra factor of 2 in (5.46) compared to (5.42) is a feature, not a bug. If you have done Exercise 3.1 (or even just paid attention in §2.1), you know that a Langmuir oscillations involve some mean (oscillating) flows of the plasma, and so a sloshing of energy between potential, $|\boldsymbol{E}_{\boldsymbol{k}}|^2/8\pi$, and kinetic, $n_e m_e|\boldsymbol{u}_{\boldsymbol{k}}|^2/2$, where $\boldsymbol{u}_{\boldsymbol{k}} = (1/n_e)\int\mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\,\delta f_{\boldsymbol{k}}$. These flows are contained in the non-resonant ("thermal") part of $\delta f_{\boldsymbol{k}}$, i.e., in $\delta f_{\boldsymbol{k}}$ at velocities such that $\boldsymbol{k}\cdot\boldsymbol{v} \ll \omega_{\mathrm{pe}}$. In this region of velocity space, let us write the distribution function (5.36) as follows:

$$\delta f_{\boldsymbol{k}}^{(\mathrm{th})} = \frac{e}{m_e}\underbrace{c_i e^{p_i t}}_{=\varphi_{\boldsymbol{k}}}\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-ip_i}\frac{2\boldsymbol{k}\cdot\boldsymbol{v}}{v_{\mathrm{th}e}^2}f_0 + e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}h_{\boldsymbol{k}} \approx -\frac{e\varphi_{\boldsymbol{k}}}{T}\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega_{\mathrm{pe}}}f_0 + e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}h_{\boldsymbol{k}}, \tag{5.47}$$

where $h_{\boldsymbol{k}}$ denotes everything in (5.36) that multiplies $e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$. This should remind you of Exercise 5.2. The first term in (5.47) is precisely the plasma flow:

$$\boldsymbol{u}_{\boldsymbol{k}} = \frac{1}{n_e}\int\mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\,\delta f_{\boldsymbol{k}}^{(\mathrm{th})} = -\frac{e\varphi_{\boldsymbol{k}}}{T\omega_{\mathrm{pe}}n_e}\boldsymbol{k}\cdot\underbrace{\int\mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}f_0}_{=\mathsf{I}n_e v_{\mathrm{th}e}^2/2} = \frac{e\boldsymbol{E}_{\boldsymbol{k}}}{i\omega_{\mathrm{pe}}m_e}. \tag{5.48}$$

The contribution from the second term in (5.47) to the velocity integral has vanished in the limit $(\boldsymbol{k}\cdot\boldsymbol{v})t \gg 1$. Note that (5.48) just says that $m_e\dot{\boldsymbol{u}}_{\boldsymbol{k}} = -e\boldsymbol{E}_{\boldsymbol{k}}$, as indeed is the case in a Langmuir oscillation. It is not hard to check that $m_e n_e|\boldsymbol{u}_{\boldsymbol{k}}|^2/2 = |\boldsymbol{E}_{\boldsymbol{k}}|^2/8\pi$.

Let us now work out the contribution of (5.47) to the free energy [cf. (5.24)]:

$$\begin{aligned}\int\mathrm{d}\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{th})}|^2}{2f_0} &= \int\mathrm{d}\boldsymbol{v}\left[\frac{e^2|\varphi_{\boldsymbol{k}}|^2}{2T\omega_{\mathrm{pe}}^2}(\boldsymbol{k}\cdot\boldsymbol{v})^2 f_0 + \frac{T|h_{\boldsymbol{k}}|^2}{2f_0} + e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}(\dots) + e^{i\boldsymbol{k}\cdot\boldsymbol{v}t}(\dots)^*\right]\\ &= \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} + \int\mathrm{d}\boldsymbol{v}\,\frac{T|h_{\boldsymbol{k}}|^2}{2f_0},\end{aligned} \tag{5.49}$$

where the velocity integral has been done in the same way as in (5.48) and the contribution from the terms that oscillate in $\boldsymbol{v}$ has been integrated away. The salient property of $h_{\boldsymbol{k}}$ is that

[47] I am grateful to Glenn Wagner for making me practice what I preach and work out this derivation correctly, with all the meaningful factors of 2.

it does not depend on time. Therefore, its contribution to the time derivative of the free energy vanishes and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{th})}|^2}{2f_0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}, \tag{5.50}$$

i.e., the kinetic energy of the Langmuir oscillations decays at the same rate as their potential (electric) energy.

Finally, adding (5.46) and (5.50), we get (5.42), q.e.d.

Exercise 5.4. Free-energy conservation for sound waves. Consider an ion-acoustic wave (§3.8) damped on electrons according to (3.75) (with $u_e = 0$). Work out the contributions to the free energy from the ions and from the electrons. Check that free energy is conserved.

---

6. Quasilinear Theory

6.1. *General Scheme of QLT*

In §§3 and 5, I discussed at length the structure of the linear solution corresponding to a Landau-damped initial perturbation. This could be adequately done for a Maxwellian plasma and the result was that, after some interesting transient time-dependent phase-space dynamics, perturbations damped away and their energy turned into heat, increasing somewhat the temperature of the equilibrium (see, however, Q-9).

Let us now turn to a different problem: an *unstable* (and so decidedly non-Maxwellian) equilibrium distribution giving rise to exponentially growing perturbations. The specific example on which we shall focus is the *bump-on-tail instability,* which involves generation of unstable Langmuir waves with phase velocities corresponding to instances of positive derivative of the equilibrium distribution function (Fig. 28). The energy of the waves grows exponentially:

$$\frac{\partial|\boldsymbol{E}_{\boldsymbol{k}}|^2}{\partial t} = 2\gamma_{\boldsymbol{k}}|\boldsymbol{E}_{\boldsymbol{k}}|^2, \quad \gamma_{\boldsymbol{k}} = \frac{\pi}{2}\frac{\omega_{\mathrm{pe}}^3}{k^2}\frac{1}{n_e}F'\left(\frac{\omega_{\mathrm{pe}}}{k}\right), \tag{6.1}$$

where $F(v_z) = \iint \mathrm{d}v_x \mathrm{d}v_y\, f_0(\boldsymbol{v})$ [see (3.41)]. In the absence of collisions, the only way for the system to achieve a nontrivial steady state (i.e., such that $|\boldsymbol{E}_{\boldsymbol{k}}|^2$ is not just zero everywhere) is by adjusting the equilibrium distribution so that

$$\gamma_{\boldsymbol{k}} = 0 \quad \Leftrightarrow \quad F'\left(\frac{\omega_{\mathrm{pe}}}{k}\right) = 0 \tag{6.2}$$

at all k where $|\boldsymbol{E}_{\boldsymbol{k}}|^2 \neq 0$, say, $k \in [k_2, k_1]$. If we translate this range into velocities, $v = \omega_{\mathrm{pe}}/k$, we see that the equilibrium must develop a flat spot:

$$F'(v) = 0 \quad \text{for} \quad v \in [v_1, v_2] = \left[\frac{\omega_{\mathrm{pe}}}{k_1}, \frac{\omega_{\mathrm{pe}}}{k_2}\right]. \tag{6.3}$$

This is called a *quasilinear plateau* (§6.4). Obviously, the rest of the equilibrium distribution may (and will) also be modified in some, to be determined, way (§§6.6, 6.7).

These modifications of the original (initial) equilibrium distribution can be accomplished by the growing fluctuations via the feedback mechanism already discussed in §2.3, namely, the equilibrium distribution will evolve slowly according to (2.11):

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{q}{m}\sum_{\boldsymbol{k}} i\boldsymbol{k}\,\langle\varphi_{\boldsymbol{k}}^*\delta f_{\boldsymbol{k}}\rangle\right). \tag{6.4}$$

The time averaging here [see (2.7)] is over $\omega_{\mathrm{pe}}^{-1} \ll \Delta t \ll \gamma_{\boldsymbol{k}}^{-1}$.

The general scheme of QLT is:

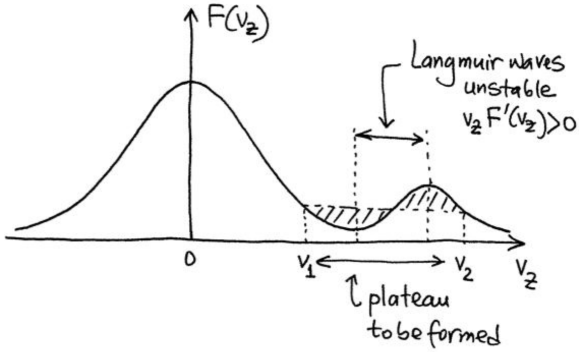


FIGURE 28. An unstable distribution with a bump on its tail.

• start with an unstable equilibrium f_0;
• use the linearised equations (3.1) and (3.2) to work out the linear solution for the growing perturbations $\varphi_{\boldsymbol{k}}$ and $\delta f_{\boldsymbol{k}}$ in terms of f_0;
• use this solution in (6.4) to evolve f_0, leading, if everything works as it is supposed to, to an ever-less-unstable equilibrium.

We shall keep only the fastest growing mode (all others are exponentially small after a while), and so the solution (3.16) for the electric perturbations is

$$\varphi_{\boldsymbol{k}} = c_{\boldsymbol{k}} e^{(-i\omega_{\boldsymbol{k}}+\gamma_{\boldsymbol{k}})t}. \tag{6.5}$$

In the solution (5.26) for the perturbed distribution function, we may ignore the ballistic term because the exponentially growing piece (the first term) will eventually leave all this velocity-space structure behind,[48] so

$$\delta f_{\boldsymbol{k}} = \frac{q}{m}\frac{\varphi_{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}. \tag{6.6}$$

Substituting (6.6) into (6.4), we get

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial\boldsymbol{v}}\cdot\left(\frac{q^2}{m^2}\sum_{\boldsymbol{k}} i\boldsymbol{k}|\varphi_{\boldsymbol{k}}|^2\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}\right) = \frac{\partial}{\partial\boldsymbol{v}}\cdot\mathsf{D}(\boldsymbol{v})\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}. \tag{6.7}$$

[48]See, however, Q-9 on how to avoid having to wait for this to happen: in fact, the results below are valid for $\gamma_k t \lesssim 1$ as well.

This is a diffusion equation in velocity space, with a velocity-dependent diffusion matrix

$$
\begin{aligned}
\mathsf{D}(\boldsymbol{v}) &= -\frac{q^2}{m^2}\sum_{\boldsymbol{k}} i\boldsymbol{k}\boldsymbol{k}|\varphi_{\boldsymbol{k}}|^2 \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}} \\
&= -\frac{q^2}{m^2}\sum_{\boldsymbol{k}} i\boldsymbol{k}\boldsymbol{k}|\varphi_{\boldsymbol{k}}|^2 \frac{1}{2}\left(\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}} + \underbrace{\frac{1}{-\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{-\boldsymbol{k}}-i\gamma_{-\boldsymbol{k}}}}_{\substack{\text{here I changed} \\ \text{variables } \boldsymbol{k}\to-\boldsymbol{k}}}\right) \\
&= -\frac{q^2}{m^2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \frac{i}{2}\left(\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}} - \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}+i\gamma_{\boldsymbol{k}}}\right) \\
&= \frac{q^2}{m^2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \,\mathrm{Im}\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}} \\
&= \frac{q^2}{m^2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \frac{\gamma_{\boldsymbol{k}}}{(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})^2+\gamma_{\boldsymbol{k}}^2}.
\end{aligned}
\tag{6.8}
$$

To obtain these expressions, I used the fact that the wave-number sum could just as well be over $-\boldsymbol{k}$ instead of $\boldsymbol{k}$ and that $\omega_{-\boldsymbol{k}} = -\omega_{\boldsymbol{k}}$, $\gamma_{-\boldsymbol{k}} = \gamma_{\boldsymbol{k}}$ [because $\varphi_{-\boldsymbol{k}} = \varphi^*_{\boldsymbol{k}}$, where $\varphi_{\boldsymbol{k}}$ is given by (6.5)]. The matrix D is manifestly positive definite—this adds credence to the *a priori* expectation that a plateau will form: diffusion will smooth the bump in the equilibrium distribution function.

The question of validity of the QL approximation is quite nontrivial and rife with subtle issues, all of which I have swept under the carpet. They mostly have to do with whether coupling between waves [the last term in (2.12)] truly remains unimportant throughout the quasilinear evolution, especially as the plateau regime is approached and the growth rate of the waves becomes infinitesimally small. If you wish to investigate further—and in the process gain a finer appreciation of nonlinear plasma theory,—the article by Besse *et al.* (2011) is a good starting point, from which you can follow the paper trail backwards in time and decide for yourself whether you trust the QLT.

6.2. *Conservation Laws*

When we get to the stage of solving a specific problem (§6.3), we shall see that paying attention to energy and momentum budgets leads one to important discoveries about the QL evolution of the particle distribution. With this prospect in mind, as well as by way of a consistency check, let us check that the quasilinear kinetic equation (6.7) conserves energy and momentum.

6.2.1. *Energy Conservation*

The rate of change of the particle energy associated with the equilibrium distribution is

$$
\begin{aligned}
\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2} f_{0\alpha} = V\sum_\alpha \int \mathrm{d}\boldsymbol{v}\,\frac{m_\alpha v^2}{2}\frac{\partial}{\partial \boldsymbol{v}}\cdot \mathsf{D}_\alpha(\boldsymbol{v})\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha \boldsymbol{v}\cdot \mathsf{D}_\alpha(\boldsymbol{v})\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\sum_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{k^2}\int \mathrm{d}\boldsymbol{v}\,\mathrm{Im}\underbrace{\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}}_{\substack{\text{add and substract}\\ \omega_{\boldsymbol{k}}+i\gamma_{\boldsymbol{k}}\text{ in the}\\ \text{numerator}}}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi}\mathrm{Im}\Bigg[(\omega_{\boldsymbol{k}}+i\gamma_{\boldsymbol{k}})\underbrace{\sum_\alpha\frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}}_{\substack{=1-\epsilon(-i\omega_{\boldsymbol{k}}+\gamma_{\boldsymbol{k}},\boldsymbol{k})=1\\ \text{because } -i\omega_{\boldsymbol{k}}+i\gamma_{\boldsymbol{k}}\text{ is a solution of}\\ \text{dispersion relation } \epsilon=0}}\Bigg]\\
&= -V\sum_{\boldsymbol{k}}2\gamma_{\boldsymbol{k}}\frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{r}\,\frac{E^2}{8\pi} = -\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t},\quad\text{q.e.d.},
\end{aligned}
\tag{6.9}
$$

viz., the total energy $\mathscr{K}+\mathscr{U}=\mathrm{const}$. This will motivate §6.6.

6.2.2. *Momentum Conservation*

Since unstable distributions like the one with a bump on its tail can carry net momentum, it is useful to calculate its rate of change:

$$
\begin{aligned}
\frac{\mathrm{d}\mathscr{P}}{\mathrm{d}t} &\equiv \frac{\mathrm{d}}{\mathrm{d}t}\sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, m_\alpha \boldsymbol{v} f_{0\alpha} = V\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha\boldsymbol{v}\frac{\partial}{\partial \boldsymbol{v}}\cdot \mathsf{D}_\alpha(\boldsymbol{v})\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_\alpha \int \mathrm{d}\boldsymbol{v}\, m_\alpha \mathsf{D}_\alpha(\boldsymbol{v})\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_\alpha \frac{q_\alpha^2}{m_\alpha}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}|\boldsymbol{E}_{\boldsymbol{k}}|^2}{k^2}\int \mathrm{d}\boldsymbol{v}\,\mathrm{Im}\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}\\
&= -V\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi}\mathrm{Im}\underbrace{\sum_\alpha\frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}-i\gamma_{\boldsymbol{k}}}\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}}_{=1-\epsilon(-i\omega_{\boldsymbol{k}}+\gamma_{\boldsymbol{k}},\boldsymbol{k})=1} = 0,\quad\text{q.e.d.},
\end{aligned}
\tag{6.10}
$$

so momentum can only be redistributed between particles. This will motivate §6.7.

6.3. *Quasilinear Equations for the Bump-on-Tail Instability in 1D*

What follows is the iconic QL calculation due to Vedenov *et al.* (1962) and Drummond & Pines (1962).

These two papers, published in the same year, are a spectacular example of the "great minds think alike" principle. They both appeared in the Proceedings of the 1961 IAEA conference in Salzburg, one of those early international gatherings in which the Soviets (grudgingly allowed out) and the Westerners (eager to meet the aliens) told each other about their achievements in the recently declassified controlled-nuclear-fusion research. The entire Proceedings are now online (`http://www-naweb.iaea.org/napc/physics/FEC/1961.pdf`)—a remarkable historical document and a great read, containing, besides the papers (in three languages), a record of the discussions that were held. The Vedenov *et al.* (1962) paper is in Russian, but you will find a very similar

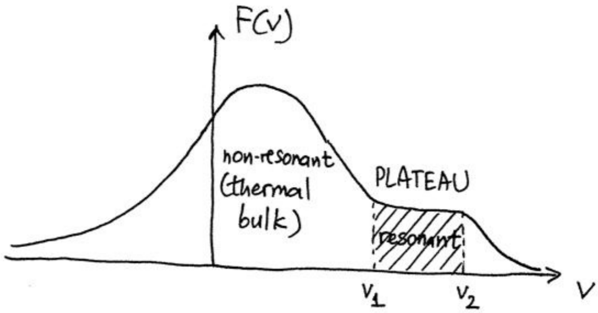


FIGURE 29. Quasilinear plateau.

exposition in English in the review by Vedenov (1963) published shortly thereafter. Two other lucid accounts of quasilinear theory belonging to the same historical (and historic!) period are in the books by Kadomtsev (1965) and by Sagdeev & Galeev (1969).

As promised in §6.1, I shall consider electron Langmuir oscillations in 1D, triggered by the bump-on-tail instability, so $\boldsymbol{k} = k\hat{\boldsymbol{z}}$, $\omega_k = \omega_{\rm pe}$, γ_k is given by (6.1), and the QL diffusion equation (6.7) becomes

$$\boxed{\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v}}\,, \tag{6.11}$$

where $F(v)$ is the 1D version of the distribution function, $v = v_z$ and the diffusion coefficient, now a scalar, is given by

$$\boxed{D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2\, \mathrm{Im} \frac{1}{kv - \omega_{\rm pe} - i\gamma_k}}\,. \tag{6.12}$$

As I explained when discussing (6.1), if the fluctuation field has reached a steady state, it must be the case that

$$\frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2 = 0 \quad \Leftrightarrow \quad |E_k|^2 = 0 \quad \text{or} \quad \gamma_k = 0, \tag{6.13}$$

i.e., either there are no fluctuations or there is no growth (or damping) rate. The result is a non-zero spectrum of fluctuations in the interval $k \in [k_2, k_1]$ and a plateau in the distribution function in the corresponding velocity interval $v \in [v_1, v_2] = [\omega_{\rm pe}/k_1, \omega_{\rm pe}/k_2]$ [see (6.3) and Fig. 29]. The particles in this interval are *resonant* with Langmuir waves; those in the ("thermal") bulk of the distribution outside this interval are *non-resonant*. We will have solved the problem completely if we find

- $F^{\rm plateau}$, the value of the distribution function in the interval $[v_1, v_2]$,
- the extent of the plateau $[v_1, v_2]$,
- the functional form of the spectrum $|E_k|^2$ in the interval $[k_2, k_1]$,
- any modifications of the distribution function $F(v)$ of the nonresonant particles.

6.4. *Resonant Region: QL Plateau and Spectrum*

Consider first the velocities $v \in [v_1, v_2]$ for which $|E_{k=\omega_{\rm pe}/v}|^2 \neq 0$. If L is the linear size of the system, the wave-number sum in (6.12) can be replaced by an integral according to

$$\sum_k = \sum_k \frac{\Delta k}{2\pi/L} = \frac{L}{2\pi} \int \mathrm{d}k. \tag{6.14}$$

Defining the continuous energy spectrum of the Langmuir waves[49]

$$W(k) = \frac{L}{2\pi}\frac{|E_k|^2}{4\pi}, \tag{6.15}$$

we rewrite the QL diffusion coefficient (6.12) in the following form:

$$D(v) = \frac{e^2}{m_e^2}\frac{1}{v}\,\mathrm{Im}\int \mathrm{d}k\,\frac{4\pi W(k)}{k - \omega_{\mathrm{pe}}/v - i\gamma_k/v} = \frac{e^2}{m_e^2}\frac{4\pi^2}{v}\,W\left(\frac{\omega_{\mathrm{pe}}}{v}\right). \tag{6.16}$$

The last expression is obtained by applying Plemelj's formula (3.25) to the wave-number integral taken in the limit $\gamma_k/v \to +0$.[50] Substituting now this expression into (6.11) and using also (6.1) to express

$$\gamma_k = \frac{\pi}{2}\frac{\omega_{\mathrm{pe}}^3}{k^2}\frac{1}{n_e}F'\left(\frac{\omega_{\mathrm{pe}}}{k}\right) \quad\Rightarrow\quad \frac{\partial F}{\partial v} = \left[\frac{2}{\pi}\frac{k^2}{\omega_{\mathrm{pe}}^3}\,n_e\gamma_k\right]_{k=\omega_{\mathrm{pe}}/v}, \tag{6.17}$$

we get

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v}\frac{e^2}{m_e^2}\frac{4\pi^2}{v}\,W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\left[\frac{2}{\pi}\frac{k^2}{\omega_{\mathrm{pe}}^3}\,n_e\gamma_k\right]_{k=\omega_{\mathrm{pe}}/v} = \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}\,\underbrace{2\gamma_{\omega_{\mathrm{pe}}/v}W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)}_{=\,\partial W/\partial t}. \tag{6.18}$$

Rearranging, we arrive at

$$\frac{\partial}{\partial t}\left[F - \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\right] = 0. \tag{6.19}$$

Thus, during QL evolution, the expression in the square brackets stays constant in time. Since at $t = 0$, there are no waves, $W = 0$, we find

$$F(0, v) + \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}W\left(t, \frac{\omega_{\mathrm{pe}}}{v}\right) = F(t, v) \to F^{\mathrm{plateau}} \quad \text{as} \quad t \to \infty. \tag{6.20}$$

In the saturated state ($t \to \infty$), $W(\omega_{\mathrm{pe}}/v) = 0$ outside the interval $v \in [v_1, v_2]$. Therefore, (6.20) gives us two implicit equations for v_1 and v_2:

$$\boxed{F(0, v_1) = F(0, v_2) = F^{\mathrm{plateau}}} \tag{6.21}$$

and, after integration over velocities, also an equation for F^{plateau}:[51]

$$\int_{v_1}^{v_2}\mathrm{d}v\,\left[F^{\mathrm{plateau}} - F(0, v)\right] = 0 \quad\Rightarrow\quad \boxed{F^{\mathrm{plateau}} = \frac{1}{v_2 - v_1}\int_{v_1}^{v_2}\mathrm{d}v\,F(0, v)}\,. \tag{6.22}$$

Finally, integrating (6.20) with respect to v and using the boundary condition $W(\omega_{\mathrm{pe}}/v_1) = 0$, we get, at $t \to \infty$,

$$W\left(\frac{\omega_{\mathrm{pe}}}{v}\right) = \frac{m_e v^3}{\omega_{\mathrm{pe}}}\int_{v_1}^{v}\mathrm{d}v'\,\left[F^{\mathrm{plateau}} - F(0, v')\right]. \tag{6.23}$$

---

[49] Why the prefactor is $1/4\pi$, rather than $1/8\pi$, will become clear at the end of §6.5.

[50] In fact, the wave-number integral must be taken along the Landau contour (i.e., keeping the contour below the pole) regardless of the sign of γ_k: see Q-9, where you get to work out the QLT for Landau-damped, rather than growing, perturbations.

[51] This is somewhat reminiscent of the "Maxwell construction" in thermodynamics of real gases: the plateau sits at such a level that the integral under it, i.e., the number of particles involved, stays the same as it was for the same velocities in the initial state; see Fig. 28.

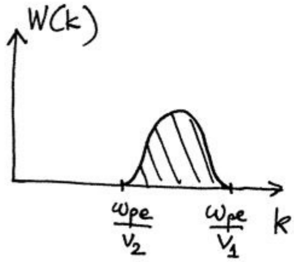


FIGURE 30. Quasilinear spectrum.

Hence the spectrum is

$$\boxed{W(k) = \frac{m_e\omega_{\rm pe}^2}{k^3}\int_{v_1}^{\omega_{\rm pe}/k} \mathrm{d}v\left[F^{\rm plateau} - F(0,v)\right]} \quad \text{for} \quad k \in \left[\frac{\omega_{\rm pe}}{v_2}, \frac{\omega_{\rm pe}}{v_1}\right] \tag{6.24}$$

and $W(k) = 0$ everywhere else (Fig. 30).

Thus, we have completed the first three items of the programme formulated at the end of §6.3. What about the particle distribution outside the resonant region? How is it modified by the quasilinear evolution? Is it modified at all? The following calculation shows that it must be.

6.5. *Energy of Resonant Particles*

Since feeding the instability requires extracting energy from the resonant particles, their energy must change. We calculate this change by taking the $m_e v^2/2$ moment of (6.20):

$$\begin{aligned}
\mathscr{K}_{\rm res}(\infty) - \mathscr{K}_{\rm res}(0) &= \int_{v_1}^{v_2} \mathrm{d}v\, \frac{m_e v^2}{2}\left[F^{\rm plateau} - F(0,v)\right] \\
&= \int_{v_1}^{v_2} \mathrm{d}v\, \frac{m_e v^2}{2}\frac{\partial}{\partial v}\frac{\omega_{\rm pe}}{m_e v^3} W\left(\frac{\omega_{\rm pe}}{v}\right) \\
&= -\omega_{\rm pe}\int_{v_1}^{v_2}\mathrm{d}v\,\frac{1}{v^2}\,W\left(\frac{\omega_{\rm pe}}{v}\right) \\
&= -\int_{\omega_{\rm pe}/v_2}^{\omega_{\rm pe}/v_1}\mathrm{d}k\, W(k) = -2\sum_{\boldsymbol{k}}\frac{|E_k|^2}{8\pi} \equiv -2\,\mathscr{U}(\infty).
\end{aligned} \tag{6.25}$$

Thus, only half of the energy lost by the resonant particles goes into the electric-field energy of the waves,

$$\boxed{\mathscr{U}(\infty) = \frac{\mathscr{K}_{\rm res}(0) - \mathscr{K}_{\rm res}(\infty)}{2}}\,. \tag{6.26}$$

Since the energy must be conserved overall [see (6.9)], we must account for the missing half: this is easy to do physically, as, obviously, the electric energy of the waves is their potential energy, which is half of their total energy—the other half being the kinetic energy of the oscillatory plasma motions associated with the wave (in §5.7, this was worked out explicitly). These oscillations are enabled by the non-resonant, "thermal-bulk" particles, and so we must be able to show that, as a result of QL evolution, these particles pick up the total of $\mathscr{U}(\infty)$ of energy—one might say that the plasma is *heated*.

This may appear to be slightly loose language. Technically speaking, since there are no collisions, this is not "true" heating in the sense described in §5, i.e., the exact total entropy does not increase. The "thermal" energy that increases is the energy of plasma oscillations, which are mean "fluid" motions of the plasma, whereas "true" heating would involve an increase in the energy of particle motions around the mean. However, if you view the separation of f_0 from δf and the subsequent averaging over the latter as another "coarse-graining" operation akin to the separation of collisions from collisionless dynamics in §§1.4–1.7, then it should start making sense that the increase of the kinetic energy of the fluctuating "fluid" motions is indeed tantamount to heating the mean distribution f_0. It is not hard to prove that this process is irreversible, i.e., that the QL "collision operator" (6.7) has an entropy and an "H-theorem".

Exercise 6.1. Do this.

6.6. *Heating of Non-Resonant Particles*

Consider the thermal bulk of the distribution, $v \ll v_1$ (assuming that the bump is indeed far out in the tail of the distribution). The QL diffusion coefficient (6.12) becomes, assuming now $\gamma_k, kv \ll \omega_{\rm pe}$ and using the last expression in (6.8),

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2}\sum_k |E_k|^2 \frac{\gamma_k}{(kv-\omega_{\rm pe})^2+\gamma_k^2} \approx \frac{e^2}{m_e^2}\sum_k |E_k|^2\frac{\gamma_k}{\omega_{\rm pe}^2} \\ &= \frac{e^2}{m_e^2\omega_{\rm pe}^2}\sum_k \frac{1}{2}\frac{\partial |E_k|^2}{\partial t} = \frac{4\pi e^2}{m_e^2\omega_{\rm pe}^2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_k\frac{|E_k|^2}{8\pi} = \frac{1}{m_e n_e}\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}, \end{aligned} \tag{6.27}$$

independent of v. The QL evolution equation (6.11) for the bulk distribution is then[52]

$$\frac{\partial F}{\partial t} = \frac{1}{m_e n_e}\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}\frac{\partial^2 F}{\partial v^2}. \tag{6.28}$$

Equation (6.28) describes slow diffusion of the bulk distribution, i.e., as the wave field grows, the bulk distribution gets a little broader (which is what heating is). Namely, the "thermal" energy satisfies

$$\frac{\mathrm{d}\mathscr{K}_{\rm th}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\frac{m_e v^2}{2}F = \frac{1}{m_e n_e}\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}\underbrace{\int\mathrm{d}v\,\frac{m_e v^2}{2}\frac{\partial^2 F}{\partial v^2}}_{\substack{= m_e n_e \\ \text{(by parts twice)}}} = \frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}. \tag{6.29}$$

Integrating this with respect to time, we find that the missing half of the energy lost by the resonant particles indeed goes into the thermal bulk:

$$\boxed{\mathscr{K}_{\rm th}(\infty) - \mathscr{K}_{\rm th}(0) = \mathscr{U}(\infty) = \frac{\mathscr{K}_{\rm res}(0)-\mathscr{K}_{\rm res}(\infty)}{2}}\,. \tag{6.30}$$

Overall, the energy is, of course, conserved:

$$\mathscr{K}_{\rm th}(\infty) + \mathscr{K}_{\rm res}(\infty) + \mathscr{U}(\infty) = \mathscr{K}_{\rm th}(0) + \mathscr{K}_{\rm res}(0), \tag{6.31}$$

as it shoud be, in accordance with (6.9).

Equation (6.28) can be explicitly solved: changing the time variable to $\tau = \mathscr{U}(t)/m_e n_e$ turns it into a simple diffusion equation

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial v^2}. \tag{6.32}$$

[52]Note that this implies $\mathrm{d}\int\mathrm{d}vF(v)/\mathrm{d}t = 0$, so the number of these particles is conserved, there is no exchange between the non-resonant and resonant populations.

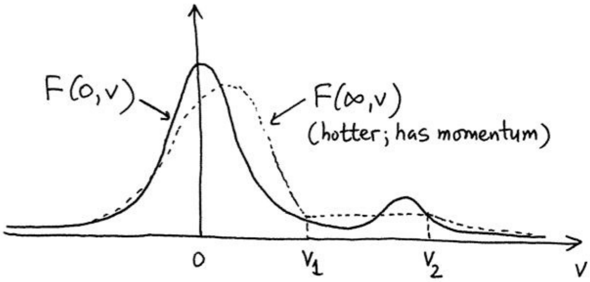


FIGURE 31. The initial distribution and the final outcome of the QL evolution: its bulk hotter and shifted towards the plateau in the tail.

If we let the initial distribution be a Maxwellian and ignore the bump on its tail, the solution is

$$F(\tau,v) = \int \mathrm{d}v' F(0,v')\,\frac{e^{-(v-v')^2/4\tau}}{\sqrt{4\pi\tau}} = \int \mathrm{d}v'\,\frac{n_e}{\sqrt{\pi v_{\mathrm{the}}^2 4\pi\tau}}\exp\left[-\frac{v'^2}{v_{\mathrm{the}}^2} - \frac{(v-v')^2}{4\tau}\right]$$
$$= \frac{n_e}{\sqrt{\pi(v_{\mathrm{the}}^2+4\tau)}}\exp\left(-\frac{v^2}{v_{\mathrm{the}}^2+4\tau}\right). \tag{6.33}$$

Since

$$v_{\mathrm{the}}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{4\mathscr{U}(t)}{m_e n_e} = \frac{2}{m_e}\left[T_e + \frac{2\mathscr{U}(t)}{n_e}\right], \tag{6.34}$$

one concludes that an initially Maxwellian bulk stays Maxwellian but its temperature grows as the wave energy grows, reaching in saturation

$$\boxed{T_e(\infty) = T_e(0) + \frac{2\mathscr{U}(\infty)}{n_e}}\,. \tag{6.35}$$

6.7. *Momentum Conservation*

The bump-on-tail configuration is in general asymmetric in v and so the particles in the bump carry a net mean momentum. Let us find out whether this momentum changes. Taking the $m_e v$ moment of (6.20), we calculate the total momentum lost by the resonant particles:

$$\begin{aligned}\mathscr{P}_{\mathrm{res}}(\infty) - \mathscr{P}_{\mathrm{res}}(0) &= \int_{v_1}^{v_2} \mathrm{d}v\, m_e v\left[F^{\mathrm{plateau}} - F(0,v)\right]\\ &= \int_{v_1}^{v_2} \mathrm{d}v\, m_e v\,\frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3} W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\\ &= -\omega_{\mathrm{pe}}\int_{v_1}^{v_2}\mathrm{d}v\,\frac{1}{v^3}\,W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\\ &= -\int_{\omega_{\mathrm{pe}}/v_2}^{\omega_{\mathrm{pe}}/v_1}\mathrm{d}k\,\frac{kW(k)}{\omega_{\mathrm{pe}}} < 0.\end{aligned} \tag{6.36}$$

This is negative, so momentum is indeed lost. Since it cannot go into electric field [see (6.10)], it must all get transferred to the thermal particles. Let us confirm this.

Going back to the QL diffusion equation (6.28) for the non-resonant particles, at first glance, we have a problem: the diffusion coefficient is independent of v and so momentum is conserved. However, one should never take zero for an answer when dealing with asymptotic expansions—indeed, it turns out here that we ought to work to higher order in our calculation of $D(v)$. Keeping next-order terms in (6.27), we get

$$\begin{aligned} D(v) &= \frac{e^2}{m_e^2}\sum_k |E_k|^2 \frac{\gamma_k}{(kv-\omega_{\rm pe})^2+\gamma_k^2} = \frac{e^2}{m_e^2}\sum_k |E_k|^2 \frac{\gamma_k}{\omega_{\rm pe}^2}\left(1+\frac{2kv}{\omega_{\rm pe}}+\dots\right) \\ &\approx \frac{4\pi e^2}{m_e^2\omega_{\rm pe}^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_k\frac{|E_k|^2}{8\pi}+v\sum_k\frac{k|E_k|^2}{4\pi\omega_{\rm pe}}\right] = \frac{1}{m_e n_e}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{U}+v\int\mathrm{d}k\,\frac{kW(k)}{\omega_{\rm pe}}\right]. \end{aligned} \tag{6.37}$$

Thus, there is a wave-induced drag term in the QL diffusion equation (6.11), which indeed turns out to impart to the thermal particles the small additional momentum that, according to (6.36), the resonant particles lose when rearranging themselves to produce the QL plateau:

$$\begin{aligned} \frac{\mathrm{d}\mathscr{P}_{\rm th}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}v\, m_e v F = \int\mathrm{d}v\, m_e v\frac{\partial}{\partial v}D(v)\frac{\partial F}{\partial v} = -m_e\int\mathrm{d}v\, D(v)\frac{\partial F}{\partial v} \\ &= -\left[\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}k\,\frac{kW(k)}{\omega_{\rm pe}}\right]\frac{1}{n_e}\int\mathrm{d}v\, v\frac{\partial F}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}k\,\frac{kW(k)}{\omega_{\rm pe}}, \end{aligned} \tag{6.38}$$

whence, integrating and comparing with (6.36),

$$\boxed{\mathscr{P}_{\rm th}(\infty)-\mathscr{P}_{\rm th}(0) = \int\mathrm{d}k\,\frac{kW(k)}{\omega_{\rm pe}} = \mathscr{P}_{\rm res}(0)-\mathscr{P}_{\rm res}(\infty)}\,. \tag{6.39}$$

This means that *the thermal bulk of the final distribution is not only slightly broader (hotter) than that of the initial one (§6.6), but it is also slightly shifted towards the plateau* (Fig. 31).

In a collisionless plasma, this is the steady state. However, as this steady state is approached, $\gamma_k \to 0$, so the QL evolution becomes ever slower and even a very small collision frequency can become important.[53] Eventually, collisions will erode the plateau (even if with great difficulty) and return the plasma to a global Maxwellian equilibrium—which is the fate of all things.

[53]Note that collision operators themselves represent QL dynamics (of linearly stable f_0 interacting with electric noise), and their derivation is a version of QLT: see §11.

Plasma Kinetics Problem Set

1. Industrialised linear theory with the Z function. Consider a two-species plasma close to Maxwellian equilibrium. Rederive all the results obtained in §§3.4, 3.5, 3.8, 3.9, 3.10 starting from (3.90) and using the asymptotic expansions (3.97) and (3.98) of the plasma dispersion function.

Namely, consider the limits $\zeta_e \gg 1$ or $\zeta_e \ll 1$ and $\zeta_i \gg 1$, find solutions in these limits and establish the conditions on the wave number of the perturbations and on the equilibrium parameters under which these solutions are valid.

In particular, for the case of $\zeta_e \ll 1$ and $\zeta_i \gg 1$, obtain general expressions for the wave frequency and damping without assuming $k\lambda_{\mathrm{D}e}$ to be either small or large. Recover from your solution the cases considered in §§3.8, 3.9 and 3.10.

Find also the ion contribution to the damping of the ion acoustic and Langmuir waves and comment on the circumstances in which it might be important to know what it is.

Convince yourself that you believe the sketch of longitudinal plasma waves in Fig. 16. If you feel computationally inclined, solve the plasma dispersion relation (3.90) numerically [using, e.g., (3.96)] and see if you can reproduce Fig. 16.

You may wish to check your results against some textbook: e.g., Krall & Trivelpiece (1973) and Alexandrov *et al.* (1984) give very thorough treatments of the linear theory (although in rather different styles than I did).

2. Transverse plasma waves. Go back to the Vlasov–Maxwell, rather then Vlasov–Poisson, system and consider *electromagnetic* perturbations in a Maxwellian unmagnetised plasma (unmagnetised in the sense that in equilibrium, $\boldsymbol{B}_0 = 0$):

$$\frac{\partial \delta f_\alpha}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} = 0, \tag{6.40}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy Maxwell's equations (1.23–1.26) with charge and current densities determined by the perturbed distribution function δf_α.

(a) Consider an initial-value problem for such perturbations and show that the equation for the Laplace transform of $\boldsymbol{E}$ can be written in the form[54]

$$\hat{\epsilon}(p,\boldsymbol{k})\cdot\hat{\boldsymbol{E}}(p) = \left(\begin{array}{c}\text{terms associated with initial}\\ \text{perturbations of } \delta f_\alpha,\ \boldsymbol{E} \text{ and } \boldsymbol{B}\end{array}\right), \tag{6.41}$$

where the *dielectric tensor* $\hat{\epsilon}(p,\boldsymbol{k})$ is, in tensor notation,

$$\epsilon_{ij}(p,\boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\epsilon_{\mathrm{TT}}(p,k) + \frac{k_i k_j}{k^2}\,\epsilon_{\mathrm{LL}}(p,k) \tag{6.42}$$

and the *longitudinal dielectric function* $\epsilon_{\mathrm{LL}}(p,k)$ is the familiar electrostatic one, given by (3.90), while the *transverse dielectric function* is

$$\boxed{\epsilon_{\mathrm{TT}}(p,k) = 1 + \frac{1}{p^2}\left[k^2c^2 - \sum_\alpha \omega_{\mathrm{p}\alpha}^2 \zeta_\alpha Z(\zeta_\alpha)\right]}\,. \tag{6.43}$$

[54] In Q-3, dealing with the Weibel instability, you will have to do essentially the same calculation, but with a non-Maxwellian equilibrium. To avoid doing the work twice, you could do that question first and then specialise to a Maxwellian $f_{0\alpha}$. However, the algebra is a bit hairier for the non-Maxwellian case, so it may be useful to do the simpler case first, to train your hand—and also to have a way to cross-check the later calculation.

(b) Hence solve the transverse dispersion relation, $\epsilon_{\rm TT}(p,k)=0$, and show that, in the high-frequency limit ($|\zeta_e|\gg 1$), the resulting waves are simply the light waves, which, at long wave lengths, turn into plasma oscillations, viz.,

$$\boxed{\omega^2=k^2c^2+\omega_{\rm pe}^2}\,. \tag{6.44}$$

What is the wave length above which light can "feel" that it is propagating through plasma?—this is called the *plasma (electron) skin depth*, d_e. Are these waves damped?

(c) In the low-frequency limit ($|\zeta_e|\ll 1$), show that perturbations are aperiodic (have zero frequency) and damped. Find their damping rate and show that this result is valid for perturbations with wave lengths longer than the plasma skin depth ($kd_e\ll 1$). Explain physically why these perturbations fail to propagate.

Do one of Q-3, Q-4, or Q-5 (but do not pick Q-5 if you are planning to do Q-11).

---

3. Weibel instability. Weibel (1958) realised that transverse plasma perturbations can go unstable if the equilibrium distribution is anisotropic with respect to some special direction $\hat{\boldsymbol{n}}$, namely if $f_{0\alpha}=f_{0\alpha}(v_\perp,v_\parallel)$, where $v_\parallel=\boldsymbol{v}\cdot\hat{\boldsymbol{n}}$, $v_\perp=|\boldsymbol{v}_\perp|$ and $\boldsymbol{v}_\perp=\boldsymbol{v}-v_\parallel\hat{\boldsymbol{n}}$. The anisotropy can be due to some beam or stream of particles injected into the plasma, it also arises in collisionless shocks or, generically, when plasma is sheared or non-isotropically compressed by some external force. The simplest model for an anisotropic distribution of the required type is a *bi-Maxwellian*[55]:

$$f_{0\alpha}=\frac{n_\alpha}{\pi^{3/2}v_{{\rm th}\perp\alpha}^2v_{{\rm th}\parallel\alpha}}\exp\left(-\frac{v_\perp^2}{v_{{\rm th}\perp\alpha}^2}-\frac{v_\parallel^2}{v_{{\rm th}\parallel\alpha}^2}\right), \tag{6.45}$$

where, formally, $v_{{\rm th}\perp\alpha}=\sqrt{2T_{\perp\alpha}/m_\alpha}$ and $v_{{\rm th}\parallel\alpha}=\sqrt{2T_{\parallel\alpha}/m_\alpha}$ are the two "thermal speeds" in a plasma characterised by two effective temperatures $T_{\perp\alpha}$ and $T_{\parallel\alpha}$ (for each species).

(a) Using exactly the same method as in Q-2, consider electromagnetic perturbations in a bi-Maxwellian plasma, assuming their wave vectors to be parallel to the direction of anisotropy, $\boldsymbol{k}\parallel\hat{\boldsymbol{n}}$. Show that the dielectric tensor again has the form (6.42) and the longitudial dielectric function is again given by (3.90), while the transverse dielectric function is

$$\boxed{\epsilon_{\rm TT}(p,k)=1+\frac{1}{p^2}\left[k^2c^2+\sum_\alpha\omega_{{\rm p}\alpha}^2\left(1-\frac{T_{\perp\alpha}}{T_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]\right)\right]}\,. \tag{6.46}$$

(b) Show that in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

$$\boxed{\gamma=\frac{kv_{{\rm th}\parallel e}}{\sqrt{\pi}}\frac{T_{\parallel e}}{T_{\perp e}}\left(\varDelta_e-k^2d_e^2\right)}\,, \tag{6.47}$$

where $\varDelta_e=T_{\perp e}/T_{\parallel e}-1$ is the fractional temperature anisotropy, which must be positive

---

[55]In Exercise 4.8, you need the dielectric tensor in terms of a general equilibrium distribution $f_{0\alpha}(v_x,v_y,v_z)$. If you are planning to do that exercise, it may save time (at the price of a very slight increase in algebra) to do the derivation with a general $f_{0\alpha}$ and then specialise to the bi-Maxwellian (6.45). You can check your algebra by looking up the result in Krall & Trivelpiece (1973) or in Davidson (1983).

in order for the instability to occur. Find the maximum growth rate and the corresponding wave number. Under what condition(s) is the asymptotic limit in which you worked indeed a valid approximation for this solution?

(c) Are there any other unstable solutions? (cf. Weibel 1958)

(d) What happens if the electrons are isotropic but ions are not?

(e*) If you want a challenge and a test of stamina, work out the case of perturbations whose wave number is *not* necessarily in the direction of the anisotropy ($\boldsymbol{k}\times\hat{\boldsymbol{n}}\neq 0$). Are the $\boldsymbol{k}\parallel\hat{\boldsymbol{n}}$ or some oblique perturbations the fastest growing? This is a lot of algebra, so only do it if you enjoy this sort of thing. The dispersion relation for this general case appears to be in the Appendix of Ruyer *et al.* (2015), but they only solve it numerically; no one seems to have looked at asymptotic limits. This could be the start of a dissertation.

4. Two-stream instability.[56] Consider one-dimensional, electrostatic perturbations in a two-species (electron-ion) plasma. Let the electron distribution function with respect to velocities in the direction (z) of the spatial variation of perturbations be a "double Lorentzian" consisting of two counterpropagating beams with velocity $u_{\rm b}$ and width $v_{\rm b}$, viz.,

$$F_e(v_z) = \frac{n_e v_{\rm b}}{2\pi}\left[\frac{1}{(v_z-u_{\rm b})^2+v_{\rm b}^2}+\frac{1}{(v_z+u_{\rm b})^2+v_{\rm b}^2}\right] \tag{6.48}$$

(see Fig. 14b), while the ions are Maxwellian with thermal speed $v_{{\rm th}i}\ll u_{\rm b}$. Assume also that the phase velocity (p/k) will be of the same order as $u_{\rm b}$ and hence that the ion contribution to the dielectric function (3.26) is negligible.

(a) By integrating by parts and then choosing the integration contour judiciously, or otherwise, calculate the dielectric function $\epsilon(p,k)$ for this plasma and hence show that the dispersion relation is

$$\sigma^4 + (2u_{\rm b}^2+v_{\rm p}^2)\sigma^2 + u_{\rm b}^2(u_{\rm b}^2-v_{\rm p}^2) = 0, \tag{6.49}$$

where $\sigma = v_{\rm b}+p/k$ and $v_{\rm p}=\omega_{{\rm p}e}/k$.

(b) In the long-wavelength limit, viz., $k\ll\omega_{{\rm p}e}/u_{\rm b}$, find the condition for an instability to exist and calculate the growth rate of this instability. Is the nature of this instability kinetic (due to Landau resonance) or hydrodynamic?

(c) Consider the case of cold beams, $v_{\rm b}=0$. Without making any *a priori* assumptions about k, calculate the maximum growth rate of the instability. Sketch the growth rate as a function of k.

(d) Allowing warm beams, $v_{\rm b}>0$, show that the system is unstable provided

$$u_{\rm b}>v_{\rm b} \quad\text{and}\quad k<\omega_{{\rm p}e}\frac{\sqrt{u_{\rm b}^2-v_{\rm b}^2}}{u_{\rm b}^2+v_{\rm b}^2}. \tag{6.50}$$

What is the effect that a finite beam width has on the stability of the system and on the kind of perturbations that can grow?

In §4.4, you might find it instructive to compare the results that you have just obtained by solving the dispersion relation (6.49) directly with what can be inferred via Penrose's criterion and Nyquist's method.

---

[56] This is based on the 2019 exam question.

5. Buneman (1958) instability.[57] Consider a plasma consisting of Maxwellian ions and a cold electron beam (velocity $u_e \gg v_{\text{the}}$—much greater than the width of the distribution).

(a) Starting from the standard expression for the plasma dielectric function describing infinitesimal electrostatic perturbations with wave vector k in the direction of the beam and $\propto e^{pt}$ (where p is, in general, complex), assume $|p/k| \gg v_{\text{th}i}$ and show that the dispersion relation is

$$1 + \frac{\omega_{\text{p}i}^2}{p^2} - \frac{\omega_{\text{pe}}^2}{(ku_e - ip)^2} = 0. \tag{6.51}$$

(b) Looking for solutions with $\omega_{\text{pe}} \gg |p| \gg \omega_{\text{p}i}$, show that there is an instability that attains its maximum growth rate at $k = \omega_{\text{pe}}/u_e$. *Hint.* You can do this either by identifying the dominant balance in the dispersion relation or by looking for a solution in the form $p = |p|e^{i\theta}$ and maximising the growth rate with respect to θ.

(c) Show that the maximum growth rate is

$$\boxed{\gamma = \frac{\sqrt{3}}{2^{4/3}}\,\omega_{\text{pe}}^{1/3}\omega_{\text{p}i}^{2/3}}\,. \tag{6.52}$$

Is this instability kinetic or hydrodynamic? (I.e., do Landau resonances with either ions or electrons play a role?)

6. Free energy and stability. (a) Starting from the linearised Vlasov–Poisson system and assuming a Maxwellian equilibrium, show by direct calculation from the equations, rather then via expansion of the entropy function and the use of energy conservation (as was done in §5.2), that free energy is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\left[\sum_\alpha \int \mathrm{d}\boldsymbol{v}\,\frac{T_\alpha \delta f_\alpha^2}{2f_{0\alpha}} + \frac{|\boldsymbol{\nabla}\varphi|^2}{8\pi}\right] = 0. \tag{6.53}$$

This is an exercise in integrating by parts.

(b) Now consider the full Vlasov–Maxwell equations and prove, again for a Maxwellian plasma plus small perturbations,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\left[\sum_\alpha \int \mathrm{d}\boldsymbol{v}\,\frac{T_\alpha \delta f_\alpha^2}{2f_{0\alpha}} + \frac{|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2}{8\pi}\right] = 0}\,. \tag{6.54}$$

(c) Consider the same problem, this time with an equilibrium that is not Maxwellian, but merely isotropic, i.e., $f_{0\alpha} = f_{0\alpha}(v)$, or, in what will prove to be a more convenient form,

$$f_{0\alpha} = f_{0\alpha}(\varepsilon_\alpha), \tag{6.55}$$

where $\varepsilon_\alpha = m_\alpha v^2/2$ is the particle energy. Find an integral quantity quadratic in perturbed fields and distributions that is conserved by the Vlasov–Maxwell system under these circumstances and that turns into the free energy (6.54) in the case of a Maxwellian equilibrium (if in difficulty, you will find the answer in, e.g., Davidson 1983 or in Kruskal

[57] This is based on the 2021 exam question. The first half of Q-11 is very similar to this (if a little easier algebraically), so doing both of them might be tedious for you.

& Oberman 1958, which appears to be the original source). Argue that

$$\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \tag{6.56}$$

is a sufficient condition for stability of small ($\delta f_\alpha \ll f_{0\alpha}$, but not necessarily infinitesimal) perturbations in such a plasma.

7. Fluctuation-dissipation relation for a collisionless plasma. Let us consider a linear kinetic system in which perturbations are stirred up by an external force, which we can think of as an imposed (time-dependent) electric field $\boldsymbol{E}_{\rm ext} = -\boldsymbol{\nabla}\chi$. The perturbed distribution function then satisfies

$$\frac{\partial \delta f_\alpha}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\delta f_\alpha - \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{\nabla}\varphi_{\rm tot}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} = 0, \tag{6.57}$$

where $\varphi_{\rm tot} = \varphi + \chi$ is the total potential, whose self-consistent part, φ, obeys the usual Poisson equation

$$-\nabla^2\varphi = 4\pi\sum_\alpha q_\alpha \int {\rm d}\boldsymbol{v}\,\delta f_\alpha \tag{6.58}$$

and the equilibrium $f_{0\alpha}$ is assumed to be Maxwellian.

(a) By considering an initial-value problem for (6.57) and (6.58) with zero initial perturbation, show that the Laplace transforms of $\varphi_{\rm tot}$ and χ are related by

$$\hat\varphi_{\rm tot}(p) = \frac{\hat\chi(p)}{\epsilon(p)}, \tag{6.59}$$

where $\epsilon(p)$ is the dielectric function given by (3.90).

(b) Consider a time-periodic external force,

$$\chi(t) = \chi_0 e^{-i\omega_0 t}. \tag{6.60}$$

Working out the relevant Laplace transforms and their inverses [see (3.14)], show that, after transients have decayed, the total electric field in the system will oscillate at the same frequency as the external force and be given by

$$\varphi_{\rm tot}(t) = \frac{\chi_0\, e^{-i\omega_0 t}}{\epsilon(-i\omega_0)}. \tag{6.61}$$

(c) Now consider the *plasma-kinetic Langevin problem:* assume the external force to be a white noise, i.e., a random process with the time-correlation function

$$\langle \chi(t)\chi^*(t')\rangle = 2A\,\delta(t-t'). \tag{6.62}$$

Show that the resulting steady-state mean-square fluctuation level in the plasma will be

$$\boxed{\langle|\varphi_{\rm tot}(t)|^2\rangle = \frac{A}{\pi}\int_{-\infty}^{+\infty}\frac{{\rm d}\omega}{|\epsilon(-i\omega)|^2}}\,. \tag{6.63}$$

This is a kinetic *fluctuation-dissipation relation:* given a certain level of external stirring, parametrised by A, this formula predicts the fluctuation energy in terms of A and of the internal dissipative properties of the plasma, encoded by its dielectric function.

(d) For a system in which the Landau damping is weak, $|\gamma| \ll kv_{{\rm th}\alpha}$, calculate the

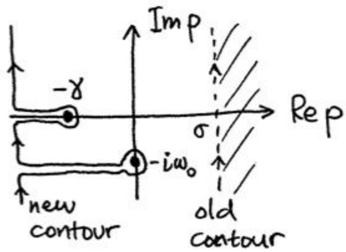


FIGURE 32. Shifting the integration contour in (6.68).

integral (6.63) using Plemelj's formula (3.25) to show that

$$\langle|\varphi_{\rm tot}(t)|^2\rangle = A\sum_i \frac{1}{|\gamma_i|}\left[\frac{\partial\,{\rm Re}\,\epsilon(-i\omega)}{\partial\omega}\right]^{-2}_{\omega=\omega_i}, \tag{6.64}$$

where $p_i = -i\omega_i + \gamma_i$ are the weak-damping roots of the dispersion relation.

Here is a reminder of how the *standard Langevin problem* can be solved using Laplace transforms. The Langevin equation is

$$\boxed{\frac{\partial\varphi}{\partial t} + \gamma\varphi = \chi(t)}\,, \tag{6.65}$$

where φ describes some quantity, e.g., the velocity of a Brownian particle, subject to a damping rate γ and an external force χ. In the case of a Brownian particle, χ is assumed to be a white noise, as per (6.62). Assuming $\varphi(t=0)=0$, the Laplace-trasform solution of (6.65) is

$$\hat\varphi(p) = \frac{\hat\chi(p)}{p+\gamma}. \tag{6.66}$$

Considering first a non-random oscillatory force (6.60), we have

$$\hat\chi(p) = \int_0^\infty {\rm d}t\, e^{-pt}\chi(t) = \frac{\chi_0}{p+i\omega_0} \quad\Rightarrow\quad \hat\varphi(p) = \frac{\chi_0}{(p+\gamma)(p+i\omega_0)}. \tag{6.67}$$

The inverse Laplace transform of $\hat\varphi$ is calculated by shifting the integration contour to large negative ${\rm Re}\,p$ while not allowing it to cross the two poles, $p=-\gamma$ and $p=-i\omega_0$, in a manner analogous to that explained in §3.1 (Fig. 6) and shown in Fig. 32. The integral is then dominated by the contributions from the poles:

$$\varphi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} {\rm d}p\, e^{pt}\hat\varphi(p) = \chi_0\left(\frac{e^{-i\omega_0 t}}{-i\omega_0+\gamma} + \frac{e^{-\gamma t}}{-\gamma+i\omega_0}\right) \to \frac{\chi_0\, e^{-i\omega_0 t}}{-i\omega_0+\gamma} \quad \text{as} \quad t\to\infty, \tag{6.68}$$

which is quite obviously the right solution of (6.65) with a periodic force (the second term in the brackets is the decaying transient needed to enforce the zero initial condition).

In the more complicated case of a white-noise force [see (6.62)],

$$\begin{aligned}\langle|\varphi(t)|^2\rangle &= \frac{1}{(2\pi)^2}\left\langle\left|\int_{-i\infty+\sigma}^{i\infty+\sigma}{\rm d}p\, e^{pt}\,\frac{\hat\chi(p)}{p+\gamma}\right|^2\right\rangle\\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}{\rm d}\omega\int_{-\infty}^{+\infty}{\rm d}\omega' e^{[-i(\omega-\omega')+2\sigma]t}\,\frac{\langle\hat\chi(-i\omega+\sigma)\hat\chi^*(-i\omega'+\sigma)\rangle}{(-i\omega+\sigma+\gamma)(i\omega'+\sigma+\gamma)},\end{aligned} \tag{6.69}$$

where we have changed variables $p=-i\omega+\sigma$ and similarly for the second integral. The correlation function of the Laplace-transformed force is, using (6.62),

$$\langle\hat\chi(p)\hat\chi^*(p')\rangle = \int_0^\infty {\rm d}t\int_0^\infty {\rm d}t' e^{-pt-p'^*t'}\langle\chi(t)\chi^*(t')\rangle = 2A\int_0^\infty {\rm d}t\, e^{-(p+p'^*)t} = \frac{2A}{p+p'^*}, \tag{6.70}$$

provided $\mathrm{Re}\, p > 0$ and $\mathrm{Re}\, p' > 0$. Then (6.69) becomes

$$\begin{aligned}\langle|\varphi(t)|^2\rangle &= \frac{A}{2\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_{-\infty}^{+\infty}\mathrm{d}\omega'\,\frac{e^{[-i(\omega-\omega')+2\sigma]t}}{[-i(\omega-\omega')+2\sigma]\,(-i\omega+\sigma+\gamma)(i\omega'+\sigma+\gamma)}\\ &= \frac{A}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega'\,\frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)}\,\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathrm{d}p\,\frac{e^{pt}}{(p+i\omega'+\sigma)(p+\gamma)}\\ &= \frac{A}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega'\,\frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)}\left[\frac{e^{-(i\omega'+\sigma)t}}{-i\omega'-\sigma+\gamma}+\frac{e^{-\gamma t}}{\gamma+i\omega'+\sigma}\right],\end{aligned} \tag{6.71}$$

where we have reverted to the p variable in one of the integrals and then performed the integration by the same manipulation of the contour as in (6.68). We now note that, since there are no exponentially growing solutions in this system, $\sigma > 0$ can be chosen arbitrarily small. Taking $\sigma \to +0$ and neglecting the decaying transient in (6.71), we get, in the limit $t \to \infty$,

$$\langle|\varphi(t)|^2\rangle = \frac{A}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{|-i\omega'+\gamma|^2} = \frac{A}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega'^2+\gamma^2} = \frac{A}{\gamma}. \tag{6.72}$$

Note that, while the integral in (6.72) is doable exactly, it can, for the case of weak damping, also be computed via Plemelj's formula.

Equation (6.72) is the standard Langevin fluctuation-dissipation relation. It can also be obtained without Laplace transforms, either by directly integrating (6.65) and correlating $\varphi(t)$ with itself or by noticing that

$$\frac{\partial}{\partial t}\frac{\langle\varphi^2\rangle}{2} + \gamma\langle\varphi^2\rangle = \langle\chi(t)\varphi(t)\rangle = \left\langle\chi(t)\int_0^t\mathrm{d}t'\left[-\gamma\varphi(t')+\chi(t')\right]\right\rangle = A, \tag{6.73}$$

where we have used (6.62) and the fact that $\langle\chi(t)\varphi(t')\rangle = 0$ for $t' \leqslant t$, by causality. Equation (6.72) is the steady-state solution to the above, but (6.73) also teaches us that, if we interpret $\langle\varphi^2\rangle/2$ as energy, A is the power that is injected into the system by the external force. Thus, fluctuation-dissipation relations such as (6.72) tells us what fluctuation *energy* will persist in a dissipative system if a certain amount of *power* is pumped in.

Q-8 is optional.

---

8. Phase-mixing spectrum. Here we study the velocity-space structure of the perturbed distribution function δf derived in Q-7.

In order to do this, we need to review the *Hermite transform:*

$$\delta f_m = \frac{1}{n}\int\mathrm{d}v_z\,\frac{H_m(u)\delta f(v_z)}{\sqrt{2^m m!}}, \quad u = \frac{v_z}{v_{\mathrm{th}}}, \quad H_m(u) = (-1)^m e^{u^2}\frac{\mathrm{d}^m}{\mathrm{d}u^m}\,e^{-u^2}, \tag{6.74}$$

where H_m is the Hermite polynomial of (integer) order m. We are only concerned with the v_z dependence of δf (where z, as always, is along the wave number of the perturbations—in this case set by the wave number of the force); all v_x and v_y dependence is Maxwellian and can be integrated out. The inverse transform is given by

$$\delta f(v_z) = \sum_{m=0}^{\infty}\frac{H_m(u)F(v_z)}{\sqrt{2^m m!}}\,\delta f_m, \quad F(v_z) = \frac{n}{\sqrt{\pi}\,v_{\mathrm{th}}}\,e^{-u^2}. \tag{6.75}$$

Because $H_m(u)$ are orthogonal polynomials, viz.,

$$\frac{1}{n}\int\mathrm{d}v_z\,H_m(u)H_{m'}(u)F(v_z) = 2^m m!\,\delta_{mm'}, \tag{6.76}$$

they have a Parseval theorem and so the contribution of the perturbed distribution function to the free energy [see (5.19)] can be written as

$$\int\mathrm{d}\boldsymbol{v}\,\frac{T|\delta f|^2}{2f_0} = \frac{nT}{2}\sum_m|\delta f_m|^2. \tag{6.77}$$

In a plasma where perturbations are constantly stirred up by a force, Landau damping must be operating all the time, removing energy from φ to provide "dissipation" of the injected power. The process of phase mixing that accompanies Landau damping must then lead to a certain fluctuation level $\langle|\delta f_m|^2\rangle$ in the Hermite moments of δf. Lower m's correspond to "fluid" quantities: density ($m = 0$), flow velocity ($m = 1$), temperature ($m = 2$). Higher m's correspond to finer structure in velocity space: indeed, for $m \gg 1$, the Hermite polynomials can be approximated by trigonometric functions,

$$H_m(u) \approx \sqrt{2}\left(\frac{2m}{e}\right)^{m/2}\cos\left(\sqrt{2m}\,u - \frac{\pi m}{2}\right)e^{u^2/2}, \tag{6.78}$$

and so the Hermite transform is somewhat analogous to a Fourier transform in velocity space with "frequency" $\sqrt{2m}/v_{\mathrm{th}}$.

(a) Show that in the kinetic Langevin problem described in Q-7(c), the mean square fluctuation level of the m-th Hermite moment of the perturbed distribution function is given by

$$\left\langle|\delta f_m(t)|^2\right\rangle = \frac{q^2 D}{T^2\pi 2^m m!}\int_{-\infty}^{+\infty}\mathrm{d}\omega\left|\frac{\zeta Z^{(m)}(\zeta)}{\epsilon(-i\omega)}\right|^2, \quad \zeta = \frac{\omega}{kv_{\mathrm{th}}}, \tag{6.79}$$

where $Z^{(m)}(\zeta)$ is the m-th derivative of the plasma dispersion function [note (3.99)].

(b*) Show that, assuming $m \gg 1$ and $\zeta \ll \sqrt{2m}$,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi}\,i^{m+1}\left(\frac{2m}{e}\right)^{m/2}e^{i\zeta\sqrt{2m}-\zeta^2/2} \tag{6.80}$$

and, therefore, that

$$\boxed{\left\langle|\delta f_m(t)|^2\right\rangle \approx \frac{\mathrm{const}}{\sqrt{m}}}\,. \tag{6.81}$$

Thus, the *Hermite spectrum* of the free energy is shallow and, in particular, the total free energy diverges—it has to be regularised by collisions. This is a manifestation of a copious amount of fine-scale structure in velocity space (note also how this shows that Landau-damped perturbations involve all Hermite moments, not just the "fluid" ones).

Deriving (6.80) is a (reasonably hard) mathematical exercise: it involves using (3.99) and (6.78) and manipulating contours in the complex plane. This is a treat for those who like this sort of thing. Getting to (6.81) will also require the use of Stirling's formula.

The Hermite order at which the spectrum (6.81) must be cut off due to collisions can be quickly deduced as follows. We saw in §5.5 that the typical velocity derivative of δf can be estimated according to (5.27) and the time it takes for this perturbation to be wiped out by collisions is given by (5.33). But, in view of (6.78), the velocity gradients probed by the Hermite moment m are of order $\sqrt{2m}/v_{\mathrm{th}}$. The collisional cutoff m_{c} in Hermite space can then be estimated so:

$$m_{\mathrm{c}} \sim v_{\mathrm{th}}^2\frac{\partial^2}{\partial v^2} \sim (kv_{\mathrm{th}}t_{\mathrm{c}})^2 \sim \left(\frac{kv_{\mathrm{th}}}{\nu}\right)^{2/3}. \tag{6.82}$$

Therefore, the total free energy stored in phase space diverges: using (6.77) and (6.81),

$$\frac{1}{n}\int\mathrm{d}\boldsymbol{v}\,\frac{\delta f^2}{2f_0} = \frac{1}{2}\sum_m\left\langle|\delta f_m|^2\right\rangle \sim \int^{m_{\mathrm{c}}}\mathrm{d}m\,\frac{\mathrm{const}}{\sqrt{m}} \propto \nu^{-1/3} \to \infty \quad \text{as} \quad \nu \to +0. \tag{6.83}$$

In contrast, the total free-energy dissipation rate is finite, however small is the collision frequency: estimating the right-hand side of (5.19), we get

$$\frac{1}{n}\int\mathrm{d}\boldsymbol{v}\,\frac{\delta f}{f_0}\left(\frac{\partial\delta f}{\partial t}\right)_{\mathrm{c}} \sim -\nu\sum_m m\left\langle|\delta f_m|^2\right\rangle \propto \nu\int^{m_{\mathrm{c}}}\mathrm{d}m\,\sqrt{m} \sim kv_{\mathrm{th}}. \tag{6.84}$$

Thus, the kinetic system can collisionally produce entropy at a rate that is entirely independent of the collision frequency.

If you find phase-space turbulence and generally life in Hermite space as fascinating as I do, you can learn more from Kanekar *et al.* (2015) (on fluctuation-dissipation relations and Hermite spectra) and from §§12.2–12.4 (on what happens when nonlinearity strikes).

Do two of Q-9, Q-10, Q-11, Q-12 or Q-13 (do not pick Q-11 if you have already done Q-5; pick only one, or none, of Q-12 and Q-13).

---

9. QLT of Landau damping. In §6, we discussed the QLT of an unstable system, in which, whatever the size of the initial electric perturbations, they eventually grew large enough to affect the equilibrium distribution and modified it so as to suppress further growth. In a *stable* equilibrium, any initial perturbations will be Landau-damped, but, if they are sufficiently large to start with, they can also affect f_0 quasilinearly in a way that will slow down this damping (this calculation, due to Vedenov *et al.* 1961, appears to pre-date slightly the classic 1962 bump-on-tail papers cited in §6.3).

Consider, in 1D, an initial spectrum $W(0, k)$ of plasma oscillations (waves) excited in the wave-number range $[k_2, k_1] = [\omega_{\rm pe}/v_2, \omega_{\rm pe}/v_1] \ll \lambda_{\rm De}^{-1}$, with total electric energy $\mathscr{U}(0)$. Modify the QLT of §6 to show the following.

(a) A steady state can be achieved in which the distribution develops a plateau in the velocity interval $[v_1, v_2]$ (Fig. 33). Find $F^{\rm plateau}$ in terms of v_1, v_2 and the initial distribution $F(0, v)$. What is the energy of the waves in this steady state? What is the lower bound on initial electric energy $\mathscr{U}(0)$ below which the perturbations would just decay without forming a fully-fledged plateau?

(b) Derive the evolution equation for the thermal (nonresonant) bulk of the distribution and show that it cools during the QL evolution, with the total thermal energy declining by the same amount as the electric energy of the waves:

$$\mathscr{K}_{\rm th}(t) - \mathscr{K}_{\rm th}(0) = -\left[\mathscr{U}(0) - \mathscr{U}(t)\right]. \tag{6.85}$$

Identify where all the energy lost by the thermal particles and the waves goes and thus confirm that the total energy in the system is conserved. Why, physically, do thermal particles lose energy?

(c) Show that we must have

$$\frac{\mathscr{U}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{\rm pe}}\frac{\delta v}{v} \tag{6.86}$$

in order for the wave energy to change only by a small fraction before saturating and

$$\frac{\mathscr{U}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{\rm pe}}\left(\frac{\delta v}{v}\right)^3 \frac{1}{(k\lambda_{\rm De})^2} \tag{6.87}$$

in order for the QL evolution to be faster than the damping. Here $\delta v = v_2 - v_1$ and $v \sim v_1 \sim v_2$.

This question requires some nuance in handling the calculation of the QL diffusion coefficient. In §6.1, we used the expression (6.6) for $\delta f_{\boldsymbol{k}}$ in which only the eigenmode-like part was retained, while the phase-mixing terms were dropped on the grounds that we could always just wait long enough for them to be eclipsed by the term containing an exponentially growing factor $e^{\gamma_{\boldsymbol{k}} t}$. When we are dealing with *damped* perturbations, there is no point in waiting because the exponential term is getting smaller, while the phase-mixing terms do not decay (except by collisions, see §§5.3 and 5.5, but we are not prepared to wait for that—note that, according to Catto 2024, perhaps we should be).

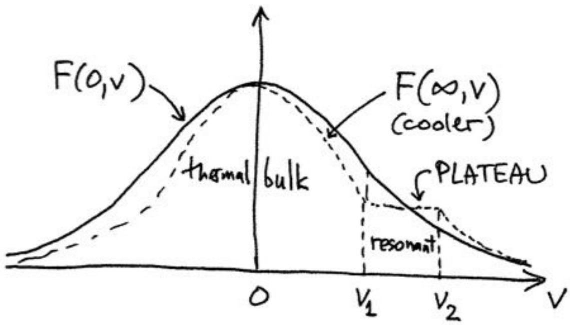


FIGURE 33. Initial stable equilibrium distribution and the final outcome of the QL evolution of a system with Landau-damped electric perturbations.

Let us, therefore, bite the bullet and use the full expression (5.26) for the perturbed distribution function, where we single out the slowest-damped mode and assume that all others, if any, will be damped fast enough never to produce significant QL effects:

$$\delta f_{\boldsymbol{k}} = \frac{q}{m}\varphi_{\boldsymbol{k}} \frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t - \gamma_{\boldsymbol{k}} t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}}\, \boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}\left(g_{\boldsymbol{k}} + \dots\right), \tag{6.88}$$

where "..." stand for any possible undamped, phase-mixing remnants of other modes. When the solution (6.88) is substituted into (6.4), where it is multiplied by $\varphi^*_{\boldsymbol{k}}$ and time averaged [according to (2.7)], the second term vanishes because, for resonant particles ($\boldsymbol{k}\cdot\boldsymbol{v} \approx \omega_{\boldsymbol{k}}$), it contains no resonant denominators and so is smaller than the first term, whereas for the nonresonant particles, it is removed by time averaging (check that this works at least for $|\gamma_{\boldsymbol{k}}|t \lesssim 1$ and indeed beyond that). Keeping only the first term in the expression (6.88), substituting it into (6.4) and going through a calculation analogous to that given in (6.8), we find that the diffusion matrix is (check this)

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \mathrm{Im}\left\langle \frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}})t - \gamma_{\boldsymbol{k}} t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}}\right\rangle, \tag{6.89}$$

which is a generalisation of the penultimate line of (6.8). For nonresonant particles, the phase-mixing term is eliminated by time averaging and we end up with the old result: the last line of (6.8). For resonant particles, assuming $|\gamma_{\boldsymbol{k}}| \ll |\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}} \sim \boldsymbol{k}\cdot\boldsymbol{v}$ and $|\gamma_{\boldsymbol{k}}|t \ll 1$, we may adopt the approximation (5.41), which we have previously used to analyse the structure of δf. This gives us

$$\boxed{\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2}\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \pi\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})}\,, \tag{6.90}$$

which is the same result as (6.16)—including, importantly, the sign, which we would have gotten wrong had we just mechanically applied Plemelj's formula to (6.12) with $\gamma_{\boldsymbol{k}} < 0$. This is equivalent to saying that the k integral in (6.16) should be taken along the Landau contour, rather than simply along the real line.

Note that the above construction was done assuming $|\gamma_{\boldsymbol{k}}|t \ll 1$, i.e., all the QL action has to occur before the initial perturbations decay away (which is reasonable). Note also that there is nothing above that would not apply to the case of unstable perturbations ($\gamma_{\boldsymbol{k}} > 0$) and so we conclude that results of §6, derived formally for $\gamma_{\boldsymbol{k}} t \gg 1$, in fact also hold on shorter time scales ($\gamma_{\boldsymbol{k}} t \ll 1$, but, obviously, still $\omega_{\boldsymbol{k}} t \gg 1$).

---

10. QLT of Weibel instability. (a) Starting from the Vlasov equations including magnetic perturbations, show that the slow evolution of the equilibrium distribution function is described by the diffusion equation

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}(\boldsymbol{v}) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} \tag{6.91}$$

with the QL diffusion matrix

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{1}{i(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}}) + \gamma_{\boldsymbol{k}}} \left(\boldsymbol{E}^*_{\boldsymbol{k}} + \frac{\boldsymbol{v}\times\boldsymbol{B}^*_{\boldsymbol{k}}}{c}\right)\left(\boldsymbol{E}_{\boldsymbol{k}} + \frac{\boldsymbol{v}\times\boldsymbol{B}_{\boldsymbol{k}}}{c}\right), \tag{6.92}$$

where $\omega_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}}$ are the frequency and the growth rate, respectively, of the fastest-growing mode.

(b) Consider the example of the low-frequency electron Weibel instability with wave numbers $\boldsymbol{k}$ parallel to the anisotropy direction [see (6.47)]. Take $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ and $\boldsymbol{B}_{\boldsymbol{k}} = B_{\boldsymbol{k}}\hat{\boldsymbol{y}}$ and, denoting $\Omega_{\boldsymbol{k}} = eB_{\boldsymbol{k}}/m_e c$ (the Larmor frequency associated with the perturbed magnetic field), show that (6.91) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_x}\left(D_{xx}\frac{\partial f_0}{\partial v_x} + D_{xz}\frac{\partial f_0}{\partial v_z}\right) + \frac{\partial}{\partial v_z} D_{zz}\frac{\partial f_0}{\partial v_z}, \tag{6.93}$$

where the coefficients of the QL diffusion tensor are

$$D_{xx} = \sum_k \frac{\gamma_k}{k^2}\,|\Omega_k|^2, \quad D_{xz} = -\sum_k \frac{2\gamma_k v_x v_z}{k^2 v_z^2 + \gamma_k^2}\,|\Omega_k|^2, \quad D_{zz} = \sum_k \frac{\gamma_k v_x^2}{k^2 v_z^2 + \gamma_k^2}\,|\Omega_k|^2. \tag{6.94}$$

(c) Suppose the electron distribution function f_0 is initially the bi-Maxwellian (6.45) with $0 < T_\perp/T_\parallel - 1 \ll 1$ (as should be the case for this instability to work). As QL evolution starts, we may define the temperatures of the evolving distribution according to

$$T_\perp = \frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, \frac{m(v_x^2+v_y^2)}{2}\, f_0, \quad T_\parallel = \frac{1}{n}\int \mathrm{d}\boldsymbol{v}\, m v_z^2 f_0. \tag{6.95}$$

Show that initially, viz., before f_0 has time to change shape significantly so as no longer to be representable as a bi-Maxwellian, the two temperatures will evolve approximately (using $\gamma_k \ll k v_{\mathrm{th}}$) according to

$$\boxed{\frac{\partial T_\perp}{\partial t} = -\lambda T_\perp, \quad \frac{\partial T_\parallel}{\partial t} = 2\lambda T_\perp}, \quad \text{where} \quad \lambda(T_\perp, T_\parallel) = \sum_k \frac{2\gamma_k|\Omega_k|^2}{k^2 v_{\mathrm{th}\parallel}^2}. \tag{6.96}$$

Thus, QL evolution will lead, at least initially, to the reduction of the temperature anisotropy, thus weaking the instability (these equations should not be used to trace $T_\perp/T_\parallel - 1$ all the way to zero because there is no reason why the QL evolution should preserve the bi-Maxwellian shape of f_0).

Note that, even modulo the caveat about the bi-Maxwellian not being a long-term solution, this does not give us a way to estimate (or even guess) what the saturated fluctuation level will be. The standard Weibel lore is that saturation occurs when the approximations that were used to derive the linear theory (Q-3) break down, namely, when magnetic field becomes strong enough to magnetise the plasma, rendering the Larmor scale $\rho_e = v_{\mathrm{th}e}/\Omega_k$ associated with the fluctuations small enough to be comparable to the latter's wavelengths $\sim k^{-1}$. Using the typical values of k from (6.47), we can write this condition as follows

$$\Omega_k \sim k v_{\mathrm{th}e} \sim \sqrt{\Delta_e}\,\frac{v_{\mathrm{th}e}}{d_e} \quad \Leftrightarrow \quad \boxed{\frac{1}{\beta_e} \equiv \frac{B^2/8\pi}{n_e T_e} \sim \Delta_e}. \tag{6.97}$$

Thus, Weibel instability will produce fluctuations the ratio of whose magnetic-energy density to the electron-thermal-energy density (customarily referred to as the inverse of "plasma beta," $1/\beta_e$) is comparable to the electron pressure anisotropy Δ_e. Because at that point the fluctuations will be relaxing this pressure anisotropy at the same rate as they can grow in the first place [in (6.96), $\lambda \sim \gamma_k$], the QL approach is not valid anymore.

These considerations are, however, usually assumed to be qualitatively sound and lead people to believe that, even in collisionless plasmas, the anisotropy of the electron distribution must be largely self-regulating, with unstable Weibel fluctuations engendered by the anisotropy quickly acting to isotropise the plasma (or at least the electrons).

This is all currently very topical in the part of the plasma-astrophysics world preoccupied with collisionless shocks, origin of the cosmic magnetism, hot weakly collisional environments such as the intergalactic medium (in galaxy clusters) or accretion flows around black holes and many other interesting subjects.

(d) Equations (6.96) say that the total mean kinetic energy,

$$\int \mathrm{d}\boldsymbol{v}\, \frac{mv^2}{2} f_0 = n\left(T_\perp + \frac{T_\parallel}{2}\right), \tag{6.98}$$

does not change. But fluctuations are generated and grow at the rate γ_k! Without much further algebra, can you tell whether you should therefore doubt the result (6.96)?

11. Linear and QL theory of ion-electron streaming instability.[58] Consider a plasma consisting of Maxwellian electrons and a cold ion beam (velocity $u_i \gg v_{\mathrm{th}e}$—much greater than the width of the electron distribution; cf. Q-5, where the two species' roles were reversed). The corresponding distribution functions, which can be considered one-dimensional (in the direction of the beam), are

$$F_e(v) = \frac{n_e}{\sqrt{\pi}\, v_{\mathrm{th}e}} e^{-v^2/v_{\mathrm{th}e}^2}, \qquad F_i(v) = n_i \delta(v - u_i). \tag{6.99}$$

(a) Starting, as usual, with the dispersion relation that determines the complex increment p of linear perturbations in an electrostatic plasma and anticipating that an instability will be possible for $p/k \gg v_{\mathrm{th}e}$, show that, in this limit, the dispersion relation is

$$1 + \frac{\omega_{\mathrm{pe}}^2}{p^2} - \frac{\omega_{\mathrm{pi}}^2}{(ku_i - ip)^2} = 0. \tag{6.100}$$

(b) Show that there is an instability that attains its fastest growth at $k = \omega_{\mathrm{pe}}/u_i$ and produces, at that wave number, waves with the real frequency and growth rate

$$\boxed{\omega_k = \left[1 - \frac{1}{2}\left(\frac{Zm_e}{2m_i}\right)^{1/3}\right]\omega_{\mathrm{pe}} \quad \text{and} \quad \gamma_k = \frac{\sqrt{3}}{2}\left(\frac{Zm_e}{2m_i}\right)^{1/3}\omega_{\mathrm{pe}}}\,. \tag{6.101}$$

(c) The equations of QLT for the distribution functions and the wave spectrum are (see §6.1)

$$\frac{\partial F_\alpha}{\partial t} = \frac{\partial}{\partial v}\left[\frac{q_\alpha^2}{m_\alpha^2}\sum_k \frac{\gamma_k |E_k|^2}{(kv-\omega_k)^2 + \gamma_k^2}\right]\frac{\partial F_\alpha}{\partial v}, \qquad \frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2, \tag{6.102}$$

where, as usual, E_k is the electric-field amplitude at wave number k, ω_k is the real frequency of the waves, and γ_k is their growth rate at that wave number. Assuming that, at least in the initial stage of the quasilinear evolution, the distribution functions will

[58]This is based on the 2022 exam question.

preserve their functional form (6.99), but T_e and u_i will change slowly with time, show that the kinetic-energy densities of the electrons and ions will satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\frac{m_e v^2}{2}F_e = \frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{m_i n_i u_i^2}{2} = -2\frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t}, \tag{6.103}$$

where $\mathscr{U} = \sum_k |E_k|^2/8\pi$ is the energy density of the electric field. Despite their rather innocuous appearance, deriving these equations (especially the second) does require some mildly nontrivial calculations. In carrying them out, you may assume that the wave-number sums in (6.102) are dominated by the fastest-growing mode described by (6.101).

(d) Based on (6.103), describe the energy flows in the system and, qualitatively, the evolution of the distribution functions. How do the mean momenta of these distributions evolve? (Only twiddle algebra—rough estimates—are called for here.)

(e) Make an educated qualitative guess about the typical size of $\mathscr{U}$ at which the instability saturates, i.e., the distribution becomes stable. The answer is

$$\mathscr{U} \sim \frac{m_e n_e u_i^2}{2}. \tag{6.104}$$

If you are struggling with any of this, especially the qualitative arguments at the end of the question, you may find inspiration in Krall & Trivelpiece (1973).

---

12. QLT of stochastic acceleration.[59] Consider a population of particles of charge q and mass m. Assume that collisions are entirely negligible. Assume further that an electrostatic fluctuation field $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$ (with zero spatial mean) is present and that this field is given and externally determined, i.e., it is unaffected by the particles that are under consideration. This might happen physically if, for example, the particles are a low-density admixture in a plasma consisting of some more numerous species of ions and electrons, which dominate the plasma's dielectric response.

As usual, we assume that the distribution function can be represented as $f = f_0(t,\boldsymbol{v}) + \delta f(t,\boldsymbol{r},\boldsymbol{v})$, where f_0 is spatially homogeneous and changes slowly in time compared to the perturbed distribution $\delta f \ll f_0$. Its evolution is described by (2.11), where angle brackets again denote the time average over the fast variation of the fluctuation field.

(a) Assume that φ is sufficiently small for it to be possible to determine δf from the linearised kinetic equation. Let $\delta f = 0$ at $t = 0$. Show that f_0 satisfies a QL diffusion equation with the diffusion matrix

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2}\sum_{\boldsymbol{k}} \boldsymbol{k}\boldsymbol{k}\frac{1}{2\pi i}\int \mathrm{d}p\,\frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\int_{-\infty}^{t}\mathrm{d}\tau\, e^{p\tau} C_{\boldsymbol{k}}(\tau), \tag{6.105}$$

where the p integration is along a contour appropriate for an inverse Laplace transform and $C_{\boldsymbol{k}}(t-t') = \langle\varphi^*_{\boldsymbol{k}}(t)\varphi_{\boldsymbol{k}}(t')\rangle$ is the correlation function of the fluctuation field (which is taken to be statistically stationary, so $C_{\boldsymbol{k}}$ depends only on the time difference $t - t'$).

(b) Let the correlation function have the form

$$\frac{q^2}{m^2}C_{\boldsymbol{k}}(\tau) = \frac{A_{\boldsymbol{k}}}{\tau_{\boldsymbol{k}}}\,e^{-|\tau|/\tau_{\boldsymbol{k}}}, \tag{6.106}$$

i.e., $\tau_{\boldsymbol{k}}$ is the correlation time of the fluctuation field and $A_{\boldsymbol{k}}\tau_{\boldsymbol{k}}^{-1}$ its spectrum (normalised

[59] Except for part (d), this is based on the 2018 exam question.

in this way so that $A_{\boldsymbol{k}}$ is the time integral of the correlation function, for consistency of notation with Q-13 and elsewhere in these Lectures, e.g., §12); assume $\tau_{-\boldsymbol{k}} = \tau_{\boldsymbol{k}}$. Do the integrals in (6.105) and show that, at times $t \gg \tau_{\boldsymbol{k}}$,

$$\boxed{\mathsf{D}(\boldsymbol{v}) = \sum_{\boldsymbol{k}} \boldsymbol{k}\boldsymbol{k} \frac{A_{\boldsymbol{k}}}{1 + (\tau_{\boldsymbol{k}} \boldsymbol{k} \cdot \boldsymbol{v})^2}}\,. \tag{6.107}$$

(c) Restrict consideration to 1D in space and to the limit in which $\tau_k \ll (kv)^{-1}$ for typical wave numbers of the fluctuations and typical particle velocities (i.e., the fluctuation field is short-time correlated). Assuming that f_0 at $t = 0$ is a Maxwellian with temperature T_0, predict the evolution of f_0 with time. Discuss what physically is happening to the particles. Discuss the validity of the short-correlation-time approximation and of the assumption of slow evolution of f_0. What is, roughly, the condition on the amplitude and the correlation time of the fluctuation field that makes these assumptions compatible?

(d) When the distribution "heats up" sufficiently, the short-correlation-time approximation will be broken. Staying in 1D, consider the opposite limit, $\tau_k \gg (kv)^{-1}$. Show that the resulting QL equation admits a subdiffusive solution, with

$$f_0(t, v) \propto \frac{e^{-v^4/\alpha t}}{t^{1/4}}, \quad \alpha = 16 \sum_k \frac{A_k}{\tau_k^2}. \tag{6.108}$$

In view of this result and of (c), discuss qualitatively how an initially "cold" particle distribution would evolve with time.

The original, classic paper on stochastic acceleration is Sturrock (1966). Note that the velocity dependence of the diffusion matrix (6.107) is determined by the functional form of $C_{\boldsymbol{k}}(\tau)$, so interesting τ dependences of the latter can lead to interesting distributions f_0 of the accelerated particles. The result (6.108) might actually have been new at the time when it was derived for this question (2017)—at least I am not aware of any prior reference for it. Since then, it has been confirmed (and further studied) numerically by Nastac *et al.* (2025).

---

13. QLT of randomly forced Vlasov–Poisson plasma.[60] This question brings together some of the strands previously explored in Q-7, Q-9 and Q-12 (but can be done independently of them).

Consider a 1D plasma in which, as usual, the electron distribution is $f(t, z, v) = f_0(t, v) + \delta f(t, z, v)$. Any perturbations of the ion distribution are to be neglected. Let the (Fourier-transformed) electrostatic potential associated with the perturbations satisfy

$$\varphi_k = \chi_k(t) - \frac{4\pi e}{k^2} \int \mathrm{d}v\, \delta f_k(t, v), \tag{6.109}$$

where $\chi_k(t)$ is an externally imposed random noise, white in time, viz.,

$$\frac{e^2}{m_e^2} \langle \chi_k(t) \chi_{k'}^*(t') \rangle = 2A_k \delta_{k,k'} \delta(t - t'), \tag{6.110}$$

and the noise spectrum A_k is independent of time [cf. (6.62) and (6.106)].

(a) Assuming $\delta f_k(t = 0, v) = 0$ and using the QL approximation, show that

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial f_0}{\partial v} \tag{6.111}$$

---

[60] Except for part (f), this is based on the 2024 exam question.

with the QL diffusion coefficient

$$D(v) = \sum_k \frac{k^2 A_k}{2\pi^2} \int \mathrm{d}p \int (\mathrm{d}p')^* \frac{e^{(p+p'^*)t}}{(p+ikv)(p+p'^*)\epsilon(p,k)\epsilon(p',k)^*}, \tag{6.112}$$

where the integration contours in p and p' are ones associated with the inverse Laplace transform, viz., $p, p' \in (-i\infty + \sigma, +i\infty + \sigma)$ with a real constant $\sigma > 0$ to the right of any poles in the complex p plane; $\epsilon(p,k)$ is the dielectric function (3.26).

(b) Assume that f_0 is a stable distribution and hence argue that you are allowed to take $\sigma \to +0$. Show that, if we wait long enough for any damped perturbations to decay, the quasilinear diffusion coefficient is [cf. (6.107)]

$$\boxed{D(v) = \sum_k \frac{k^2 A_k}{|\epsilon(-ikv,k)|^2}}\,. \tag{6.113}$$

(c) Consider a noise spectrum $k^2 A_k$ concentrated at wavenumbers $k \gg \lambda_{\mathrm{De}}^{-1}$. Show that the rate of growth of the total electron kinetic energy per unit volume (the heating rate) is

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = m_e n_e \sum_k k^2 A_k. \tag{6.114}$$

How does f_0 evolve in time in this case?

(d) Now consider $k^2 A_k$ concentrated at $k \ll \lambda_{\mathrm{De}}^{-1}$. Prove that the heating rate is the same as (6.114), even though the approximation for the dielectric function given by (3.38) and (3.41) now has to be used.[61] You may also wish to use the formula [cf. (6.72)]

$$\frac{1}{(kv-\omega)^2 + \gamma^2} \approx \frac{\pi\delta(kv-\omega)}{|\gamma|} \quad \text{if} \quad \gamma \ll kv \sim \omega. \tag{6.115}$$

(e) The heating in one of the limits considered above is called "resonant heating" and in the other, "stochastic heating" (cf. Q-12). Which one is which and why? Explain the physical mechanism in each case.

(f*) For the case (d), show that, at $v \sim \omega_{\mathrm{pe}}/k$,

$$\frac{\partial f_0}{\partial t} \approx \frac{L}{2\pi} n_e \omega_{\mathrm{pe}}^3 \mathrm{sgn}[f_0'(v)] \frac{\partial}{\partial v} \frac{A_{k=\omega_{\mathrm{pe}}/v}}{v^5}, \tag{6.116}$$

where L is the size of the (1D) box. What is the form of the QL diffusion equation at $v \ll \omega_{\mathrm{pe}}/k$? In view of these results, describe qualitatively (and, if you have it in yourself, quantitatively) the evolution of f_0 with time and, in particular, prove that

$$f_0(v) \propto v^{-3}, \qquad v_{\mathrm{the}} \ll v \ll \frac{\omega_{\mathrm{pe}}}{k}, \tag{6.117}$$

where v_{the} is the (time-dependent) typical velocity scale where the bulk of f_0 resides.

If you fancy doing some more QL calculations, work through all of the above again but now in 3D and for an external noise $\chi_{\boldsymbol{k}}(t)$ that is finite-time correlated with the correlation function (6.106) ("red noise").

You will find some numerical experiments illustrating (6.113) (as well as its derivation) in Nastac *et al.* (2025)—this paper will also take you beyond QLT, to the material of §12.

---

[61] In Exercise 12.4, you will have an opportunity to prove that the heating rate is the same no matter what, as long as the external forcing is white.

The result (6.117) was first spotted by Banik *et al.* (2024). This result is generalisable to d dimemsions (in which case it becomes $f_0 \propto v^{-d-2}$) and corresponds, for any d, to the distribution function of particle energies $\varepsilon = m_e v^2/2$ scaling as ε^{-2}—reminiscent of, but not clear whether directly related to, the universal collisionless equilibria derived by Ewart (§10.4). It is, in fact, also not clear whether any realistic situation with energy injection at $k\lambda_{\mathrm{D}e} \ll 1$ can be adequately described by QL theory: you will learn why in §8, where we shall study Langmuir turbulence, with its propensity for transferring energy to large scales. A comprehensive treatment of this type of stirred plasma still awaits an ambitious student in search of a dissertation topic.

IUCUNDI ACTI LABORES.

PART II
Advanced Kinetic Theory

In this "advanced" part of these kinetic-theory lectures, I will try to make some inroads into nonlinear theory beyond QLT—this means not (entirely) neglecting the last term in (2.12), which is responsible for the nonlinear interactions between the perturbed electric field ($\varphi_{\boldsymbol{k}'}$) and the perturbed distribution function ($\delta f_{\boldsymbol{k}-\boldsymbol{k}'}$). Nonlinear theory of anything is, of course, hard—indeed, in most cases, intractable. These days, an impatient researcher's answer to being faced with a hard question is to outsource it to a computer. This sometimes leads to spectacular successes, but also, somewhat more frequently, to spectacular confusion about how to assess the output. In dealing with a steady stream of data produced by ever more powerful machines, one is often helped by the residual memory of analytical results obtained in the prehistoric era when computation was harder than theory and plasma physicists had to find ingenious ways to solve nonlinear problems "by hand"—which usually required finding ingenious ways of posing problems that were solvable. These can be separated into three broad categories: (ii) interesting particular cases of nonlinear behaviour involving just a few waves (two pretty, and classic, examples of this are §§12.1 and 13.1), (ii) looking at systems of very many waves amenable to some approximate statistical treatment (§§7–8), closure assumptions, or scaling arguments (§§11–12), and (iii) applying overarching physical principles, such as minimum energy (§9) or maximum entropy (§10).

7. Quasiparticle Kinetics

7.1. *QLT in the Language of Quasiparticles*

First I would like to outline a neat way of reformulating the QLT, which both sheds some light on the meaning of what was done in §6 and opens up promising avenues for theorising further about nonlinear plasma states.

Let us re-imagine our system of particles and waves as a mixture of two interacting gases: "true" particles (electrons) and *quasiparticles*, or *plasmons*, which will be the "quantised" version of Langmuir waves. If each of these plasmons has momentum $\hbar\boldsymbol{k}$ and energy $\hbar\omega_{\boldsymbol{k}}$, we can declare

$$N_{\boldsymbol{k}} = \frac{V|\boldsymbol{E}_{\boldsymbol{k}}|^2/4\pi}{\hbar\omega_{\boldsymbol{k}}} \tag{7.1}$$

to be the mean occupation number of plasmons with wave number $\boldsymbol{k}$ in a box of volume V. The total energy of these plasmons is then

$$\sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} N_{\boldsymbol{k}} = V\sum_{\boldsymbol{k}} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi}, \tag{7.2}$$

twice the total electric energy in the system (twice because it includes the energy of the mean oscillatory motion of electrons within a wave; see discussion at the end of §6.5 and/or in §5.7). Similarly, the total momentum of the plasmons is

$$\sum_{\boldsymbol{k}} \hbar\boldsymbol{k} N_{\boldsymbol{k}} = V\sum_{\boldsymbol{k}} \frac{\boldsymbol{k}|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi\omega_{\boldsymbol{k}}}. \tag{7.3}$$

This is indeed in line with our previous calculations [see (6.39)]. Note that the role of $\hbar$ here is simply to define a splitting of wave energy into individual plasmons—this can be

(a) absorption of a plasmon by an electron (b) emission of a plasmon by an electron

FIGURE 34. Diagrams for (7.6).

done in an arbitrary way, provided $\hbar$ is small enough to ensure $N_{\boldsymbol{k}} \gg 1$. Since there is nothing quantum-mechanical about our system, all our results will in the end have to be independent of $\hbar$, so we will use $\hbar$ as an arbitrarily small parmeter, in which it will be convenient to expand, while expecting it eventually to cancel out in all physically meaningful relationships.

We may now think of the QL evolution (or indeed generally of the nonlinear evolution) of our plasma in terms of interactions between plasmons and electrons. These are *resonant* electrons; the thermal bulk only participates via its supporting role of enabling oscillatory plasma motions associated with plasmons. The electrons are described by their distribution function $f_0(\boldsymbol{v})$, which we can, to make our formalism nicely uniform, recast in terms of occupation numbers: if the wave number corresponding to velocity $\boldsymbol{v}$ is $\boldsymbol{p} = m_e\boldsymbol{v}/\hbar$, then its occupation number is

$$n_{\boldsymbol{p}} = \left(\frac{2\pi\hbar}{m_e}\right)^3 f_0(\boldsymbol{v}) \quad \Rightarrow \quad \sum_{\boldsymbol{p}} n_{\boldsymbol{p}} = \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{p}\, n_{\boldsymbol{p}} = V\int \mathrm{d}\boldsymbol{v}\, f_0(\boldsymbol{v}) = Vn_e. \tag{7.4}$$

It is understood that $n_{\boldsymbol{p}} \ll 1$ (our electron gas is non-degenerate).

The QL evolution of the plasmon and electron distributions is controlled by two processes: absorption or emission of a plasmon by an electron (known as Cherenkov absorption/emission). Diagrammatically, these can be depicted as shown in Fig. 34. As we know from §6.2, they are subject to momentum conservation, $\boldsymbol{p} = \boldsymbol{k} + (\boldsymbol{p} - \boldsymbol{k})$, and energy conservation:

$$\begin{aligned} 0 = \varepsilon^e_{\boldsymbol{p}} - \varepsilon^l_{\boldsymbol{k}} - \varepsilon^e_{\boldsymbol{p}-\boldsymbol{k}} = \frac{\hbar^2 p^2}{2m_e} - \hbar\omega_{\boldsymbol{k}} - \frac{\hbar^2|\boldsymbol{p}-\boldsymbol{k}|^2}{2m_e} = \hbar\left(-\omega_{\boldsymbol{k}} + \frac{\hbar\boldsymbol{p}\cdot\boldsymbol{k}}{m_e} - \frac{\hbar k^2}{2m_e}\right) \\ = \hbar(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}}) + O(\hbar^2). \end{aligned} \tag{7.5}$$

This is the familiar resonance condition $\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} = 0$. The superscripts e and l stand for electrons and (Langmuir) plasmons.

7.1.1. *Plasmon Distribution*

We may now write an equation for the evolution of the plasmon occupation number:[62]

$$\frac{\partial N_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p}}\Big[\underbrace{-\,\mathfrak{w}_{(\boldsymbol{p}-\boldsymbol{k},\boldsymbol{k})\to\boldsymbol{p}}\, n_{\boldsymbol{p}-\boldsymbol{k}} N_{\boldsymbol{k}}}_{\text{Fig. 34(a)}} + \underbrace{\mathfrak{w}_{\boldsymbol{p}\to(\boldsymbol{k},\boldsymbol{p}-\boldsymbol{k})}\, n_{\boldsymbol{p}}(N_{\boldsymbol{k}}+1)}_{\text{Fig. 34(b)}}\Big], \tag{7.6}$$

[62]Technically speaking, the first term in this equation should be multiplied by $(1 - n_{\boldsymbol{p}})$ and the second by $(1 - n_{\boldsymbol{p}-\boldsymbol{k}})$ because electrons are fermions, and if our electron, having absorbed or emitted a plasmon, tries to emerge with a momentum that another electron already has, the probability of such an event must be zero. However, since our plasma is classical, $n_{\boldsymbol{p}} \ll 1$ and so these extra factors can be approximated by unity. The same comment applies to calculations in §§7.1.2, 7.2.4, and 7.2.5.

where $\mathfrak{w}$ are the probabilities of absorption and emission and must be equal:[63]

$$\mathfrak{w}_{(\boldsymbol{p}-\boldsymbol{k},\boldsymbol{k})\to\boldsymbol{p}} = \mathfrak{w}_{\boldsymbol{p}\to(\boldsymbol{k},\boldsymbol{p}-\boldsymbol{k})} \equiv \mathfrak{w}(\boldsymbol{p},\boldsymbol{k}) = w(\boldsymbol{p},\boldsymbol{k})\delta(\varepsilon^e_{\boldsymbol{p}} - \varepsilon^l_{\boldsymbol{k}} - \varepsilon^e_{\boldsymbol{p}-\boldsymbol{k}}). \tag{7.7}$$

The first term in the right-hand side of (7.6) describes the absorption of one of (indistinguishable) $N_{\boldsymbol{k}}$ plasmons by one of $n_{\boldsymbol{p}-\boldsymbol{k}}$ electrons, the second term desribes the emission by one of $n_{\boldsymbol{p}}$ electrons of one of $N_{\boldsymbol{k}}+1$ plasmons. The $+1$ is, of course, a small correction to $N_{\boldsymbol{k}} \gg 1$ and can be neglected, although sometimes, in analogous but more complicated calculations, it has to be kept because lowest-order terms cancel (see, e.g., §§7.2.1 and 7.2.2). Using (7.7), (7.5) and (7.4), we find

$$\begin{aligned}
\frac{\partial N_{\boldsymbol{k}}}{\partial t} &\approx \sum_{\boldsymbol{p}} \delta(\varepsilon^e_{\boldsymbol{p}} - \varepsilon^l_{\boldsymbol{k}} - \varepsilon^e_{\boldsymbol{p}-\boldsymbol{k}})\, w(\boldsymbol{p},\boldsymbol{k})(n_{\boldsymbol{p}} - n_{\boldsymbol{p}-\boldsymbol{k}})N_{\boldsymbol{k}} \\
&= V\int \mathrm{d}\boldsymbol{v}\,\delta\big(\hbar(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})\big)\, w\Big(\frac{m_e\boldsymbol{v}}{\hbar},\boldsymbol{k}\Big)\left[f_0(\boldsymbol{v}) - f_0\Big(\boldsymbol{v} - \frac{\hbar\boldsymbol{k}}{m_e}\Big)\right] N_{\boldsymbol{k}} \\
&\approx V\int \mathrm{d}\boldsymbol{v}\,\frac{1}{\cancel{\hbar}}\,\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})\, w\Big(\frac{m_e\boldsymbol{v}}{\hbar},\boldsymbol{k}\Big)\,\frac{\cancel{\hbar}}{m_e}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}\,N_{\boldsymbol{k}} \\
&= \frac{V}{m_e} w\Big(\frac{m_e\omega_{\mathrm{pe}}}{\hbar k^2}\boldsymbol{k},\boldsymbol{k}\Big)\,F'\Big(\frac{\omega_{\mathrm{pe}}}{k}\Big)\,N_{\boldsymbol{k}} \equiv 2\gamma_{\boldsymbol{k}}N_{\boldsymbol{k}}
\end{aligned} \tag{7.8}$$

Note that, as expected, $\hbar$ has disappeared from our equations, after having being used as an expansion parameter.

Since $N_{\boldsymbol{k}} \propto |\boldsymbol{E}_{\boldsymbol{k}}|^2$ [see (7.1)], the prefactor in (7.8) is clearly just the (twice) growth or damping rate of the waves. Comparing with (6.1), we read off the expression for the absorption/emission probability:

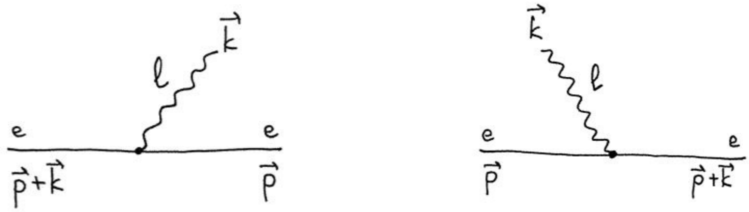
$$w\Big(\frac{m_e\omega_{\mathrm{pe}}}{\hbar k^2}\boldsymbol{k},\boldsymbol{k}\Big) = \frac{\pi m_e\omega^3_{\mathrm{pe}}}{V n_e k^2}. \tag{7.9}$$

Thus, our calculation of Landau damping in §3.5 could be thought of as a calculation of this probability. Whether there is damping or an instability is decided by whether it is absorption or emission of plasmons that occurs more frequently—and that depends on whether, for any given $\boldsymbol{k}$, there are more electrons that are slightly slower or slightly faster than the plasmons with wave number $\boldsymbol{k}$. Note that getting the correct sign of the damping rate is automatic in this approach, since the probability w must obviously be positive.

7.1.2. *Electron Distribution*

The evolution equation for the occupation number of electrons can be derived in a similar fashion, if we itemise the processes that lead to an electron ending up in a state with a given wave number $\boldsymbol{p} = m_e\boldsymbol{v}/\hbar$ or moving from this state to one with a different wave number. The four relevant diagrams are the two in Fig. 34 and the additional two shown in Fig. 35. The absorption and emission probabilities are the same as before and

[63]You might wonder why I call them probabilities, as they clearly are not normalised as ones. How they are normalised is not too important because, to work out specific expressions for them, we will have to redo the calculation (partially) via a different route—as in (7.9)—at which point the right units and prefactors will emerge organically. The important thing is the relationships between these probabilities, as well as the energy-conservation constraint, which determine the structure of the "kinetic equations" for occupation numbers, derived in what follows.



(a) emission of a plasmon by an electron (b) absorption of a plasmon by an electron

FIGURE 35. Additional diagrams for (7.10).

so are the energy-conservation conditions. Therefore,

$$\begin{aligned}\frac{\partial n_{\boldsymbol{p}}}{\partial t} &= \sum_{\boldsymbol{k}}\Big[\underbrace{\mathfrak{w}_{(\boldsymbol{p}-\boldsymbol{k},\boldsymbol{k})\to\boldsymbol{p}}n_{\boldsymbol{p}-\boldsymbol{k}}N_{\boldsymbol{k}}}_{\text{Fig. 34(a)}} + \underbrace{\mathfrak{w}_{\boldsymbol{p}+\boldsymbol{k}\to(\boldsymbol{k},\boldsymbol{p})}n_{\boldsymbol{p}+\boldsymbol{k}}(N_{\boldsymbol{k}}+1)}_{\text{Fig. 35(a)}} \\ &\qquad - \underbrace{\mathfrak{w}_{\boldsymbol{p}\to(\boldsymbol{k},\boldsymbol{p}-\boldsymbol{k})}n_{\boldsymbol{p}}(N_{\boldsymbol{k}}+1)}_{\text{Fig. 34(b)}} - \underbrace{\mathfrak{w}_{(\boldsymbol{p},\boldsymbol{k})\to\boldsymbol{p}+\boldsymbol{k}}n_{\boldsymbol{p}}N_{\boldsymbol{k}}}_{\text{Fig. 35(b)}}\Big] \\ &\approx \sum_{\boldsymbol{k}}\big[\mathfrak{w}(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k})(n_{\boldsymbol{p}+\boldsymbol{k}}-n_{\boldsymbol{p}}) - \mathfrak{w}(\boldsymbol{p},\boldsymbol{k})(n_{\boldsymbol{p}}-n_{\boldsymbol{p}-\boldsymbol{k}})\big]N_{\boldsymbol{k}} \\ &\approx \frac{\partial}{\partial\boldsymbol{p}}\cdot\sum_{\boldsymbol{k}}\mathfrak{w}(\boldsymbol{p},\boldsymbol{k})N_{\boldsymbol{k}}\boldsymbol{k}\boldsymbol{k}\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial\boldsymbol{p}},\end{aligned} \tag{7.10}$$

where I have expanded twice in small $\boldsymbol{k}$ (i.e., in $\hbar$). This is a diffusion equation in $\boldsymbol{p}$ (or, equivalently, in $\boldsymbol{v} = \hbar\boldsymbol{p}/m_e$) space. In view of (7.4), (7.10) has the same form as (6.7), viz.,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\mathsf{D}(\boldsymbol{v})\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}, \tag{7.11}$$

where the diffusion matrix is

$$\mathsf{D}(\boldsymbol{v}) = \sum_{\boldsymbol{k}}\boldsymbol{k}\boldsymbol{k}\,\frac{\hbar N_{\boldsymbol{k}}}{m_e^2}\,w\Big(\frac{m_e\boldsymbol{v}}{\hbar},\boldsymbol{k}\Big)\,\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}) = \frac{e^2}{m_e^2}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v}-\omega_{\boldsymbol{k}}). \tag{7.12}$$

The last expression is identical to the resonant form of the QL diffusion matrix (6.8) [cf. (6.16) and (6.90)]. To derive it, we used the definition (7.1) of $N_{\boldsymbol{k}}$ and the absorption/emission probability (7.7), already known from linear theory.

Thus, we are able to recover the (resonant part of the) QLT from our new electron-plasmon interaction approach. There is more to this approach than a pretty "field-theoretic" reformulation of already-derived earlier results. The diagram technique and the interpretation of the nonlinear state of the plasma as arising from interactions between particles and quasiparticles can be readily generalised to situations in which the nonlinear interactions in (2.12) cannot be neglected and/or more than one type of waves is present. In this new language, the nonlinear interactions would be manifested as interactions between plasmons (rather than only between plasmons and electrons) contributing to the rate of change of $N_{\boldsymbol{k}}$. There are many possibilities: four-plasmon interactions, interactions between plasmons and phonons (sound waves), as well as between the latter and electrons and/or ions, etc. A comprehensive monograph on this subject is Tsytovich (1995).[64] The lectures by Kingsep (1996) are a more human-scale, and humane-style, pedagogical

[64]It is a terrifying remake of his two earlier, equally deadly, books Tsytovich (1970, 1977). He also has a short book of lectures, Tsytovich (1972), which are useful as a kind of extended

(a) decay of plasmon a into b and c (b) merger of plasmons b and c into a

FIGURE 36. Diagrams for (7.15) (three-wave interaction).

exposition, but are, alas, only available in the original Russian. In §7.2, I will attempt a basic introduction and give a few examples.

I have introduced the language of kinetics of quasiparticles and their interactions with "true" particles as a reformulation of QLT for plasmas. The method is much more general and originates, as far as I know, from condensed-matter physics, the classic problem being the kinetics of electrons and phonons in metals—the founding texts on this subject are Peierls (1955) and Ziman (1960).

7.2. *Weak Turbulence*

If we can think about plasmon-electron interactions in terms of occupation numbers and probabilities of absorption/emission, we can also think this way of interactions between plasmons, not necessarily involving electrons or ions directly.

7.2.1. *Three–Wave Interactions*

A plasmon with momentum $\hbar\boldsymbol{k}$ and energy $\varepsilon^a_{\boldsymbol{k}} = \hbar\omega^a_{\boldsymbol{k}}$ might "decay" into two other plasmons, with momenta and energies $\hbar\boldsymbol{p}$, $\varepsilon^b_{\boldsymbol{p}} = \hbar\omega^b_{\boldsymbol{p}}$ and $\hbar\boldsymbol{q}$, $\varepsilon^c_{\boldsymbol{q}} = \hbar\omega^c_{\boldsymbol{q}}$ (Fig. 36a), or, conversely, the latter two plasmons might merge into one (Fig. 36b). The superscripts a, b, and c are there to indicate that these plasmons can be of different kinds: e.g., longitudinal Langmuir waves considered in §7.1, sound waves (*phonons*; see §3.8), transverse electromagnetic waves (*photons*; see Q-2). Both the decay and merger processes must conserve momentum and energy, viz.,

$$\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}, \quad \omega^a_{\boldsymbol{k}} = \omega^b_{\boldsymbol{p}} + \omega^c_{\boldsymbol{q}}, \tag{7.13}$$

and occur with equal probabilities

$$\mathfrak{w}^{a\to b+c}_{\boldsymbol{k}\to(\boldsymbol{p},\boldsymbol{q})} = \mathfrak{w}^{b+c\to a}_{(\boldsymbol{p},\boldsymbol{q})\to\boldsymbol{k}} \equiv \mathfrak{w}^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p}) = w^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p})\delta(\omega^a_{\boldsymbol{k}} - \omega^b_{\boldsymbol{p}} - \omega^c_{\boldsymbol{q}}). \tag{7.14}$$

These constraints are enough for us to infer the general architecture of the evolution equations for the plasmon occupation numbers. Assuming a, b, and c are all different, the rate of decrease of $N^a_{\boldsymbol{k}}$ comes from the first of these processes and the rate of its increase from the second, so, analogously to (7.6),

$$\frac{\partial N^a_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}} \Big[\underbrace{-\,\mathfrak{w}^{a\to b+c}_{\boldsymbol{k}\to(\boldsymbol{p},\boldsymbol{q})} N^a_{\boldsymbol{k}}(N^b_{\boldsymbol{p}}+1)(N^c_{\boldsymbol{q}}+1)}_{\text{Fig. 36(a)}} + \underbrace{\mathfrak{w}^{b+c\to a}_{(\boldsymbol{p},\boldsymbol{q})\to\boldsymbol{k}} N^b_{\boldsymbol{p}} N^c_{\boldsymbol{q}}(N^a_{\boldsymbol{k}}+1)}_{\text{Fig. 36(b)}} \Big]. \tag{7.15}$$

---

abstract of his canon, but are not fully self-contained. Two excellent texts on WT that speak a slightly different mathematical language are Zakharov *et al.* (1992) and Nazarenko (2011) (my §§7.3, 7.4, 8.3 and 8.4 were largely inspired by the first of these and the papers that it was based on). Two early-day classics written by the Founding Fathers are Kadomtsev (1965) and Sagdeev & Galeev (1969), documenting how it all started. Specifically on weak turbulence of Langmuir waves, there is a long, mushy review by Musher *et al.* (1995).

The rates are proportional to the products of the occupation numbers of all the participants in the three-plasmon interaction, so they are cubic. The plasmon numbers are not conserved, and, just like in (7.6), the $+1$'s in (7.15) appear where the occupation numbers are incremented as a result of the interaction. However, unlike in (7.6), these small corrections are not negligible because if they were neglected, the right-hand of (7.15) would vanish. In view of (7.14), we have (assuming all occupation numbers to be $\gg 1$)[65]

$$\boxed{\frac{\partial N_{\boldsymbol{k}}^a}{\partial t} \approx \sum_{\boldsymbol{p}} \delta(\omega_{\boldsymbol{k}}^a - \omega_{\boldsymbol{p}}^b - \omega_{\boldsymbol{k}-\boldsymbol{p}}^c) w^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p}) \left(N_{\boldsymbol{p}}^b N_{\boldsymbol{k}-\boldsymbol{p}}^c - N_{\boldsymbol{k}}^a N_{\boldsymbol{p}}^b - N_{\boldsymbol{k}}^a N_{\boldsymbol{k}-\boldsymbol{p}}^c\right)}\,. \quad (7.16)$$

This is a *kinetic equation* for plasmons of species a in a system where three-wave interactions are permitted between this species and two others.

Exercise 7.1. (a) Derive the kinetic equations for $N_{\boldsymbol{p}}^b$ and $N_{\boldsymbol{q}}^c$, still asuming that a, b, c are all different. Why does this assumption matter? What are the kinetic equations if $a = b \neq c$?

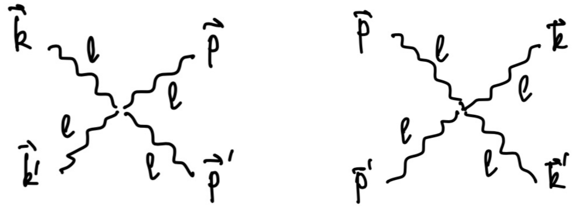
(b) Confirm that the total energy and momentum of this gas of plasmons, defined analogously to (7.2) and (7.3), respectively, are conserved by the kinetic equations that you have derived.

When would a contribution to the rate of change of $N_{\boldsymbol{k}}^a$ such as the right-hand side of (7.16) be important, in comparison with, e.g., the quasilinear rate (7.8)? Obviously, when the quasilinear rate is small in comparison, e.g., when the waves interact with very few resonant particles in the tail of the distribution, $\gamma_{\boldsymbol{k}}$ in (7.8) is exponentially small, and/or $N_{\boldsymbol{k}}^a$ fails to saturate quasilinearly before the right-hand side of (7.16) gets larger than $\gamma_{\boldsymbol{k}} N_{\boldsymbol{k}}^a$. This should be checked in each particular case—I shall talk later on about what kind of particular cases are possible.

It is clear that the right-hand side of (7.16) must originate from the nonlinear term in the kinetic equation (2.12) because it involves coupling between field perturbations at different wave numbers—indeed, the momentum conservation law (7.13) is just the restriction on wave numbers arising in the Fourier representation of a quadratic nonlinearity.

How does one complete the derivation of (7.16), i.e., calculate $w^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p})$? I am afraid the only way to do it is perturbatively, starting from the Vlasov–Poisson system (2.12) and (2.9) (or some approximation thereof), expanding $\varphi_{\boldsymbol{k}}$ into the linear part plus at least one iteration involving the nonlinear term, then working out the rate of change of $N_{\boldsymbol{k}}^a \propto k^2|\varphi_{\boldsymbol{k}}|^2$, comparing the result with (7.16) and reading off $w^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p})$—just as we did in (7.9). What then, might you ask, is the point of the quasiparticle formalism if in the end we must revert back to laborious perturbation theory? Well, apart from giving one copious amounts of physical insight and a pleasing impression of doing field-theoretic calculations, this approach actually reduces the amount of hard labour involved in that perturbation theory: having applied the constraints imposed by the conservation laws and by the principle of equal probabilities (7.14), we were able to establish the general structure of the answer, viz., that the coefficients in front of the three terms in (7.16) must be the same—the same coefficients also turn up in the kinetic equations for $N_{\boldsymbol{p}}^b$ and $N_{\boldsymbol{q}}^c$ derived in Exercise 7.1. This means that, in ploughing through the perturbation theory, you only need to keep track of contributions that give rise to

[65]Note that if $N_{\boldsymbol{k}}^a$ is much larger than all other occupation numbers, the first term in (7.16) can be neglected and the equation takes the form $\partial N_{\boldsymbol{k}}^a/\partial t = -\gamma_{\boldsymbol{k}} N_{\boldsymbol{k}}^a$. This is known as a "decay instability".



(a) (b)

FIGURE 37. Diagrams for (7.20) (four-wave interaction): plasmons scattering off each other.

just one of these terms, thus cutting down the amount of algebra by a factor of 9. Once you try these calculations, you will be very grateful for such a concession![66]

Since the nonlinearity in (2.12) is quadratic, does this mean that the rate of change of $N_{\boldsymbol{k}}^a$ must always be quadratic in plasmon occupation numbers? Not at all, because the probability $w^{a\leftrightarrow b+c}(\boldsymbol{k},\boldsymbol{p})$ of any particular three-wave process can easily turn out to be zero. Indeed, the conservation laws (7.13) are not necessarily that easy to satisfy. For example, consider the case of three long-wavelength Langmuir plasmons, all with $k\lambda_{\mathrm{De}} \ll 1$. Their frequency is given by (3.39), viz.,

$$\omega_{\boldsymbol{k}} \approx \omega_{\mathrm{pe}}\left(1 + \frac{3}{4}k^2\lambda_{\mathrm{De}}^2\right). \tag{7.17}$$

Clearly, the frequency condition in (7.13) is unfulfillable—you cannot couple two long-wavelength Langmuir plasmons and get a third such plasmon. What happens then, in a gas of Langmuir plasmons? There are two ways of getting a non-zero rate of change of $N_{\boldsymbol{k}}$ in such a situation.

Exercise 7.2. It turns out that a process involving three ion acoustic waves (§3.8) is also impossible. Why is this?

7.2.2. *Four–Wave Interactions*

The first possibility is to go to next order: while three plasmons cannot couple, four can:

$$\boldsymbol{k} + \boldsymbol{k}' = \boldsymbol{p} + \boldsymbol{p}', \quad \omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'} = \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{p}'}. \tag{7.18}$$

The probability of this "plasmon-scattering" process, or its inverse, is

$$\mathfrak{w}_{(\boldsymbol{k},\boldsymbol{k}')\to(\boldsymbol{p},\boldsymbol{p}')} = \mathfrak{w}_{(\boldsymbol{p},\boldsymbol{p}')\to(\boldsymbol{k},\boldsymbol{k}')} = w(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}')\delta(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'} - \omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}'}). \tag{7.19}$$

The kinetic equation is a sum of the two diagrams in Fig. 37:

$$\begin{aligned}\frac{\partial N_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k}',\boldsymbol{p},\boldsymbol{p}'} &\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{p}+\boldsymbol{p}'}\delta(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'} - \omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}'})w(\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}') \\ &\times \underbrace{\left[-N_{\boldsymbol{k}}N_{\boldsymbol{k}'}(N_{\boldsymbol{p}}+1)(N_{\boldsymbol{p}'}+1) + N_{\boldsymbol{p}}N_{\boldsymbol{p}'}(N_{\boldsymbol{k}}+1)(N_{\boldsymbol{k}'}+1)\right]}_{= N_{\boldsymbol{p}}N_{\boldsymbol{p}'}\left(N_{\boldsymbol{k}}+N_{\boldsymbol{k}'}\right) - N_{\boldsymbol{k}}N_{\boldsymbol{k}'}\left(N_{\boldsymbol{p}}+N_{\boldsymbol{p}'}\right)}.\end{aligned} \tag{7.20}$$

[66] I will provide one such calculation in §8.4 and ask you to do another one in Exercises 8.7 and 8.9 (if you are brave, do also the open-ended Exercise 8.10). My vehicle for those will be a simplified set of equations describing the coupled dynamics of plasmons, phonons, and ions. An ultimate repository (one might say graveyard) of many such calculations, done in a fully general setting, is Tsytovich (1995).

The lowest-order terms have again cancelled out. The algebra discount for doing the perturbation theory is buy one—get three free (see Exercise 8.7).

7.2.3. *Langmuir–Sound Turbulence*

The second possibility is that a Langmuir plasmon decays into something else. One obviously attractive, and realisable, option is that it should decay into a phonon (s) and another, slightly lower-frequency Langmuir plasmon (l) (Fig. 38a):

$$\omega^l_{\boldsymbol{p}} = \omega^s_{\boldsymbol{k}} + \omega^l_{\boldsymbol{p}-\boldsymbol{k}}. \tag{7.21}$$

Indeed this turns out to be the only three-wave process allowed for longitudinal plasma waves. Even this is only allowed—obviously—provided sound waves can actually exist, i.e., if $T_i \ll T_e$ (see §3.8).[67]

Note that this process is formally very similar to the one that was considered in §7.1: there, an electron emitted or absorbed a plasmon; here, a plasmon emits or absorbs a phonon. The difference is that in §7.1, we could assume that the electron occupation numbers were small but electron momenta large in comparison with the plasmons, whereas here the occupation numbers and momenta of the plasmons and the phonons can be in any relationship to each other (although the plasmons' energies are, of course, much greater than the phonons').

Exercise 7.3. (a) Work out under what assumption the energy conservation condition (7.21) can be turned into the wave-resonance condition $\omega^s_{\boldsymbol{k}} = \boldsymbol{k}\cdot\boldsymbol{v}^l_{\boldsymbol{k}}$, where $\boldsymbol{v}^l_{\boldsymbol{k}}$ is the group velocity of the Langmuir wave.

(b) Show, in fact, that the "optimal" phonon for a plasmon with momentum $\hbar\boldsymbol{p}$ to emit has $\boldsymbol{k} \approx 2\boldsymbol{p}$. Thus, phonons are pushed to larger k's (smaller scales).

Let the probability of the plasmon-decay/phonon-emission process, or its inverse (Fig. 38b), be

$$\mathfrak{w}(\boldsymbol{p},\boldsymbol{k}) = w(\boldsymbol{p},\boldsymbol{k})\delta(\omega^l_{\boldsymbol{p}} - \omega^l_{\boldsymbol{p}-\boldsymbol{k}} - \omega^s_{\boldsymbol{k}}). \tag{7.22}$$

Tallying up the relevant diagrams gets us the results that were, in fact, already anticipated in Exercise 7.1.

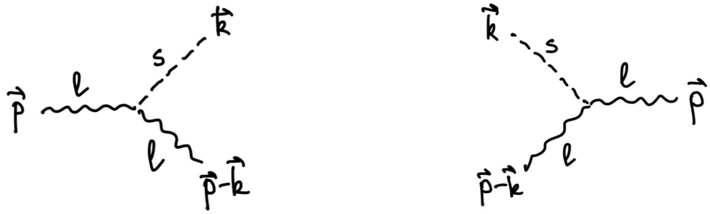
The two diagrams in Fig. 38 give us the rate of change of the number of phonons:

$$\frac{\partial N^s_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p}} \mathfrak{w}(\boldsymbol{p},\boldsymbol{k})\left(N^l_{\boldsymbol{p}}N^l_{\boldsymbol{p}-\boldsymbol{k}} + N^l_{\boldsymbol{p}}N^s_{\boldsymbol{k}} - N^l_{\boldsymbol{p}-\boldsymbol{k}}N^s_{\boldsymbol{k}}\right) \equiv S_{\boldsymbol{k}} + \gamma^s_{\boldsymbol{k}}N^s_{\boldsymbol{k}}. \tag{7.23}$$

Thus, in the initial absence of phonons, there is a source of them ($S_{\boldsymbol{k}} > 0$), due to plasmon decay. In their presence, there is a certain rate at which they might be reabsorbed. The sign $\gamma^s_{\boldsymbol{k}}$ is not obviously fixed, but it is clear that it should be negative if there is to be a steady state. If we assumed $N^l_{\boldsymbol{p}} \ll N^s_{\boldsymbol{k}}$ and $k \ll p$, we would end up with a very similar situation to (7.8): a "Landau damping" of phonons on plasmons (see Exercise 8.13)—or an instability, depending on the sign of $\gamma^s_{\boldsymbol{k}}$, which is determined by the functional form of the plasmon spectrum $N^l_{\boldsymbol{p}}$.

To work out the rate of change of $N^l_{\boldsymbol{p}}$, we need to take into account the diagrams that went into (7.23) plus two more, shown in Fig. 39,—entirely analogously to the derivation of the QL equation for electrons in §7.1.2 (cf. Fig. 35). The difference now is that, in general, the occupation numbers of plasmons are not much smaller than those of the

[67] If sound waves are strongly damped (e.g., at $T_i = T_e$), four-wave interactions (§7.2.2) and/or induced-scattering processes of the kind considered in §7.2.4 have to be invoked to describe Langmuir turbulence: see §8.2.7 and Exercises 8.7 and 8.10.



(a) decay (emission) of a plasmon (phonon) (b) absorption of a phonon

FIGURE 38. Diagrams for (7.23).

(a) decay (emission) of a plasmon (phonon) (b) absorption of a phonon

FIGURE 39. Additional diagrams for (7.24).

phonons, and also their momenta $\boldsymbol{p}$ and $\boldsymbol{k}$ are of the same order, rather than $k \ll p$. The result is

$$\frac{\partial N^l_{\boldsymbol{p}}}{\partial t} = \sum_{\boldsymbol{k}} \Big[\underbrace{\mathfrak{w}(\boldsymbol{p},\boldsymbol{k}) \left(N^l_{\boldsymbol{p}-\boldsymbol{k}} N^s_{\boldsymbol{k}} - N^l_{\boldsymbol{p}-\boldsymbol{k}} N^l_{\boldsymbol{p}} - N^s_{\boldsymbol{k}} N^l_{\boldsymbol{p}} \right)}_{\text{Fig. 38}} + \underbrace{\mathfrak{w}(\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}) \left(N^l_{\boldsymbol{p}+\boldsymbol{k}} N^s_{\boldsymbol{k}} + N^l_{\boldsymbol{p}+\boldsymbol{k}} N^l_{\boldsymbol{p}} - N^s_{\boldsymbol{k}} N^l_{\boldsymbol{p}} \right)}_{\text{Fig. 39}} \Big] \equiv M_{\boldsymbol{p}} + \gamma^l_{\boldsymbol{p}} N^l_{\boldsymbol{p}}. \tag{7.24}$$

In the initial absence of phonons, we find an exponential change, at the rate $\gamma^l_{\boldsymbol{p}}$, in the number of plasmons at a given $\boldsymbol{p}$, due to emission of phonons—the counterpart to the source term in (7.23). In the presence of phonons, this rate is modified (downwards—the sign of this contribution *is* definite), to account for the reabsorption of the phonons, and there is also a mode-coupling term $M_{\boldsymbol{p}}$ (containing the $N^l_{\boldsymbol{p}-\boldsymbol{k}} N^s_{\boldsymbol{k}}$ and $N^l_{\boldsymbol{p}+\boldsymbol{k}} N^s_{\boldsymbol{k}}$ contributions). Again, assuming $N^l_{\boldsymbol{p}} \ll N^s_{\boldsymbol{k}}$ and $k \ll p$ would turn (7.24) into something very similar to the QL diffusion equation (7.10) (see Exercise 8.14), but this is not, generally speaking, the right regime this time (I will return to it in §8.5).

The Langmuir–sound system will be my workhorse example, so, to reduce clutter, let me simplify notation a bit:

$$N^l_{\boldsymbol{p}} = N_{\boldsymbol{p}}, \qquad N^s_{\boldsymbol{k}} = n_{\boldsymbol{k}}, \qquad \omega^l_{\boldsymbol{p}} = \omega_{\boldsymbol{p}}, \qquad \omega^s_{\boldsymbol{k}} = \varOmega_{\boldsymbol{k}}. \tag{7.25}$$

The kinetic equations (7.23) and (7.24) can be written rather compactly as follows

$$\boxed{\frac{\partial n_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p}} T_{\boldsymbol{p},\boldsymbol{k}}, \qquad \frac{\partial N_{\boldsymbol{p}}}{\partial t} = \sum_{\boldsymbol{k}} \left(T_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}} - T_{\boldsymbol{p},\boldsymbol{k}} \right)}\,, \tag{7.26}$$

where $T_{\boldsymbol{p},\boldsymbol{k}}$ is the expression under the wave-number sum in (7.23), viz.,

$$T_{\boldsymbol{p},\boldsymbol{k}} = \delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}-\boldsymbol{k}} - \varOmega_{\boldsymbol{k}})\, w(\boldsymbol{p},\boldsymbol{k}) \left[N_{\boldsymbol{p}} N_{\boldsymbol{p}-\boldsymbol{k}} + (N_{\boldsymbol{p}} - N_{\boldsymbol{p}-\boldsymbol{k}})\, n_{\boldsymbol{k}} \right]. \tag{7.27}$$

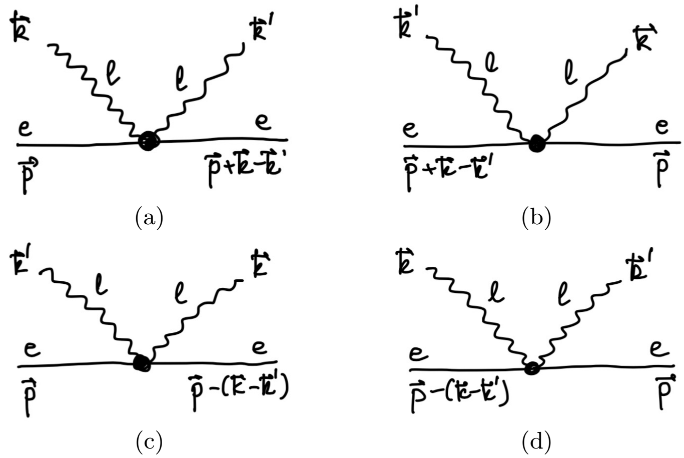


FIGURE 40. Diagrams for (7.31) and (7.33): induced scattering.

I will call $T_{\boldsymbol{p},\boldsymbol{k}}$ the *transfer function* as it quantifies the shuffling of energy (and of plasmon number) between wave numbers.

Exercise 7.4. Confirm by direct calculation the fact (obvious from the diagrams) that the kinetic equations for the Langmuir–sound turbulence (§7.2.3) and for the Langmuir turbulence with four-wave interactions (§7.2.2) conserve the total number of plasmons $N = \sum_{\boldsymbol{p}} N_{\boldsymbol{p}}$, as well as the total energy and the total momentum of the quasiparticle gas.

The quasiparticle formalism expounded above gives "kinetic theory" an extended meaning—kinetics both of "real" particles and the waves. Interestingly, for the interactions that involve exclusively waves, the underlying dynamics are in fact fluid, at least approximately: one does not need to worry about weak kinetic damping effects on Langmuir or sound waves in order to derive the "kinetic equations" (7.23) and (7.24) (I will show this formally in §8). But not all nonlinear effects, and not all WT, are fluid-dynamical in this sense. There are nonlinear interactions that can be described by WT and that involve "true" kinetic effects—i.e., wave-particle resonances. The best known example of these is presented in the next section.

7.2.4. *Wave–Particle Interactions: Induced Scattering*

In §7.1, we saw how to handle the interactions involving one plasmon and one "real" particle (an electron). There is a nonlinear version of these interactions, involving two plasmons: a plasmon with momentum $\hbar\boldsymbol{k}$ meets a particle with momentum $\hbar\boldsymbol{p} = m\boldsymbol{v}$, gets absorbed, particle gets excited, cannot contain itself, immediately emits another plasmon with momentum $\hbar\boldsymbol{k}'$, and speeds away with momentum $\hbar\boldsymbol{p}'$ (Fig. 40a). This is called *induced scattering* (or *stimulated emission*). The conservation laws are [cf. (7.5)][68]

$$\boldsymbol{k} + \boldsymbol{p} = \boldsymbol{k}' + \boldsymbol{p}' \quad \Rightarrow \quad \boldsymbol{p}' = \boldsymbol{p} + \boldsymbol{k} - \boldsymbol{k}', \tag{7.28}$$

$$0 = \varepsilon^l_{\boldsymbol{k}} + \varepsilon^e_{\boldsymbol{p}} - \varepsilon^l_{\boldsymbol{k}'} - \varepsilon^e_{\boldsymbol{p}'} = \hbar\omega_{\boldsymbol{k}} + \frac{\hbar^2 p^2}{2m} - \hbar\omega_{\boldsymbol{k}'} - \frac{\hbar^2|\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{k}'|^2}{2m} \approx \hbar\left[\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'} - (\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}\right]. \tag{7.29}$$

The probability of this process, and of its inverse (Fig. 40b), is

$$\mathfrak{w}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}') = w(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')\,\delta\left(\varepsilon^l_{\boldsymbol{k}} + \varepsilon^e_{\boldsymbol{p}} - \varepsilon^l_{\boldsymbol{k}'} - \varepsilon^e_{\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{k}'}\right). \tag{7.30}$$

---

[68] There is nothing in this consideration that requires the particle to be an electron, it can perfectly well be an ion. Indeed, for Langmuir waves, the latter possibility is a more relevant one, and will be considered in Exercise 8.10.

We are ready to calculate. The diagrams in Fig. 40(a,b) give us

$$\frac{\partial N_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{p},\boldsymbol{k}'} \mathfrak{w}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')\left[-N_{\boldsymbol{k}} n_{\boldsymbol{p}}(N_{\boldsymbol{k}'}+1) + N_{\boldsymbol{k}'} n_{\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{k}'}(N_{\boldsymbol{k}}+1)\right]$$
$$\approx \sum_{\boldsymbol{p},\boldsymbol{k}'} \mathfrak{w}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')(\boldsymbol{k}-\boldsymbol{k}')\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} N_{\boldsymbol{k}'} N_{\boldsymbol{k}} \equiv 2\gamma_{\boldsymbol{k}}^{\mathrm{nl}} N_{\boldsymbol{k}}. \tag{7.31}$$

This is rather similar to the Landau-damping equation (7.8), except with a new rate, which, expressed in terms of $f_0(\boldsymbol{v})$ using (7.4), works out to be

$$2\gamma_{\boldsymbol{k}}^{\mathrm{nl}} = \sum_{\boldsymbol{k}'} N_{\boldsymbol{k}'} \int \mathrm{d}\boldsymbol{v}\, \delta\big(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'} - (\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}\big) \frac{V}{m}\, w\left(\frac{m\boldsymbol{v}}{\hbar}, \boldsymbol{k}, \boldsymbol{k}'\right)(\boldsymbol{k}-\boldsymbol{k}')\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{7.32}$$

This is sometimes called "nonlinear Landau damping"—confusingly, because it has nothing to do with the suppression of Landau damping by particle trapping (O'Neil 1965; Mazitov 1965), usually called that.

Finally, adding up the diagrams in Fig. 40(a,b) and two similar ones in Fig. 40(c,d), we get the equation for the electron distribution function: denoting $\boldsymbol{k}'' = \boldsymbol{k} - \boldsymbol{k}'$,

$$\frac{\partial n_{\boldsymbol{p}}}{\partial t} \approx \sum_{\boldsymbol{k},\boldsymbol{k}'} N_{\boldsymbol{k}} N_{\boldsymbol{k}'}\left[\mathfrak{w}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')(n_{\boldsymbol{p}+\boldsymbol{k}''} - n_{\boldsymbol{p}}) - \mathfrak{w}(\boldsymbol{p}-\boldsymbol{k}'',\boldsymbol{k},\boldsymbol{k}')(n_{\boldsymbol{p}} - n_{\boldsymbol{p}-\boldsymbol{k}''})\right]$$
$$\approx \frac{\partial}{\partial \boldsymbol{p}}\cdot\sum_{\boldsymbol{k},\boldsymbol{k}'} \mathfrak{w}(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}') N_{\boldsymbol{k}} N_{\boldsymbol{k}'} \boldsymbol{k}''\boldsymbol{k}''\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \quad\Leftrightarrow\quad \frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\mathsf{D}^{\mathrm{nl}}(\boldsymbol{v})\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{7.33}$$

This is a diffusion equation, like (7.11), but now with a "nonlinear" diffusion coefficient that depends quadratically on the electric-field spectrum. Note that since this effect is second order in $N_{\boldsymbol{k}}$, it should, formally, be subdominant to the QL diffusion (7.11), which is first order. However, the resonance (7.29) is much easier to achieve because, rather than matching $\omega_{\boldsymbol{k}}$ to $\boldsymbol{k}\cdot\boldsymbol{v}$, which might push you far into the tail of the distribution, here all you need to do is match the frequency difference $\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'}$ for some pair of wave numbers to $(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}$ (so, roughly, you are matching the particle's velocity not to the wave's phase velocity but to its group velocity). This accommodating nature of induced scattering is its claim to ubiquitous relevance.

There is an important observation that one can make without knowing $w(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')$. Since (7.33) is a diffusion equation, it should cause $f_0(\boldsymbol{v})$ to broaden with time, i.e., the particle distribution will heat up (cf. Q-12 and Q-13)—this is called *turbulent heating*. Since energy is conserved, the plasmon population should on average lose energy, shifting to lower frequencies and lower wave numbers—this is corroborated by quantitative calculations (see Tsytovich's books or the original papers by Pikelner & Tsytovich 1969, Liperovskii & Tsytovich 1970, and Breĭzman *et al.* 1973, as well as Musher *et al.* 1995, who review further interesting complications). Therefore, WT will push waves into a region of $\boldsymbol{k}$ space where the WT approximation ceases to be valid and the turbulence becomes strong—I will come back to this point in §7.5.

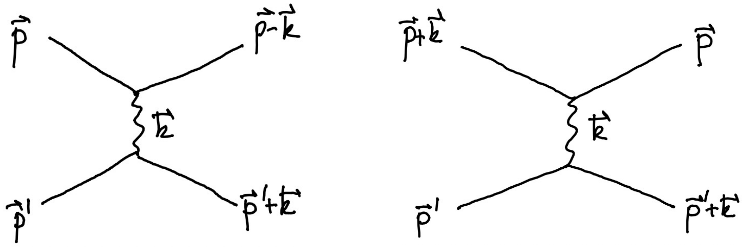
Working out $w(\boldsymbol{p},\boldsymbol{k},\boldsymbol{k}')$ for this problem is a bit of a nightmare—or, depending on your attitude, a character-building endeavour (Exercise 8.10). You will find the answer in Kadomtsev (1965), Kingsep (1996), or, alongside 1001 similar calculations of probabilities of WT processes, in Tsytovich (1995),

7.2.5. *A Digression: "Real" Collisions*

Finally, let me give you another example of the use of the diagram technique, which, while not properly a WT calculation, illustrates the unifying nature of the formalism. If we can talk about interactions between waves and particles, why not use the same language for interactions between particles and thus work out the general form of the collision integral previewed in (1.47).

A binary Coulomb collision can be thought of as two particles meeting, exchanging a plasmon with a certain energy and momentum, and parting ways (Fig. 41a). For simplicity, consider two particles of the same species. The energy-conservation requirement is

$$0 = \varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}'} - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}} - \varepsilon_{\boldsymbol{p}'+\boldsymbol{k}} = \frac{\hbar^2}{2m}\left(p^2 + p'^2 - |\boldsymbol{p}-\boldsymbol{k}|^2 - |\boldsymbol{p}'+\boldsymbol{k}|^2\right) \approx \frac{\hbar^2}{m}\boldsymbol{k}\cdot(\boldsymbol{p}-\boldsymbol{p}'), \tag{7.34}$$



(a) (b)

FIGURE 41. Diagrams for (7.36): binary collisions.

assuming at the last step that $k \ll p, p'$ (glancing collisions). Let the probability of this process be

$$\mathfrak{w}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k}) = w(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k})\,\delta\big(\varepsilon_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p}'} - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}} - \varepsilon_{\boldsymbol{p}+\boldsymbol{k}}\big). \tag{7.35}$$

Adding up the diagrams in Fig. 41, and their inverses, we get

$$\begin{aligned}
\frac{\partial n_{\boldsymbol{p}}}{\partial t} &= \sum_{\boldsymbol{k},\boldsymbol{p}'} \left[\mathfrak{w}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k})\,(-n_{\boldsymbol{p}} n_{\boldsymbol{p}'} + n_{\boldsymbol{p}-\boldsymbol{k}} n_{\boldsymbol{p}'+\boldsymbol{k}}) + \mathfrak{w}(\boldsymbol{p}+\boldsymbol{k}, \boldsymbol{p}', \boldsymbol{k})\,(n_{\boldsymbol{p}+\boldsymbol{k}} n_{\boldsymbol{p}'} - n_{\boldsymbol{p}} n_{\boldsymbol{p}'+\boldsymbol{k}})\right] \\
&\approx \sum_{\boldsymbol{k},\boldsymbol{p}'} \left[\mathfrak{w}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k}) \left(n_{\boldsymbol{p}} \boldsymbol{k}\cdot\frac{\partial n_{\boldsymbol{p}'}}{\partial \boldsymbol{p}'} - n_{\boldsymbol{p}'} \boldsymbol{k}\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}}\right)\right. \\
&\qquad \left. + \mathfrak{w}(\boldsymbol{p}+\boldsymbol{k}, \boldsymbol{p}', \boldsymbol{k}) \left(n_{\boldsymbol{p}'} \boldsymbol{k}\cdot\frac{\partial n_{\boldsymbol{p}+\boldsymbol{k}}}{\partial \boldsymbol{p}} - n_{\boldsymbol{p}} \boldsymbol{k}\cdot\frac{\partial n_{\boldsymbol{p}'}}{\partial \boldsymbol{p}'}\right)\right] \\
&\approx \frac{\partial}{\partial \boldsymbol{p}}\cdot\sum_{\boldsymbol{k},\boldsymbol{p}'} \mathfrak{w}(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k})\boldsymbol{k}\boldsymbol{k}\cdot\left(n_{\boldsymbol{p}'}\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} - n_{\boldsymbol{p}}\frac{\partial n_{\boldsymbol{p}'}}{\partial \boldsymbol{p}'}\right).
\end{aligned} \tag{7.36}$$

Rewriting this in terms of $f_0(\boldsymbol{v})$, we get

$$\begin{aligned}
\frac{\partial f_0(\boldsymbol{v})}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}' \sum_{\boldsymbol{k}} \delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')\big)\frac{V\hbar}{m^2}\, w\left(\frac{m\boldsymbol{v}}{\hbar}, \frac{m\boldsymbol{v}'}{\hbar}, \boldsymbol{k}\right) \\
\boldsymbol{k}\boldsymbol{k}\cdot\left[f_0(\boldsymbol{v}')\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} - f_0(\boldsymbol{v})\frac{\partial f_0(\boldsymbol{v}')}{\partial \boldsymbol{v}'}\right].
\end{aligned} \tag{7.37}$$

This is indeed the general Fokker–Planck form (1.47) of the collision integral. The probability w will be worked out in §§11.5 and 11.6.

Exercise 7.5. Multispecies collisions. Generalise this construction to binary collisions between particles of different species.

You will find a diagrammatic derivation of the collision integral for gravitating systems, analogous to (7.37), in a paper by Hamilton (2021), who claims to have been inspired by these Lectures—from classroom to research frontier in one quick leap, a good example to follow!

7.3. *Statistical Mechanics of Quasiparticles*

7.3.1. *Entropy*

The expressions for the rates of change of quasiparticle occupation numbers are a kind of collision integrals for the quasiparticles (the analogy reinforced by §7.2.5). It is not hard to see that they have an entropy and an H-theorem. The quasiparticles are indistinguishable bosons (no exclusion principle), so their entropy ought to be

$$\boxed{S = -\sum_{\boldsymbol{k}} \left[N_{\boldsymbol{k}} \ln N_{\boldsymbol{k}} - (1+N_{\boldsymbol{k}})\ln(1+N_{\boldsymbol{k}})\right] \approx \sum_{\boldsymbol{k}} \ln N_{\boldsymbol{k}}}. \tag{7.38}$$

The simple approximate expression arises because all occupation numbers are $N_{\boldsymbol{k}} \gg 1$. In other words, the entropy of the quasiparticle gas is just the sum of Boltzmann entropies associated with each $\boldsymbol{k}$.

Let us prove that this entropy cannot decrease: e.g., for the Langmuir–sound system (7.26),

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \sum_{\boldsymbol{k}} \frac{1}{n_{\boldsymbol{k}}}\frac{\partial n_{\boldsymbol{k}}}{\partial t} + \sum_{\boldsymbol{p}} \frac{1}{N_{\boldsymbol{p}}}\frac{\partial N_{\boldsymbol{p}}}{\partial t} = \sum_{\boldsymbol{k},\boldsymbol{p}} \left(\frac{T_{\boldsymbol{p},\boldsymbol{k}}}{n_{\boldsymbol{k}}} + \frac{T_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}}}{N_{\boldsymbol{p}}} - \frac{T_{\boldsymbol{p},\boldsymbol{k}}}{N_{\boldsymbol{p}}}\right)\\ &= \sum_{\boldsymbol{k},\boldsymbol{p}} T_{\boldsymbol{p},\boldsymbol{k}} \left(\frac{1}{n_{\boldsymbol{k}}} + \frac{1}{N_{\boldsymbol{p}-\boldsymbol{k}}} - \frac{1}{N_{\boldsymbol{p}}}\right)\\ &= \sum_{\boldsymbol{k},\boldsymbol{p}} \mathfrak{w}(\boldsymbol{p},\boldsymbol{k}) \frac{\left[N_{\boldsymbol{p}}N_{\boldsymbol{p}-\boldsymbol{k}} + \left(N_{\boldsymbol{p}} - N_{\boldsymbol{p}-\boldsymbol{k}}\right)n_{\boldsymbol{k}}\right]^2}{n_{\boldsymbol{k}}N_{\boldsymbol{p}}N_{\boldsymbol{p}-\boldsymbol{k}}} \geqslant 0, \qquad \text{q.e.d.}\end{aligned} \tag{7.39}$$

Exercise 7.6. Prove that the H-theorem holds for the four-wave kinetic equation (7.20).

You might be wondering—legitimately—how it happened that, treating a collisionless and, therefore, reversible, system, we have ended up with an H-theorem and, therefore, irreversibility. Obviously, an implicit assumption has been made that must be the quasiparticle equivalent of the hypothesis of molecular chaos—in WT, this is the so called "random-phase approximation", which I will introduce in §8.4.2, once I have a specific example of underlying dynamical equations to work with.

7.3.2. *Thermodynamical Equilibrium Distributions*

Where there is entropy increase, there is thermodynamical equilibrium. The standard operating procedure is to maximise entropy subject to holding constant whatever invariants the system has. The Langmuir–sound system has two: the total energy $\mathscr{E}$ and the total number of plasmons N (see Exercise 7.4).[69] Therefore, we must solve the maximisation problem

$$S - \beta\left(\sum_{\boldsymbol{k}} \hbar\Omega_{\boldsymbol{k}} n_{\boldsymbol{k}} + \sum_{\boldsymbol{p}} \hbar\omega_{\boldsymbol{p}} N_{\boldsymbol{p}} - \mathscr{E}\right) + \beta\mu\left(\sum_{\boldsymbol{p}} N_{\boldsymbol{p}} - N\right) \to \max, \tag{7.40}$$

where $\beta = 1/T$ and μ are Lagrange multipliers (temperature and chemical potential). Varying the above expression with respect to $n_{\boldsymbol{k}}$ and $N_{\boldsymbol{p}}$, we get the equilibrium distributions of the plasmon and phonon fields:

$$\boxed{n_{\boldsymbol{k}} = \frac{T}{\hbar\Omega_{\boldsymbol{k}}}, \qquad N_{\boldsymbol{p}} = \frac{T}{\hbar\omega_{\boldsymbol{p}} - \mu}}\,. \tag{7.41}$$

The distributions (7.41) are *Rayleigh–Jeans distributions*, familiar from the statistical mechanics of radiation.

Exercise 7.7. Verify by direct calculation that the Rayleigh–Jeans distributions (7.41) are indeed stationary solutions of the kinetic equations (7.26).

Exercise 7.8. An attempt to integrate these distributions to find T and μ in terms of $\mathscr{E}$ and N produces divergent results. Is this a problem? How can it be fixed?

---

[69] As usual in statistical mechanics, we do not need to worry about the total momentum because we can always trasfer ourselves into the reference frame moving with the total momentum of our system.

7.4. *Stationary Nonequilibrium Distributions*

I have kept using the word "turbulence", but I actually I have not done any turbulence so far—just derived generic kinetic equations for weakly interacting quasiparticles (§7.2) and shown that they support some fairly bland thermal equilibria (§7.3). The interesting applications are indeed to turbulence—and that means to systems that are *not* in thermal equilibrium.

Since our kinetic equations conserve energy, that means that their right-hand sides cannot either generate or dissipate any net energy—they can only move energy around between different wave numbers or exchange it between different species of waves. Not worrying about the latter for a moment, assuming wave-number isotropy, and defining the k-shell-integrated total-energy spectrum for the Langmuir–sound system as[70]

$$\bar{E}_k = \frac{Vk^2}{2\pi^2}\left(\hbar\Omega_{\boldsymbol{k}} n_{\boldsymbol{k}} + \hbar\omega_{\boldsymbol{k}} N_{\boldsymbol{k}}\right), \qquad \int \mathrm{d}k\, \bar{E}_k = \mathscr{E}, \tag{7.42}$$

it must be possible to write the evolution equation for it in terms of an energy flux ε_k:

$$\frac{\partial \bar{E}_k}{\partial t} = \frac{Vk^2}{2\pi^2}\sum_{\boldsymbol{p}}\left[\hbar\Omega_{\boldsymbol{k}} T_{\boldsymbol{p},\boldsymbol{k}} + \hbar\omega_{\boldsymbol{k}}\left(T_{\boldsymbol{k}+\boldsymbol{p},\boldsymbol{p}} - T_{\boldsymbol{k},\boldsymbol{p}}\right)\right] \equiv -\frac{\partial \varepsilon_k}{\partial k}. \tag{7.43}$$

Since the plasmon number $N = \sum_{\boldsymbol{p}} N_{\boldsymbol{p}} \equiv \int \mathrm{d}p\, \bar{N}_p$ is also conserved, $\bar{N}_p = (Vp^2/2\pi^2)N_{\boldsymbol{p}}$ satisfies

$$\frac{\partial \bar{N}_p}{\partial t} = \frac{Vp^2}{2\pi^2}\sum_{\boldsymbol{k}}\left(T_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}} - T_{\boldsymbol{p},\boldsymbol{k}}\right) \equiv -\frac{\partial \varGamma_p}{\partial p}, \tag{7.44}$$

where $\varGamma_p$ is the plasmon-number flux (also known as the "wave-action flux"). Stationary solutions can be of two flavours: those with zero flux and those with a constant flux. The Rayleigh–Jeans solutions (7.41) are of the former kind: nothing flows anywhere because there are no sources or sinks of either energy or particles. If there were, there would not be an equilibrium, and a (statistically) stationary state would have to feature the invariants constantly flowing from sources to sinks. This is exactly what a turbulent system is: a chaotic system in which energy and other invariants, if they are there, are constantly injected at some scales (i.e., some k's) and flow to other scales where they can be dissipated. The sources and sinks are not in the kinetic equations that we have derived—they must be added as they represent some additional physics, not covered by nonlinear interactions, which are conservative. Injection can be (and usually is in plasma physics) done by some instability; dissipation is ultimately always by collisions, but, in "collisionless" (i.e., weakly collisional) systems, those are usually accessed via phase mixing (Landau damping)—this was discussed quite thoroughly in §5, although to what extent that discussion applies to turbulent systems is a tricky question (see §12). If the injection and dissipation are concentrated at sufficiently disparate wave numbers, there will be scale-invariant k-space intervals in between that energy and other invariants must flow through in order to get dissipated—it is in these intervals that constant-flux solutions will materialise. Working out these solutions, by setting

$$\varepsilon_k = \varepsilon = \text{const}, \qquad \varGamma_p = \varGamma = \text{const}, \tag{7.45}$$

and the directions of the corresponding fluxes (the signs of ε and $\varGamma$) is what we shall need the kinetic equations for (see §§8.4.3–8.4.6).

[70]The prefactor comes from $\sum_{\boldsymbol{k}} = V\int \mathrm{d}\boldsymbol{k}/(2\pi)^3 = V\int \mathrm{d}k\, 4\pi k^2/(2\pi)^3 = (V/2\pi^2)\int \mathrm{d}k\, k^2$.

There are three further nuances. First, valid constant-flux solutions will only be obtainable from kinetic equations without the need to include sources and sinks if the interactions are *local* in k, i.e., if the sources and the sinks cannot couple to each other directly (otherwise energy might just "leap" directly from sources to sinks). There will be a scheme for checking whether that is true (see the end of §8.4.6). Secondly, stationary solutions of the kinetic equations are only of any use if they are *stable*. That too must be, and can be, checked within the WT formalism (I shall not deal with this; see Zakharov *et al.* 1992). Thirdly, the $\boldsymbol{k}$-space *isotropy* assumption is not always good, and there is an interesting zoo of non-isotropic possibilities (I will ignore them all; see, e.g., Musher *et al.* 1995).

Let us keep all this in mind, but this is about as far as one can get without actually calculating $w(\boldsymbol{p}, \boldsymbol{k})$. In §8.4, I shall calculate it for the Langmuir-sound turbulence (§7.2.3), after I have derived the dynamical equations for it. Once this is done, I will show you how one can solve the WT kinetic equations and obtain steady-state, constant-flux, nonequilibrium distributions and spectra.

7.5. *Validity of WT Approximation*

Since I have not yet fully revealed what the WT approximation consists of, I cannot just yet give you a full account of the conditions for its validity. These will emerge piecemeal here and in §§8.1 and 8.4, as some specific meat is put on the generic bones that have emerged so far.

What should be clear at this point is that we have confined ourselves to a situation where we must be able to think of our plasma as a gas of monocromatic waves with definite wave numbers (momenta) and frequencies (energies), which interact subject to conservation rules on both, e.g., (7.13) or (7.18). Matching wave numbers is a straightforward requirement that mathematically is traceable to the simple fact that the nonlinear terms in the underlying dynamical equations [e.g., (2.12)], when Fourier transformed, turn into convolution sums over wave numbers. In contrast, the matching of frequencies is a more nontrivial proposition. At the very least, in order to be meaningful, it requires that the wave frequencies be identifiable, i.e., that the waves survive (much) longer than their own period of oscillation. In other words, the typical nonlinear interaction time is long compared to the wave period:

$$\boxed{\omega_{\boldsymbol{k}} t_{\rm nl} \gg 1}\,, \tag{7.46}$$

as promised in §2.4.3. It is perhaps intuitive then that the delta functions constraining the WT interaction probabilities, such as $\delta\big(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}-\boldsymbol{k}} - \Omega_{\boldsymbol{k}}\big)$ in (7.27), in fact have an effective width of order $\sim t_{\rm nl}^{-1}$, which is assumed small compared to any of the frequencies.

How well this assumption is satisfied depends on the amplitudes of the perturbations (i.e., on the overall fluctuation level), which must remain sufficiently small. This has to be checked for each particular system—see, e.g., §8.4.7,[71] where I will finally bid farewell to WT and move on to stronger stuff.

It happens quite often—indeed, more often than not—that a WT system drives itself towards such (low) wave numbers, frequencies, and/or (high) amplitudes that the WT applicability condition (7.46) gets broken, i.e., the frequency-matching delta functions are broadened to the point that they no longer constrain interactions (there was a glimpse of this at the end of §7.2.4). This turns the system from a weakly coupled to a strongly

[71] Another key assumption that enables one to write the kinetic equations in the form I did above is the random-phase approximation, already alluded to at the end of §7.3.1 to excuse entropy nonconservation and introduced properly in §8.4.2.

coupled one, from weak turbulence to strong turbulence. In such a regime, it is no longer true that monochromatic plasma waves are the natural elementary building blocks of the nonlinear system, and some other entities, each formally composed of many Fourier modes, emerge to claim pre-eminence. What they are is not always (indeed, not usually) clear. I will attempt to illustrate this point in a specific way in §§8.5 and 8.6.

8. Langmuir Turbulence

This section is dedicated to one particular kind of plasma turbulence—probably the best-, or at any rate the earliest-, studied kind: turbulence of coupled Langmuir and sound waves. This is not the most relevant situation for most fusion (except perhaps inertial-fusion) or astrophysical plasmas, but it has the advantage of involving the two most basic plasma waves that we have already studied above, at length, and it also gives me an opportunity to illustrate some concepts and methods that find their application in many other plasma problems: how separation of scales enables one to derive reduced systems of equations (§§8.1 and 8.2),[72] how they turn out to have Hamiltonian structure (§8.3), how one can use them to calculate probabilities of basic WT processes and thus complete, and solve, the kinetic equations derived in §7.2 (§8.4), how the WT approximation breaks down and waves (quasiparticles) cease to be the basic actors of plasma dynamics, ceding ground to essentially nonlinear structures and how then a strongly turbulent regime might be tackled, with great difficulty but also great ingenuity (§§8.5 and 8.6).

8.1. *Zakharov's Equations*

So let me pick up where I left off in §7.2.3 and ask what dynamical equations are satisfied by the plasmons and the phonons—so I can use these equations to calculate $w(\boldsymbol{p},\boldsymbol{k})$ (and, it will turn out, do much more besides). First, I would like to derive these equations in a physically intuitive, if non-rigorous, way—I will clean up my act in §8.2.

8.1.1. *Langmuir Waves*

In Exercise 3.1, you had an opportunity to derive the linearised fluid equations for Langmuir waves—they are just the density and velocity moments of the electron kinetic equation, ignoring ions entirely:

$$\frac{\partial \delta\tilde{n}_e}{\partial t} + \boldsymbol{\nabla}\cdot(\bar{n}_e\tilde{\boldsymbol{u}}_e) = 0, \tag{8.1}$$

$$m_e\bar{n}_e\frac{\partial \tilde{\boldsymbol{u}}_e}{\partial t} = -e\bar{n}_e\widetilde{\boldsymbol{E}} - \boldsymbol{\nabla}\delta\tilde{p}_e. \tag{8.2}$$

The overtildes mark the quantities that will turn out to oscillate as a Langmuir wave. The overbars designate the fields that are much slower than that—in the above, $\bar{n}_e$ so far just means the equilibrium density on top of which the perturbations occur. The last term in (8.2) contains the electron pressure perturbation, which, in Exercise 3.1, you found to be

$$\delta\tilde{p}_e = 3T_e\delta\tilde{n}_e = \frac{3}{2}\,m_e v_{\text{th}e}^2\delta\tilde{n}_e \tag{8.3}$$

at long wavelengths, i.e., if $kv_{\text{th}e} \ll \omega_{\text{pe}}$ (and so kinetic effects could be ignored). The electric field is $\widetilde{\boldsymbol{E}} = -\boldsymbol{\nabla}\tilde{\varphi}$, and the Poisson equation tells us that $4\pi e\delta\tilde{n}_e = \nabla^2\tilde{\varphi}$. Taking the time derivative of (8.1) and using (8.2) and (8.3), we thus arrive at a closed equation

[72] In a different setting, this theme will be taken up again in §§16.2 and 16.3.

for $\tilde{\varphi}$:

$$\frac{\partial^2}{\partial t^2}\nabla^2\tilde{\varphi} + \boldsymbol{\nabla}\cdot\left(\frac{4\pi e^2\bar{n}_e}{m_e}\boldsymbol{\nabla}\tilde{\varphi}\right) - \frac{3}{2}v_{\mathrm{th}e}^2\nabla^4\tilde{\varphi} = 0. \tag{8.4}$$

In the linear problem, the dimensional factor in the second term is just ω_{pe}^2, and then from (8.4) we can promptly read off the dispersion relation (3.39) for a Langmuir wave:

$$\omega^2 = \omega_{\mathrm{pe}}^2 + \frac{3}{2}\,k^2 v_{\mathrm{th}e}^2. \tag{8.5}$$

Let me now posit that in a nonlinear system where the dominant interactions are between Langmuir and sound waves, the nature of these interactions must be "modulational": sound waves are much slower than Langmuir waves and so the latter will "see" the former simply as slow density modulation on top of the equilibrium. Mathematically this is captured if we let

$$\bar{n}_e = n_{0e} + \delta\bar{n}_e, \tag{8.6}$$

where n_{0e} is the homogeneous equilibrium density and $\delta\bar{n}_e$ the density perturbation associated with sound. This gives us just the right kind of nonlinearity: (8.4) becomes

$$\boxed{\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathrm{pe}}^2 - \frac{3}{2}v_{\mathrm{th}e}^2\nabla^2\right)\nabla^2\tilde{\varphi} = -\omega_{\mathrm{pe}}^2\boldsymbol{\nabla}\cdot\left(\frac{\delta\bar{n}_e}{n_{0e}}\boldsymbol{\nabla}\tilde{\varphi}\right)}\,. \tag{8.7}$$

In order to close the system, we need an equation for $\delta\bar{n}_e$, the electron-density perturbation associated with sound waves.

8.1.2. *Sound Waves*

Let us work in the limit of cold ions, $T_i \ll T_e$, when sound waves can thrive without being bothered by Landau damping (§3.8). In this limit, they too can be described by simple linearised fluid equations, derived in Exercise 3.6:

$$\frac{\partial\delta n_i}{\partial t} + n_{0i}\boldsymbol{\nabla}\cdot\boldsymbol{u}_i = 0, \tag{8.8}$$

$$m_i n_{0i}\frac{\partial\boldsymbol{u}_i}{\partial t} = -\boldsymbol{\nabla}\delta\bar{p}_e. \tag{8.9}$$

Since ions are cold, only the electron pressure matters. We do not need to have bars over the ion quantities because they do not have Langmuir-frequency variation, but the electron pressure here is the slow, averaged part that the ions can "see". If you did Exercise 3.6, you know that electrons are isothermal at this slow time scale, viz., $\delta\bar{p}_e = T_e\delta\bar{n}_e$. Also, the Poisson equation for these perturbations just turns into a statement of quasineutrality: $Z\delta n_i = \delta\bar{n}_e$ (there cannot be any deviations from that on time scales longer than the inverse plasma frequency; see §2.1); the same relationship is also of course true for the equilibrium densities: $Zn_{0i} = n_{0e}$. The equation for sound waves follows immediately:

$$\frac{\partial^2\delta\bar{n}_e}{\partial t^2} = c_{\mathrm{s}}^2\nabla^2\delta\bar{n}_e. \tag{8.10}$$

Now we need to find a way to modify this equation to account for the emission (or absorption) of sound waves by Langmuir waves. The answer is intuitive: the average effect of a fast-oscillating electric field on particles amounts to an additional effective pressure equal to the energy density of the field—this is known as the *ponderomotive force*. Here is a quick derivation. An electron's equation of motion is

$$m_e\ddot{\boldsymbol{r}} = -e\widetilde{\boldsymbol{E}}(\boldsymbol{r}). \tag{8.11}$$

FIGURE 42. Vladimir Zakharov (1939-2023), Soviet/Russian mathematician and poet. Photo taken from Falkovich *et al.* (2023).

Now split this motion into a fast-oscillating part and a slowly drifting part, $\boldsymbol{r} = \bar{\boldsymbol{r}} + \tilde{\boldsymbol{r}}$, and assume $|\tilde{\boldsymbol{r}}| \ll |\bar{\boldsymbol{r}}|$. The fast-oscillating part satisfies

$$m_e \ddot{\tilde{\boldsymbol{r}}} \approx -e\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}}) \quad \Rightarrow \quad \tilde{\boldsymbol{r}} \approx \frac{e}{m_e\omega_{\rm pe}^2}\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}}), \tag{8.12}$$

assuming that $\widetilde{\boldsymbol{E}}$ oscillates at the single frequency $\omega_{\rm pe}$. Time-averaging (8.11) over the fast oscillations and expanding $\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}} + \tilde{\boldsymbol{r}})$ in small $\tilde{\boldsymbol{r}}$, we get

$$m_e \ddot{\bar{\boldsymbol{r}}} \approx -e\overline{\left[\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}}) + \tilde{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{E}}(\bar{\boldsymbol{r}})\right]} = -\frac{e^2}{m_e\omega_{\rm pe}^2}\overline{\widetilde{\boldsymbol{E}}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{E}}} = -\frac{1}{n_{0e}}\boldsymbol{\nabla}\frac{\overline{|\boldsymbol{\nabla}\tilde{\varphi}|^2}}{8\pi}. \tag{8.13}$$

Adding this effective electron pressure $\overline{|\boldsymbol{\nabla}\tilde{\varphi}|^2}/8\pi$ (or, equivalently, an effective electric potential) to (8.9) and retracing the route that led to (8.10), we get the desired equation featuring a nonlinear coupling of sound waves to Langmuir waves:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c_{\rm s}^2\nabla^2\right)\delta\bar{n}_e = c_{\rm s}^2\boldsymbol{\nabla}\frac{\overline{|\boldsymbol{\nabla}\tilde{\varphi}|^2}}{8\pi T_e}}\,. \tag{8.14}$$

Together with (8.7), this forms a closed system of equations.

Exercise 8.1. Is there a ponderomotive contribution to the effective ion pressure and need it be taken into account?

8.1.3. *Final Form of Zakharov's Equations*

We are one step away from the final form of the Zakharov (1972) equations. A further simplification is achieved if, in (8.16), we take out the $\omega_{\rm pe}$ oscillations:

$$\tilde{\varphi} = \frac{1}{2}\left(\psi e^{-i\omega_{\rm pe}t} + \psi^* e^{i\omega_{\rm pe}t}\right), \tag{8.15}$$

where ψ varies on a time scale much longer than $\omega_{\rm pe}^{-1}$. Substituting (8.15) into (8.7) and neglecting $\partial_t^2\psi \ll i\omega_{\rm pe}\partial_t\psi$, dividing through by $-\omega_{\rm pe}e^{-i\omega_{\rm pe}t}$, and averaging out oscillatory terms, we get Zakharov's first equation:

$$\boxed{\nabla^2\left(i\omega_{\rm pe}^{-1}\frac{\partial\psi}{\partial t} + \frac{3}{2}\lambda_{\rm De}^2\nabla^2\psi\right) = \frac{1}{2}\boldsymbol{\nabla}\cdot\left(\frac{\delta\bar{n}_e}{n_{0e}}\boldsymbol{\nabla}\psi\right)}\,. \tag{8.16}$$

Zakharov's second equation is the same as (8.14), but with the substitution of (8.15):

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\delta\bar{n}_e = c_s^2\nabla^2\frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi T_e}}. \tag{8.17}$$

We have our closed system—and even a cursory glance at the structure of these equations confirms that they should capture just the kind of interactions anticipated in §7.2.3: the right-hand side of (8.17) is quadratic in ψ and so describes two plasmons coupling to generate a phonon (or, rather, a plasmon emitting/absorbing a phonon and becoming a different plasmon); the right-hand side of (8.16) contains a product of $\delta\bar{n}_e$ and ψ and so describes the modification of a plasmon by the emission or absoption of a phonon.

Exercise 8.2. Secondary instabilities of a Langmuir wave. That plasmons emit phonons—i.e., that decay instabilities exist—can be demonstrated directly from the dynamical equations (8.16) and (8.17). Consider a monochromatic Langmuir wave as the lowest-order solution of Zakharov's equations in the absence of nonlinearity, then perturb around it, linearise, and work out the *secondary instabilities* of the perturbations: what are the growth rates, the peak-growth wave numbers, various parameter regimes, etc.? The best treatment of this topic in the literature that I am aware of is the review by Thornhill & ter Haar (1978, §3) (although these results already appear, presented with extreme pith, in Zakharov 1972). This is not really kinetic theory, more fluid dynamics, but it is a challenging yet doable calculation that one can sink one's teeth into and enjoy. Note that some of the results of this calculation directly anticipate §8.5.2.

8.1.4. *Quasistatic Limit*

The linear frequency of the variation of ψ in (8.16) is

$$\delta\omega_{\boldsymbol{k}} = \frac{3}{2}\,k^2\lambda_{\mathrm{De}}^2\omega_{\mathrm{pe}} = \frac{3}{4}\frac{k^2v_{\mathrm{the}}^2}{\omega_{\mathrm{pe}}}. \tag{8.18}$$

This frequency will still be greater than the sound frequency, enabling the ponderomotive source term in the right-hand side of (8.17) to excite sound waves, provided

$$\frac{\delta\omega_{\boldsymbol{k}}}{kc_s} \sim k\lambda_{\mathrm{De}}\frac{v_{\mathrm{the}}}{c_s} \gg 1 \quad\Leftrightarrow\quad k\lambda_{\mathrm{De}} \gg \sqrt{\frac{m_e}{m_i}}. \tag{8.19}$$

In the opposite limit,

$$\boxed{k\lambda_{\mathrm{De}} \ll \sqrt{\frac{m_e}{m_i}}}, \tag{8.20}$$

the source term is quasistatic, no sound waves can be excited and (8.17) simply enforces a balance between the electron pressure and the ponderomotive force:[73]

$$\delta\bar{n}_e = -\frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi T_e}. \tag{8.21}$$

Putting this into (8.16) gets us a closed equation for ψ:[74]

$$\boxed{\nabla^2\left(i\omega_{\mathrm{pe}}^{-1}\frac{\partial\psi}{\partial t} + \frac{3}{2}\lambda_{\mathrm{De}}^2\nabla^2\psi\right) = -\frac{\boldsymbol{\nabla}\cdot\left(|\boldsymbol{\nabla}\psi|^2\boldsymbol{\nabla}\psi\right)}{32\pi n_{0e}T_e}}. \tag{8.22}$$

The nonlinearity is cubic, so this is actually an example of a situation where four-wave interactions will dominate nonlinear dynamics (see §7.2.2).

---

[73] One might call this the incompressible limit of Zakharov's equations.

[74] A mathematically identical equation, but with T_e replaced by $T_e + T_i/Z$ in the right-hand side, arises in the limit of hot ions—hot enough to stream quickly and thus cause the ion density to have a simple Boltzmann response. This approximation is derived in §8.2.7.

Let us check that the nonlinear term in (8.22) does not spoil the assumption that ψ changes slowly compared to sound. The characteristic nonlinear time is

$$t_{\rm nl}^{-1} \sim \omega_{\rm pe} W, \quad W = \frac{\overline{|\widetilde{\boldsymbol{E}}|^2}}{8\pi n_e T_e}, \tag{8.23}$$

where W is the nondimensionalised mean square amplitude of the Langmuir oscillations. Then

$$t_{\rm nl} k c_{\rm s} \gg 1 \quad \Leftrightarrow \quad \boxed{W \ll k\lambda_{\rm De}\sqrt{\frac{m_e}{m_i}}}. \tag{8.24}$$

If we wanted to break this restriction, i.e., if ψ were to be allowed to saturate at higher amplitudes than this, we would have to go back to (8.17).

If you are completely happy with the derivation in §§8.1.1 and 8.1.2, skip the next section, but if you feel that the approximations and assumptions hoisted upon you are difficult to accept at this level of (non-)rigour, read on: in §8.2, I provide a formal perturbative derivation of the Zakharov (1972) equations (8.16) and (8.17), which is surprisingly difficult to find in the literature.

8.2. *Formal Derivation of Zakharov's Equations*

8.2.1. *Scale Separations*

The problem has four characteristic time scales: the plasma oscillation frequency, the electron streaming rate, the ion sound frequency and the ion streaming rate:

$$\omega_{\rm pe} \gg k v_{\rm the} \gg k c_{\rm s} \sim k v_{{\rm th}i}. \tag{8.25}$$

The relative size of these frequencies is controlled by the following three independent parameters:

$$\frac{k v_{\rm the}}{\omega_{\rm pe}} \sim k\lambda_{\rm De} \ll 1, \quad \frac{k c_{\rm s}}{k v_{\rm the}} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \quad \frac{k v_{{\rm th}i}}{k c_{\rm s}} \sim \sqrt{\frac{T_i}{T_e}} \sim 1. \tag{8.26}$$

The scale separation between ions and electrons is non-negotiable as the mass ratio is always small. As long as $k\lambda_{\rm De} \ll 1$, which we will assume here, the electron Landau damping is exponentially small and the electrons will be fluid (as we will see shortly; it is no surprise, given what we know from §3.5). Ions too behave as a fluid if they are cold ($T_i \ll T_e$; cf. §3.8), which is the limit most often considered in the context of Zakharov's equations, if not necessarily one that is most relevant physically.

8.2.2. *Electron Kinetics and Ordering*

As anticipated in §8.1, let us split the electron distribution function and the electrostatic potential into two parts: the time-averaged ("slow", denoted by overbars) and fluctuating ("fast", denoted by overtildes):

$$f_e = \bar{f}_e + \tilde{f}_e, \quad \varphi = \bar{\varphi} + \tilde{\varphi}. \tag{8.27}$$

The time average is taken over time scales longer than both $\omega_{\rm pe}^{-1}$ and $(k v_{\rm the})^{-1}$ but shorter than $(k c_{\rm s})^{-1}$ or $(k v_{{\rm th}i})^{-1}$, i.e., $\bar{f}_e$ and $\bar{\varphi}$ are the electron distribution and potential that the ions can "see". The slow part of the electron distribution is assumed to consist of a homogeneous Maxwellian equilibrium (5.6) plus a perturbation:

$$\bar{f}_e = f_{0e} + \delta\bar{f}_e. \tag{8.28}$$

The slow and fast distribution functions satisfy the following equations, which are obtained by time averaging the Vlasov equation (1.50) for electrons ($\alpha = e$, $q_\alpha = -e$) and subtracting the average from the exact equation:

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}\delta\bar{f}_e+\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}+\frac{e}{m_e}\overline{(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial\tilde{f}_e}{\partial\boldsymbol{v}}}=0,\tag{8.29}$$

$$\frac{\partial\tilde{f}_e}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}\tilde{f}_e+\frac{e}{m_e}(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}+\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial\tilde{f}_e}{\partial\boldsymbol{v}}+\frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial\tilde{f}_e}{\partial\boldsymbol{v}}}=0,\tag{8.30}$$

where all time evolution on ion scales is neglected because $kv_{\rm the}$ is large. The slow and fast parts of the Poisson equation (1.51) are

$$-\nabla^2\bar{\varphi}=4\pi e(Z\delta n_i-\delta\bar{n}_e)=4\pi e\left(Z\int\mathrm{d}\boldsymbol{v}\,\delta f_i-\int\mathrm{d}\boldsymbol{v}\,\delta\bar{f}_e\right),\tag{8.31}$$

$$-\nabla^2\tilde{\varphi}=-4\pi e\tilde{n}_e=-4\pi e\int\mathrm{d}\boldsymbol{v}\,\tilde{f}_e,\tag{8.32}$$

where δf_i is the perturbed ion distribution function and δn_i its density. We shall solve (8.30) and (8.32) for $\tilde{\varphi}$ and $\tilde{f}_e$, use that to calculate the last term in (8.29), which will give rise to the ponderomotive force, then solve (8.29) for $\bar{f}_e$ in terms of $\bar{\varphi}$, and finally use that solution in (8.31) to get an expression for $\bar{\varphi}$ in terms of f_i. The latter can then be coupled with the ion Vlasov–Landau equation (5.1) ($\alpha=i$, $q_\alpha=Ze$), giving rise to a closed "hybrid" system for kinetic ions and "fluid" electrons.

In order to implement this plan, we shall carry out a perturbation expansion of the above equations in the small parameter

$$\epsilon=k\lambda_{\rm De}.\tag{8.33}$$

The algebra becomes more compact if we first make the following ansatz, designed to remove the third (the largest, as we will see) term in (8.30):[75]

$$\tilde{f}_e=-\boldsymbol{u}\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}+h,\tag{8.34}$$

where $\boldsymbol{u}$ is, by definition, the velocity associated with the plasma oscillation [cf. (5.48)]:

$$\frac{\partial\boldsymbol{u}}{\partial t}=\frac{e}{m_e}\boldsymbol{\nabla}\tilde{\varphi}.\tag{8.35}$$

Then the fast electron kinetic equation (8.30) becomes

$$\frac{\partial h}{\partial t}=\underbrace{\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\boldsymbol{u}\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}\right)}_{\epsilon^2}-\underbrace{\boldsymbol{v}\cdot\boldsymbol{\nabla}h}_{\epsilon^3}+\underbrace{\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial}{\partial\boldsymbol{v}}\boldsymbol{u}\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}}_{\epsilon^4}-\underbrace{\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial h}{\partial\boldsymbol{v}}}_{\epsilon^5}$$
$$+\underbrace{\frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial}{\partial\boldsymbol{v}}}\boldsymbol{u}\cdot\frac{\partial\bar{f}_e}{\partial\boldsymbol{v}}}_{\epsilon^2}-\underbrace{\frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial h}{\partial\boldsymbol{v}}}}_{\epsilon^3},\tag{8.36}$$

where the ordering of each term in the small parameter (8.33) has been determined, as indicated above, based on the following assumptions. The plasma-oscillation velocity (8.35) is

$$\frac{u}{v_{\rm the}}\sim\frac{ke\tilde{\varphi}}{m_e v_{\rm the}\omega_{\rm pe}}\sim k\lambda_{\rm De}\frac{e\tilde{\varphi}}{T_e}\sim\epsilon,\tag{8.37}$$

[75] This is equivalent to splitting the electron distribution function into fast and slow parts using as the velocity variable of $\bar{f}_e$ the peculiar velocity of the particle around a centre oscillating with velocity $\boldsymbol{u}$ (cf. DuBois *et al.* 1995): namely, set $f_e=\bar{f}_e(\boldsymbol{r},\boldsymbol{v}-\boldsymbol{u}(t,\boldsymbol{r}))+h(t,\boldsymbol{r},\boldsymbol{v})$ and expand in small $\boldsymbol{u}$.

if, in general,[76]

$$\frac{e\tilde{\varphi}}{T_e} \sim 1. \tag{8.38}$$

Anticipating that the ponderomotive "potential" [see (8.13)] will enter on equal footing with the slow potential and that the slow perturbed electron distribution will express the Boltzmann response to the latter modified by the former, let us adopt the ordering

$$\frac{\delta\bar{f}_e}{f_{0e}} \sim \frac{e\bar{\varphi}}{T_e} \sim \frac{e^2\overline{|\widetilde{\boldsymbol{E}}|^2}}{m_e\omega_{\rm pe}^2 T_e} \sim (k\lambda_{\rm De})^2\left(\frac{e\tilde{\varphi}}{T_e}\right)^2 \sim \epsilon^2. \tag{8.39}$$

Since the inhomogeneous terms in (8.36) are, thus, $O(\epsilon^2)$, it follows that $h \sim \epsilon^2 f_{0e}$ and, since the first term in (8.34) has no density moment, $\tilde{n}_e \sim \epsilon^2 n_{0e}$.

From (8.36), to lowest order,

$$\begin{aligned}\frac{\partial h^{(2)}}{\partial t} &= \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{u}\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}} + \frac{\partial \boldsymbol{u}}{\partial t}\cdot\widetilde{\frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{u}}\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}}\\ &= -\frac{2}{v_{\rm the}^2}\left[v_iv_j\partial_i u_j + \left(\delta_{ij} - \frac{2v_iv_j}{v_{\rm the}^2}\right)\frac{\partial u_i}{\partial t}u_j\right]f_{0e},\end{aligned} \tag{8.40}$$

where I have used (8.35) and the fact that f_{0e} is a Maxwellian.

8.2.3. *Ponderomotive Response*

With (8.40) in hand, we are now in a position to calculate the last term in (8.29). First, using (8.34) and (8.35) and keeping terms of order ϵ^2 and ϵ^3,

$$\begin{aligned}\frac{e}{m_e}\overline{(\boldsymbol{\nabla}\tilde{\varphi})\cdot\frac{\partial\tilde{f}_e}{\partial \boldsymbol{v}}} &= \overline{\frac{\partial \boldsymbol{u}}{\partial t}\cdot\frac{\partial}{\partial \boldsymbol{v}}\left(-\boldsymbol{u}\cdot\frac{\partial f_{0e}}{\partial \boldsymbol{v}} + h^{(2)}\right)}\\ &= \frac{2}{v_{\rm the}^2}\overline{\left(\delta_{ij} - \frac{2v_iv_j}{v_{\rm the}^2}\right)\frac{\partial u_i}{\partial t}u_j} f_{0e} + \overline{\frac{\partial \boldsymbol{u}}{\partial t}\cdot\frac{\partial h^{(2)}}{\partial \boldsymbol{v}}}\\ &= \overline{\frac{\partial}{\partial t}\left\{\frac{2}{v_{\rm the}^2}\left[\frac{u^2}{2} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{v_{\rm the}^2}\right]f_{0e} + \boldsymbol{u}\cdot\frac{\partial h^{(2)}}{\partial \boldsymbol{v}}\right\}} - \overline{\boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial h^{(2)}}{\partial t}}.\end{aligned} \tag{8.41}$$

The first term is a full time derivative and so vanishes under averaging, whereas the second term can be calculated using (8.40):

$$\begin{aligned}-\overline{\boldsymbol{u}\cdot\frac{\partial}{\partial \boldsymbol{v}}\frac{\partial h^{(2)}}{\partial t}} &= \frac{2}{v_{\rm the}^2}\left[\overline{v_ju_l\partial_l u_j + v_iu_l\partial_i u_l - \frac{2v_iv_jv_l}{v_{\rm the}^2}u_l\partial_i u_j}\right.\\ &\qquad \left. - \frac{2}{v_{\rm the}^2}\overline{\left(v_lu_l\frac{\partial u_i}{\partial t}u_i + v_ju_l\frac{\partial u_l}{\partial t}u_j + v_iu_l\frac{\partial u_i}{\partial t}u_l - \frac{2v_iv_jv_l}{v_{\rm the}^2}u_l\frac{\partial u_i}{\partial t}u_j\right)}\right]f_{0e}\\ &= \frac{2}{v_{\rm the}^2}\left\{\overline{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\left[\frac{u^2}{2} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{v_{\rm the}^2}\right]}\right.\\ &\qquad \left. - \frac{2}{v_{\rm the}^2}\overline{\frac{\partial}{\partial t}\left[u^2\boldsymbol{u}\cdot\boldsymbol{v} - \frac{2(\boldsymbol{u}\cdot\boldsymbol{v})^3}{3v_{\rm the}^2}\right]}\right\}f_{0e}\\ &= 2\boldsymbol{v}\cdot\boldsymbol{\nabla}\overline{\left[\frac{u^2}{v_{\rm the}^2} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{v_{\rm the}^4}\right]}f_{0e}.\end{aligned} \tag{8.42}$$

The last expression was obtained after noticing that any full time derivative vanishes under averaging and that, $\boldsymbol{u}$ defined by (8.35) being a potential field, one could rewrite $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \boldsymbol{\nabla}|\boldsymbol{u}|^2/2$.

Note that (8.42) is $O(\epsilon^3)$, as are the other two terms in (8.29). Inserting (8.42) into (8.29),

---

[76]Note that this ordering means that the equations that are being derived will be, in principle, suitable both for weak- and strong-turbulence regimes (see §8.4.7).

we obtain the following solution for the slow part of the perturbed elecron distribution:

$$\delta\bar{f}_e = \left\{\frac{e\bar{\varphi}}{T_e} - 2\overline{\left[\frac{u^2}{v_{\text{the}}^2} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})^2}{v_{\text{the}}^4}\right]}\right\} f_{0e}. \tag{8.43}$$

The first term is the Boltzmann response, the second the ponderomotive one. The resulting electron density perturbation is

$$\frac{\delta\bar{n}_e}{n_{0e}} = \frac{e\bar{\varphi}}{T_e} - \frac{\overline{u^2}}{v_{\text{the}}^2}. \tag{8.44}$$

8.2.4. *Electron Fluid Dynamics*

In order to obtain the evolution equation for $\tilde{\varphi}$, we will need to solve (8.36), coupled to (8.32), to higher order than the lowest, namely, up to ϵ^4. Rather than solving the kinetic equation (8.36) order by order, it turns out to be a faster procedure to take moments of it exactly and then close the resulting hierarchy by calculating the second moment using $h^{(2)}$ given by (8.40).

The zeroth (density) moment of (8.36) is

$$\frac{\partial\tilde{n}_e}{\partial t} + \boldsymbol{\nabla}\cdot(\bar{n}_e\boldsymbol{u}) + \boldsymbol{\nabla}\cdot\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}h = 0, \tag{8.45}$$

the continuity equation. The first moment is

$$\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}h = -\boldsymbol{\nabla}\cdot\int \mathrm{d}\boldsymbol{v}\,(\boldsymbol{u}\boldsymbol{v}+\boldsymbol{v}\boldsymbol{u})\bar{f}_e - \boldsymbol{\nabla}\cdot\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h + \frac{e}{m_e}\tilde{n}_e\boldsymbol{\nabla}\bar{\varphi} + \frac{e}{m_e}\widetilde{\tilde{n}_e\boldsymbol{\nabla}\tilde{\varphi}}. \tag{8.46}$$

The first term on the right-hand side is zero to all orders up to at least ϵ^4 because, according to (8.43), $\bar{f}_e$ is even in $\boldsymbol{v}$ up to second order. The remaining terms are $O(\epsilon^3)$, except the penultimate one, which is $O(\epsilon^5)$ and can be safely dropped. Combining (8.45) with (8.46) and using (8.35), we have

$$\frac{\partial^2\tilde{n}_e}{\partial t^2} + \boldsymbol{\nabla}\cdot\left(\frac{e}{m_e}\bar{n}_e\boldsymbol{\nabla}\tilde{\varphi}\right) - \boldsymbol{\nabla}\boldsymbol{\nabla} : \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h + \boldsymbol{\nabla}\cdot\left(\frac{e}{m_e}\widetilde{\tilde{n}_e\boldsymbol{\nabla}\tilde{\varphi}}\right) = 0. \tag{8.47}$$

This equation is valid up to and including terms of order ϵ^4.

Note that, in order to maintain this level of precision, we need to keep the lowest-order contribution to h in the second velocity moment. This satisfies (8.40), which it is now convenient to rewrite as

$$\frac{\partial h^{(2)}}{\partial t} = -\frac{2}{v_{\text{the}}^2}\left[v_i v_j \partial_i u_j + \frac{\partial}{\partial t}\left(\frac{u^2}{2} - \frac{u_i u_j v_i v_j}{v_{\text{the}}^2}\right)\right] f_{0e}. \tag{8.48}$$

The stress tensor satisfies

$$\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{v}\,v_i v_j h = -\frac{n_{0e}v_{\text{the}}^2}{2}\left(\partial_i u_j + \partial_j u_i + \delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \frac{\partial}{\partial t}n_{0e}u_i u_j. \tag{8.49}$$

Therefore,

$$\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\boldsymbol{\nabla} : \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h\right) = -\frac{3}{2}v_{\text{the}}^2\nabla^2\boldsymbol{\nabla}\cdot(n_{0e}\boldsymbol{u}) + \frac{\partial}{\partial t}n_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla} : \boldsymbol{u}\boldsymbol{u}. \tag{8.50}$$

From (8.45), to lowest order, $\boldsymbol{\nabla}\cdot(n_{0e}\boldsymbol{u}) = -\partial\tilde{n}_e/\partial t$ and so the above equation can be integrated in time:

$$\boldsymbol{\nabla}\boldsymbol{\nabla} : \int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h = \frac{3}{2}v_{\text{the}}^2\nabla^2\tilde{n}_e + n_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla} : \widetilde{\boldsymbol{u}\boldsymbol{u}}. \tag{8.51}$$

Inserting (8.51) into (8.47) and using also the fast Poisson equation (8.32) to express $\tilde{n}_e$ via $\tilde{\varphi}$, we obtain

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla^2\tilde{\varphi} + \boldsymbol{\nabla}\cdot\left(\omega_{\text{pe}}^2\frac{\bar{n}_e}{n_{0e}}\boldsymbol{\nabla}\tilde{\varphi}\right) - \frac{3}{2}v_{\text{the}}^2\nabla^4\tilde{\varphi} = 4\pi e n_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla} : \widetilde{\boldsymbol{u}\boldsymbol{u}} - \boldsymbol{\nabla}\cdot\left[\frac{e}{m_e}\widetilde{(\nabla^2\tilde{\varphi})\boldsymbol{\nabla}\tilde{\varphi}}\right]}. \tag{8.52}$$

The left-hand side of this equation is what was intuited in (8.7). The terms on the right-hand side of (8.52), not captured in (8.7), are nonlinear interactions between Langmuir waves, which will disappear in a moment.

The dominant $\omega_{\rm pe}$ oscillation in (8.52) can now be taken out in the way that was already anticipated in §8.1.3, by introducing, via (8.15), the slow-varying complex amplitude ψ. Then, from (8.35), to lowest order in ϵ,

$$\boldsymbol{u} = i\frac{e}{2m_e\omega_{\rm pe}}\boldsymbol{\nabla}\left(\psi e^{-i\omega_{\rm pe}t} - \psi^* e^{i\omega_{\rm pe}t}\right). \tag{8.53}$$

Substituting (8.15) and (8.53) into (8.52), we get Zakharov's first equation (8.16) in the same way as we did in §8.1.3. The right-hand side of (8.52) disappears because we can average out the oscillatory terms with frequencies $\omega_{\rm pe}$ and $2\omega_{\rm pe}$. Finally, substituting (8.53) into (8.44), we have, for the slow density perturbation,

$$\boxed{\frac{\delta\bar{n}_e}{n_{0e}} = \frac{e\bar{\varphi}}{T_e} - \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e}}\,. \tag{8.54}$$

To get $\bar{\varphi}$, we need to bring in the ions.

8.2.5. *Ion Kinetics*

Since the left-hand side of the slow Poisson equation (8.31) is $O(\epsilon^4)$, while its right-hand side is $O(\epsilon^2)$, (8.31) predictably turns into the quasineutrality equation

$$\delta\bar{n}_e = Z\delta n_i. \tag{8.55}$$

Combined with (8.54), this becomes

$$\frac{e\bar{\varphi}}{T_e} = \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e} + \frac{1}{n_{0i}}\int \mathrm{d}\boldsymbol{v}\,\delta f_i, \tag{8.56}$$

where ψ obeys (8.16). The first term in (8.56) is the effective ponderomotive "potential" (in the fluid-dynamical language, it can also be thought of, perhaps more physically, as the effective ponderomotive "pressure": see §8.1.2). The ion distribution function $f_i = f_{0i} + \delta f_i$ is found from the ion Vlasov–Landau equation (5.1) with the slow potential $\bar{\varphi}$:

$$\boxed{\frac{\partial f_i}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_i - \frac{Ze}{m_i}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial f_i}{\partial \boldsymbol{v}} = \left(\frac{\partial f_i}{\partial t}\right)_{\rm c}}\,. \tag{8.57}$$

Together with (8.16), (8.57) and (8.56) make up a closed hybrid system describing kinetic ions and fluid electrons. The electrons affect the ions via the ponderomotive nonlinearity in (8.56), while the ions modulate the plasma frequency and thereby the dynamics of the electrons.

8.2.6. *Ion Fluid Dynamics*

Let me now show how ions can become fluid, giving rise to the second equation (8.17) in the classic Zakharov (1972) system.

The zeroth and first moments of (8.57) are

$$\frac{\partial \delta n_i}{\partial t} + \boldsymbol{\nabla}\cdot\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\delta f_i = 0, \tag{8.58}$$

$$\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\delta f_i + \boldsymbol{\nabla}\cdot\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}\delta f_i = -\frac{Ze}{m_i}n_i\boldsymbol{\nabla}\bar{\varphi} = -c_{\rm s}^2 n_i\boldsymbol{\nabla}\left(\frac{\delta n_i}{n_{0i}} + \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e}\right), \tag{8.59}$$

where the last expression was obtained with the aid of (8.56). Combining these two equations and keeping only the lowest-order terms, both in ϵ and in T_i/T_e, which is now assumed small so as to allow the ion pressure (stress) tensor in the left-hand side of (8.59) to be neglected, and replacing $\delta n_i/n_{0i}$ with $\delta\bar{n}_e/n_{0e}$ (by quasineutrality), we get Zakharov's second equation (8.17), describing sound waves excited by the ponderomotive force.

8.2.7. *Boltzmann Quasistatics*

When the ions are not cold (T_i/T_e is not small), (8.17) regains the ion pressure term, via which it couples to the rest of the moments of δf_i. This is a dissipation channel for the sound waves, via Landau damping, at a typical rate $\sim kv_{{\rm th}i}$. In the WT language, this is a regime in which induced scattering of plasmons by ions (§7.2.4) should become the dominant process.

There is, however, a very simple limiting case where kinetic physics again becomes irrelevant. The characteristic frequency $\delta\omega_{\boldsymbol{k}}$ of ψ is given by (8.18). Consider the limit in which the ion streaming rate greatly exceeds that frequency:

$$k v_{\mathrm{th}i} \gg \delta\omega_{\boldsymbol{k}} \sim k^2\lambda_{\mathrm{De}}^2\omega_{\mathrm{pe}} \quad \Leftrightarrow \quad k\lambda_{\mathrm{De}} \ll \sqrt{\frac{T_i}{T_e}\frac{m_e}{m_i}}. \tag{8.60}$$

This is best achievable (obviously) when ions are hot, the opposite limit to the one considered in §8.2.6. In this limit, the $\partial f_i/\partial t$ term in the ion kinetic equation (8.57) can be neglected. It is then easy to verify, by direct substitution, that the resulting equation is solved by the Maxwell–Boltzmann distribution:

$$f_i = \frac{n_{0i}}{(\pi v_{\mathrm{th}i}^2)^{3/2}}\exp\left(-\frac{v^2}{v_{\mathrm{th}i}^2} - \frac{Ze\bar{\varphi}}{T_i}\right) \quad \Rightarrow \quad \delta f_i = -\frac{Ze\bar{\varphi}}{T_i}f_{0i} \quad \Rightarrow \quad \frac{\delta n_i}{n_{0i}} = -\frac{Ze\bar{\varphi}}{T_i}. \tag{8.61}$$

Thus, the ion response is simply Boltzmann. So is the (slow) electron response, except with an additional effective potential associated with the ponderomotive force—we already know this from (8.54). Combining (8.61) with that and with quasineutrality, via (8.56), we get

$$\boxed{\delta\bar{n}_e = -\frac{|\boldsymbol{\nabla}\psi|^2}{16\pi(T_e + T_i/Z)}}\,. \tag{8.62}$$

Interestingly, this is mathematically the same result as (8.21) was for *cold* ions and $k\lambda_{\mathrm{De}} \ll \sqrt{m_e/m_i}$. Just like in that cold-ion quasistatic limit, in this, hot-ion one, everything is now wrapped up in a single Zakharov equation—(8.22) with T_e replaced by $T_e + T_i/Z$,—which describes Langmuir dynamics controlled by four-wave interactions (§7.2.2).

8.3. *Hamiltonian Form of Zakharov's Equations*

In order to make a transition from the dynamical equations for the fields ψ and $\delta\bar{n}_e$ to kinetic equations for the occupation numbers of plasmons and phonons, we need to recast the former in terms of variables that we can directly connect to the energies and, therefore, occupation numbers, of our "particles", in the way it was done in (7.2) for the quasiparticle formulation of QLT. Thus, for the plasmons, we again have the total energy[77]

$$H^l = \sum_{\boldsymbol{p}}\hbar\omega_{\boldsymbol{p}}N_{\boldsymbol{p}} = V\sum_{\boldsymbol{p}}\frac{p^2|\psi_{\boldsymbol{p}}|^2}{8\pi} \equiv \sum_{\boldsymbol{p}}\hbar\omega_{\boldsymbol{p}}a_{\boldsymbol{p}}^*a_{\boldsymbol{p}}, \tag{8.63}$$

where $\psi_{\boldsymbol{p}}$ is the Fourier transform of ψ defined by (8.15), so $p^2|\psi_{\boldsymbol{p}}|^2 = 2|\widetilde{\boldsymbol{E}}_{\boldsymbol{p}}|^2$, averaged over oscillations of frequency ω_{pe} and above. A new variable

$$\boxed{a_{\boldsymbol{p}} = p\sqrt{\frac{V}{8\pi\hbar\omega_{\boldsymbol{p}}}}\,\psi_{\boldsymbol{p}}} \tag{8.64}$$

has been introduced, which makes (8.63) look like a Hamiltonian of a system of particles created or annihilated by the "operators" $a_{\boldsymbol{p}}^*$ and $a_{\boldsymbol{p}}$. We shall see in a moment that this is not a coincidence.

Let us now write the first Zakharov equation (8.16) in wave-number space:

$$p^2\left(i\omega_{\mathrm{pe}}^{-1}\frac{\partial}{\partial t} - \frac{3}{2}p^2\lambda_{\mathrm{De}}^2\right)\psi_{\boldsymbol{p}} = \frac{1}{2}\boldsymbol{p}\cdot\sum_{\boldsymbol{k}}(\boldsymbol{p}-\boldsymbol{k})\psi_{\boldsymbol{p}-\boldsymbol{k}}\xi_{\boldsymbol{k}}, \qquad \xi_{\boldsymbol{k}} \equiv \left(\frac{\delta\bar{n}_e}{n_{0e}}\right)_{\boldsymbol{k}}, \tag{8.65}$$

[77] If the appearance of $\hbar$ in a purely classical calculation triggers you, you may safely set $\hbar = 1$ everywhere. I carry it in order to keep track of dimensions, and for the uniformity of the formalism between here and §7.

or, in terms of $a_{\boldsymbol{p}}$,

$$i\frac{\partial a_{\boldsymbol{p}}}{\partial t} - \delta\omega_{\boldsymbol{p}} a_{\boldsymbol{p}} = \frac{\omega_{\rm pe}}{2}\sum_{\boldsymbol{k},\boldsymbol{q}} \delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}} \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\, a_{\boldsymbol{q}}\xi_{\boldsymbol{k}}. \tag{8.66}$$

I have made use of the previously introduced notation (8.18) for the "residual" frequency $\delta\omega_{\boldsymbol{p}}$ and also approximated $\omega_{\boldsymbol{p}} \approx \omega_{\boldsymbol{p}-\boldsymbol{k}} \approx \omega_{\rm pe}$ in the nonlinear term.

Now let us turn to the second Zakharov equation (8.17), describing sound waves. In fact, I need to walk back to the two fluid equations from which it originated: the ion continuity equation (8.8) and the ion momentum equation (8.9), the former recast in terms of $\xi = \delta\bar{n}_e/n_{0e}$ using quasineutrality and the latter with $\delta\bar{p}_e = T_e\delta\bar{n}_e$ + an "effective pressure" term representing the ponderomotive force [see (8.13)]:

$$\frac{\partial\xi_{\boldsymbol{k}}}{\partial t} = -i\boldsymbol{k}\cdot\boldsymbol{u}_{i,\boldsymbol{k}}, \qquad \frac{\partial\boldsymbol{u}_{i,\boldsymbol{k}}}{\partial t} = -i\boldsymbol{k}\left(c_{\rm s}^2\xi_{\boldsymbol{k}} + \chi_{\boldsymbol{k}}\right), \tag{8.67}$$

where, expanding the quadratic expression for the ponderomotive force as a Fourier convolution and using (8.64),

$$\chi_{\boldsymbol{k}} \equiv \frac{\overline{(|\boldsymbol{\nabla}\psi|^2)}_{\boldsymbol{k}}}{16\pi m_i n_{0i}} = -\sum_{\boldsymbol{p}} \frac{\boldsymbol{p}\cdot(\boldsymbol{k}-\boldsymbol{p})\overline{\psi_{\boldsymbol{p}}\psi^*_{\boldsymbol{k}-\boldsymbol{p}}}}{16\pi m_i n_{0i}} = \frac{\hbar\omega_{\rm pe}}{2m_i n_{0i}V}\sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k},\boldsymbol{p}-\boldsymbol{q}}\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}\,\overline{a_{\boldsymbol{p}}a^*_{\boldsymbol{q}}}. \tag{8.68}$$

At the last step, I made use of the fact that $\psi_{\boldsymbol{q}} = \psi_{-\boldsymbol{q}}$—this follows from (8.15) and the reality condition $\tilde{\varphi}_{-\boldsymbol{q}} = \tilde{\varphi}^*_{\boldsymbol{q}}$. Now, clearly, the flow is potential, $\boldsymbol{u}_{i,\boldsymbol{k}} = i\boldsymbol{k}\phi_{\boldsymbol{k}}$, so our equations become

$$\frac{\partial\xi_{\boldsymbol{k}}}{\partial t} = k^2\phi_{\boldsymbol{k}}, \qquad \frac{\partial\phi_{\boldsymbol{k}}}{\partial t} = -c_{\rm s}^2\xi_{\boldsymbol{k}} - \chi_{\boldsymbol{k}}. \tag{8.69}$$

The linear part of these equations is Hamiltonian, in canonical variables $(\xi_{\boldsymbol{k}}, \phi_{\boldsymbol{k}})$, with the Hamiltonian being (up to a multiplicative constant) the energy of the sound waves (cf. Exercise 5.4):

$$H^s = \int {\rm d}\boldsymbol{r}\left(\frac{m_i n_{0i}u_i^2}{2} + \frac{n_{0e}T_e}{2}\frac{\delta\bar{n}_e^2}{n_{0e}^2}\right) = \frac{m_i n_{0i}V}{2}\sum_{\boldsymbol{k}}\left(k^2\phi_{\boldsymbol{k}}^2 + c_{\rm s}^2\xi_{\boldsymbol{k}}^2\right) \equiv \sum_{\boldsymbol{k}}\hbar\Omega_{\boldsymbol{k}}b^*_{\boldsymbol{k}}b_{\boldsymbol{k}}, \tag{8.70}$$

where $\Omega_{\boldsymbol{k}} = kc_{\rm s}$. The last step is accomplished by the standard transformation

$$\boxed{b_{\boldsymbol{k}} = \sqrt{\frac{m_i n_{0i}V}{2\hbar}}\left(\sqrt{\frac{c_{\rm s}}{k}}\,\xi_{\boldsymbol{k}} + i\sqrt{\frac{k}{c_{\rm s}}}\,\phi_{\boldsymbol{k}}\right)}\,. \tag{8.71}$$

Note that, while $\xi_{\boldsymbol{k}}$ and $\phi_{\boldsymbol{k}}$ are Fourier transforms of real fields (so $\xi_{-\boldsymbol{k}} = \xi^*_{\boldsymbol{k}}$ and $\phi_{-\boldsymbol{k}} = \phi^*_{\boldsymbol{k}}$), the new variable $b_{\boldsymbol{k}}$ is genuinely complex and so contains full information about both fields, which can be extracted back from it:

$$\xi_{\boldsymbol{k}} = \sqrt{\frac{\hbar k}{2m_i n_{0i}Vc_{\rm s}}}\left(b_{\boldsymbol{k}} + b^*_{-\boldsymbol{k}}\right), \qquad \phi_{\boldsymbol{k}} = -i\sqrt{\frac{\hbar c_{\rm s}}{2m_i n_{0i}Vk}}\left(b_{\boldsymbol{k}} - b^*_{-\boldsymbol{k}}\right). \tag{8.72}$$

The two equations (8.69) can, therefore, be wrapped into one:

$$i\frac{\partial b_{\boldsymbol{k}}}{\partial t} - \Omega_{\boldsymbol{k}}b_{\boldsymbol{k}} = \sqrt{\frac{m_i n_{0i}Vk}{2\hbar c_{\rm s}}}\,\chi_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{q}} M_{\boldsymbol{k}\boldsymbol{p}\boldsymbol{q}}\delta_{\boldsymbol{k},\boldsymbol{p}-\boldsymbol{q}}\overline{a_{\boldsymbol{p}}a^*_{\boldsymbol{q}}}, \tag{8.73}$$

where the mode-coupling coefficients ("matrix elements") are

$$\boxed{M_{\boldsymbol{kpq}} = \frac{\omega_{\mathrm{pe}}}{2}\sqrt{\frac{\hbar k}{2m_i n_{0i} V c_{\mathrm{s}}}}\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{pq}}\,. \tag{8.74}$$

Using (8.72) in (8.66), we can get the first Zakharov equation into a form that features the same coupling coefficients:

$$i\frac{\partial a_{\boldsymbol{p}}}{\partial t} - \delta\omega_{\boldsymbol{p}} a_{\boldsymbol{p}} = \sum_{\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{kpq}}\delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}} a_{\boldsymbol{q}}\left(b_{\boldsymbol{k}} + b^*_{-\boldsymbol{k}}\right) = \sum_{\boldsymbol{k},\boldsymbol{q}} M_{\boldsymbol{kpq}} a_{\boldsymbol{q}}\left(b_{\boldsymbol{k}}\delta_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{k}} + b^*_{\boldsymbol{k}}\delta_{\boldsymbol{p},\boldsymbol{q}-\boldsymbol{k}}\right). \tag{8.75}$$

It is now manifest that (8.75) and (8.73) are Hamiltonian equations in the form they are usually written for creation-annihilation variables:

$$i\hbar\frac{\partial a_{\boldsymbol{p}}}{\partial t} = \frac{\partial H}{\partial a^*_{\boldsymbol{p}}}, \quad i\hbar\frac{\partial b_{\boldsymbol{k}}}{\partial t} = \frac{\partial H}{\partial b^*_{\boldsymbol{k}}}, \tag{8.76}$$

where the Hamiltonian consists of the free-plasmon energy (8.63) (with the constant part of the frequency ω_{pe} taken out), the free-phonon energy (8.70), and the interaction Hamiltonian describing the emission/absorption of phonons by plasmons:

$$\boxed{H = \hbar\sum_{\boldsymbol{k}}\left(\delta\omega_{\boldsymbol{k}} a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + \Omega_{\boldsymbol{k}} b^*_{\boldsymbol{k}} b_{\boldsymbol{k}}\right) + \hbar\sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}}\delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}} M_{\boldsymbol{kpq}}\left(b^*_{\boldsymbol{k}} a^*_{\boldsymbol{q}} a_{\boldsymbol{p}} + a^*_{\boldsymbol{p}} a_{\boldsymbol{q}} b_{\boldsymbol{k}}\right)}\,. \tag{8.77}$$

The two terms in the interaction Hamiltonian manifestly correspond to the two diagrams in Fig. 38.

Exercise 8.3. Plasmon number conservation. Prove (and note) that our equations also conserve the total number of plasmons $N = \sum_{\boldsymbol{p}} a^*_{\boldsymbol{p}} a_{\boldsymbol{p}}$ (which is why it was OK to take the constant-frequency part out of the Hamiltonian). In Exercise 7.4, you already showed that the kinetic equation for $N_{\boldsymbol{p}}$ also had this property. Note also that, unlike energy, N is not an invariant of (8.7), so it is an *adiabatic invariant* whose conservation depends on the assumption $\partial/\partial t \ll \omega_{\mathrm{pe}}$, which got us from (8.7) to (8.16).

That our system ought to be Hamiltonian, and that the Hamiltonian for the three-wave interactions described by the diagrams in Fig. 38 should have the general form (8.77), could have been argued *a priori*, so the added value from the above derivations is that we have also worked out the coupling coefficients $M_{\boldsymbol{kpq}}$ and the correspondence between the creation-annihilation variables $a_{\boldsymbol{k}}$, $b_{\boldsymbol{k}}$ and the physical fields $\tilde{\varphi}$, $\delta\bar{n}_e$. If you like the Hamiltonian approach, you will find a very extensive treatment of its application to plasma turbulence in the review by Zakharov *et al.* (1985)—and an even more general (not specially focused on plasma physics), and more pedagogical, exposition in the now-classic book by Zakharov *et al.* (1992).

Exercise 8.4. Hamiltonian for four-wave interactions. Consider the quasistatic version (8.22) of the Zakharov system. By deriving the evolution equation for $a_{\boldsymbol{k}}$, or otherwise, show that the plasmon Hamiltonian with four-wave interactions is

$$\boxed{H = \hbar\sum_{\boldsymbol{k}}\delta\omega_{\boldsymbol{k}} a^*_{\boldsymbol{k}} a_{\boldsymbol{k}} + \hbar\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{p},\boldsymbol{p}'}\delta_{\boldsymbol{k}+\boldsymbol{k}',\boldsymbol{p}+\boldsymbol{p}'} M_{\boldsymbol{kk}'\boldsymbol{pp}'} a^*_{\boldsymbol{p}} a^*_{\boldsymbol{p}'} a_{\boldsymbol{k}} a_{\boldsymbol{k}'}}\,, \tag{8.78}$$

and calculate the coupling coefficients $M_{\boldsymbol{kk}'\boldsymbol{pp}'}$. You will find the answers to this Exercise, as well as to Exercises 8.5, 8.7, 8.9, and 8.11 in the literature cited above, or in the original, famous paper by Zakharov (1972)—but following his derivations will not necessarily be easier than working them out by yourself.

8.4. *Weak Langmuir Turbulence*

Let me now show you how to derive (§§8.4.1–8.4.2) and solve (§§8.4.3–8.4.6) the kinetic equations (7.23) and (7.24) starting from the dynamical equations (8.73) and (8.75). As far as I know, the example of WT considered here was first worked out by Zakharov & Kuznetsov (1978) and became a minor classic of the WT genre.

Note that because our starting point is a fairly generic Hamiltonian system (with the nature of the specific problem hidden in $M_{\boldsymbol{kpq}}$), the basic strategies and ideas laid out below will have much more general applicability than to the specific, narrow, and perhaps even irrelevant, problem of Langmuir–sound turbulence at $T_i \ll T_e$.

8.4.1. *Perturbation Theory*

We need evolution equations for $N_{\boldsymbol{p}} = \langle |a_{\boldsymbol{p}}|^2 \rangle$ and $n_{\boldsymbol{k}} = \langle |b_{\boldsymbol{k}}|^2 \rangle$, so let us multiply (8.75) and (8.73) by $a^*_{\boldsymbol{k}}$ and $b^*_{\boldsymbol{k}}$, respectively, subtract the complex conjugates of the same, average[78] and see what happens. In fact, we only need to do one equation, e.g., the sound-wave one (8.73), because we know that it will have the structure (7.23) and, once we work out $w(\boldsymbol{p}, \boldsymbol{k})$, we will be able to write the Langmuir-wave kinetic equation (7.24) without any further effort.

Thus, we have for the phonon occupation number:[79]

$$\frac{\partial n_{\boldsymbol{k}}}{\partial t} = 2\,\mathrm{Im} \sum_{\boldsymbol{p},\boldsymbol{q}} \delta_{\boldsymbol{k},\boldsymbol{p}-\boldsymbol{q}} M_{\boldsymbol{kpq}} \left\langle b^*_{\boldsymbol{k}} a_{\boldsymbol{p}} a^*_{\boldsymbol{q}} \right\rangle . \tag{8.79}$$

As always in nonlinear problems, the second-order correlator depends on the third-order one—the so-called *closure problem*. The perturbative solution to this problem is to assume the nonlinearity to be weak and so to truncate the expansion for the fields in the higher-order correlator at the linear level. As we shall see shortly, if we do this at the level of the third-order correlator, everything will vanish, so we need to iterate once more.

Using (8.75) for $a_{\boldsymbol{p}}$ and $a^*_{\boldsymbol{q}}$ and (8.73) for $b^*_{\boldsymbol{k}}$, we get

$$\begin{aligned}\left[\frac{\partial}{\partial t} + i\left(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{k}}\right)\right] \left\langle b^*_{\boldsymbol{k}} a_{\boldsymbol{p}} a^*_{\boldsymbol{q}} \right\rangle = i \sum_{\boldsymbol{k}',\boldsymbol{k}''} \Big[M_{\boldsymbol{kk}'\boldsymbol{k}''} \delta_{\boldsymbol{k},\boldsymbol{k}'-\boldsymbol{k}''} \left\langle a^*_{\boldsymbol{k}'} a_{\boldsymbol{k}''} a_{\boldsymbol{p}} a^*_{\boldsymbol{q}} \right\rangle \\ - M_{\boldsymbol{k}'\boldsymbol{p}\boldsymbol{k}''} \left(\delta_{\boldsymbol{p},\boldsymbol{k}''+\boldsymbol{k}'} \left\langle b^*_{\boldsymbol{k}} b_{\boldsymbol{k}'} a_{\boldsymbol{k}''} a^*_{\boldsymbol{q}} \right\rangle + \delta_{\boldsymbol{p},\boldsymbol{k}''-\boldsymbol{k}'} \left\langle b^*_{\boldsymbol{k}} b^*_{\boldsymbol{k}'} a_{\boldsymbol{k}''} a^*_{\boldsymbol{q}} \right\rangle \right) \\ + M_{\boldsymbol{k}'\boldsymbol{q}\boldsymbol{k}''} \left(\delta_{\boldsymbol{q},\boldsymbol{k}''+\boldsymbol{k}'} \left\langle b^*_{\boldsymbol{k}} b^*_{\boldsymbol{k}'} a_{\boldsymbol{p}} a^*_{\boldsymbol{k}''} \right\rangle + \delta_{\boldsymbol{q},\boldsymbol{k}''-\boldsymbol{k}'} \left\langle b^*_{\boldsymbol{k}} b_{\boldsymbol{k}'} a_{\boldsymbol{p}} a^*_{\boldsymbol{k}''} \right\rangle \right) \Big] \equiv i A_{\boldsymbol{kpq}}.\end{aligned} \tag{8.80}$$

Note that, since only the difference of Langmuir frequencies enters, I have, for brevity of notation, replaced $\delta\omega_{\boldsymbol{p}}$ with $\omega_{\boldsymbol{p}}$. We do not need to go any further in perturbation theory and so can solve the above equation treating the right-hand side as a constant (being an average, it changes slowly in time compared to any of the waves' oscillations):

$$\left\langle b^*_{\boldsymbol{k}} a_{\boldsymbol{p}} a^*_{\boldsymbol{q}} \right\rangle = \frac{1 - e^{-i(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{k}})t}}{\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{k}}} A_{\boldsymbol{kpq}} \to i\pi\delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} - \Omega_{\boldsymbol{k}})\, A_{\boldsymbol{kpq}} \tag{8.81}$$

as $t \to \infty$ (i.e., after many wave periods), by the same token as in (5.41)—there will be a lot of these delta functions in kinetic theory!

[78] What "average" means will be explained in §8.4.2.

[79] I have removed the time averaging (denoted by overbars) that was needed to iron out variation of the Langmuir electric fields on time scales shorter than the sound frequency. Any sensible averaging here will take care of that automatically.

Now we need to calculate the fourth-order correlation functions inside $A_{\boldsymbol{kpq}}$. That will require another conceptual step.

8.4.2. *Random-Phase Approximation*

Our dynamical equations (8.73) and (8.75), with their small nonlinearities, describe the evolution of amplitudes and phases of the dynamical variables: $a_{\boldsymbol{k}} = |a_{\boldsymbol{k}}|(t)e^{i\theta_{\boldsymbol{k}}(t)}$ and similarly for $b_{\boldsymbol{k}}$. The amplitudes evolve slowly (to lowest order in the nonlinearity, not at all) and phases quickly: $\theta_{\boldsymbol{k}}(t) \approx -\omega_{\boldsymbol{k}}t$. This is not unlike (indeed, exactly analogous) to ballistic motion of particles in an ideal gas. When nonlinearity is introduced, even if small, the trajectories are perturbed and phases quickly stochasticised—this too is analogous to molecular chaos arising even from infrequent collisions between particles.[80]

Thus, we shall assume the randomness of phases and consequent vanishing (to lowest order in the perturbation theory) of correlations between the dynamical variables except when their phases manage to cancel exactly.[81] To wit:

$$\langle a_{\boldsymbol{k}}\rangle = \left\langle |a_{\boldsymbol{k}}|e^{i\theta_{\boldsymbol{k}}}\right\rangle = 0, \tag{8.82}$$

$$\langle a_{\boldsymbol{k}}a_{\boldsymbol{k}'}\rangle = \left\langle |a_{\boldsymbol{k}}||a_{\boldsymbol{k}'}|e^{i(\theta_{\boldsymbol{k}}+\theta_{\boldsymbol{k}'})}\right\rangle = 0, \tag{8.83}$$

$$\langle a_{\boldsymbol{k}}a^*_{\boldsymbol{k}'}\rangle = \left\langle |a_{\boldsymbol{k}}||a_{\boldsymbol{k}'}|e^{i(\theta_{\boldsymbol{k}}-\theta_{\boldsymbol{k}'})}\right\rangle = N_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{k}'}. \tag{8.84}$$

By the same token, all odd-order correlators vanish, as do all even-order ones that do not have an equal number of occurrences of $a_{\boldsymbol{k}}$ and $a^*_{\boldsymbol{k}}$. Thus, taking, e.g., one of the fourth-order correlators from (8.80):

$$\left\langle a^*_{\boldsymbol{k}'}a_{\boldsymbol{k}''}a_{\boldsymbol{p}}a^*_{\boldsymbol{q}}\right\rangle = N_{\boldsymbol{k}'}N_{\boldsymbol{p}}\left(\delta_{\boldsymbol{k}',\boldsymbol{k}''}\delta_{\boldsymbol{p},\boldsymbol{q}} + \delta_{\boldsymbol{p},\boldsymbol{k}'}\delta_{\boldsymbol{q},\boldsymbol{k}''}\right). \tag{8.85}$$

Also, in a system where frequencies of different types of waves are well separated ($\omega_{\boldsymbol{p}} \gg \Omega_{\boldsymbol{k}}$ always), only correlations between the same species of waves survive because otherwise phases cannot be matched. Let us do two other correlators in (8.80):

$$\left\langle b^*_{\boldsymbol{k}}b_{\boldsymbol{k}'}a_{\boldsymbol{k}''}a^*_{\boldsymbol{q}}\right\rangle = n_{\boldsymbol{k}}N_{\boldsymbol{q}}\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\boldsymbol{q},\boldsymbol{k}''}, \qquad \left\langle b^*_{\boldsymbol{k}}b_{\boldsymbol{k}'}a_{\boldsymbol{p}}a^*_{\boldsymbol{k}''}\right\rangle = n_{\boldsymbol{k}}N_{\boldsymbol{p}}\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta_{\boldsymbol{p},\boldsymbol{k}''}. \tag{8.86}$$

The rest of the correlators in (8.80) are zero.

Exercise 8.5. Work out, under the same assumptions, the 6th-order correlator $\left\langle a_{\boldsymbol{k}}a_{\boldsymbol{p}}a_{\boldsymbol{q}}a^*_{\boldsymbol{k}'}a^*_{\boldsymbol{p}'}a^*_{\boldsymbol{q}'}\right\rangle$ in terms of the second-order ones ($N_{\boldsymbol{k}} = \langle|a_{\boldsymbol{k}}|^2\rangle$). This will come useful in Exercise 8.7.

We are done: the right-hand side of (8.80) is

$$\begin{aligned} A_{\boldsymbol{kpq}} &= M_{\boldsymbol{kpq}}N_{\boldsymbol{p}}N_{\boldsymbol{q}} - M_{\boldsymbol{kpq}}n_{\boldsymbol{k}}N_{\boldsymbol{q}} + M_{\boldsymbol{kqp}}n_{\boldsymbol{k}}N_{\boldsymbol{p}} \\ &= M_{\boldsymbol{kpq}}\left[N_{\boldsymbol{p}}N_{\boldsymbol{q}} + \left(N_{\boldsymbol{p}} - N_{\boldsymbol{q}}\right)n_{\boldsymbol{k}}\right]. \end{aligned} \tag{8.87}$$

Note that the first term in (8.85) did not contribute because $M_{0\boldsymbol{k}'\boldsymbol{k}'} = 0$, and that $M_{\boldsymbol{kpq}} = M_{\boldsymbol{kqp}}$ was used to simplify things. Putting (8.87) back into (8.79), via (8.81), we recover, triumphantly, the anticipated kinetic equation in the form (7.23):

$$\boxed{\frac{\partial n_{\boldsymbol{k}}}{\partial t} = 2\pi\sum_{\boldsymbol{p}} M^2_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}-\boldsymbol{k}}\delta(\omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}-\boldsymbol{k}} - \Omega_{\boldsymbol{k}})\left[N_{\boldsymbol{p}}N_{\boldsymbol{p}-\boldsymbol{k}} + \left(N_{\boldsymbol{p}} - N_{\boldsymbol{p}-\boldsymbol{k}}\right)n_{\boldsymbol{k}}\right] \equiv \sum_{\boldsymbol{p}} T_{\boldsymbol{p},\boldsymbol{k}}}\,. \tag{8.88}$$

---

[80]The fact that the wave frequencies are dispersive helps: as time goes on, different $\boldsymbol{k}$'s decorrelate—this is similar to phase mixing.

[81]"Exactly" means *exactly* here: this is not just about matching linear frequencies!

Manifestly, by comparison with (7.23) [or (7.27)],

$$w(\boldsymbol{p},\boldsymbol{k}) = 2\pi M^2_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{p}-\boldsymbol{k}} = w_0 k\left(\frac{\boldsymbol{p}}{p}\cdot\frac{\boldsymbol{p}-\boldsymbol{k}}{|\boldsymbol{p}-\boldsymbol{k}|}\right)^2, \qquad w_0 = \frac{\pi\hbar\omega_{\rm pe}^2}{4m_i n_{0i} V c_{\rm s}}. \tag{8.89}$$

In fact, to calculate this, we did not even need to follow all of the terms in (8.80), it was enough just to work out one of them, e.g., $\langle a^*_{\boldsymbol{k}'} a_{\boldsymbol{k}''} a_{\boldsymbol{p}} a^*_{\boldsymbol{q}}\rangle \propto N_{\boldsymbol{p}} N_{\boldsymbol{p}-\boldsymbol{k}}$. With (8.89) in hand, we also have the kinetic equation (7.24) for the plasmons: to write it in a short form anticipated in (7.26),

$$\boxed{\frac{\partial N_{\boldsymbol{p}}}{\partial t} = \sum_{\boldsymbol{k}} \left(T_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}} - T_{\boldsymbol{p},\boldsymbol{k}}\right)}\,, \tag{8.90}$$

where $T_{\boldsymbol{p},\boldsymbol{k}}$ is the transfer function (7.27), now fully specified by the expression under the wave-number sum in (8.88).

The rules (8.82–8.86), and similar, for calculating correlators are generally known as the *random-phase approximation (RPA).* Since they amount to splitting higher-order correlators into second-order ones, they are mathematically equivalent to assuming the fields to be Gaussian to lowest order in the perturbation theory (the dynamical variables are an accumulation of a sequence of random, independent kicks due to nonlinear interactions with each other—an instance of Central Limit Theorem, one might argue). Recalling the calculations in §7.2, it is now obvious that this assumption was implicitly already made when the rates of change of occupation numbers were written as products of these numbers—a quasiparticle version of Boltzmann's (in fact, Maxwell's and Ehrenfest's) *Stosszahlansatz* (no wonder then that the standard form of the collision integral emerged so easily in §7.2.5). Just how valid, or otherwise, this assumption might be, is, as often with such things, a subtler question than it looks. I will sweep it resolutely under the carpet, while referring you to extended discussions of the matter in the textbooks by Zakharov *et al.* (1992) and Nazarenko (2011), and move on happily to play with the kinetic equations (8.88) and (8.90).

Exercise 8.6. To practice perturbation theory and the use of RPA, write, starting from (8.75), the evolution equation for $N_{\boldsymbol{p}}$ in terms of third-order correlators, then solve for the latter in terms of fourth-order ones, split the averages, and verify that the kinetic equation for the plasmons is indeed (8.90).

Exercise 8.7. Kinetic equation for weak Langmuir turbulence with four-wave interactions. Starting from the Hamiltonian derived in Exercise 8.4, work out, along the same lines as done above, the kinetic equation for the plasmon occupation number in Langmuir turbulence with four-wave interactions.

8.4.3. *Further Simplifications and Approximations*

As I promised in §7.4, I shall look for *isotropic* solutions of the kinetic equations (8.88) and (8.90), so $N_{\boldsymbol{p}}$ depends only on p and $n_{\boldsymbol{k}}$ on k. This is sensible because the frequencies only depend on the magnitudes of the wave numbers, $\omega_{\boldsymbol{p}} = \omega_p$, $\varOmega_{\boldsymbol{k}} = \varOmega_k$—there are no special directions in the system.[82] Since both ω_p and $\varOmega_k$ are simple power functions,

$$\omega_p = \omega_{\rm pe} + \frac{3v_{\rm the}^2}{4\omega_{\rm pe}}\,p^2 \equiv \omega_{\rm pe} + \alpha p^2 \equiv \omega_{\rm pe} + \delta\omega_p, \qquad \varOmega_k = c_{\rm s}k, \tag{8.91}$$

[82]This will become spectacularly untrue in systems with a background magnetic field: see §§16.3 and 16.4.

there are one-to-one correspondences between wave numbers and frequencies, a feature that will greatly simplify calculations. Indeed, we can consider the occupation numbers to be functions of the frequencies only: $N_{\boldsymbol{p}} = N(\omega_p)$ and $n_{\boldsymbol{k}} = n(\Omega_k)$. Then the transfer function is also a function of frequencies only:

$$\begin{aligned} T_{\boldsymbol{p},\boldsymbol{k}} &= w_0 k \left(\frac{p^2 - \boldsymbol{p}\cdot\boldsymbol{k}}{p|\boldsymbol{p}-\boldsymbol{k}|}\right)^2 \delta\big(\omega_p - \omega_{|\boldsymbol{p}-\boldsymbol{k}|} - \Omega_k\big)\left[N_{\boldsymbol{p}}N_{\boldsymbol{p}-\boldsymbol{k}} + \left(N_{\boldsymbol{p}} - N_{\boldsymbol{p}-\boldsymbol{k}}\right) n_{\boldsymbol{k}}\right] \\ &\equiv \delta\big(\omega_p - \omega_{|\boldsymbol{p}-\boldsymbol{k}|} - \Omega_k\big)\, \tilde{T}(\omega_p, \Omega_k). \end{aligned} \tag{8.92}$$

Everything inside $\tilde{T}(\omega_p, \Omega_k)$ that depends on $\omega_{|\boldsymbol{p}-\boldsymbol{k}|}$ is reconstructible in terms of ω_p and Ω_k via the delta function. This includes $\boldsymbol{p}\cdot\boldsymbol{k}$, which appears in the prefactor: indeed, the argument of the delta function is[83]

$$\alpha p^2 - \alpha|\boldsymbol{p}-\boldsymbol{k}|^2 - \Omega_k = \alpha(2\boldsymbol{p}\cdot\boldsymbol{k} - k^2) - \Omega_k = 0 \quad \Rightarrow \quad \boldsymbol{p}\cdot\boldsymbol{k} = \frac{k^2}{2} + \frac{\Omega_k}{\alpha}. \tag{8.93}$$

A very attractive further simplification is now possible if we assume $\delta\omega_p \gg \Omega_k$ (which is true outside the quasistatic limit of §8.1.4):

$$\begin{aligned} T_{\boldsymbol{p}+\boldsymbol{k},\boldsymbol{k}} &= \delta\left(\omega_{|\boldsymbol{p}+\boldsymbol{k}|} - \omega_p - \Omega_k\right)\tilde{T}(\omega_{|\boldsymbol{p}+\boldsymbol{k}|}, \Omega_k) \\ &\approx \delta\left(\omega_{|\boldsymbol{p}+\boldsymbol{k}|} - \omega_p - \Omega_k\right)\left[\tilde{T}(\omega_p, \Omega_k) + \underbrace{\left(\omega_{|\boldsymbol{p}+\boldsymbol{k}|} - \omega_p\right)}_{=\,\Omega_k}\frac{\partial\tilde{T}}{\partial\omega_p}\right] \end{aligned} \tag{8.94}$$

In the same vein, expanding $N_{\boldsymbol{p}-\boldsymbol{k}} = N(\omega_{|\boldsymbol{p}-\boldsymbol{k}|})$ in (8.92) in $\omega_p - \omega_{|\boldsymbol{p}-\boldsymbol{k}|} = \Omega_k$ and using (8.93) in the prefactor, we find

$$\tilde{T}(\omega_p, \Omega_k) \approx w_0 k \left(1 - \frac{k^2}{2p^2}\right)^2 \left(N_{\boldsymbol{p}}^2 + \Omega_k n_{\boldsymbol{k}} \frac{\partial N_{\boldsymbol{p}}}{\partial\omega_p}\right). \tag{8.95}$$

Note that $\Omega_k \partial N_{\boldsymbol{p}}/\partial\omega_p = (c_{\rm s}k/2\alpha p)\partial N_{\boldsymbol{p}}/\partial p$.

From (8.95), we see immediately that for the Rayleigh–Jeans distributions (7.41), $\tilde{T}(\omega_p, \Omega_k) = 0$, so they are (still) legitimate stationary solutions of our equations. It makes sense that these solutions should be there, but they are not what we are after—we should be looking for the constant-flux solutions promised in §7.4.

8.4.4. *Plasmon Flux*

Let us prepare to use (8.94) and (8.92) in the plasmon kinetic equation (8.90), where we can simplify the $\boldsymbol{k}$ sum by noticing that the dependence on the direction of $\boldsymbol{k}$ (with respect to $\boldsymbol{p}$) is only left inside the delta functions:

$$\begin{aligned} \sum_{\boldsymbol{k}} \delta\big(\alpha(2\boldsymbol{p}\cdot\boldsymbol{k} \pm k^2) - c_{\rm s}k\big) &= \frac{V}{(2\pi)^3}\int_0^\infty \mathrm{d}k\, k^2\, 2\pi \int_{-1}^1 \mathrm{d}\cos\theta\, \delta\big(\alpha(2pk\cos\theta \pm k^2) - c_{\rm s}k\big)\,, \\ &= \frac{V}{4\pi^2}\int_0^{2p\pm c_{\rm s}/\alpha} \frac{\mathrm{d}k\, k}{2\alpha p}, \end{aligned} \tag{8.96}$$

where "+" applies to the integral of (8.94) and "−" to the integral of (8.92). Note that $c_{\rm s}/\alpha \ll 2p$ because $\Omega_p \ll \delta\omega_p$. We can now turn the plasmon kinetic equation (8.90) into the conservative form (7.44) by defining, as was done in §7.4, $\bar{N}_p = (Vp^2/2\pi^2)N_{\boldsymbol{p}}$ and

[83] So an "optimal" phonon for a plasmon to emit is one with $\boldsymbol{k} \approx 2\boldsymbol{p}$ (this was Exercise 7.3b).

using (8.96):

$$\frac{\partial \bar{N}_p}{\partial t} \approx \left(\frac{V}{2\pi^2}\right)^2 \frac{p}{4\alpha}\left\{\int_0^{2p+c_s/\alpha} \mathrm{d}k\left[k\tilde{T}(\omega_p,\Omega_k) + k\Omega_k\frac{\partial\tilde{T}}{\partial\omega_p}\right] - \int_0^{2p-c_s/\alpha}\mathrm{d}k\, k\tilde{T}(\omega_p,\Omega_k)\right\}$$
$$\approx \left(\frac{V}{2\pi^2}\right)^2 \frac{p}{4\alpha}\left(\frac{2c_s}{\alpha}\,2p\tilde{T} + \int_0^{2p}\mathrm{d}k\,\frac{c_s k^2}{2\alpha p}\frac{\partial\tilde{T}}{\partial p}\right)$$
$$= \left(\frac{V}{2\pi^2}\right)^2 \frac{c_s}{8\alpha^2}\left(8p^2\tilde{T} + \int_0^{2p}\mathrm{d}k\,k^2\frac{\partial\tilde{T}}{\partial p}\right) = -\frac{\partial\Gamma_p}{\partial p}, \tag{8.97}$$

where the plasmon-number flux is

$$\Gamma_p = -\left(\frac{V}{2\pi^2}\right)^2 \frac{c_s}{8\alpha^2}\int_0^{2p}\mathrm{d}k\,k^2\tilde{T}(\omega_p,\Omega_k)$$
$$= -\left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_s}{8\alpha^2}\int_0^{2p}\mathrm{d}k\,k^3\left(1-\frac{k^2}{2p^2}\right)^2\left(N_{\boldsymbol{p}}^2 + \frac{c_s k n_{\boldsymbol{k}}}{2\alpha p}\frac{\partial N_{\boldsymbol{p}}}{\partial p}\right). \tag{8.98}$$

I have used the approximate expression (8.95) for $\tilde{T}(\omega_p,\Omega_k)$. This approximation is called the "diffusion approximation" because under it, (8.97) has turned into a diffusion equation for $\bar{N}_p$ in wave-number space.[84]

So this is what I promised you in §7.4: a specific expression for a flux, setting which to be constant will yield stationary nonequilibrium distributions. Here is how to find them. Let us look for power-law solutions:

$$N_{\boldsymbol{p}} = Ap^{-x}, \qquad n_{\boldsymbol{k}} = Bk^{-y}. \tag{8.99}$$

Putting these into (8.98) and rescaling p out of the integral by introducing a new integration variable $\xi = k/p$, we get

$$\Gamma_p = -\left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_s A^2}{8\alpha^2}\,p^{4-2x}\underbrace{\int_0^2 \mathrm{d}\xi\,\xi^3\left(1-\frac{\xi^2}{2}\right)^2\left(1-\frac{c_s Bx}{2\alpha A}\,\xi^{-y+1}p^{x-y-1}\right)}_{\equiv I_1(p)}. \tag{8.100}$$

This can only be independent of p if the powers of p in both terms vanish, giving $x = 2$ and $y = x - 1 = 1$, so

$$\boxed{N_{\boldsymbol{p}} = A_\Gamma p^{-2}, \qquad n_{\boldsymbol{p}} = B_\Gamma p^{-1}}\,. \tag{8.101}$$

I have equipped the constants with a subscript Γ to emphasise that they are specific to the regime in which the distributions are determined by the constancy of the plasmon flux.

Substituting (8.101) back into (8.100), we get, after doing the integral $I_1 = 4/3$:

$$\boxed{\Gamma = -\left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_s A_\Gamma^2}{6\alpha^2}\left(1-\frac{c_s B_\Gamma}{\alpha A_\Gamma}\right)}\,. \tag{8.102}$$

The sign of this expression is not definite, so we do not yet have a way of knowing

[84]That this simplification should be possible is a particular feature of the problem at hand, not a generic property of WT. Neither is it essential to have such an approximation in order to be able to find constant-flux, power-law solutions as I shall do below. Such solutions (can) also exist in cases where the rates of change of quasiparticle occupation numbers are only explicitly expressible as wave-number integrals—a famous example is Exercise 8.9.

whether the plasmon number flows to smaller or larger p. The answer is that it will flow to smaller p, but we will only be able to deduce this formally once we know what happens with the energy flux.

8.4.5. *Energy Flux*

Let us turn to the phonon kinetic equation (8.88). This contains a similar integral to those we encountered above:

$$\begin{aligned}
\frac{\partial n_{\boldsymbol{k}}}{\partial t} &= \sum_{\boldsymbol{p}} \delta\big(\omega_p - \omega_{|\boldsymbol{p}-\boldsymbol{k}|} - \Omega_k\big)\, \tilde{T}(\omega_p, \Omega_k) \\
&= \frac{V}{(2\pi)^3} \int_0^\infty \mathrm{d}p\, p^2 \tilde{T}(\omega_p, \Omega_k)\, 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta\, \delta\big(\alpha(2pk\cos\theta - k^2) - c_{\mathrm{s}} k\big) \\
&\approx \frac{V}{2\pi^2} \frac{w_0}{4\alpha} \int_{k/2}^\infty \mathrm{d}p\, p \left(1 - \frac{k^2}{2p^2}\right)^2 \left(N_{\boldsymbol{p}}^2 + \frac{c_{\mathrm{s}} k n_{\boldsymbol{k}}}{2\alpha p} \frac{\partial N_{\boldsymbol{p}}}{\partial p}\right) \\
&= \frac{V}{2\pi^2} \frac{w_0 A^2}{4\alpha}\, k^{2-2x} \underbrace{\int_{1/2}^\infty \mathrm{d}\zeta\, \zeta^{1-2x} \left(1 - \frac{1}{2\zeta^2}\right)^2 \left(1 - \frac{c_{\mathrm{s}} B x}{2\alpha A}\, \zeta^{x-2} k^{x-y-1}\right)}_{\equiv I_2(k)}.
\end{aligned} \tag{8.103}$$

At the last step, I substituted the power-law solutions (8.99) and rescaled k out of the integral by changing the integration variable to $\zeta = p/k$.

We are now ready to construct an equation for the evolution of the total-energy spectrum

$$\bar{E}_k = \frac{V k^2}{2\pi^2} \left(\hbar \Omega_k n_{\boldsymbol{k}} + \hbar \delta\omega_k N_{\boldsymbol{k}}\right). \tag{8.104}$$

This was already defined in (7.42), but here I have taken out the bit corresponding to the constant part of the plasmon frequency, $\hbar\omega_{\mathrm{pe}} N_{\boldsymbol{k}}$, because that has the same evolution as the plasmon number, governed by (8.97). I shall call $\bar{E}_k$ "residual energy". Using (8.103), (8.97) and (8.100), we get

$$\begin{aligned}
\frac{\partial \bar{E}_k}{\partial t} &= \left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 A^2}{4\alpha}\, \hbar \Omega_k k^{4-2x} I_2(k) - \hbar \delta\omega_k \frac{\partial \Gamma_k}{\partial k} \\
&= \left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_{\mathrm{s}} A^2}{4\alpha} \underbrace{\left[k^{5-2x} I_2(k) + \frac{1}{2} k^2 \frac{\partial}{\partial k} k^{4-2x} I_1(k)\right]}_{= k^{5-2x}\left[I_2 + (2-x) I_1\right]} \equiv -\frac{\partial \varepsilon_k}{\partial k}.
\end{aligned} \tag{8.105}$$

This expression can only be scale-invariant if the k dependence in I_1 and I_2 disappears. The condition for that is

$$y = x - 1. \tag{8.106}$$

In fact, this could have perhaps been guessed *a priori*: it is just the statement that the contributions to $\bar{E}_k$ from the phonons and the plasmons have the same scaling: $\Omega_k n_{\boldsymbol{k}} \sim \delta\omega_k N_{\boldsymbol{k}}$.

Adopting this assumption, we conclude immediately that a stationary solution requires

$$I_2 + (2-x) I_1 = 0, \tag{8.107}$$

where the integrals, which now depend only on the exponent x and on the constants A

and B, are

$$I_1 = \int_0^2 \mathrm{d}\xi\, \xi^3 \left(1 - \frac{\xi^2}{2}\right)^2 \left(1 - \frac{c_\mathrm{s} B x}{2\alpha A}\, \xi^{-x+2}\right), \tag{8.108}$$

$$I_2 = \int_{1/2}^\infty \mathrm{d}\zeta\, \zeta^{1-2x} \left(1 - \frac{1}{2\zeta^2}\right)^2 \left(1 - \frac{c_\mathrm{s} B x}{2\alpha A}\, \zeta^{x-2}\right)$$
$$= \int_0^2 \mathrm{d}\xi\, \xi^{2x-3} \left(1 - \frac{\xi^2}{2}\right)^2 \left(1 - \frac{c_\mathrm{s} B x}{2\alpha A}\, \xi^{-x+2}\right). \tag{8.109}$$

The last transformation of I_2 was accomplished by changing the integration variable to $\xi = 1/\zeta$, mappping the integration domain of I_2 onto that of I_1. The two integrals become identical at $x = 3$, which then conveniently solves (8.107). Thus, we have obtained the constant-residual-energy-flux solutions:

$$\boxed{N_{\boldsymbol{k}} = A_\varepsilon k^{-3}, \qquad n_{\boldsymbol{k}} = B_\varepsilon k^{-2}}\,. \tag{8.110}$$

As in (8.101), the constants have acquired a subscript indicating that they pertain to the constant-ε regime only.

Let us now look at the residual-energy flux. Unlike for Γ_p in (8.98), we do not have a nice general expression for ε_k in terms of $N_{\boldsymbol{k}}$ and $n_{\boldsymbol{k}}$. For power-law solutions, however, we can easily get ε_k from (8.105) by direct integration:

$$\varepsilon_k = -\left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_\mathrm{s} A^2}{4\alpha} \frac{I_2 + (2-x) I_1}{6 - 2x}\, k^{6-2x}. \tag{8.111}$$

The k dependence does indeed disappear at $x = 3$, but the remaining expression has a $0/0$ indeterminacy. By L'Hôpital's rule,

$$-\left[\frac{I_2 + (2-x) I_1}{6-2x}\right]_{x=3} = \frac{1}{2}\left[\frac{\partial I_2}{\partial x} + (2-x)\frac{\partial I_1}{\partial x} - I_1\right]_{x=3} = c_1 + c_2 \frac{c_\mathrm{s} B_\varepsilon}{\alpha A_\varepsilon}, \tag{8.112}$$

where the numerical coefficients, arising from the ugly melée of fractions and logs that is the exact calculation of I_1, I_2, and their derivatives, are

$$c_1 = \frac{4(3\ln 2 - 1)}{9} \approx 0.035, \qquad c_2 = \frac{3754}{3675} - \frac{44 \ln 2}{35} \approx 0.150. \tag{8.113}$$

For the record,

$$I_1 = \frac{4}{3} + \frac{2^{5-x} x\,[32 + x(x-10)]}{(x-6)(x-8)(x-10)} \frac{c_\mathrm{s} B}{\alpha A}, \tag{8.114}$$

$$I_2 = \frac{2^{2x-3}\,[4 + x(x-3)]}{x(x^2-1)} - \frac{2^{x-1}\,[8 + x(x-2)]}{(x+2)(x+4)} \frac{c_\mathrm{s} B}{\alpha A}, \tag{8.115}$$

hence, after differentiation and substitution of $x = 3$, the values (8.113).

The useful takeway is that the coefficients (8.113) are positive, and so then is the residual-energy flux (8.111), always:

$$\boxed{\varepsilon = \left(\frac{V}{2\pi^2}\right)^2 \frac{\hbar w_0 c_\mathrm{s} A_\varepsilon^2}{4\alpha} \left(c_1 + c_2 \frac{c_\mathrm{s} B_\varepsilon}{\alpha A_\varepsilon}\right) > 0}\,. \tag{8.116}$$

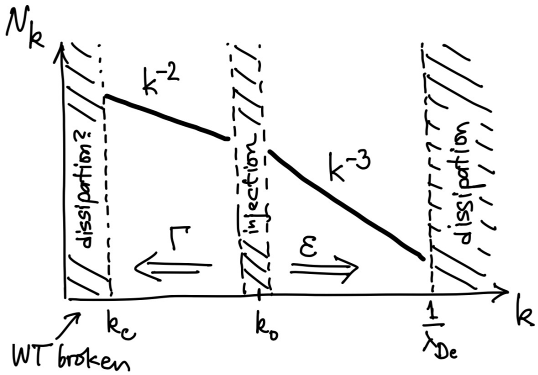


FIGURE 43. Direct and inverse cascades. Life in the shaded regions is not described by the scale-invariant WT calculation, but the spectra and cascades in between are. The dissipation region at small scales starts at $k\lambda_{\mathrm{De}} \sim 1$, where Langmuir waves start getting heavily Landau damped (cf. Fig. 16). The (presumed) dissipation region at large scales, where WT breaks down, should start around $k \sim k_{\mathrm{c}}$ given by (8.127).

8.4.6. *Direct and Inverse Cascades*

Thus, if energy is injected into a plasmon–phonon gas at some wave number k_0 and at the rate ε, it will be *cascaded directly*, i.e., flow to larger wave numbers (smaller scales) $k > k_0$, leaving in its wake the distributions (8.110). The energy here is the residual energy (8.104), from which the constant part of the plasmon frequency has been taken out.

The latter (in fact, dominant) part of the total energy, $\sum_{\boldsymbol{p}} \hbar\omega_{\mathrm{pe}} N_{\boldsymbol{p}}$, is proportional to the plasmon number N and conserved separately. It cannot also have a direct cascade. Indeed, the plasmon flux is given by (8.100) and we can easily calculate it for the spectra (8.110), i.e., for $x = 3$ and $y = 2$: using (8.114),

$$\Gamma_k = -\left(\frac{V}{2\pi^2}\right)^2 \frac{w_0 c_{\mathrm{s}} A_\varepsilon^2}{8\alpha^2} \left(\frac{4}{3} - \frac{44}{35}\frac{c_{\mathrm{s}} B_\varepsilon}{\alpha A_\varepsilon}\right) k^{-2} \to 0 \quad \text{as} \quad k \to \infty. \tag{8.117}$$

So it peters out at small scales. The natural conclusion is, therefore, that the plasmon number will *cascade inversely*, i.e., flow to smaller wave numbers (larger scales) $k < k_0$. The expression (8.102) must then be negative—not being sign-definite, it can be, implying $c_{\mathrm{s}} B_\Gamma < \alpha A_\Gamma$ for the distributions (8.101).

The dual-cascade picture just described is illustrated in Fig. 43.

Exercise 8.8. What is the residual-energy flux at $k < k_0$? Does it matter in this region?

There is a very simple dimensional argument, invented by Fjørtoft (1953) in the context of 2D turbulence, that allows one to predict which invariant will flow to small scales and which to large. Dimensionally, the constant-flux distributions must satisfy

$$\frac{k\bar{N}_k}{\tau_k} \sim \Gamma, \qquad \frac{k\bar{E}_k}{\tau_k} \sim \frac{\hbar\alpha k^3 \bar{N}_k}{\tau_k} \sim \varepsilon, \tag{8.118}$$

where τ_k is the effective rate at which nonlinear interactions happen at the scale k^{-1}. At the injection wave number k_0, this implies that

$$\varepsilon \sim \hbar\alpha k_0^2 \Gamma. \tag{8.119}$$

If there were a direct cascade of plasmon number, this would imply, at $k \gg k_0$, an arriving residual-energy flux

$$\frac{k\bar{E}_k}{\tau_k} \sim \hbar\alpha k^2 \Gamma \gg \varepsilon. \tag{8.120}$$

No such supply of residual energy is available from the injection scale, so this scenario cannot be realised. In contrast, if the plasmon number cascades inversely, to $k \ll k_0$, the amount of residual energy arriving there is minuscule, so everything is fine. The basic conclusion is that the invariant that has more powers of k cascades directly and the one that has fewer inversely.

Let me summarise our formal progress. We have found the scaling of the distributions, the direction of cascades, and the relationships (8.102) and (8.116) between the constant fluxes corresponding to these cascades and the constants A_Γ, B_Γ, A_ε, and B_ε in the distributions (8.101) and (8.110). Thus, we have a viable solution.[85] Note that the fact that the integrals (8.108) and (8.109) that went into the calculation of the fluxes converged is the confirmation of *locality* of fluxes—a blow up would have indicated that the whole approach of solving in scale-invariant ranges of k away from injection and dissipation scales had, in fact, been inapplicable.

There is one outstanding task that I will not take up. Since the constants A_Γ, B_Γ and A_ε, B_ε are in general different, we have four constants to determine and so far only two equations relating them to the fluxes: (8.102) and (8.116). The additional constraints will come from matching the fluxes to the injection region around k_0, where the (in general, nonuniversal) physics of energy injection will connect Γ and ε to the perturbation amplitudes—and thus to k_0 and the constants.

An elegant way to sort this out would be to demand that the plasmon flux (8.117) at $k > k_0$ and the energy flux at $k < k_0$ (calculated in Exercise 8.8) be exactly zero. The former condition gives a nontrivial relationship between A_ε and B_ε, but in trying to enforce the latter, one runs into the problem that setting $\varepsilon_k = 0$ also makes $\Gamma = 0$ at $k < k_0$. This exhumes from under the rug a subtlety going rather beyond the range of issues that I wish to enagage with here. The problem at hand has an annoying degeneracy: the constant-Γ scalings (8.101) actually coincide with the scalings of the equilibrium Rayleigh–Jeans distributions (7.41). Does this mean that they cannot support anything other than zero fluxes? This wrinkle was ironed out by Kanashov & Rubenchik (1980), whose solution is reproduced at the end of §3.2.2 of Zakharov *et al.* (1992): the gist of it is that the true solutions are, in fact, ever so slightly different from the power laws (8.101) (there are logarithmic corrections) and can support non-zero fluxes after all. Do investigate if you are intrigued.

This is as much as I shall say about the formal WT theory. The subject is much vaster than this, and full of deep insights, subtle nuances, clever tricks, twists and turns. I have chosen to work through one complete calculation, which features many of the key ideas and methods of the WT theory, but if you wish to become an expert, read the textbooks by Zakharov *et al.* (1992) and Nazarenko (2011) and the rest of the literature cited above.[86] Either of Exercises 8.9 or 8.10 offers a proactive independent-study path that should prove both challenging and exciting.

Exercise 8.9. Spectra of Langmuir turbulence with four-wave interactions. Starting from the kinetic equation derived in Exercise 8.7, work out the Langmuir-wave spectra arising from four-wave interactions. Your conclusion should be that plasmons will flow to larger scales,

[85] It is perhaps worth spelling out that this is *a* solution. No proof has been provided, and there is none, that the isotropic solution is the only one possible. Indeed, anisotropic solutions do exist. I again refer you to Musher *et al.* (1995) for a taste of them.

[86] The example that I have chosen is not, in fact, (in my view) particularly clearly presented in Zakharov *et al.* (1992), or in the original paper by Zakharov & Kuznetsov (1978), so I hope this section has provided some added value.

leaving behind a distribution $N_{\boldsymbol{k}} \propto k^{-7/3}$. The answers are in Zakharov (1972) or Zakharov *et al.* (1992). In getting there, you will learn about *Zakharov transformations*, an inspired feat of analytical skulduggery, of which the mapping $\zeta = 1/\xi$ in §8.4.5 was an almost trivial example. You will also experience locality not being always guaranteed.

Exercise 8.10. Induced scattering of Langmuir waves on ions. This is an independent-study topic. Consider the regime ($T_i \sim T_e$) in which sound waves are strongly damped but not so strongly as for the hot-ion quasistatic limit (§8.2.7) to apply. That is, the system is described by the first Zakharov equation (8.16) coupled via $\delta\bar{n}_e$ to the ion kinetic equation (8.57). In this regime, the dominant nonlinear process is the induced scattering of plasmons on ions (analogous to what was considered in §7.2.4). Work out what the scattering probability is, what, therefore, are the kinetic equations for the plasmons and the ions, and study what kind of solutions they might have. In the literature, the common monicker for this regime is "isothermal Langmuir turbulence" ("isothermal" in the sense that $T_i = T_e$; the Langmuir–sound turbulence discussed above is, accordingly, "nonisothermal Langmuir turbulence"). In navigating the literature, you may choose to follow the Tsytovich approach, featuring heavy-duty perturbative kinetic calculations (see the books cited at the end of §7.1 or the original papers cited in §7.2.4), or (or and?) the Zakharov one, with a more Hamiltonian flavour (see reviews, and references therein, by Zakharov *et al.* 1985 and Musher *et al.* 1995, the latter focussing specially on WT). If you plough through even some of it, you have my respect, but you may well emerge with an enduring hatred of the Soviet scientific writing style.[87]

8.4.7. *Breakdown of WT Approximation*

Let me focus on a key finding of the above WT calculation: both the plasmons' number N and, to lowest approximation, their energy $\hbar\omega_{\rm pe}N$ are pushed by the WT interactions to ever larger scales. As phonons have a shallower distribution at these scales than the plasmons, $n_{\boldsymbol{k}} \propto kN_{\boldsymbol{k}}$ (and an even shallower energy spectrum $\hbar k c_{\rm s} n_{\boldsymbol{k}}$), everything is dominated by the plasmons there—a phenomenon sometimes referred to as the formation of a "Langmuir condensate". I shall now argue that the WT approximation will break down for this condensate, below a certain critical wave number.

The condition for a system to be in the WT regime is that quasiparticles endure over a number of interactions, i.e., their frequencies must be large compared to the characteristic rate at which the nonlinearity acts—in §7.5, I wrote this as (7.46). Let us check when this is satisfied for Zakharov's equations. For the first of them, (8.16), describing the evolution of the plasmon field, this requirement takes the form

$$\delta\omega_{\boldsymbol{k}} = \frac{3}{2}\,k^2\lambda_{\rm De}^2\omega_{\rm pe} \gg t_{\rm nl}^{-1} \sim \omega_{\rm pe}\frac{\delta\bar{n}_e}{n_{0e}} \quad\Leftrightarrow\quad \boxed{\frac{\delta\bar{n}_e}{n_{0e}} \ll k^2\lambda_{\rm De}^2}\,, \tag{8.121}$$

an upper limit on the perturbed density amplitude, or, equivalently, a lower limit on the wave number.

Now consider the second Zakharov equation (8.17). The condition for the nonlinear term in this equation to be small compared to the linear terms is

$$\frac{\delta\bar{n}_e}{n_{0e}} \gg W = \frac{\overline{|\widetilde{\boldsymbol{E}}|^2}}{8\pi n_e T_e}, \tag{8.122}$$

where W is the ratio of electric-field energy density to the electron thermal pressure—a measure of the energy in the plasmon field. Obviously, in order for the conditions (8.122) and (8.121) to be realisable simultaneously, it must be the case that

$$\boxed{W \ll k^2\lambda_{\rm De}^2}\,. \tag{8.123}$$

---

[87]This Exercise, alongside Exercise 8.9 and its prequels, has only been completed once since the inception of this course, by Agi Wierzchucka as part of her MMathPhys dissertation (2024).

Note that since $\widetilde{\boldsymbol{E}} = -\boldsymbol{\nabla}\tilde{\varphi}$, this condition is equivalent to

$$\frac{e\tilde{\varphi}}{T_e} \ll 1, \tag{8.124}$$

whereas Zakharov's equations are perfectly valid when $e\tilde{\varphi}/T_e \sim 1$ [see (8.38)], so they are able to describe non-WT dynamics.

As Langmuir excitations move to smaller k's, the condition (8.123) of WT's validity for those excitations at those scales, viz.,

$$\frac{1}{n_e T_e}\int_0^k \mathrm{d}p\,\hbar\omega_{\mathrm{pe}}\bar{N}_p \ll k^2\lambda_{\mathrm{D}e}^2, \tag{8.125}$$

becomes ever more stringent and eventually untenable. I have integrated the plasmon energy only up to k so as to include just the larger-scale excitations. Let us estimate when (8.125) becomes untenable. Since, according to (8.101), $N_{\boldsymbol{p}} \propto p^{-2}$, we have[88]

$$\bar{N}_p \propto p^2 N_{\boldsymbol{p}} \propto p^0 \quad \Rightarrow \quad N \sim \int_0^{k_0}\mathrm{d}p\,\bar{N}_p \sim k_0\bar{N}_p \quad \Rightarrow \quad \int_0^k \mathrm{d}p\,\hbar\omega_{\mathrm{pe}}\bar{N}_p \sim \frac{kH^l}{k_0}, \tag{8.126}$$

where $H^l \approx \hbar\omega_{\mathrm{pe}}N \equiv Wn_eT_e$ is the total plasmon energy, with W as defined in (8.122). Therefore, the critical wave number k_{c} at which WT breaks down is

$$\boxed{k_{\mathrm{c}}\lambda_{\mathrm{D}e} \sim \frac{W}{k_0\lambda_{\mathrm{D}e}}}\,. \tag{8.127}$$

Thus, WT drives itself into a strong-turbulence regime. Something strongly nonlinear must happen at $k \lesssim k_{\mathrm{c}}$ for these scales to receive the plasmon flux supplied to them by the WT cascade and, if there is to be a stationary state, to dissipate it. What that might be is the subject to which I shall now turn.

Exercise 8.11. Breakdown of WT approximation for Langmuir turbulence in the quasistatic limit. In §8.1.4, we saw that at $k\lambda_{\mathrm{D}e} \ll \sqrt{m_e/m_i}$, the ponderomotive force arising from the plasmon field becomes too slow to generate sound waves and the ion dynamics instead become quasistatic. Then there are no longer sound waves, and the inequality (8.122) turns into equality. Together with (8.121), it again gives (8.123) as the validity condition for WT—but this is a different WT, one in which four-wave interactions dominate. As shown in Exercise 8.9, this too will support an inverse plasmon cascade (indeed it is the sole physically realisable constant-flux solution for that kind of turbulence). Suppose the transition to this WT regime happens before the critical wave number (8.127) is reached. Estimate the critical wave number at which WT will break down in this case.

Exercise 8.12. Weak MHD turbulence. If you have already followed §16.3, this is a good place to do Exercise 16.12. Note in particular the rather subtle set of issues around the breakdown of WT approximation for weak RMHD turbulence—you can think about that on your own and/or read Schekochihin (2022, §4 and Appendix A).

---

[88]The contribution to N from $p > k_0$ is not, strictly speaking, negligible. Since $N_{\boldsymbol{p}} \propto p^{-3}$ in that range [see (8.110)], the shell-integrated distribution is $\bar{N}_p \propto p^{-1}$. This will give an additional contribution to N of the order of $\ln(1/k_0\lambda_{\mathrm{D}e})$ (assuming dissipation kicks in at $k\lambda_{\mathrm{D}e} \sim 1$). This is a log of a largish number, but it is not a big crime to view it as order unity. Note that unlike the plasmon energy, the energy of the phonons is very heavily dominated by the smallest scales ($\lambda_{\mathrm{D}e}$) because, at $k > k_0$, $n_{\boldsymbol{k}} \propto k^{-2}$ and so the corresponding 1D energy spectrum is $\propto \varOmega_k k^2 n_{\boldsymbol{k}} \propto k$.

8.5. *Kinetics and Fluid Dynamics of Langmuir Condensate*

The WT inverse cascade has filled large scales with a Langmuir condensate, which we no longer assume satisfies the WT assumptions. Phonons are scarce. What can be done to describe this situation? Here is another flavour of quasiparticle kinetics that is both useful and interesting to play with.

8.5.1. *Spatially Inhomogeneous Quasiparticle Kinetics*

Let us consider the interaction of the condensate with a very slow and very large-scale (much larger than the plasmon scales) density field—a gentle modulation of the plasmon frequency. Zakharov's equations (8.16) and (8.17) are still valid. We shall still want to treat the Langmuir oscillations as quasiparticles but now let us think of them as living in a spatially varying density field (equivalently, potential) determined by the ion motions, with the latter not necessarily quantisable into phonons—meaning that the nonlinearity in (8.17) no longer needs to be small. This breaks the restriction (8.122) and, therefore, (8.123) no longer needs to be satisfied, even though (8.121) can still hold (and so plasmons are still plasmons). This partial liberation from WT will allow us to derive again the kinetic equation for the plasmons by working out an evolution equation for their average energy at each wave number, but this time treating the slow density field $\delta\bar{n}_e$ as non-random, and thus not involved in the averaging. This approach and the results of it that I will show below (§8.5.2 and Exercises 8.13 and 8.14), were pioneered, early in the WT game, by Vedenov & Rudakov (1965) and pedagogically expanded in the follow-up paper by Vedenov *et al.* (1967).

Mathematically, the new trick is to introduce a two-point correlation function of the plasmon field

$$\langle\psi(\boldsymbol{r})\psi^*(\boldsymbol{r}')\rangle = C(\boldsymbol{R},\boldsymbol{\rho}), \qquad \boldsymbol{R} = \frac{\boldsymbol{r}+\boldsymbol{r}'}{2}, \qquad \boldsymbol{\rho} = \boldsymbol{r}-\boldsymbol{r}', \tag{8.128}$$

and treat it as a function not of $\boldsymbol{r}$ and $\boldsymbol{r}'$ but the two new variables $\boldsymbol{R}$ and $\boldsymbol{\rho}$ introduced above. We shall assume that $\boldsymbol{\nabla}_{\boldsymbol{\rho}}C \gg \boldsymbol{\nabla}_{\boldsymbol{R}}C$, i.e., that the $\boldsymbol{\rho}$ variable captures the smaller scales at which plasmons exist, while $\boldsymbol{R}$ picks up the spatial inhomogeneity associated with the gentle variation of $\delta\bar{n}_e$ (this is the same as taking the limit, which I have so far avoided, of phonon wave numbers being small compared to the plasmon ones, $k \ll p$ in the notation of §7.2.3; this limit should be recoverable from our new approach). In terms of the new variables,

$$\boldsymbol{r} = \boldsymbol{R} + \frac{\boldsymbol{\rho}}{2}, \qquad \boldsymbol{r}' = \boldsymbol{R} - \frac{\boldsymbol{\rho}}{2}, \qquad \boldsymbol{\nabla}_{\boldsymbol{r}} = \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}} + \boldsymbol{\nabla}_{\boldsymbol{\rho}}, \qquad \boldsymbol{\nabla}_{\boldsymbol{r}'} = \frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{R}} - \boldsymbol{\nabla}_{\boldsymbol{\rho}}. \tag{8.129}$$

The average energy density of the plasmon field *at any given point* $\boldsymbol{R}$ *in space* is

$$\frac{\langle|\boldsymbol{\nabla}\psi(\boldsymbol{R})|^2\rangle}{8\pi} = \left[\frac{\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\boldsymbol{\nabla}_{\boldsymbol{r}'}\langle\psi(\boldsymbol{r})\psi^*(\boldsymbol{r}')\rangle}{8\pi}\right]_{\boldsymbol{r}=\boldsymbol{r}'=\boldsymbol{R}} \approx \left[-\frac{\nabla^2_{\boldsymbol{\rho}}C(\boldsymbol{R},\boldsymbol{\rho})}{8\pi}\right]_{\boldsymbol{\rho}=0}$$
$$= \sum_{\boldsymbol{p}}\frac{p^2C_{\boldsymbol{p}}(\boldsymbol{R})}{8\pi} \equiv \sum_{\boldsymbol{p}}\hbar\omega_{\boldsymbol{p}}f_{\boldsymbol{p}} \quad\Rightarrow\quad \boxed{f_{\boldsymbol{p}}(\boldsymbol{R}) \approx \frac{p^2C_{\boldsymbol{p}}(\boldsymbol{R})}{8\pi\hbar\omega_{\mathrm{pe}}}}\,, \tag{8.130}$$

where $C_{\boldsymbol{p}}(\boldsymbol{R})$ is the Fourier transform of $C(\boldsymbol{R},\boldsymbol{\rho}) = \sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{\rho}}C_{\boldsymbol{p}}(\boldsymbol{R})$ with respect to $\boldsymbol{\rho}$. The new definition of the plasmons' *occupation-number density* $f_{\boldsymbol{p}}$ is obvious; what I previously called their occupation number is the integral of this new quantity over all space: $N_{\boldsymbol{p}} = \int\mathrm{d}\boldsymbol{R}\,f_{\boldsymbol{p}}(\boldsymbol{R})$, but the new feature now is that $f_{\boldsymbol{p}}$ is allowed to vary (slowly) in space.

Let us make an enlightened guess as to what the kinetic equation for $f_{\boldsymbol{p}}(\boldsymbol{R})$ must be. This is a distribution function of quasiparticles whose one-particle Hamiltonian is

$$H_1(\boldsymbol{R},\boldsymbol{p}) = \hbar\omega_{\boldsymbol{p}}(\boldsymbol{R}) \approx \hbar\omega_{\rm pe}\left[1 + \frac{1}{2}\frac{\delta\bar{n}_e(\boldsymbol{R})}{n_{0e}} + \frac{3}{2}\,p^2\lambda_{\rm De}^2\right]. \tag{8.131}$$

Its dependence on the quasiparticle momentum $\hbar\boldsymbol{p}$ comes from the usual wave-number dispersion in the frequency and its dependence on the particle position $\boldsymbol{R}$ from the fact that the frequency is slowly modulated in space by $\delta\bar{n}_e$. Then, by Liouville's theorem, the distribution function of these particles must satisfy (with $\hbar$'s helpfully cancelling) an equation that is quite familiar to the practitioners of "ray-tracing" optics:

$$\frac{\partial f_{\boldsymbol{p}}}{\partial t} = -\dot{\boldsymbol{R}}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}} f_{\boldsymbol{p}} - \dot{\boldsymbol{p}}\cdot\frac{\partial f_{\boldsymbol{p}}}{\partial \boldsymbol{p}} = -\frac{\partial\omega_{\boldsymbol{p}}}{\partial\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}} f_{\boldsymbol{p}} + (\boldsymbol{\nabla}_{\boldsymbol{R}}\omega_{\boldsymbol{p}})\cdot\frac{\partial f_{\boldsymbol{p}}}{\partial\boldsymbol{p}}. \tag{8.132}$$

Denoting by $\boldsymbol{v}_{\boldsymbol{p}} = 3\omega_{\rm pe}\lambda_{\rm De}^2\boldsymbol{p}$ the group velocity of a Langmuir wave and resuscitating the §8.3 notation $\xi = \delta\bar{n}_e/n_{0e}$, we get

$$\boxed{\frac{\partial f_{\boldsymbol{p}}}{\partial t} + \boldsymbol{v}_{\boldsymbol{p}}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}} f_{\boldsymbol{p}} - \frac{\omega_{\rm pe}}{2}\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\cdot\frac{\partial f_{\boldsymbol{p}}}{\partial\boldsymbol{p}} = 0}\,. \tag{8.133}$$

This looks exactly like the electrostatic Vlasov equation (1.50), with the wave's group velocity playing the part of quasiparticle's velocity and the ion-scale (electron) pressure providing the effective potential. The proof that this really works is given below, but the easily convinced and the impatient can leap over it.

Derivation of (8.133). The starting point must be Zakharov's first equation (8.16), which I would like to rewrite as follows

$$\nabla_{\boldsymbol{r}}^2\left(i\omega_{\rm pe}^{-1}\frac{\partial\psi}{\partial t} + \frac{3}{2}\,\lambda_{\rm De}^2\nabla_{\boldsymbol{r}}^2\psi - \frac{\xi\psi}{2}\right) = -\frac{1}{2}\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot(\psi\boldsymbol{\nabla}_{\boldsymbol{r}}\xi)\,. \tag{8.134}$$

Let us multiply this equation by $\nabla_{\boldsymbol{r}'}^2\psi^*(\boldsymbol{r}')$ and subtract the same equation but complex-conjugated and with $\boldsymbol{r}$ and $\boldsymbol{r}'$ swapped:

$$\begin{aligned}\nabla_{\boldsymbol{r}}^2\nabla_{\boldsymbol{r}'}^2\left[i\omega_{\rm pe}^{-1}\frac{\partial}{\partial t} + \frac{3}{2}\,\lambda_{\rm De}^2\left(\nabla_{\boldsymbol{r}}^2 - \nabla_{\boldsymbol{r}'}^2\right) - \frac{\xi(\boldsymbol{r}) - \xi(\boldsymbol{r}')}{2}\right]\psi(\boldsymbol{r})\psi^*(\boldsymbol{r}') = \\ -\frac{1}{2}\Big\{\nabla_{\boldsymbol{r}'}^2\boldsymbol{\nabla}_{\boldsymbol{r}}\cdot\left[\psi(\boldsymbol{r})\psi^*(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}}\xi(\boldsymbol{r})\right] - \nabla_{\boldsymbol{r}}^2\boldsymbol{\nabla}_{\boldsymbol{r}'}\cdot\left[\psi(\boldsymbol{r})\psi^*(\boldsymbol{r}')\boldsymbol{\nabla}_{\boldsymbol{r}'}\xi(\boldsymbol{r}')\right]\Big\}.\end{aligned} \tag{8.135}$$

Now average this equation, leaving ξ outside the averages, as promised above, and use (8.129) to transform everything to the $(\boldsymbol{R},\rho)$ variables. Assuming slow variation of ξ, we may approximate

$$\xi(\boldsymbol{r}) \approx \xi(\boldsymbol{R}) + \frac{\boldsymbol{\rho}}{2}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\xi(\boldsymbol{R}),\quad \xi(\boldsymbol{r}') \approx \xi(\boldsymbol{R}) - \frac{\boldsymbol{\rho}}{2}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\xi(\boldsymbol{R}) \quad\Rightarrow\quad \boldsymbol{\nabla}_{\boldsymbol{r}}\xi(\boldsymbol{r}) \approx \boldsymbol{\nabla}_{\boldsymbol{r}'}\xi(\boldsymbol{r}') \approx \boldsymbol{\nabla}_{\boldsymbol{R}}\xi(\boldsymbol{R}) \tag{8.136}$$

and neglect $\boldsymbol{\nabla}_{\boldsymbol{R}}C$ compared to $\boldsymbol{\nabla}_{\boldsymbol{\rho}}C$ wherever opportune. The result is

$$\nabla_{\boldsymbol{\rho}}^4\left[i\omega_{\rm pe}^{-1}\frac{\partial}{\partial t} + 3\lambda_{\rm De}^2\boldsymbol{\nabla}_{\boldsymbol{\rho}}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}} - \frac{1}{2}\,(\boldsymbol{\rho}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\right]C(\boldsymbol{R},\boldsymbol{\rho}) = -\nabla_{\boldsymbol{\rho}}^2\left[(\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\cdot\boldsymbol{\nabla}_{\boldsymbol{\rho}}C(\boldsymbol{R},\boldsymbol{\rho})\right]. \tag{8.137}$$

Finally, Fourier transform in $\boldsymbol{\rho}$ and divide through by $-ip^2$:

$$\left(\omega_{\rm pe}^{-1}\frac{\partial}{\partial t} + 3\lambda_{\rm De}^2\boldsymbol{p}\cdot\boldsymbol{\nabla}_{\boldsymbol{R}}\right)p^2C_{\boldsymbol{k}} = \frac{p^2}{2}\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\cdot\frac{\partial C_{\boldsymbol{p}}}{\partial\boldsymbol{p}} + (\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\cdot\boldsymbol{p}C_{\boldsymbol{p}} = \frac{1}{2}\,(\boldsymbol{\nabla}_{\boldsymbol{R}}\xi)\cdot\frac{\partial}{\partial\boldsymbol{p}}p^2C_{\boldsymbol{p}}. \tag{8.138}$$

Since $C_{\boldsymbol{p}}$ is $f_{\boldsymbol{p}}$ times a constant [see (8.130)], (8.138) only needs to be divided by $8\pi\hbar$ to become the anticipated kinetic equation (8.133), q.e.d.

To (8.133) must be appended an equation for $\xi(\boldsymbol{R})$, which is just the second Zakharov equation (8.17), whereby ξ is coupled to the plasmon distribution via the ponderomotive force:

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla_{\boldsymbol{R}}^2\right)\xi(\boldsymbol{R}) = \nabla_{\boldsymbol{R}}^2 \frac{\hbar\omega_{\rm pe}}{2m_i n_{0i}}\sum_{\boldsymbol{p}} f_{\boldsymbol{p}}(\boldsymbol{R})}\,. \tag{8.139}$$

In our new kinetic system, where $f_{\boldsymbol{p}}(\boldsymbol{R})$ is coupled to the field $\xi(\boldsymbol{R})$, (8.139) plays the role of the Poisson equation (1.51).

We can do with the Vedenov–Rudakov system (8.133) and (8.139) all the same things that we know how to do with the Vlasov–Poisson system (1.50) and (1.51), e.g., derive the "Landau damping" of a sound wave in a plasmon gas (Exercise 8.13) or work out the QL diffusion of the latter in a random field of sound waves (Exercise 8.14), but the really interesting new phenomenon for the sake of which this has all been done is that the plasmon gas—the Langmuir condensate pumped up by the WT cascade—is unstable. This is an effect in which the ability to handle spatially inhomogeneous quasiparticle distributions provided to us by this new formalism turns out to be essential.

Exercise 8.13. Damping of a sound wave in a plasmon gas. (a) Via a calculation analogous to those in §3, work out the dispersion relation for a weakly damped sound wave propagating through plasmon gas and calculate its Landau-damping rate.

(b) Show that this is the same result as can be derived from the phonon kinetic equation (7.23) under appropriate assumptions. What is, therefore, $w(\boldsymbol{p},\boldsymbol{k})$ and does it agree with (8.89)?

Exercise 8.14. QL diffusion of plasmon gas. (a) Construct a QLT, analogously to §6, of plasmon diffusion in a stochastic field of sound waves.

(b) Show that this is also derivable from the plasmon kinetic equation (7.24) under appropriate assumptions. Again make sure that you have calculated $w(\boldsymbol{p},\boldsymbol{k})$ correctly.

Exercise 8.15. Plasmon–ion kinetics. Lifting the cold-ion approximation, i.e., allowing $T_i \sim T_e$, write a closed set of equations for coupled dynamics of plasmons and ions.

8.5.2. *Modulational Instability of Plasmon Gas*

Let us take the analogy between quasiparticle kinetics and "real" kinetics another logical step further and derive the "fluid dynamics" of the plasmon gas. Namely, define the density and velocity of the "plasmon flow",

$$N(\boldsymbol{R}) = \sum_{\boldsymbol{p}} f_{\boldsymbol{p}}(\boldsymbol{R}), \qquad \boldsymbol{U}(\boldsymbol{R}) = \frac{1}{N(\boldsymbol{R})}\sum_{\boldsymbol{p}} \boldsymbol{v}_{\boldsymbol{p}} f_{\boldsymbol{p}}, \tag{8.140}$$

and work out the evolution equations for them by taking moments of (8.133). As usual,

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla}\cdot(N\boldsymbol{U}) = 0, \tag{8.141}$$

$$N\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U}\right) = -\boldsymbol{\nabla}\cdot\mathsf{P} - \frac{3}{2}\,\omega_{\rm pe}^2\lambda_{\rm De}^2 N\boldsymbol{\nabla}\xi, \tag{8.142}$$

where the "plasmon pressure"

$$\mathsf{P} = \sum_{\boldsymbol{p}} (\boldsymbol{v}_{\boldsymbol{p}} - \boldsymbol{U})(\boldsymbol{v}_{\boldsymbol{p}} - \boldsymbol{U}) f_{\boldsymbol{p}} \tag{8.143}$$

will be neglected, subject to *a posteriori* confirmation in Exercise 8.16.

This assumption of "cold" plasmons gives us a closed set of the plasmon-fluid equations, which we can now linearise around a homogeneous, static state with a constant density

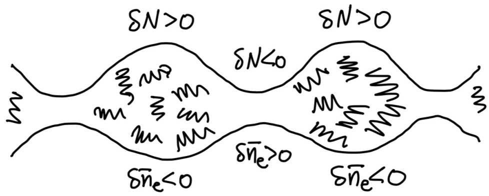


FIGURE 44. Modulational instability creates "caverns" filled with Langmuir oscillations ($\delta N > 0$), expelling plasma particles ($\delta \bar{n}_e < 0$).

N_0 and velocity $\boldsymbol{U}_0 = 0$:

$$\frac{\partial\,\delta N}{\partial t} + N_0 \boldsymbol{\nabla}\cdot\boldsymbol{U} = 0, \tag{8.144}$$

$$\frac{\partial \boldsymbol{U}}{\partial t} = -\frac{3}{4}\, v_{\rm the}^2 \boldsymbol{\nabla}\xi, \tag{8.145}$$

$$\frac{\partial^2\xi}{\partial t^2} - c_{\rm s}^2\nabla^2\xi = \frac{\hbar\omega_{\rm pe}}{2m_i n_{0i}}\nabla^2\delta N, \tag{8.146}$$

the last equation being (8.139). Assuming perturbations $\propto e^{-i\Omega t + i\boldsymbol{k}\cdot\boldsymbol{R}}$ converts these equation into the dispersion relation

$$\Omega^4 - k^2c_{\rm s}^2\Omega^2 - k^4v_0^4 = 0, \qquad v_0^4 = \frac{3\hbar\omega_{\rm pe}N_0v_{\rm the}^2}{8m_i n_{0i}} = \frac{3}{8}Wv_{\rm the}^2c_{\rm s}^2, \tag{8.147}$$

where $W = \hbar\omega_{\rm pe}N_0/n_{0e}T_e$ is the dimensionless Langmuir exitation level, already defined in (8.122). The solution of this dispersion relation is

$$\boxed{\Omega_k^2 = \frac{k^2c_{\rm s}^2}{2}\left(1 \pm \sqrt{1 + 4\frac{v_0^4}{c_{\rm s}^4}}\right)}\,. \tag{8.148}$$

The "+" root is the sound wave with a nonlinear modification of the frequency due to finite W (i.e., due to the plasmon gas supplying additional background pressure). The much more exciting "−" root is always unstable ($\Omega_k^2 < 0$)—this is called the *modulational instability*. The two interesting limits are the subsonic and the supersonic:

$$v_0 \ll c_{\rm s} \quad\Leftrightarrow\quad W \ll \frac{m_e}{m_i} \quad\Rightarrow\quad \Omega_k \approx i\frac{kv_0^2}{c_{\rm s}}, \tag{8.149}$$

$$v_0 \gg c_{\rm s} \quad\Leftrightarrow\quad W \gg \frac{m_e}{m_i} \quad\Rightarrow\quad \Omega_k \approx ikv_0. \tag{8.150}$$

The nature of the unstable perturbation becomes transparent if we work out the perturbation of the density of plasmons vs. that of ions and electrons: using (8.144) and (8.145), for $\Omega_k^2 < 0$,

$$\frac{\delta N_{\boldsymbol{k}}}{N_0} = -\frac{3k^2v_{\rm the}^2}{4|\Omega_k^2|}\left(\frac{\delta\bar{n}_e}{n_{0e}}\right)_{\boldsymbol{k}}. \tag{8.151}$$

Thus, the system breaks up into regions of low plasma density with high concentration of plasmons and vice versa (Fig. 44). The mechanism for the instability is quite simple: since ξ has the opposite sign to δN, the force in (8.145) is a negative pressure (on plasmons). An initial upward perturbation of the plasmon density in a region will act to suck in more plasmons while expelling plasma particles from that region by ponderomotive force. In other words, regions of high electric-energy (plasmon) density push out the plasma and regions of high plasma density push out the plasmons. Thus, the system breaks up into so-called "caverns" filled with Langmuir-oscillating electric fields. Inside these caverns, the plasma frequency is smaller than outside (because $\delta\bar{n}_e < 0$), so Langmuir waves will have a tendency to get trapped in them when the perturbation amplitude becomes finite and the modulational instability saturates.

Thus, the Langmuir condensate is not made up of monochromatic waves freely running around the system but rather of spatially localised plasmon bunches, known as *cavitons*. It is tempting to think of them as the new quasiparticles—and indeed in 1D you can do that because stable 1D soliton solutions to Zakharov's equations exist (§8.5.3). In 3D, however, the situation is trickier (§8.5.4).

Exercise 8.16. Validity of the cold-plasmon approximation. Work out the condition under which the results above are consistent with the neglect of the plasmon pressure in (8.142). You should find

$$\boxed{\Delta p^2 \lambda_{\mathrm{D}e}^2 \ll \begin{cases} W & \text{if} \quad W \ll \dfrac{m_e}{m_i}, \\ \sqrt{W\dfrac{m_e}{m_i}} & \text{if} \quad W \gg \dfrac{m_e}{m_i}, \end{cases}} \tag{8.152}$$

where Δp^2 is the mean square width of the plasmon distribution [cf. the WT validity condition (8.123)].

Exercise 8.17. Kinetic modulational instability. Relax the cold-plasmon assumption and work out a kinetic theory of the modulational instability by linearising (8.133) and (8.139) directly, without recourse to fluid equations (this is actually the same calculation as in Exercise 8.13, just with a different focus).

8.5.3. *Langmuir Solitons*

Coming soon, but for now, see Gorev *et al.* (1976), Thornhill & ter Haar (1978), and Kingsep (1996).

8.5.4. *Langmuir Collapse*

In 3D, caverns filled with plasmons (cavitons) will collapse self-similarly, in finite time. Here is the simplest, although nonrigorous, way to see this.

Let us imagine that modulational instability has resulted in the Langmuir condensate breaking up in many isolated cavitons, so we can treat each as a separate entity. Zakharov's equation (8.16) conserves the plasmon number, and we shall assume (somewhat cavalierly) that it does so in each individual caviton. Then, whatever the caviton does, it must be the case that

$$N \sim E^2 l^d \sim \text{const} \quad \Rightarrow \quad E^2 \sim \frac{1}{l^d}, \tag{8.153}$$

where $E^2/8\pi$ is the energy density of the Langmuir electric fields inside the caviton, l its characteristic size and d its dimension. This implies that if the caviton is not stationary, either E inside it will grow while its size diminishes, or vice versa. The possibility that E decays is not interesting because it would take us back to a homogeneous Langmuir condensate, which we know is unstable.

Let us then explore what kind of self-similar solutions of Zakharov's equations might exist that accommodate growing E. The two equations (8.16) and (8.17) can be written schematically in a "scaling" form keeping only the powers of the characteristic scale l

and characteristic time τ associated with each term, viz.,

$$\nabla^2\left(i\omega_{\rm pe}^{-1}\frac{\partial\psi}{\partial t}+\frac{3}{2}\lambda_{\rm De}^2\nabla^2\psi\right)=\frac{1}{2}\boldsymbol{\nabla}\cdot(\xi\boldsymbol{\nabla}\psi)\quad\Rightarrow\quad\left[\frac{E}{\tau}\right]+\left[\frac{E}{l^2}\right]=\left[\xi E\right],\tag{8.154}$$

$$\left(\frac{\partial^2}{\partial t^2}-c_{\rm s}^2\nabla^2\right)\xi=c_{\rm s}^2\nabla^2\frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e}\quad\Rightarrow\quad\left[\frac{\xi}{\tau^2}\right]+\left[\frac{\xi}{l^2}\right]=\left[\frac{E^2}{l^2}\right],\tag{8.155}$$

where $E\sim\boldsymbol{\nabla}\psi$ and I continue using the notation $\xi=\delta\bar{n}_e/n_{0e}$. Each of these two equations will boil down to a balance between two of its terms, with the third one negligible. At least one term involving τ must survive somewhere if there is to be a time-dependent solution. One can systematically go through all possible options as to which terms to neglect, hence work out the scalings of E, ξ, and τ with l and then check that the neglected terms do indeed vanish in comparison with the retained ones as $l\to 0$. It is not hard to convince oneself that the only viable possibility is to drop the first term on the left-hand side of (8.154) (i.e., the first Zakharov equation turns into static relation) and the second term on the left-hand side of (8.155) (i.e., the dynamics are supersonic).

Exercise 8.18. Check this. Show that the physical regime in which these approximations hold is

$$\boxed{W\gg\frac{m_e}{m_i}},\tag{8.156}$$

the supersonic limit that already made an appearance in (8.150).

The remaining balances then imply

$$\xi\sim\frac{1}{l^2},\qquad E^2\sim\frac{\xi l^2}{\tau^2}\sim\frac{1}{\tau^2}\quad\Rightarrow\quad\tau\sim l^{d/2},\tag{8.157}$$

where (8.153) was used at the last step. Knowing this, we see that it was fine to neglect the first term in (8.154) if

$$\frac{l^2}{\tau}\to 0\quad\text{as}\quad l\to 0\quad\Leftrightarrow\quad d<4,\tag{8.158}$$

and the second term in (8.155) if

$$\frac{\tau}{l}\to 0\quad\text{as}\quad l\to 0\quad\Leftrightarrow\quad d>2.\tag{8.159}$$

You can see why d matters. At $d=3$, we have a perfectly legitimate possibility.

Finally, what is the relationship of τ to time? Since $l\to 0$ with advancing time, we must have $\tau\to 0$ as well. The only way to achieve that is $\tau=t_0-t$, where t_0 is some fixed time. Then

$$\boxed{E^2\sim\frac{1}{(t_0-t)^2},\qquad\frac{\delta\bar{n}_e}{n_{0e}}\sim\frac{1}{(t_0-t)^{4/3}},\qquad l\sim(t_0-t)^{2/3}}.\tag{8.160}$$

Thus, the caviton collapses to a point and fields inside it go to infinity in finite time. The collapse is explosive.

This is, of course, not actually a solution of anything yet, only a piece of initial back-of-the-matchbox scoping out. At the very least, three further things must be done.

(i) The scalings (8.160) can be used to design the self-similar solution of Zakharov's equations (8.154) and (8.155), where the spatial dependence can be made spherically symmetric (since we have implicitly assumed the caviton to have only one scale), the new similarity variable is $r/(t_0-t)^{2/3}$, and the terms anticipated to be negligible should be neglected (note, however, that ψ is a complex field and can have a time-dependent phase, which we have learned nothing about in the above consideration, but which can be chosen to have such a time dependence that self-similarity is not destroyed). One could then see if the resulting equations have a reasonable solution.

(ii) What if $W \ll m_e/m_i$, i.e., as you saw in Exercise 8.18, the supersonic regime does not apply? It turns out that one can show that there is still a collapse, which will eventually take the plasmon energy beyond the supersonic limit, returning the above solution to relevance.

(iii) How inevitable is the collapsing solution? Can one demonstrate formally that it cannot be avoided? It turns out that one can.

Zakharov (1972) did all of those things, so I refer you to his paper for further mathematical enlightenment (this will require effort: his paper is not an easy read).

Why does Langmuir collapse matter? It matters because it provides a scenario for how energy pumped by WT from small scales into the Langmuir condensate at large ones might find a way to get dissipated. If the condensate breaks up into cavitons, which then collapse in finite time down to scales as small as $\lambda_{\mathrm{D}e}$ (which is the limit of validity of the equations predicting the collapse), then it is reasonable to think that the electric energy trapped in the cavitons will then be deposited into electrons via Landau damping, or some nonlinear version thereof (also a tricky subject: see §§12.2–12.4). Thus, the nonlinear physics finally accomplishes, by a circuitous route, what nonlinear physics is supposed to accomplish: bring energy from injection to thermalisation.

8.6. *Strong Langmuir Turbulence*

The task of the strong-turbulence theory then is to work out the distribution of energy in $\boldsymbol{k}$ space that is left in the wake of (or as signature of) its journey from cavitons formed by the modulational instability out of the Langmuir condensate at large scales, through their collapse to small scales, and finally to electron heat. There is no textbook-level certainty about the answers here, but there are many clever ideas.

8.6.1. *Soliton Turbulence*

Coming soon; see Kingsep *et al.* (1973), Gorev *et al.* (1976).

8.6.2. *Caviton Turbulence*

Coming soon. See reviews by Gorev *et al.* (1976), Thornhill & ter Haar (1978), Rudakov & Tsytovich (1978), Goldman (1984), Zakharov *et al.* (1985), and Robinson (1997), which document the accumulation and evolution of views over time, until saturation was reached.

Exercise 8.19. Statistical mechanics of Langmuir condensate. Pelletier (1980*a*,*b*) made a clever attempt to deploy the machinery of condensed matter physics to describe the statistical ensemble of *strongly interacting* plasmons with the Hamiltonian (8.78). If you like that sort of cross-cultural spirit, work through his calculation and see what you think.

Much of what I have dealt with in §§7.2–8.6 has been basically fluid turbulence—the kinetics was of quasiparticles. The fluctuating fields were perturbations on top of an equilibrium, which was hovering in the background and assumed to be Maxwellian whenever it needed to be specified. At no point after §7.1, with the brief exceptions of §§7.2.4, 7.2.5 and Exercise 8.10, did I engage with the question of the evolution of f_0, promised in §2 to be amongst the central ones in these Lectures. I am now going to come back to it from a number of very different angles than I have presented so far.

9. Nonlinear Stability and Thermodynamics of Collisionless Plasma

Let me go back to the generalist agenda first articulated at the beginning of §4: What kind of equilibria are stable?[89] Are there universal distributions to which a collisionless plasma will relax? This time I shall ask the stability question while eschewing any recourse to linear theory. Later on, this will nudge us towards certain distributions that will turn out to make some sense statistical-mechanically (§10) and that I will then show to be obtainable via QLT (§11).

I will start by presenting this material as a formal template for proofs of nonlinear stability (§9.1), but, after a revealing example (§9.2), it will become clear why I keep referring to it as "thermodynamics" (§9.3).

9.1. *Nonlinear Stability Theory: Thermodynamical Method*

The general idea of the method is to find, for a given initial equilibrium distribution $f(0)$, an upper bound on the amount of energy that might be transferred into electromagnetic perturbations (not necessarily small). If that bound is zero, the system is stable; if it is not zero but is sharp enough to be nontrivial, it gives us a constraint on the amplitude of the perturbations in the saturated state.

Here is how it is done.[90] Let us introduce a functional

$$\mathscr{F} = \underbrace{\int \mathrm{d}\boldsymbol{r}\,\frac{E^2+B^2}{8\pi}}_{\equiv\,\mathscr{U}} + \underbrace{\int \mathrm{d}\boldsymbol{Q}\,\left[A(\boldsymbol{Q},f) - A(\boldsymbol{Q},f_*)\right]}_{\equiv\,\mathscr{A}[f,f_*]} = \mathscr{U} + \mathscr{A}[f,f_*], \tag{9.1}$$

where $\boldsymbol{Q} = (\boldsymbol{r},\boldsymbol{v})$ is the phase-space variable and f_* is some trial distribution, which will sometimes represent our best guess about the properties of the stable distribution towards which the system wants to evolve and/or in the general vicinity of which we are interested in investigating stability.[91] The function $A(\boldsymbol{Q},f)$ is chosen in such a way that, for any f,

$$\mathscr{A}[f,f_*] \geqslant 0, \tag{9.2}$$

with the equality reached at $f = f_*$. If $\mathscr{A}[f,f_*]$ is also chosen so that $\mathscr{F}$ is conserved by the (collisionless) Vlasov–Maxwell equations, then $\mathscr{F}(t) = \mathscr{F}(0)$ and the inequality (9.2) gives us an upper bound on the field energy at time t:

$$\boxed{\mathscr{U}(t) - \mathscr{U}(0) = \mathscr{A}\big[f(0),f_*\big] - \mathscr{A}\big[f(t),f_*\big] \leqslant \mathscr{A}\big[f(0),f_*\big]}\,, \tag{9.3}$$

[89] In Q-6, isotropic, monotonically decreasing equilibria were found to be stable not just against infinitesimal (linear), electrostatic perturbations, but also against small but finite electromagnetic ones, giving us a taste of a powerful nonlinear constraint.

[90] These ideas appear to have crystallised in the papers by T. K. Fowler in the early 1960s (see his review, Fowler 1968; his reminiscences and speculations on the subject 50 years later can be found in Fowler 2016). A number of founding fathers of plasma physics were thinking along these lines around the same time (references are given in opportune places below).

[91] I use f_* instead of f_0 to denote this trial distribution for fear of confusion with the initial distribution $f(0)$ and because formally f_* need not be the coarse-grained mean equilibrium that f_0 has been defined to be everywhere since §2.2. But of course, f_0 is what you should be thinking about (certainly when $f - f_* \ll f_*$, as, e.g., in §9.4), and we shall be back with f_0 from §10 onwards.

where $f(0)$ is the initial ($t = 0$) equilibrium whose stability is under investigation. The bound (9.3) implies stability if $\mathscr{A}[f(0), f_*] = 0$, i.e., certainly for $f(0) = f_*$. This guarantees stability of any f_* for which a functional $\mathscr{A}[f, f_*]$ satisfying (9.2) and giving a conserved $\mathscr{F}$ can be produced.

This argument is based on the assumption that if the total electromagnetic enegy decreases, that corresponds to initial perturbations decaying. You might wonder what happens if $\mathscr{U}(0)$ contains some equilibrium magnetic field and if that equilibrium is unstable: can the equilibrium field's energy be tapped and transferred partially into unstable perturbations of kinetic energy in such a way that $\mathscr{U}(t) < \mathscr{U}(0)$ even though the system is unstable? I do not know how to isolate formally the set of conditions under which this is impossible (you may wish to think about this question; §18 might help). To avoid this problem, we could either restrict all considerations in this section to unmagnetised initial equilibria or consider initial states that are small perturbations over stable magnetised equilibria (e.g., a straight homogeneous magnetic field (16.1), or force-free, minimum-energy equilibria, §17.3), with the understanding that $\mathscr{U}(0)$ contains only the perturbed part of the magnetic energy. The latter approach connects $\mathscr{F}$, via §9.4, to free energy and its generalisations in drift-kinetic and gyrokinetic theories (Schekochihin *et al.* 2009; Kunz *et al.* 2015, 2018)—see Exercise 22.4.

Physically, the above construction is nontrivial if the bound (9.3) is smaller than the total initial kinetic energy of the particles:

$$\mathscr{A}\big[f(0), f_*\big] < \sum_\alpha \int \mathrm{d}\boldsymbol{Q}\, \frac{m_\alpha v^2}{2} f_\alpha(0) \equiv \mathscr{K}(0). \tag{9.4}$$

It is obvious that one cannot extract from a distribution more energy than $\mathscr{K}(0)$, but the above tells us that, in fact, one might only be able to extract substantially less. $\mathscr{A}\big[f(0), f_*\big]$ is an upper bound on the *available energy* of the distribution $f(0)$. The sharper it can be made, the closer we are to learning something useful. Thus, the idea is to identify some suitable functional $\mathscr{A}[f, f_*]$ for which $\mathscr{F}$ is conserved, and some class of trial distributions f_* for which (9.2) holds, then minimise $\mathscr{A}\big[f(0), f_*\big]$ within that class, subject to whatever physical constraints one can reasonably expect to hold: e.g., conservation of particles, momentum, and/or any other (possibly approximate) invariants of the system (e.g., its adiabatic invariants; see Helander 2017, 2020).[92]

To make some steps towards a practical implementation of this programme, let us first investigate how to choose A in such way as to ensure conservation of $\mathscr{F}$:

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = \frac{\mathrm{d}\mathscr{U}}{\mathrm{d}t} + \sum_\alpha \int \mathrm{d}\boldsymbol{Q}\, \frac{\partial A}{\partial f_\alpha}\frac{\partial f_\alpha}{\partial t} = \sum_\alpha \int \mathrm{d}\boldsymbol{Q} \left(\frac{\partial A}{\partial f_\alpha} - \frac{m_\alpha v^2}{2}\right)\frac{\partial f_\alpha}{\partial t} = 0. \tag{9.5}$$

The second equality was obtained by using the conservation of total energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{U} + \mathscr{K}) = 0, \quad \mathscr{K} = \sum_\alpha \int \mathrm{d}\boldsymbol{Q}\, \frac{m_\alpha v^2}{2} f_\alpha, \tag{9.6}$$

where $\mathscr{K}$ is the kinetic energy of the particles. Now (9.5) tells us how to choose A:

$$A(\boldsymbol{Q}, f) = \sum_\alpha \left[\frac{m_\alpha v^2}{2} f_\alpha + G_\alpha(f_\alpha)\right], \tag{9.7}$$

---

[92]Krall & Trivelpiece (1973) comment with a slight air of resignation that, with the rules of the game much vaguer than in linear theory, the thermodynamical approach to stability is "more art than science". In the Russian translation of their textbook, this statement provoked a disapproving footnote from the scientific editor (A. M. Dykhne), who observed that the *right* way to put it would have been "more art than craft".

where $G_\alpha(f_\alpha)$ are arbitrary functions of f_α. These can be added to A because Vlasov's equation has an infinite number of invariants: for any (sufficiently smooth) $G_\alpha(f_\alpha)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{Q}\, G_\alpha(f_\alpha) = 0. \tag{9.8}$$

This follows from the fact that, in the absence of collisions, the kinetic equation (1.30) expresses the conservation of phase volume in $(\boldsymbol{r}, \boldsymbol{v})$ space (the flow in this phase space is divergence-free).

Exercise 9.1. Prove the conservation law (9.8), assuming that the system is isolated.

These conserved quantities are sometimes referred to as *Casimir invariants.* The existence of an infinite number of conservation laws suggests that the evolution of a collisionless system in phase space is much more constrained than that of a collisional one, when the evolution is constrained only by conservation of particles, momentum and energy and the requirement (5.13) that entropy must not decrease. These constraints will play a major role in the discussion of collisionless relaxation in §§10 and 11. Keep in mind, however—and continue keeping in mind throughout that discussion—that these constraints do not, in fact, survive for very long in a real plasma. We saw in §5.5 that, in the presence of perturbations with $k \neq 0$, phase mixing would activate collisions after a time $t \propto \nu^{-1/3}$—much shorter than the collision time ν^{-1}, but, formally, still asymptotically long in the limit $\nu \to +0$. However, in turbulent plasmas, the collisional phase-space scales are, in fact, reached over times that are (nearly) independent of ν as $\nu \to +0$—just like in hydrodynamic turbulence, viscous ("Kolmogorov") scales are reached on time scales independent of viscosity as the latter tends to $+0$, a property know as the "dissipative anomaly". For now, I shall push on while ignoring this issue, but will return to it in §12.4.4 onwards.

Thus, the bound on the available energy is

$$\boxed{\mathscr{A}[f, f_*] = \mathscr{K} - \mathscr{K}_* + \sum_\alpha \int \mathrm{d}\boldsymbol{Q}\, \left[G_\alpha(f_\alpha) - G_\alpha(f_{\alpha *})\right]}\,, \tag{9.9}$$

provided f_* and G are chosen so that $\mathscr{A}[f, f_*] \geqslant 0$ for all f; here $\mathscr{K}_*$ is the kinetic energy of the trial distribution f_*. A quick sanity check is to try $G(f) = 0$. The inequality (9.2), $\mathscr{A}[f, f_*] \geqslant 0$, is then certainly satisfied for $f_* \propto \delta(\boldsymbol{v})$. The bound (9.3) becomes

$$\mathscr{U}(t) - \mathscr{U}(0) \leqslant \mathscr{K}(0), \tag{9.10}$$

i.e., one cannot extract any more energy than the total energy contained in the distribution—indeed, one cannot. Let us now see how to learn something more nontrivial.

9.2. *Fowler's Thermodynamics*

The formula (9.7) guarantees that the functional $\mathscr{F}$ given by (9.1) will be conserved by a collisionless plasma, but we still have the challenge of choosing $G(f)$ and f_* in such a way that (9.2) is satisfied, i.e., that f_* is the maximiser of $-\int \mathrm{d}\boldsymbol{Q}\, A(\boldsymbol{Q}, f)$. It is hard not to think immediately, as Fowler (1963, 1968) did, of thermodynamic entropy. As we know from statistical mechanics, the solution to the problem of maximising the entropy (5.12) subject to fixed kinetic energy and particle number,

$$-\int \mathrm{d}\boldsymbol{Q}\, f \ln f - \beta\left(\int \mathrm{d}\boldsymbol{Q}\, \frac{mv^2}{2} f - \mathscr{K}\right) - \lambda\left(\int \mathrm{d}\boldsymbol{Q}\, f - N\right) \to \max, \tag{9.11}$$

is the Maxwellian distribution

$$f_* = e^{-1-\lambda-\beta mv^2/2} \equiv C \exp\left(-\frac{mv^2}{2T}\right), \tag{9.12}$$

where I have renamed the Lagrange multipliers $\beta = 1/T$, $\lambda = -1 - \ln C$, and dropped the species index (all of this works species by species). From (9.9), it is then quite obviously the case that if we let

$$\boxed{G(f) = Tf\left(\ln\frac{f}{C} - 1\right)}\,, \tag{9.13}$$

then, for any f and any constants C and T, the required inequality (9.2), $\mathscr{A}[f, f_*] \geqslant 0$, is satisfied by (9.9). We have got our available-energy bounding function! Note that this argument also implies stability of any Maxwellian distribution (9.12) to finite perturbations (no available energy in f_*).

With this choice of $\mathscr{A}[f, f_*]$, (9.3) now provides an upper bound on the energy of the electromagnetic fields that can be extracted from any given initial distribution $f(0)$. In order to make this bound as sharp as possible, one picks the constants C and T (and, therefore, determines f_*) so as to minimise $\mathscr{A}[f(0), f_*]$.

As far as C is concerned, it is not hard to see that

$$\frac{\partial\mathscr{A}[f(0), f_*]}{\partial C} = \frac{T}{C}\int \mathrm{d}\boldsymbol{Q}\ [f_* - f(0)] = 0 \quad\Rightarrow\quad C = \frac{N}{V}\left(\frac{m}{2\pi T}\right)^{3/2} \equiv C_*(T), \tag{9.14}$$

i.e., the number of particles $N = \int \mathrm{d}\boldsymbol{Q}\, f(0)$ (of each species) must be the same for the initial distribution and for f_*.

Setting $C = C_*(T)$, let us now minimise $\mathscr{A}[f(0), f_*]$ with respect to T. Observe first that (9.9) can be slightly simplified:

$$\mathscr{A}[f(0), f_*] = \mathscr{K}(0) + T\int \mathrm{d}\boldsymbol{Q}\, f(0)\ln\frac{f(0)}{C_*(T)}. \tag{9.15}$$

Minimising this gives us

$$\frac{\partial\mathscr{A}[f(0), f_*]}{\partial T} = \int \mathrm{d}\boldsymbol{Q}\, f(0)\ln\frac{f(0)}{C_*(T)} + \frac{3}{2}N = 0. \tag{9.16}$$

The root T_* of this equation is straighforward to extract. It is then rather illuminating to observe that, with $T = T_*$, (9.15) becomes simply

$$\min_{C,T}\mathscr{A}[f(0), f_*] = \mathscr{K}(0) - \frac{3}{2}NT_* = \mathscr{K}(0) - \mathscr{K}_*, \tag{9.17}$$

i.e., just the difference between the kinetic energies of the initial distribution and the target Maxwellian distribution, whose temperature is set by (9.16).

9.2.1. *Anisotropic Equilibria*

To test-drive Fowler's method, let me consider an anisotropic initial distribution—the case that, at the end of §4, I had to relegate to Exercise 4.8 as it needed substantial extra work if it were to be handled by the method developed there. The *bi-Maxwellian distribution* is a useful and certainly the simplest model for anisotropic equilibria:

$$f(0) = \overline{C}\exp\left(-\frac{mv_\perp^2}{2T_\perp} - \frac{mv_\parallel^2}{2T_\parallel}\right), \quad \overline{C} = n\left(\frac{m}{2\pi\overline{T}}\right)^{3/2}, \tag{9.18}$$

where $\overline{T} = T_\perp^{2/3}T_\parallel^{1/3}$ and $T_\perp$ and $T_\parallel$ are the "temperatures" of particles' motion perpendicular and parallel to some special direction. Is this distribution unstable? (Yes: see Q-3.) To obtain an upper bound on the energy available for extraction from it, compute

$$\int \mathrm{d}\boldsymbol{Q}\, f(0) \ln \frac{f(0)}{C_*(T)} = \frac{3}{2} N \left(\ln \frac{T_*}{\overline{T}} - 1 \right). \tag{9.19}$$

Substituting this into (9.16), we see that the latter equation is satisfied for $T_* = \overline{T}$. The resulting estimate (9.17) of the available energy is

$$\mathscr{U}(t) - \mathscr{U}(0) \leqslant \min_{C,T} \mathscr{A}[f(0), f_*] = \frac{3}{2} N \left(\frac{2}{3} T_\perp + \frac{1}{3} T_\parallel - T_\perp^{2/3} T_\parallel^{1/3} \right). \tag{9.20}$$

The bound is zero when $T_\perp = T_\parallel$ and is always positive otherwise (because it is the difference between an arithmetic and a geometric mean of the two temperatures). We do not, of course, have any way of knowing how good an approximation this is to the true saturated level of whatever instability (if any) might exist here in any particular physical regime,[93] but this does show that temperature anisotropy is a source of free energy.

Further examples of such calculations can be found in Krall & Trivelpiece (1973, §9.14) and Fowler (1968). A certain further development of the methodology discussed above allows one to derive upper bounds not just on the energy of perturbations but also on their growth rates (Fowler 1964, 1968). If you feel that Fowler's privileging of the Maxwellian is arbitrary and unjustified, Exercise 10.1 is for you, but first do this one.

Exercise 9.2. Fowler's thermodynamics for beams. For some conveniently chosen model of a two-beam distribution (see §§3.7, 4.4, 10.1.3, and Q-4 for inspiration), calculate the upper bound on the available energy by Fowler's method.

9.3. *Why This is Thermodynamics*

As already hinted by my language and notation, $\mathscr{F}$ is the generalised version of the free energy (5.20), which was, in fact, the difference between the free energy of a given state of plasma and the free energy of a state characterised by f_*: in thermodynamic notation, we have

$$\boxed{\mathscr{F} = \mathscr{U} + \delta\mathscr{K} - \sum_\alpha T_\alpha \delta S_\alpha}\,, \tag{9.21}$$

where $\delta\mathscr{K} = \mathscr{K} - \mathscr{K}_*$, $\delta S_\alpha = S_\alpha[f_\alpha] - S_\alpha[f_{*\alpha}]$, and $S_\alpha[f_\alpha]$ is the entropy associated with the distribution f_α. Comparing this with (9.1) and (9.9), we see that our $\mathscr{F} = \mathscr{U} + \mathscr{A}$ matches (9.21) with the entropy

$$S_\alpha[f_\alpha] = -\frac{1}{T_\alpha} \int \mathrm{d}\boldsymbol{Q}\, G_\alpha(f_\alpha). \tag{9.22}$$

Thus, picking $G(f)$ is equivalent to picking an "effective entropy" function for the collisionless system and thereby choosing a particular "thermodynamics". If one then computes f_* by maximising this entropy subject to fixed $\mathscr{K}$, the inequality (9.2), $\mathscr{A}[f, f_*] \geqslant 0$, is guaranteed. All Fowler did was use the standard thermodynamic

---

[93]Nominally, this calculation applies with equal validity to many different instabilities that can be triggered by temperature anisotropy in both unmagnetised and magnetised plasmas—and indeed also to some anisotropic distributions that are, in fact, stable (which is common in magnetised plasmas where the externally imposed magnetic field is sufficiently large). Well, the fact that some energy is available does not mean that it will always be extracted. Indeed, Exercises 9.2, 9.3, 10.1 and 11.8 suggest that there are situations of interest in which none of the available-energy bounds derived in §9 represents an adequate estimate of what will actually happen.

entropy (5.12) in this argument. Clearly, this was begging to be done, but, in fact, why should this be the entropy for a collisionless plasma? Let us see if we can do anything more general, or better justified.

This may be an opportune place to pause and explain why entropy in a collisionless plasma is not "obviously" always just $-\int \mathrm{d}\boldsymbol{Q}\, f \ln f$. Indeed, is this not the expression that Shannon (1948) and Jaynes (2003) teach us that we must always go for, as we did in (5.12)? The crucial thing to realise is that the distribution function $f(\boldsymbol{Q})$ is *not* the probability that is supposed to go into Shannon's entropy, it is just the continuous version of the occupation number of a single-particle state given by the phase-space location $\boldsymbol{Q}$ (this is textbook material: see, e.g., Schekochihin 2024). The complete state of a box of classical gas or plasma is, in general, described by the *joint* probability of having a particle at $\boldsymbol{Q}_1$, another particle at $\boldsymbol{Q}_2$, and so on for all N particles in the system. Let me call this object $f_N(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N)$, a probability-density function in a $6N$-dimensional space. The Shannon entropy of this system is

$$S = -\int \cdots \int \mathrm{d}\boldsymbol{Q}_1 \dots \mathrm{d}\boldsymbol{Q}_N f_N(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N) \ln f_N(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N). \tag{9.23}$$

Now, since we assume, for a non-degenerate, classical plasma that particles do not know about each other and have identical distributions,

$$f_N(\boldsymbol{Q}_1, \dots, \boldsymbol{Q}_N) = f(\boldsymbol{Q}_1) \cdots f(\boldsymbol{Q}_N), \tag{9.24}$$

we can immediately conclude that

$$S = -N \int \mathrm{d}\boldsymbol{Q}\, f(\boldsymbol{Q}) \ln f(\boldsymbol{Q}), \tag{9.25}$$

recovering (5.12) up to an irrelevant multiplicative constant. For a collisionless plasma, no such simplification is possible. Its state is characterised, by definition, by some particular instantiation of the distribution function (phase density) $f(\boldsymbol{Q})$, which, in a highly phase-mixed, turbulent system, is a random variable. In principle, if we imagine an ensemble of such systems, there is a probability measure on the space of these functions, $P[f]$. Then the Shannon entropy for the system is

$$S = -\int \mathrm{D}f\, P[f] \ln P[f], \tag{9.26}$$

functionlly integrated over all possible f. This is not very helpful. The reason this does not break up into anything immediately obvious in terms of $\boldsymbol{Q}$-space integrals of functions of f is that a collisionless system is, in a manner of speaking, quantum—it conserves phase volume and so different bits of phase space cannot just be filled independently of each other, there is a kind of Pauli exclusion principle for phase volumes. This property will become explicit in §10. In §§10.3 and 11.7, I will formulate a generalised kinetic theory ("hyperkinetics") in which a version of (9.26) does indeed emerge as the right kind of entropy.

9.4. *Kruskal–Oberman Thermodynamics of Small Perturbations*

There is a neat development (due to Kruskal & Oberman 1958, but explained most clearly by Taylor 1963) of the formalism presented at the beginning of this section that allows one to dispense with Maxwellian equilibria at the price of keeping departures from the equilibrium small.

Let us investigate the stability of *isotropic*, spatially homogeneous distributions with respect to small (but not necessarily infinitesimal) perturbations, i.e., take $f(t) = f_* + \delta f(t)$, $\delta f \ll f_*$. If $f(0) = f_*$, the available-energy bound (9.3) will imply stability if we can find such a $G(f)$ that (9.2) holds.

In (9.7), we expand

$$G(f) = G(f_*) + G'(f_*)\delta f + G''(f_*)\frac{\delta f^2}{2} + \dots \tag{9.27}$$

and use this to obtain, keeping terms up to second order,

$$\mathscr{A}[f(t), f_*] = \int \mathrm{d}\boldsymbol{Q} \left\{ \left[\frac{mv^2}{2} + G'(f_*) \right] \delta f + G''(f_*) \frac{\delta f^2}{2} \right\}. \tag{9.28}$$

Suppose we contrive to pick $G(f_*)$ so as to obliterate the first-order term in (9.28):

$$G'(f_*) = -\frac{mv^2}{2} \equiv -\varepsilon. \tag{9.29}$$

This can be done because we have assumed that $f_* = f_*(\varepsilon)$ (it is isotropic) and if we also assume that it is a monotonic function of ε. Fowler's effective entropy (9.13) certainly has the property (9.29), but, hopefully, it is not the only one that does. Differentiating (9.29) with respect to f_*, we get

$$G''(f_*) = -\frac{1}{\partial f_*/\partial\varepsilon} \quad \Rightarrow \quad \mathscr{A}[f(t), f_*] = \int \mathrm{d}\boldsymbol{Q}\, \frac{\delta f^2}{2(-\partial f_*/\partial\varepsilon)}. \tag{9.30}$$

We see that $\mathscr{A}\big[f(t), f_*\big] \geqslant 0$ and, therefore, (9.3) with $f(0) = f_*$ implies stability of this equilibrium if (for every species)

$$\boxed{\frac{\partial f_*}{\partial\varepsilon} < 0 \quad \Rightarrow \quad \text{stability}}\,. \tag{9.31}$$

Besides stability, this construction has given us a conserved quadratic free energy for our system:

$$\boxed{\mathscr{F} = \mathscr{U} + \mathscr{A}[f, f_*] = \int \mathrm{d}\boldsymbol{r}\, \frac{E^2 + B^2}{8\pi} + \sum_\alpha \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, \frac{\delta f^2}{2(-\partial f_*/\partial\varepsilon)}}\,. \tag{9.32}$$

The stability condition (9.31) makes $\mathscr{F}$ positive definite and so no wonder the system is stable: perturbations around f_* have a conserved norm! For a Maxwellian equilibrium, $-\partial f_*/\partial\varepsilon = f_*/T$, so this $\mathscr{F}$ becomes (the electromagnetic version of) the free energy (5.20). Then, in the spirit of §9.3, (9.32) is a natural generalisation of that free energy to non-Maxwellian, collisionless plasmas.

In Q-6, the results of this section are obtained in a more straightforward way, directly from the Vlasov–Maxwell equations.

An invariant analogous to (9.32) will re-appear in an interesting generalised (and not always positive-definite!) form in the discussion of firehose and mirror stability of a magnetised plasma: see Exercise 22.4. Generalised free-energy invariants like $\mathscr{F}$ are important not just for stability calculations, but also for theories of kinetic turbulence in weakly collisional environments, e.g., the solar wind (see, e.g., Schekochihin *et al.* 2009).

9.5. *Gardner's Theorem*

The stability condition (9.31), viz., the statement that *if the equilibrium distributions of all species depend only on the particle energy and decrease monotonically with it, then the system is stable*, is, in fact an extremely general and fundamental result—as was shown by Gardner (1963) in a classic two-page paper.[94] The proof is as follows.

[94]The stability of *Maxwellian* equilibria against small perturbations was first proved by W. Newcomb, whose argument was published as Appendix I of Bernstein (1958). He was followed by Fowler (1963), who proved stability against finite perturbations in the way described in §9.2. Gardner (1963) attributes the first appearance of the stability condition (9.31) to an obscure

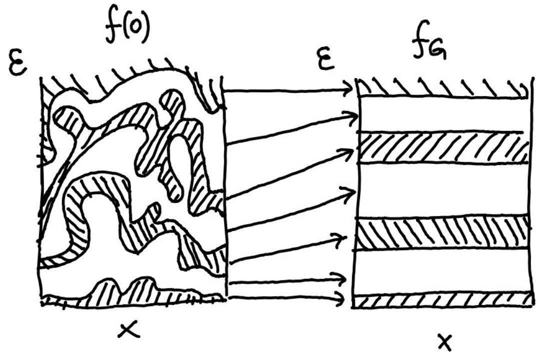


FIGURE 45. Gardner's restacking of the distribution, conserving phase-space volume.

For every species (suppressing species indices), let me take $G(f) = 0$ in (9.7), but construct a nontrivial $f_{\rm G}$ that satisfies (9.2), $\mathscr{A}\left[f(t), f_{\rm G}\right] \geqslant 0$, at every time t since the beginning of the evolution of $f(t)$ from the initial distribution $f(0)$. This amounts to adopting a "zero-entropy thermodynamics", i.e., one that minimises energy.

For any given $f(0)$, define $f_{\rm G}$ to be a monotonically decreasing function of the particle energy $\varepsilon(\boldsymbol{Q}) = mv^2/2$, such that for any $\eta > 0$, the volume of the region in the phase space $\boldsymbol{Q} = (\boldsymbol{r}, \boldsymbol{v})$ where $f_{\rm G} > \eta$ is the same as the volume of the phase-space region where $f(0) > \eta$. Then $f_{\rm G}$ is the distribution with the smallest kinetic energy, denoted here by $\mathscr{K}_{\rm G}$, that can be reached from f_0 while preserving phase-space volume:

$$\mathscr{K}(t) \geqslant \mathscr{K}_{\rm G}. \tag{9.33}$$

Indeed, while the phase-space volume occupied by any given value of the probability density is the same for $f(0)$ and for $f_{\rm G}$, the corresponding energy is always lower for $f_{\rm G}$ than for $f(0)$ or for any other f that can evolve from it, because in $f_{\rm G}$, the values of the probability density are rearranged in such a way as to put the largest of them at the lowest values of $\varepsilon(\boldsymbol{Q})$, thus minimising the kinetic energy. A vivid analogy is to think of the evolution of f under the collisionless kinetic equation (1.28) as the evolution of a mixture of "fluids" of different densities (values of f) advected in a 6D phase-space $\boldsymbol{Q} = (\boldsymbol{r}, \boldsymbol{v})$ by a divergence-free flow $\dot{\boldsymbol{Q}} = (\dot{\boldsymbol{r}}, \dot{\boldsymbol{v}})$. The lowest-energy state is the one in which these fluids are arranged in layers of density decreasing with increasing $\varepsilon(\boldsymbol{Q})$, the heaviest at the bottom, the lightest at the top (Fig. 45). This rearrangement of the distribution function is known as "Gardner restacking".

In view of (9.33), and since A is given by (9.7) with $G(f) = 0$,

$$\mathscr{A}[f, f_{\rm G}] = \mathscr{K}(t) - \mathscr{K}_{\rm G} \geqslant 0, \tag{9.34}$$

so (9.2) holds and the bound (9.3) on the available energy follows. When $f(0) = f_{\rm G}$, i.e., the equilibrium distribution satisfies (9.31), this equilibrium is stable, q.e.d. For a general initial distribution $f(0)$, the available energy is

$$\boxed{\mathscr{A}\left[f(0), f_{\rm G}\right] = \mathscr{K}(0) - \mathscr{K}_{\rm G}}\,. \tag{9.35}$$

1960 report by M. N. Rosenbluth, although the same condition was derived also by Kruskal & Oberman (1958), more or less in the manner described in §9.4. Clearly, many great minds were thinking alike.

Finally, note that the condition (9.31) is sufficient for stability, but not necessary, as we already know from, e.g., Exercise 4.2.

For those who like rigour, here's a formal proof that any minimum-energy distribution must be a function of energy only (Helander 2017).

The idea is to look for a distribution $f_{\rm G}$ such that the kinetic energy of any distribution evolving from it cannot increase. So, let us set $f(0) = f_{\rm G}$ and evolve $f(t)$ forward a short time δt. The collisionless kinetic equation can be written simply as [cf. (1.28)][95]

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{Q}}\cdot\frac{\partial f}{\partial \boldsymbol{Q}} = 0 \quad\Rightarrow\quad f(\delta t) \approx f_{\rm G} - \delta t\,\dot{\boldsymbol{Q}}\cdot\frac{\partial f_{\rm G}}{\partial \boldsymbol{Q}}. \tag{9.36}$$

The first-order (in δt) kinetic-energy change from $f_{\rm G}$ to $f(\delta t)$ is then

$$\delta\mathscr{K}[\delta\boldsymbol{Q}] = -\int \mathrm{d}\boldsymbol{Q}\,\varepsilon(\boldsymbol{Q})\delta\boldsymbol{Q}\cdot\frac{\partial f_{\rm G}}{\partial \boldsymbol{Q}}, \tag{9.37}$$

where $\delta\boldsymbol{Q} = \delta t\,\dot{\boldsymbol{Q}}$. We want to minimise $\mathscr{K}$, so we need $\delta\mathscr{K} = 0$. This will be achieved for $f_{\rm G}$ such that $\delta\mathscr{K}[\delta\boldsymbol{Q}] = 0$ for any phase-space vector $\delta\boldsymbol{Q}$ that behaves appropriately (vanishes) at the boundaries and satisfies $(\partial/\partial\boldsymbol{Q})\cdot\delta\boldsymbol{Q} = 0$. The latter condition is imposed because the phase-space velocity field in (9.36) must be divergence-free: $(\partial/\partial\boldsymbol{Q})\cdot\dot{\boldsymbol{Q}}$ (the system is Hamiltonian). This last condition can be enforced by means of a Langrange multiplier $\lambda(\boldsymbol{Q})$:

$$\delta\mathscr{K}[\delta\boldsymbol{Q}] - \int \mathrm{d}\boldsymbol{Q}\,\lambda(\boldsymbol{Q})\frac{\partial}{\partial\boldsymbol{Q}}\cdot\delta\boldsymbol{Q} = 0 \quad\Leftrightarrow\quad \varepsilon(\boldsymbol{Q})\frac{\partial f_{\rm G}}{\partial\boldsymbol{Q}} = \frac{\partial\lambda}{\partial\boldsymbol{Q}} \quad\Rightarrow\quad \frac{\partial\varepsilon}{\partial\boldsymbol{Q}}\times\frac{\partial f_{\rm G}}{\partial\boldsymbol{Q}} = 0. \tag{9.38}$$

Therefore, $f_{\rm G} = f_{\rm G}\big(\varepsilon(\boldsymbol{Q})\big)$, q.e.d.

9.5.1. *Gardner vs. Fowler*

In the proof of Gardner's theorem, I set $G(f) = 0$. In fact, there was no need to do this because Gardner restacking preserves phase volume and, therefore, cannot change the $\int \mathrm{d}\boldsymbol{Q}\,G(f)$ part of the available energy. This simple observation enables us to assess the relative quality of Fowler's (or any Fowler-like) and Gardner's methods for estimating the available energy in a distribution. Let the Gardner-restacked distribution be $f_{\rm G}$ and imagine having chosen some non-zero $G(f)$ and some target equilibrium f_*. Then, since $\int \mathrm{d}\boldsymbol{Q}\,G(f(0)) = \int \mathrm{d}\boldsymbol{Q}\,G(f_{\rm G})$, the available energy (9.9) in $f(0)$ is

$$\mathscr{A}[f(0), f_*] = \mathscr{K}(0) - \mathscr{K}_{\rm G} + \mathscr{A}[f_{\rm G}, f_*] \geqslant \mathscr{K}(0) - \mathscr{K}_{\rm G}, \tag{9.39}$$

since $\mathscr{A}[f_{\rm G}, f_*] \geqslant 0$. Thus, Gardner's bound on the available energy is always tighter than Fowler's or his ilk's. Intuitively, not increasing the entropy of the distribution gives you the best estimate of the energy that can be released from it. This result was pointed out to me by Robbie Ewart (2021), and does not, as far as I know, appear anywhere else in the literature.

A version of this result that follows from the Kruskal–Oberman formula (9.30), viz., that the Gardner available energy (9.34) in $f = f_* + \delta f$ is

$$\mathscr{A}[f_* + \delta f, f_{\rm G}] = \int \mathrm{d}\boldsymbol{Q}\,\frac{\delta f^2 - (f_{\rm G} - f_*)^2}{2(-\partial f_*/\partial\varepsilon)}, \tag{9.40}$$

---

[95]The phase-space variables do not have to be $\boldsymbol{Q} = (\boldsymbol{r}, \boldsymbol{v})$, as, e.g., they are not in such reductions of kinetic theory as gyrokinetics (see, e.g., the reviews by Howes *et al.* 2006 and Abel *et al.* 2013). These more complicated phase spaces usually describe systems in which particle motion is constrained by some adiabatic invariants (in gyrokinetics, it is the particles' first adiabatic invariant—the angular momentum of their Larmor gyration), so, effectively, the dimensionality of the phase space where particles can freely roam is lower than 6. Everything in this section can be done (and is done in Helander's paper) for general phase spaces, where the formula for $\varepsilon(\boldsymbol{Q})$ might be more complicated than $mv^2/2$.

can also be derived directly (if at the cost of more algebra) by an extension of the formalism of §9.5.2—it is so derived in Helander & Mackenbach (2024), who also give a number of instructive examples of its use. The physical reason for the correction in (9.40), and for the inequality in (9.39), is that f_* is, in general, not accessible from the perturbed state f without breaking phase-volume conservation (9.8) (the Casimirs). The smallest-energy state that *is* accessible is, by definition, $f_{\rm G}$; obviously, if $\delta f \ll f_*$, then also $f_{\rm G} - f_* \ll f_*$.

9.5.2. *Explicit Equation for Gardner Distribution*

Since Gardner restacking is as well as we can do in calculating the available energy, it is worth thinking about one how might work out Gardner-restacked distributions and hence calculate the available energy—a fancy way of thinking of this is as of seeking "ground states" (the states of minimum energy) of Vlasov's equation.

As we saw in (9.38), any distribution that is a function only of $\varepsilon(\boldsymbol{Q})$ is a minimum-energy state, but we now must find one that is accessible from a given initial distribution $f(0)$ via collisionless evolution, i.e., conserving phase-space volumes. This condition can be written in the form of the conservation law (9.8) with $G(f) = H\big(f(\boldsymbol{Q}) - \eta\big)$, where H is the Heaviside function, picking out the volume of phase space where $f > \eta$:

$$\Gamma_f(\eta) \equiv \frac{1}{V}\int \mathrm{d}\boldsymbol{Q}\, H\big(f(\boldsymbol{Q}) - \eta\big) = \mathrm{const} \tag{9.41}$$

(normalised for future convenience to the system's volume V). A further useful function, also conserved, is

$$\rho_f(\eta) \equiv \frac{1}{V}\int \mathrm{d}\boldsymbol{Q}\, \delta\big(f(\boldsymbol{Q}) - \eta\big) = -\frac{\mathrm{d}\Gamma_f}{\mathrm{d}\eta}. \tag{9.42}$$

Since $\rho_f(\eta)$ is conserved, and since the Gardner distribution $f_{\rm G}$ must be accessible from the initial distribution $f(0)$, we must have $\rho_{f_{\rm G}}(\eta) = \rho_{f(0)}(\eta)$. Therefore,

$$\rho_{f(0)}(\eta) = \int \mathrm{d}\varepsilon\, g(\varepsilon)\delta\big(f_{\rm G}(\varepsilon) - \eta\big) = -g\big(f_{\rm G}^{-1}(\eta)\big)\frac{\mathrm{d}f_{\rm G}^{-1}}{\mathrm{d}\eta}, \tag{9.43}$$

where $g(\varepsilon)$ is the density of states, i.e., $\int \mathrm{d}\boldsymbol{Q} = V\int \mathrm{d}\varepsilon\, g(\varepsilon)$. This function is entirely independent of $f(0)$ or $f_{\rm G}$: since $\varepsilon(\boldsymbol{Q}) = mv^2/2$, it is simply

$$g(\varepsilon) = \frac{2^{5/2}\pi}{m^{3/2}}\,\varepsilon^{1/2}. \tag{9.44}$$

The last step in (9.43) is made possible by the fact that the Gardner function must satisfy $f_{\rm G} = f_{\rm G}(\varepsilon(\boldsymbol{Q}))$ and $\partial f_{\rm G}/\partial\varepsilon < 0$. Finally, (9.43) gives us an explicit equation for $f_{\rm G}$ (due to Dodin & Fisch 2005):

$$\boxed{\frac{\mathrm{d}f_{\rm G}}{\mathrm{d}\varepsilon} = -\frac{g(\varepsilon)}{\rho_{f(0)}\big(f_{\rm G}(\varepsilon)\big)}}\,. \tag{9.45}$$

Helander (2017) prefers the integral version of this equation, which is as close as one can get to an actual formula for $f_{\rm G}$:

$$\boxed{\Gamma_{f(0)}\big(f_{\rm G}(\varepsilon)\big) = \int_0^\varepsilon \mathrm{d}\varepsilon'\, g(\varepsilon') \equiv \varOmega(\varepsilon)}\,. \tag{9.46}$$

9.5.3. *Anisotropic Equilibria*

Let me give an example of the use of (9.46) for the bi-Maxwellian distribution (9.18). To work out the Gardner distribution corresponding to it, observe that the volume $\Gamma[f(0),\eta]$ of the part of phase space where $f(0) > \eta$ is V times the volume of the velocity-space ellipsoid

$$\frac{mv_\perp^2}{2T_\perp} + \frac{mv_\parallel^2}{2T_\parallel} = \ln\frac{\overline{C}}{\eta} \quad\Rightarrow\quad \Gamma_{f(0)}(\eta) = \frac{4\pi}{3}\left(\frac{2\overline{T}}{m}\ln\frac{\overline{C}}{\eta}\right)^{3/2}. \tag{9.47}$$

Letting $\eta = f_{\rm G}(\varepsilon)$ and, according to (9.46), equating $\Gamma_{f(0)}\big(f_{\rm G}(\varepsilon)\big)$ to $\Omega(\varepsilon) = (4\pi/3)(2\varepsilon/m)^{3/2}$, we find

$$f_{\rm G}(\varepsilon) = \overline{C}\exp\left(-\frac{\varepsilon}{\overline{T}}\right). \tag{9.48}$$

This is an interesting, if perhaps somewhat rigged, example of a Maxwellian equilibrium having a special significance even in the absence of collisions. Unsurprisingly, therefore, the upper bound on the available energy

$$\mathscr{A}\big[f(0), f_{\rm G}\big] = \mathscr{K}(0) - \mathscr{K}_{\rm G} = \frac{3}{2}Vn\left(\frac{2}{3}T_\perp + \frac{1}{3}T_\parallel - \overline{T}\right) \tag{9.49}$$

is exactly the same as (9.20), which I found by Fowler's method. Obviously, in general, the Gardner distribution will not coincide with Fowler's target Maxwellian—and, according to §9.5.1, will give a better bound.

In Helander (2017, 2020), you will find other examples, e.g., a nice demonstration that Maxwellian equilibria with spatially dependent density and temperature have available energy.

Exercise 9.3. Gardner distribution for beams. For some conveniently chosen model of a two-beam distribution, compute the Gardner distribution and calculate the available energy. You may wish to re-use the model that you chose in Exercise 9.2 and confirm, in line with (9.39), that the bound on the available energy based on Gardner restacking is tigher than what you can get from Fowler.

---

10. Statistical Mechanics of Collisionless Plasma

The core step of the "themodynamical" method of §9 was to find a stable trial distribution f_* in some sense related to the initial state $f(0)$. If $f(0) = f_*$, then stability was guaranteed; if not, one could estimate how much energy might be releasable if $f(0)$ were to go unstable—but was there an implication that the system might actually relax to such a state? For a closed system, the answer is no. Indeed, consider the case of Gardner's distribution (§9.5). It is accessible from the intial distribution in the sense that phase-volume conservation is respected by Gardner restacking. However, it manifestly has lower energy. In a closed system, this energy can only go into fluctuating fields, but, being spatially homogeneous, Gardner's distribution can support no such fields self-consistently.[96]

Let me stress that Gardner's distribution cannot be interpreted as some mean equilibrium f_0, with the fluctuating fields hidden in a δf, because it is obtained under the requirement of phase-volume conservation, which applies only to the exact total distribution function $f = f_0 + \delta f$. As I have noted already, Gardner's thermodynamics is zero-entropy thermodynamics, so it makes sense that no separation of f into a coarse-grained and a fluctuating distribution is allowed.[97]

---

[96] The situation is different if the problem of energy extraction is formulated as it is in Dodin & Fisch (2005): an initial distribution is put into some external wave field for a while, then the waves leave the plasma, carrying with them some energy. Then one can imagine the Gardner state being accessible from the initial state, because there is no requirement that the total energy in the system be conserved.

[97] For example, the distribution with a plateau into which an initial bump-on-tail distribution relaxes quasilinearly (§6.3) is *not* the Gardner-restacked version of the latter (and neither does

FIGURE 46. Donald Lynden-Bell (1935-2018), a Cambridge astrophysicist. Photo taken from Evans (2020).

The result of §9.4, viz., conservation of the generalised free energy (9.32), suggests that a state featuring a stable Gardner-like distribution, i.e., a monotonically decreasing $f_*(\varepsilon)$, plus small perturbations, is a viable one, but it does not offer a procedure for calculating f_* if one starts with $f(0)$ somewhere far away from it. Fowler's construction (§9.2) dodges this problem by simply postulating that f_* is a Maxwellian, which allows one to use the standard formula for entropy—$G_\alpha(f_\alpha)$ in (9.13)—and prove the stability of a Maxwellian with respect to finite perturbations, but does not explain what is so special or inevitable about a Maxwellian in a collisionless plasma (something in some special cases, nothing in general—see §§10.1.2 and 10.3, respectively).

In order to determine what stable distributions are accessible to a collisionless plasma starting from a given state $f(0)$, and, therefore, what distribution it will (or may) relax to, one must do one of two things: either figure out the right entropy function and the right constraints and maximise the former subject to the latter, or derive an evolution equation for the mean distribution function f_0 directly from the Vlasov–Maxwell equations (i.e., derive a "collisionless collision integral"). The first of these approaches is explored in this section, the second in §11.

10.1. *Lynden-Bell Statistics*

Is there a way to determine universal, stable collisionless equilibria without regard to the precise nature of initial conditions, mimicking the way in which Maxwellian, Fermi–Dirac and Bose–Einstein distributions emerge in statistical mechanics as universal equilibrium states (see, e.g., Exercise 16.1 of Schekochihin 2024)? Let us work though a statistical argument proposed originally by Lynden-Bell (1967) in the context of collisionless relaxation of kinetic systems of mutually gravitating objects (e.g., stars in a

---

the QL diffusion of the mean distribution f_0 conserve phase volume; indeed, besides f_0, the system also has δf that supports a broad spectrum of electric perturbations, which help create the plateau in the first place). If you are curious, you will find an explicit comparison between the two in the paper by Kolmes & Fisch (2020), who were interested in what kind of distributions could be achieved by Gardner restacking vs. by diffusion (the latter broadly interpreted). It turns out that, by tailoring bespoke external wave fields that extract energy from particle distributions, it is possible to devise an artificial "diffusive algorithm" that recovers Gardner's distribution, but it is not one that a freely relaxing QL plasma actually follows.

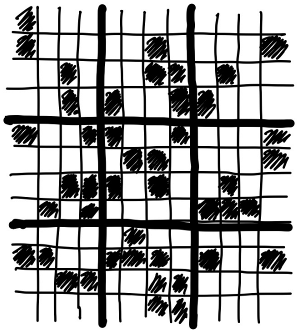


FIGURE 47. Coarse-graining micro-cells into macro-cells ($\mathcal{M} = 16$).

galaxy)—and quickly realised by Kadomtsev & Pogutse (1970) to be equally relevant to collisionless plasmas (this was one of the few historical occasions when plasma physics followed in the footsteps of galactic dynamics, rather than vice versa).

Let us start by discretising the phase space into a very large number of *micro-cells*, each with phase volume $\delta\Gamma$. Let us assume also (in what is a drastic simplifying step) that the exact distribution function in each of these micro-cells is equal to either zero or some constant, the same for all micro-cells:

$$f(\boldsymbol{Q}) = \eta \quad \text{or} \quad 0, \tag{10.1}$$

where, like in §9.1, $\boldsymbol{Q} = (\boldsymbol{r}, \boldsymbol{v})$ is the phase-space variable. This is known as a *waterbag distribution*—a constant probability density covering a finite subvolume of the phase space. Then

$$\int \mathrm{d}\boldsymbol{Q}\, f = \eta\, \delta\Gamma \mathcal{N} = N, \tag{10.2}$$

where $\mathcal{N}$ is the number of micro-cells with non-zero particle density, N is the total number of particles, and, naturally, we assume $N \gg \mathcal{N} \gg 1$. We are going to think of our plasma as a statistical-mechanical system of $\mathcal{N}$ phase-density elements, which are allowed, under collisionless evolution, to move around phase space subject to the usual constraints: conservation of energy and conservation of phase volume. The latter constraint in this language means that different phase-density elements can never cross paths to occupy the same micro-cell, i.e., that they obey a Pauli-like exclusion principle. Thus, they are fermion-like particles, albeit distinguishable by their initial position in phase space.

Let us now coarse-grain our phase space into *macro-cells*, each containing $\mathcal{M} \gg 1$ micro-cells (Fig. 47). Let $\mathcal{N}_i \leqslant \mathcal{M}$ be the occupation number of the i-th macro-cell, i.e., the number in it of micro-cells with non-zero content. Then the coarse-grained distribution f_0 can be discretised in terms of the particle density in the i-th macro-cell:

$$f_{0i} = \frac{\eta \mathcal{N}_i}{\mathcal{M}} \leqslant \eta. \tag{10.3}$$

The total number of ways of setting up a particular such distribution is

$$W = \frac{\mathcal{N}!}{\prod_i \mathcal{N}_i!} \prod_i W_i, \quad W_i = \frac{\mathcal{M}!}{(\mathcal{M} - \mathcal{N}_i)!}. \tag{10.4}$$

Here the first factor is the number of ways of distributing $\mathcal{N}$ phase-density elements amongst the macro-cells and W_i is the number of ways to distribute $\mathcal{N}_i$ distinguishable elements between the micro-cells in the i-th macro-cell. Assuming that $\mathcal{N} \gg \mathcal{M} \gg \mathcal{N}_i \gg 1$, we can use Stirling's formula ($\ln \mathcal{N}! \approx \mathcal{N} \ln \mathcal{N} - \mathcal{N}$) to find the Boltzmann entropy for our system:

$$\begin{aligned} S = \ln W &\approx \mathcal{N}(\ln \mathcal{N} - 1) - \mathcal{M} \sum_i \left[\frac{\mathcal{N}_i}{\mathcal{M}} \ln \frac{\mathcal{N}_i}{\mathcal{M}} + \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}}\right) \ln \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}}\right) \right] \\ &= \mathcal{N}(\ln \mathcal{N} - 1) - \frac{1}{\delta\Gamma} \int \mathrm{d}\boldsymbol{Q} \left[\frac{f_0}{\eta} \ln \frac{f_0}{\eta} + \left(1 - \frac{f_0}{\eta}\right) \ln \left(1 - \frac{f_0}{\eta}\right) \right], \end{aligned} \tag{10.5}$$

where $\int \mathrm{d}\boldsymbol{Q} = \mathcal{M}\delta\Gamma \sum_i$. This S is to be maximised under the constraints of a fixed number of particles N and energy $\mathscr{K}$ in the distribution. The problem is exactly the same as for a Fermi gas[98] and its solution is the Fermi–Dirac distribution:

$$\boxed{f_0(\boldsymbol{Q}) = \frac{\eta}{e^{\beta[\varepsilon(\boldsymbol{Q}) - \mu]} + 1}}\,, \tag{10.6}$$

where $\varepsilon(\boldsymbol{Q}) = mv^2/2$ is the particle energy corresponding to the given macro-cell, and β (inverse "temperature") and μ ("chemical potential") are Lagrange multipliers that are determined by fixing N and $\mathscr{K}$:

$$N = \int \mathrm{d}\boldsymbol{Q}\, f_0, \quad \mathscr{K} = \int \mathrm{d}\boldsymbol{Q}\, \varepsilon f_0. \tag{10.7}$$

There is an important nuance here that needs discussing. I derived (10.6) subject to a fixed total *kinetic* energy $\mathscr{K}$ associated with the coarse-grained distribution f_0, so I need to know $\mathscr{K}$ in order to fix the parameters of (10.6). Generally speaking, $\mathscr{K}$ is not a conserved quantity because there is energy also in the electric (and magnetic) fields. For simplicity, I consider a (statistically) homogeneous system and let each phase-space macro-cell contain the entire position space. Consequently, f_0 is independent of position. In an electrostatic system, f_0 then gives rise to no fields, so all of its energy is kinetic energy.[99] Does this mean that our fixed $\mathscr{K}$ is, in fact, conserved and so is equal to the energy of the initial state?—Only if, during the evolution of the plasma towards equilibrium, either no energy transfer from f_0 to electric fluctuations (associated with $\delta f = f - f_0$) is allowed, i.e., the initial f_0 is stable, or it is not stable, but, in the process of relaxing to a stable state (e.g., quasilinearly, as in §6), it transfers to fluctuations an amount of energy that is negligible compared to $\mathscr{K}$.

Note that since the Lynden-Bell distribution (10.6) is a maximiser of an entropy, it must be stable, by the same argument as Fowler made for a Maxwellian (§9.2). The same will be true about the more general case considered from §10.2 onwards.

10.1.1. *Degenerate Limit: the Ground State*

Every physics undergraduate knows what Fermi gas in the limit of $\beta \to \infty$ does: the distribution is a Heaviside step function collecting all the probability below the Fermi

[98] The distinguishability of the phase-density elements turns out not to matter: in (10.4), the factor of $1/\mathcal{N}_i!$ that would appear in W_i for indistinguishable fermions is recovered in the prefactor that expresses the number of ways of populating the macro-cells. For a tutorial on Fermi gases, see Schekochihin (2024, §§16–17).

[99] In this respect, my calculation is different from that of Lynden-Bell (1967), who considers a self-gravitating system, which cannot be spatially homogeneous, and so f_0 does give rise to a gravitational potential, which then gives each particle a potential energy that must be included in ε. In a 3D plasma, it is actually possible to prove that such inhomogeneous solutions with f_0 that depends only on ε do not exist (Ng & Bhattacharjee 2005).

energy μ:

$$\boxed{f_0 = \eta H(\mu - \varepsilon)}\,. \tag{10.8}$$

Therefore, from (10.7),

$$N = V\int_0^\mu \mathrm{d}\varepsilon\, g(\varepsilon)\eta \quad\Rightarrow\quad \mu = \frac{m}{2^{7/3}}\left(\frac{3n}{\pi\eta}\right)^{2/3} \equiv \varepsilon_{\mathrm{F}}, \qquad \frac{\mathscr{K}}{N} = \frac{\int_0^\mu \mathrm{d}\varepsilon\,\varepsilon g(\varepsilon)}{\int_0^\mu \mathrm{d}\varepsilon\, g(\varepsilon)} = \frac{2}{5}\,\varepsilon_{\mathrm{F}}, \tag{10.9}$$

where I used (9.44) for $g(\varepsilon)$. This f_0 is quite clearly the Gardner function corresponding to the waterbag distribution with phase-space density η: all phase elements are stacked at the lowest possible energies subject to the Pauli principle. Therefore, the energy of this distribution, $\mathscr{K}_{\mathrm{G}} = (2/5)N\varepsilon_{\mathrm{F}}$, is the lowest energy achievable at this phase-space density—in other words, (10.8) is the ground state.

If $\mathscr{K} > \mathscr{K}_{\mathrm{G}}$, β must be finite and the resulting less degenerate Lynden-Bell distributions represent "excited" states of the system, corresponding to initial waterbag distributions that occupy the phase space more sparsely.

10.1.2. *Non-Degenerate Limit: the Maxwellian Returns*

Just like in the case of Fermi–Dirac statistics, the non-degenerate limit is recovered when the initial waterbag distribution is asymptotically sparse in phase space—i.e., when it is sufficiently spread out around the part of the phase space that is accessible subject to given energy $\mathscr{K}$. Mathematically, this limit is achieved by letting

$$e^{-\beta\mu} \gg 1 \quad\Rightarrow\quad f_0 \approx \eta e^{\beta\mu}e^{-\beta\varepsilon}. \tag{10.10}$$

Then, from (10.7), after doing the integrals using $\int \mathrm{d}\boldsymbol{Q} = V\int \mathrm{d}\varepsilon\, g(\varepsilon)$ and the density of states $g(\varepsilon)$ given by (9.44), we get

$$\beta^{-1} \equiv T = \frac{2}{3}\frac{\mathscr{K}}{N}, \quad \eta e^{\beta\mu} = \frac{n}{(2\pi T/m)^{3/2}}. \tag{10.11}$$

Thus, the non-degenerate, Maxwellian limit (10.10) is

$$\frac{n}{(2\pi T/m)^{3/2}} \ll \eta \quad\Rightarrow\quad \boxed{f_0 = \frac{n}{(2\pi T/m)^{3/2}}\, e^{-\varepsilon/T}}\,. \tag{10.12}$$

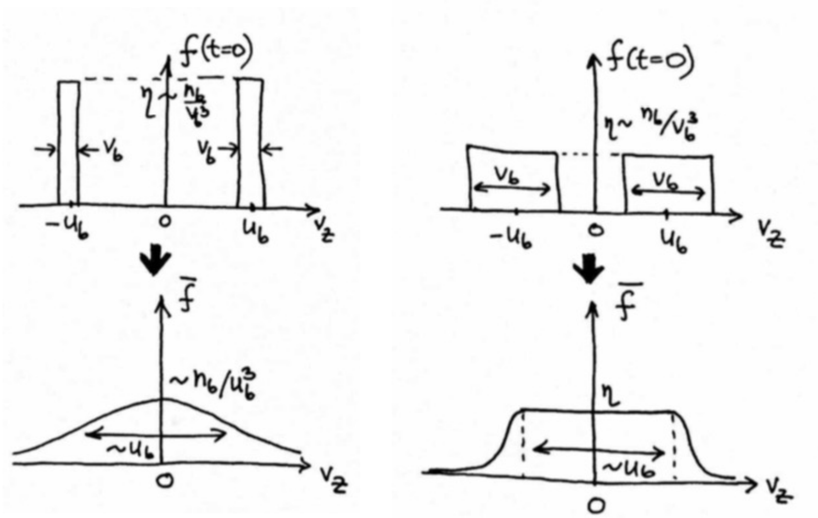
Of course it is not a particular surprise that a Maxwellian distribution emerges from a statistical mechanics of completely randomised objects with fixed overall energy. Figuring out in what way the statistical mechanics of a single waterbag is relevant to any real plasma will take some further work...

10.1.3. *Warm and Cold Beams*

Let me offer a very simple example of a physically plausible waterbag distribution: a pair of beams (streams) homogeneous in position space with number density n_{b}, velocities $\pm u_{\mathrm{b}}$ in some given direction, and constant velocity-space density over a region of width v_{b} around $\pm u_{\mathrm{b}}$, in 3D (a 1D cut of this might look like Fig. 48). This is a waterbag with

$$\eta \sim \frac{n_{\mathrm{b}}}{v_{\mathrm{b}}^3}, \quad \mathscr{K} \sim n_{\mathrm{b}} V m u_{\mathrm{b}}^2. \tag{10.13}$$

It is non-degenerate if $v_{\mathrm{b}} \ll u_{\mathrm{b}}$, i.e., if the beams are "cold"—as is obvious from (10.12). One therefore expects to see a distribution close to a Maxwellian with $T \sim m u_{\mathrm{b}}^2$ at the end of the relaxation process (Fig. 48a). In contrast, "warm" beams, $v_{\mathrm{b}} \sim u_{\mathrm{b}}$, should tend to a Fermi–Dirac flat-topped equilibrium with Fermi energy $\varepsilon_{\mathrm{F}} \sim m u_{\mathrm{b}}^2$



(a) Cold beams: $v_{\rm b} \ll u_{\rm b}$ (b) Warm beams: $v_{\rm b} \sim u_{\rm b}$

FIGURE 48. Waterbag beams and their Lynden-Bell equilibria.

(Fig. 48b). Remarkably, some recent numerical evidence may be showing something like this happening (Skoutnev *et al.* 2019).

Exercise 10.1. Lynden-Bell thermodynamics and Lynden-Bell equilibria for beams. Adapt Fowler's thermodynamics (§9.2) to accommodate Lynden-Bell's distribution as f_* and investigate the nonlinear stability of a distribution consisting of two square beams (Fig. 48). Work out and plot (or at least sketch) the parameters of the "target" Lynden-Bell distribution: T and μ as functions of $u_{\rm b}$ and $v_{\rm b}$. In the limits $v_{\rm b} \ll u_{\rm b}$ or $T \ll u_{\rm b}, v_{\rm b}$, everything should be doable analytically (in the latter case, via Sommerfeld expansion; see, e.g., §17.3.3 of Schekochihin 2024). How does the available energy in the initial distribution, liable to be transferred to the electric fluctuations, compare to $\mathscr{K}$?

10.2. *Discrete Multi-Waterbag Statistics*

The above construction contained a very restrictive assumption of an initial waterbag distribution. This restriction is, however, not hard to remove. Let us discretise the values that the distribution function can take and index them by J, so a general distribution function is represented as a superposition of waterbags:

$$f(\boldsymbol{Q}) = \sum_J f_J(\boldsymbol{Q}), \quad f_J(\boldsymbol{Q}) = \eta_J \quad \text{or} \quad 0. \tag{10.14}$$

If there are $\mathcal{N}_J$ phase elements with density η_J, then $\delta\Gamma\mathcal{N}_J$ is the phase-space volume occupied by the J-th waterbag, i.e., the phase-space volume where $f = \eta_J$. This is conserved by the collisionless evolution of f. The corresponding number of particles is $N_J = \eta_J \delta\Gamma\mathcal{N}_J$. As before, we may now coarse-grain f over groups (macro-cells) of $\mathcal{M}$ microcells and represent the resulting f_0 in terms of occupation numbers $\mathcal{N}_{iJ}$ of the i-th macro-cell by elements of phase density η_J. If one generalises straightforwardly the single-waterbag derivation presented in §10.1, it then turns out that

$$\boxed{f_0 = \sum_J \eta_J p_J(\varepsilon), \quad p_J(\varepsilon) = \frac{e^{-\beta\eta_J(\varepsilon-\mu_J)}}{1 + \sum_{J'} e^{-\beta\eta_{J'}(\varepsilon-\mu_{J'})}}}\,, \tag{10.15}$$

where μ_J and β are determined by

$$N_J = \eta_J \int \mathrm{d}\boldsymbol{Q}\, p_J, \quad \mathscr{K} = \int \mathrm{d}\boldsymbol{Q}\, \varepsilon f_0. \tag{10.16}$$

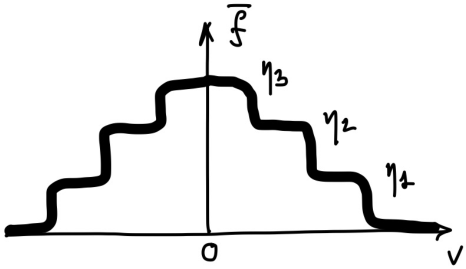


FIGURE 49. A close-to-degenerate 3-waterbag Lynden-Bell equilibrium.

Thus, the more general equilibrium distribution is a kind of superposition of many Fermi–Dirac distributions or, in the non-degenerate limit, of Maxwellians, with effective temperatures $1/\beta\eta_J$. When degenerate, they are ziggurat-like superpositions of step functions (Fig. 49). Note that, with a large number of steps, any monotonically decreasing function of ε can be approximated by such a ziggurat, so any Gardner distribution (§9.5) can be represented as a Lynden-Bell equilibrium.

Exercise 10.2. Multi-waterbag equilibrium. (a) Derive (10.15), in the way described above (Lynden-Bell 1967).

(b) Considering the Maxwellian limit, or otherwise, propose a natural way to define the overall temperature (Lynden-Bell 1967 has an answer to this, which does not seem to me to be the most natural one; hint: what is the equation of state in the non-degenerate limit?).

Exercise 10.3. Lynden-Bell distributions are Gardner distributions. Show that any Lynden-Bell distribution (10.15) is a Gardner distribition (i.e., $\partial f_0/\partial\varepsilon < 0$), and so is stable (this is easy).

Exercise 10.4. Gardner distributions are Lynden-Bell distributions. Consider a Gardner distribution, i.e., a monotonically decreasing function of ε alone. Discretise it to be a multi-waterbag distribution (10.14). Show that, under an appropriate coarse-graining, the corresponding Lynden-Bell equilibrium (10.15) is the same distribution, i.e., that Lynden-Bell's entropy-maximisation procedure does not change Gardner's distribution. Equivalently, any Gardner distribution can be represented with arbitrary precision by (10.15) (with $\beta \to \infty$). Intuitively, this is obvious, but it does, in fact, take a bit of work to prove. I owe this result to a conversation with Per Helander (2019) and the MMathPhys dissertation of Andrew Brown (2021); an advanced version of the relevant calculation (done in the continuous limit; see §10.3) is described in Appendix A of Ewart *et al.* (2023).

10.3. *Continuous Multi-Waterbag Statistics*

It is not hard to generalise all of the above to the case of a continuum of of waterbags and, therefore, to general distributions. The required formalism is motivated by the discrete multi-waterbag statistics (§10.2), but can be understood without reference to it. Let us introduce a new exact and a new coarse-grained (averaged) distribution functions:

$$P(\boldsymbol{Q},\eta) = \delta\big(f(\boldsymbol{Q}) - \eta\big), \qquad P_0(\boldsymbol{v},\eta) = \langle P(\boldsymbol{Q},\eta)\rangle\,, \tag{10.17}$$

where it is assumed that the coarse-graining average renders the distribution spatially homogeneous (dependent only on $\boldsymbol{v}$). These functions measure the (exact and average) amount of phase volume occupied by the particles whose position and velocity are $\boldsymbol{Q} = (\boldsymbol{r},\boldsymbol{v})$ and whose phase-space density is equal to η.[100] The mean phase-space density f_0,

[100] $P_0(\boldsymbol{v},\eta)$ is the continuous version of the waterbag probabilities p_J introduced in (10.15).

finding which is our primary objective, is now recoverable as the first moment of P_0,

$$\int d\eta\, \eta P(\boldsymbol{Q}, \eta) = f(\boldsymbol{Q}) \quad \Rightarrow \quad \int d\eta\, \eta P_0(\boldsymbol{v}, \eta) = f_0(\boldsymbol{v}), \tag{10.18}$$

while the zeroth moment is simply fixed by the normalisation of probabilities:

$$\int d\eta\, P(\boldsymbol{Q}, \eta) = \int d\eta\, P_0(\boldsymbol{v}, \eta) = 1. \tag{10.19}$$

The phase-volume conservation is encoded in

$$\frac{1}{V}\int d\boldsymbol{Q}\, P(\boldsymbol{Q}, \eta) = \int d\boldsymbol{v}\, P_0(\boldsymbol{v}, \eta) = \rho(\eta) = \text{const.} \tag{10.20}$$

Since this holds for every η, we have an infinite number of conservation laws (the Casimir invariants mentioned at the end of §9.1). I shall call the function $\rho(\eta)$, already introduced in (9.42), the *waterbag content* of the (exact) distribution—it (and the total energy) contains all the information that is retained from the initial state by the Lynden-Bell collisionless-relaxation process.

Technically, the function $P(\boldsymbol{Q}, \eta)$ lives in an extended 7D phase space $(\boldsymbol{r}, \boldsymbol{v}, \eta)$, but of course it is, in fact, limited by (10.20) to a 6D hypersurface within this 7D space. This formalism, which I shall call *hyperkinetics* (and which will come back with a vengeance in §11.7), originates from galactic dynamics and vortex kinetics (for which it is nicely presented in Chavanis *et al.* 1996; the review-style paper by Chavanis 2022 has a treatment of this topic that is, in certain respects, quite similar to mine in this section and in §11, and contains very many historical cross-references between different fields and lines of thought).

To determine the coarse-grained hyperkinetic distribution statistical-mechanically, it makes sense to maximise the usual Shannon (1948) entropy of P_0,[101]

$$\boxed{S = -\int d\boldsymbol{v} \int d\eta\, P_0(\boldsymbol{v}, \eta) \ln P_0(\boldsymbol{v}, \eta)}\,, \tag{10.21}$$

subject to the constraints (10.19), (10.20), and fixed energy

$$V\int d\boldsymbol{v}\, \varepsilon(\boldsymbol{v}) \int d\eta\, \eta P_0(\boldsymbol{v}, \eta) = \mathscr{K}. \tag{10.22}$$

This amounts to maximising unconditionally

$$\begin{aligned} S &- \int d\boldsymbol{v}\, \lambda(\boldsymbol{v}) \left[\int d\eta\, P_0(\boldsymbol{v}, \eta) - 1\right] - \beta\left[\int d\boldsymbol{v}\, \varepsilon(\boldsymbol{v}) \int d\eta\, \eta P_0(\boldsymbol{v}, \eta) - \frac{\mathscr{K}}{V}\right] \\ &+ \beta \int d\eta\, \eta\mu(\eta) \left[\int d\boldsymbol{v}\, P_0(\boldsymbol{v}, \eta) - \rho(\eta)\right] \to \max, \end{aligned} \tag{10.23}$$

where $\lambda(\boldsymbol{v})$, β, and $-\beta\eta\mu(\eta)$ are Lagrange multipliers. The "chemical potential" $\mu(\eta)$ is indeed the chemical potential as its job is to fix the number of particles in the waterbag η. The maximisation problem (10.23) yields the usual "grand-canonical" outcome

$$P_0(\boldsymbol{v}, \eta) = \frac{e^{-\beta\eta[\varepsilon(\boldsymbol{v}) - \mu(\eta)]}}{Z(\varepsilon(\boldsymbol{v}))} \quad \Rightarrow \quad \boxed{f_0(\boldsymbol{v}) = \frac{1}{Z(\varepsilon(\boldsymbol{v}))} \int d\eta\, \eta\, e^{-\beta\eta[\varepsilon(\boldsymbol{v}) - \mu(\eta)]}}\,, \tag{10.24}$$

[101] This is essentially the same expression as (9.26).

(a) (b) (c)

FIGURE 50. Three Oxford students who sorted out collisionless relaxation (§§10–11) and phase-space turbulence (§12)—or at least made a good start at it: (a) Toby Adkins (DPhil 2022), (b) Robbie Ewart (DPhil 2024), and (c) Michael Nastac (still around).

where the probability normalisation (10.19) is enforced by the "partition function"

$$Z(\varepsilon(\boldsymbol{v})) \equiv e^{1+\lambda(\boldsymbol{v})} = \int \mathrm{d}\eta\, e^{-\beta\eta[\varepsilon(\boldsymbol{v})-\mu(\eta)]}, \tag{10.25}$$

then β is determined by fixing the energy $\mathscr{K}$ of the distribution via (10.22), and $\mu(\eta)$ by fixing its waterbag content, expressed by $\rho(\eta)$ in (10.20). The function f_0 in (10.24) is just the continuous, multi-waterbag version of the Lynden-Bell equilibrium (10.6) [cf. (10.15)].

10.4. *Ewart's Theory of Collisionless Universality*

It is good to have an all-encompassing formalism, but it might appear to give us cause for some pessimism: via $\mu(\eta)$, everything depends on the waterbag content $\rho(\eta)$ of the initial distribution, which is, it seems, a highly non-universal and problem-dependent thing—so what hope do we have of being able to discern anything universal about the Lynden-Bell equilibria? It turns out, however, as Ewart *et al.* (2023, Fig. 50b) discovered, that something universal can, in fact, be discerned. Here is how.

Let us recall that, for any initial distribution with a given $\rho(\eta)$, we can construct, quite straightforwardly, its Gardner distribution f_{G}—the state with the lowest energy possible while preserving this $\rho(\eta)$ (§9.5.2). It is intuitively clear that this f_{G}, if it were the initial distribution, i.e., if the initial distribution had the energy $\mathscr{K} = \mathscr{K}_{\mathrm{G}}$, would also be the relaxed Lynden-Bell equilibrium, so nothing new would happen as a result of collisionless relaxation. This situation should correspond to the degenerate limit $\beta \to \infty$ of (10.24). Recalling the same story for a single waterbag (§10.1.1) makes the analogy obvious, or at least plausible (the formal proof is in Ewart *et al.* 2023; cf. Exercise 10.4). The Gardner distribution to the continuous, multi-waterbag Lynden-Bell statistics (10.24) is thus what the step distribution (10.8) is to the single-waterbag case.

Imagine now that $\mathscr{K} > \mathscr{K}_{\mathrm{G}}$. We could think of the resulting Lynden-Bell equilibrium as the same as would emerge if we took a Gardner distribution f_{G} and added to it, without breaking phase-volume conservation, $\mathscr{K} - \mathscr{K}_{\mathrm{G}}$ worth of energy. We might then expect that, if $\mathscr{K} \gg \mathscr{K}_{\mathrm{G}}$, the Lynden-Bell statistics will become non-degenerate, which, by analogy with the single-waterbag case (§10.1.2), should entail

$$Z(\varepsilon) \approx \eta_0 = \mathrm{const}, \tag{10.26}$$

where η_0 is an ε-independent constant that has dimensions of η but whose value is unimportant because the factor $e^{\beta\eta\mu(\eta)}$ appears in both the numerator and the denominator of P_0 in (10.24) and thus can be arbitrarily rescaled. Let us assume (10.26), work everything out on this basis, and then discuss whether this was a reasonable thing to do (in §10.4.3).

Instead of $\mu(\eta)$, I shall use the convenient quantity

$$F(\eta) = \frac{1}{\eta_0}\, e^{\beta\eta\mu(\eta)}, \tag{10.27}$$

called the *fugacity* of the distribution (by analogy with statistical mechanics). Then the non-degenerate limit of (10.24) is

$$P_0(\boldsymbol{v},\eta) = F(\eta)\, e^{-\beta\eta\varepsilon(\boldsymbol{v})}, \qquad f_0(\boldsymbol{v}) = \int \mathrm{d}\eta\, \eta F(\eta)\, e^{-\beta\eta\varepsilon(\boldsymbol{v})}. \tag{10.28}$$

To determine $F(\eta)$, we must enforce (10.20), which is easy in this limit: since $\int \mathrm{d}\boldsymbol{v} = \int \mathrm{d}\varepsilon\, g(\varepsilon)$, we get

$$\rho(\eta) = F(\eta) \int \mathrm{d}\varepsilon\, g(\varepsilon)\, e^{-\beta\eta\varepsilon} = \frac{A\Gamma(a+1)F(\eta)}{(\beta\eta)^{a+1}}, \tag{10.29}$$

where the density of states was set to be

$$g(\varepsilon) = A\varepsilon^a, \tag{10.30}$$

more general than (9.44) (where $a = 1/2$), but convenient to keep track of constants and powers of ε (and handy for working out analogous results for, e.g., the 2D or 1D cases, or for a relativistic plasma).

The explicit expression for $F(\eta)$ follows immediately from (10.29) and can be inserted into (10.28) to obtain the particle distribution function:

$$f_0(\boldsymbol{v}) = \frac{\beta^{a+1}}{A\Gamma(a+1)} \int \mathrm{d}\eta\, \rho(\eta)\eta^{a+2} e^{-\beta\eta\varepsilon(\boldsymbol{v})}. \tag{10.31}$$

Now recall that $\rho(\eta) = -\mathrm{d}\Gamma_f(\eta)/\mathrm{d}\eta$ [see (9.42)], use this in (10.31), and integrate by parts:

$$\begin{aligned} f_0(\boldsymbol{v}) &= \frac{\beta^{a+1}}{A\Gamma(a+1)} \int \mathrm{d}\eta\, \Gamma_f(\eta) \frac{\mathrm{d}}{\mathrm{d}\eta}\, \eta^{a+2} e^{-\beta\eta\varepsilon(\boldsymbol{v})} \\ &= \frac{1}{A\Gamma(a+1)\beta\, [\varepsilon(\boldsymbol{v})]^{a+2}} \int \mathrm{d}x\, \Gamma_f\!\left(\frac{x}{\beta\varepsilon(\boldsymbol{v})}\right) \frac{\mathrm{d}}{\mathrm{d}x}\, x^{a+2} e^{-x}. \end{aligned} \tag{10.32}$$

Simplifying a bit, this can be recast as the distribution of energies:

$$\boxed{\mathscr{N}(\varepsilon) = g(\varepsilon) f_0 = \frac{1}{\Gamma(a+1)\beta\varepsilon^2} \int \mathrm{d}x\, \Gamma_f\!\left(\frac{x}{\beta\varepsilon}\right) \frac{\mathrm{d}}{\mathrm{d}x}\, x^{a+2} e^{-x}}\,. \tag{10.33}$$

Cutting to the chase, it will turn out (in §10.4.1) that, at least at large ε, Γ_f is a very weak function of its argument, typically logarithmic. Equivalently,

$$\boxed{\rho(\eta) \sim \eta^{-1} \quad \text{as} \quad \eta \to 0} \tag{10.34}$$

[see (10.36) below]. The shape of the energy distribution is, therefore, determined by the ε^{-2} power-tail prefactor, which seems to be a fairly universal asymptotic.

This is all great news if true. Before dealing with a number of loose ends and nuances, let me remind you that whether it is true hinges on the validity of two key assumptions:

(i) The phase-density elements of which the distribution consists are independent and free to sample the entirety of the phase space without fear or prejudice. One way in which this might not be true would be if our system had further invariants that constrained its dynamics on the time scales of interest. An example is a magnetised plasma in which particles conserve adiabatic invariants (cf. Helander 2017, 2020).[102]

(ii) Phase volume is conserved by the evolution of the *exact* phase-space density $f(\boldsymbol{Q})$, i.e., by $f_0 + \delta f$. This assumption is problematic, as I warned at the end of §9.1 (after Exercise 9.1), will re-iterate in §11.8, and finally show in §12.4.4. A way to cope with that will be mooted in §11.8, but finally left hanging in §12.7, as the theory is currently incomplete.

I shall now return to the systematic exposition of Ewart's theory and sort out some details: why Γ_f is a weak function of its argument and, consequently, how universal is the ε^{-2} tail (§10.4.1); how come the energy distribution $\mathscr{N}(\varepsilon)$ appears to have a divergent mean (§10.4.2); whether the non-degenerate approximation (10.26) is justified (§10.4.3); and what happens when $\mathscr{K}$ is not asymptotically larger than $\mathscr{K}_{\rm G}$ (§10.4.4).

10.4.1. *How Universal is the ε^{-2} Tail?*

All the non-universality of our equilibrium, i.e., its dependence on initial conditions, has been stuffed into the function Γ_f sitting inside the integral in (10.33). Last time this function appeared in these Lectures was in §9.5.2, where it was used to deduce an equation for the Gardner distribution. Using Helander's version (9.46) of this equation and setting in it $\varepsilon = f_{\rm G}^{-1}(\eta)$, we get

$$\Gamma_f(\eta) = \int_0^{f_{\rm G}^{-1}(\eta)} {\rm d}\varepsilon' g(\varepsilon') = \frac{A}{a+1}\left[f_{\rm G}^{-1}(\eta)\right]^{a+1}. \tag{10.35}$$

Thus, there is a one-to-one correspondence between Γ_f and $f_{\rm G}$. If $f_{\rm G}(\varepsilon)$ decays faster at large ε than any power law (equivalently, if it has an infinite number of finite moments), then $f_{\rm G}^{-1}(\eta)$ will grow more weakly with decreasing η than any power law, and so will $\Gamma_f(\eta)$. For example, if $f_{\rm G}(\varepsilon) \sim \eta_{\max} e^{-\varepsilon/\varepsilon_{\rm G}}$, where $\varepsilon_{\rm G} \sim \mathscr{K}_{\rm G}/N$, then $f_{\rm G}^{-1}(\eta) \sim \varepsilon_{\rm G}\ln(\eta_{\max}/\eta)$. In (10.33), this is translated into[103]

$$\Gamma_f\left(\frac{x}{\beta\varepsilon}\right) \sim A\left[\varepsilon_{\rm G}\ln\left(\frac{\beta\eta_{\max}\varepsilon}{x}\right)\right]^{a+1} \tag{10.36}$$

[whence follows (10.34)]. A weak ε dependence like this will not change in a significant way the dominant ε^{-2} high-energy asymptotic in (10.33). What can change it is a power-law tail in the Gardner distribution. Consider what happens if $f_{\rm G}(\varepsilon) \propto \varepsilon^{-a-\sigma}$ at large ε. Then $f_{\rm G}^{-1}(\eta) \propto \eta^{-1/(a+\sigma)}$ at low η, and (10.33) gives us

$$\mathscr{N}(\varepsilon) \propto \varepsilon^{-2+(a+1)/(a+\sigma)}, \tag{10.37}$$

so the tail becomes "harder". This appears to be an unlikely situation.

10.4.2. *Calculating β and Dealing with Divergences*

You might have noticed that our calculation of the Lynden–Bell equilibrium is, in fact, incomplete because we still need to find β by enforcing the energy constraint (10.22). An attempt to do so leads to a nasty shock: if $\mathscr{N} \propto \varepsilon^{-2}$ at large ε, the total energy $\mathscr{K} = V\int{\rm d}\varepsilon\,\varepsilon\mathscr{N}(\varepsilon)$ diverges. There is a formal, and more general, way to see this by taking the energy moment of (10.31):

$$\begin{aligned}\frac{\mathscr{K}}{V} &= \int{\rm d}\varepsilon\,\varepsilon g(\varepsilon) f_0 = \frac{\beta^{a+1}}{\Gamma(a+1)}\int{\rm d}\eta\,\rho(\eta)\eta^{a+2}\int{\rm d}\varepsilon\,\varepsilon^{a+1}e^{-\beta\eta\varepsilon}\\ &= \frac{a+1}{\beta}\int_{\eta_{\min}}^{\eta_{\max}}{\rm d}\eta\,\rho(\eta) = \frac{a+1}{\beta}\,\Gamma_f(\eta_{\min}),\end{aligned} \tag{10.38}$$

---

[102] Another is galactic dynamics—Lynden-Bell's original preoccupation—where stars are approximately tied to the Keplerian ellipses that they follow around the central black hole (see, e.g., Hamilton *et al.* 2024).

[103] Note that $x = \beta\eta_{\max}\varepsilon$ is the upper integration limit in (10.33) because $\Gamma_f(\eta) = 0$ for $\eta > \eta_{\max}$.

where, since, self-evidently, $\Gamma_f(0) = \infty$, the only way to avoid a divergence was to introduce a lower cutoff on the phase-space density, $f(\boldsymbol{Q}) > \eta_{\min}$. Accepting this without discussion for a moment, we get

$$\boxed{\beta = \frac{a+1}{\mathscr{K}}\, V\Gamma_f(\eta_{\min})}\,. \tag{10.39}$$

The introduction of $\eta_{\min}$ sets the integration limits in (10.33) to $x \in [\beta\eta_{\min}\varepsilon, \beta\eta_{\max}\varepsilon]$ (the upper limit having always been there). The lower limit provides an exponential high-energy cutoff in $\mathscr{N}(\varepsilon)$ at

$$\varepsilon \gtrsim \frac{1}{\beta\eta_{\min}} \sim \frac{\mathscr{K}}{V\Gamma_f(\eta_{\min})\eta_{\min}}. \tag{10.40}$$

It may appear deplorable that we have a theory that requires such invasive surgery, but this cannot be helped. Note, however, that the dependence of β (and, as we shall see, of all other relevant quantities) on $\eta_{\min}$ is weak (typically logarithmic) if the argument put forward in §10.4.1 holds up, so perhaps we can live with this need to appeal to a seemingly artificial quantity. Obviously, we shall want our theory to work in the limit $\eta_{\min} \to 0$. It is not hard to estimate what the absolute lower bound for $\eta_{\min}$ is:

$$\frac{\eta_{\min}}{\eta_{\max}} \sim \frac{n_{\min}}{n} \gg \frac{1}{n\lambda_{\mathrm{D}}^3}, \tag{10.41}$$

where I assumed that the smallest reasonable value $n_{\min}$ for the particle density in a collisionless plasma must still be much larger than one particle per Debye sphere. While this gives plenty of space for making $\eta_{\min}$ small, the values of $\ln(\eta_{\max}/\eta_{\min})$ remain both reasonably finite and quite insentive to the precise choice of $\eta_{\min}$.[104]

10.4.3. *Validity of Non-degenerate Limit*

The expression (10.39), given the presence in it of $\Gamma(\eta_{\min})$, which explodes (even if logarithmically) when $\eta_{\min} \to 0$, does little to fill one with confidence that the non-degenerate limit is really valid: we see that, formally, $\beta \to \infty$ always! Let us examine then exactly why, and whether, (10.26) is satisfied.

Recall how the non-degenerate limit worked for the case of a single waterbag. The Fermi–Dirac distribution (10.6) can be written as

$$f_0 = \frac{\eta e^{-\beta(\varepsilon-\mu)}}{Z(\varepsilon)}, \qquad Z(\varepsilon) = 1 + e^{-\beta(\varepsilon-\mu)}. \tag{10.42}$$

The non-degenerate limit (§10.1.2) is obtained when the partition function $Z(\varepsilon)$ can be approximated by 1. This constant part of $Z(\varepsilon)$ corresponds to the "empty waterbag"—the fact that a fraction of the phase space contains no particles; in the non-degenerate limit, this fraction is dominant.

Similarly to (10.42), the more general partition function (10.25) must have the form

$$Z(\varepsilon) = \eta_0 + \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, e^{-\beta\eta[\varepsilon(\boldsymbol{v})-\mu(\eta)]} = \eta_0 \left[1 + \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, F(\eta)\, e^{-\beta\eta\varepsilon}\right], \tag{10.43}$$

where η_0 accounts for the empty waterbag, but its specific value, as I have argued already, is rescalable from everywhere and so does not matter. Note that (10.43) means that the chemical potential $\mu(\eta)$ must have such a form that

$$e^{\beta\eta\mu(\eta)} = \eta_0\left[\delta(\eta) + H(\eta-\eta_{\min})F(\eta)\right]. \tag{10.44}$$

Thus, in order to assess the vality of the non-degenerate approximation, one must ask if, with $F(\eta)$ given by (10.29), the second term in the square bracket in (10.43) is small compared to unity for every ε. This is the same integral as the one for f_0 in (10.28) but with one less power of η in the integrand, so it is

$$\frac{\beta^{a+1}}{A\Gamma(a+1)} \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, \Gamma_f(\eta) \frac{\mathrm{d}}{\mathrm{d}\eta}\, \eta^{a+1} e^{-\beta\eta\varepsilon} \ll 1. \tag{10.45}$$

[104]Ewart *et al.* (2023) argue that the most appropriate choice is $\eta_{\min} \sim \Delta\Gamma^{-1}$, where $\Delta\Gamma$ is another kludgy quantity, which will be introduced in §11.2 and discussed physically in §11.8.

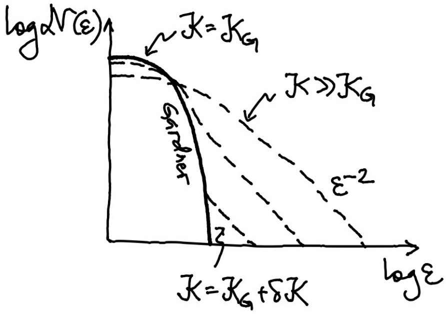


FIGURE 51. Lynden-Bell equilibria (dashed lines) with core and halo, and their Gardner distribution (thick line).

This must be satisfied at every ε, which is indeed guaranteed if it is satisfied at $\varepsilon = 0$:

$$\frac{\beta^{a+1}}{A}\int_{\eta_{\min}}^{\eta_{\max}} d\eta\, \eta^a \Gamma_f(\eta) \ll 1 \quad \Leftrightarrow \quad \mathscr{K} \gg V\left[\frac{1}{A}\int_{\eta_{\min}}^{\eta_{\max}} d\eta\, \eta^a \Gamma_f(\eta)\right]^{1/(a+1)} \Gamma_f(\eta_{\min}), \quad (10.46)$$

where I used (10.39) for β. To get a feel for what it takes to satisfy this, let $\Gamma_f(\eta) \sim A\left[\varepsilon_{\rm G} \ln(\eta_{\max}/\eta)\right]^{a+1}$, as in (10.36). Then

$$\mathscr{K} \gg V\varepsilon_{\rm G}\eta_{\max}\left[\int_{\eta_{\min}/\eta_{\max}}^{1} dx\, x^a |\ln x|^{a+1}\right]^{1/(a+1)} A\left(\varepsilon_{\rm G}\ln\frac{\eta_{\max}}{\eta_{\min}}\right)^{a+1} \sim \mathscr{K}_{\rm G}\left(\ln\frac{\eta_{\max}}{\eta_{\min}}\right)^{a+1}, \quad (10.47)$$

because $\varepsilon_{\rm G} V n \sim \mathscr{K}_{\rm G}$ and, roughly,

$$\eta_{\max} A \varepsilon_{\rm G}^{a+1} = \eta_{\max} g(\varepsilon_{\rm G})\varepsilon_{\rm G} \sim n. \quad (10.48)$$

Thus, in order for the non-degenerate approximation to hold well, the energy of the distribution has to be quite significantly larger than its minimum allowed (Gardner) value.

10.4.4. *Core and Halo*

What happens if $\mathscr{K} > \mathscr{K}_{\rm G}$, but not large enough to satisfy (10.47)? Like often in non-asymptotic situations, this cannot be answered analytically, but the Lynden-Bell equilibria can be found by solving numerically the coupled integro-transcendental equations (10.24) for P_0, (10.20) for $\mu(\eta)$, and (10.22) for β. According to Ewart *et al.* (2023), who did this (see their paper for real plots, or just glance at an artist's impression in Fig. 51), it turns out that if one adds a small amount of energy to $\mathscr{K}_{\rm G}$, the Gardner distribution will mostly survive but develop a little ε^{-2} tail at high energies (analogous to the Fermi–Dirac distribution at low temperature becoming a slightly eroded step function). If $\mathscr{K} - \mathscr{K}_{\rm G}$ is increased, the tail will grow bushier and gradually take over. A general Lynden-Bell equilibrium then conists of a non-universal "core", whose shape depends on the Gardner distribution, and a universal "halo" (the tail).

In the non-degenerate limit (10.47), it is not hard to estimate the energy at which the transition between the halo and the core occurs: in (10.33), at low ε, the Γ_f factor will start suppressing the magnitude of $\mathscr{N}(\varepsilon)$ when

$$\varepsilon \sim \frac{1}{\beta\eta_{\max}} \sim \frac{\mathscr{K}}{V\Gamma_f(\eta_{\min})\eta_{\max}}, \quad (10.49)$$

the second expression following from (10.39). Therefore, the energy content of the core is

$$\mathscr{K}_{\rm core} \sim \varepsilon V n \sim \mathscr{K}\,\frac{n}{\Gamma_f(\eta_{\min})\eta_{\max}} \sim \mathscr{K}\left(\ln\frac{\eta_{\max}}{\eta_{\min}}\right)^{-(a+1)} \ll \mathscr{K}_{\rm halo} \sim \mathscr{K}, \quad (10.50)$$

where I again used the estimate (10.36) for Γ_f and (10.48) for $\eta_{\max}$. Thus, most of the energy is in the tail of the distribution (the halo).

11. (Collisionless) Collision Integrals

Just like in the theory of gases, statistical mechanics points us to a maximum-entropy distribution, but we need kinetic theory to see if such a distribution can be reached dynamically, and under what assumptions. To address this, I am going to go back to the QL scheme (§6.1) and use it to derive a kind of "collisionless collision integral" that evolves the mean distribution function of a collisionless, homogeneous, electrostatic plasma towards some equilibrium that maximises an appropriate entropy and could perhaps be argued to be universal. For a single-waterbag initial distribution, I will recover the Fermi–Dirac distribution (10.6) as a solution of this collision integral. I will then show, in §11.7, how to generalise this constriction to the case of a continuum of waterbags that we encountered in §10.3 and recover the general Lynden-Bell equilibria (10.24) as the fixed points of this *hyperkinetic collision integral.* Getting there will require some assumptions about the nonlinear behaviour in phase space—assumptions that are neither obvious nor, possibly, always correct, and that will therefore be re-examined in §12, upon injection of some more nonlinear physics.

11.1. *General Form of Collision Integrals*

So the equilibrium (slow, space- and time-averaged) distribution f_0 again evolves according to (2.11):

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\frac{q}{m} \sum_{\boldsymbol{k}} i\boldsymbol{k} \langle \varphi_{\boldsymbol{k}}^* \delta f_{\boldsymbol{k}} \rangle \right) = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\frac{e}{m} \sum_{\boldsymbol{k}} \boldsymbol{k} \, \mathrm{Im} \langle \varphi_{\boldsymbol{k}}^* \delta f_{\boldsymbol{k}} \rangle \right), \tag{11.1}$$

where the imaginary part was distilled out of the $\boldsymbol{k}$ integral by splitting the latter in two equal parts and replacing $\boldsymbol{k} \to -\boldsymbol{k}$ in one of them [cf. (6.8)]; I have also specialised to electrons ($q = -e$) and will assume, for simplicity, that ions play no role apart from providing a homogeneous neutralising background.

Poisson's law (2.9) in this case is just

$$\varphi_{\boldsymbol{k}} = -\frac{4\pi e}{k^2} \int \mathrm{d}\boldsymbol{v}' \, \delta f_{\boldsymbol{k}}(\boldsymbol{v}'), \tag{11.2}$$

and, therefore, (11.1) turns into

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\frac{\omega_{\mathrm{pe}}^2}{n_0} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}}{k^2} \int \mathrm{d}\boldsymbol{v}' \, \mathrm{Im} \, C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') \right]}, \tag{11.3}$$

where the right-hand side—the prototypical collision integral—is expressed in terms of (the imaginary part of) the correlation function

$$C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}') = \left\langle \delta f_{\boldsymbol{k}}(\boldsymbol{v}) \delta f_{\boldsymbol{k}}^*(\boldsymbol{v}') \right\rangle. \tag{11.4}$$

This establishes an important principle—the evolution of f_0 is determined by the second-order correlator of δf, which, if we want a closed equation, it is now our task to express in terms of f_0. All derivations of collision integrals ("true" or "collisionless") are schemes for doing this, usually requiring some "closure" assumption(s) at some stage.

This is, of course, just a reiteration of the message of §2.3: to know about the equilibrium, we need to understand the fluctuations—we need a theory of phase-space turbulence.

11.2. *Microgranulation Ansatz*

We do not have it, so let us attempt to avoid needing it by instead venturing an extremely primitive guess at what the correlation function (11.4) might be. Imagine that we start the evolution of our collisionless system with some initial distribution—generally speaking, unstable—and let it proceed for a while. Instabilities might flare up and saturate, particles will stream and phase-mix the distribution, etc., so fairly quickly the exact f will become stable but extremely chopped up and fine-structured in phase space.[105] The average of this distribution over space and over "fast" times (meaning frequencies associated with any plasma processes, e.g., $\sim \omega_{\mathrm{pe}}$, and with particle streaming, $\sim \boldsymbol{k}\cdot\boldsymbol{v}$), is f_0. The remainder is $\delta f = f - f_0$. Surely it is then reasonable to think of this perturbed distribution as thoroughly mixed in phase space and essentially random?

If it is reasonable, then perhaps we can model δf by a random phase-space density field $g(\boldsymbol{r},\boldsymbol{v})$ (assigned a different letter for reasons about to transpire) whose correlation function is

$$\boxed{\left\langle g(\boldsymbol{r},\boldsymbol{v})g(\boldsymbol{r}',\boldsymbol{v}')\right\rangle = \langle g^2\rangle(\boldsymbol{v})\Delta\Gamma\,\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(\boldsymbol{v}-\boldsymbol{v}')}\,, \tag{11.5}$$

where $\Delta\Gamma$ is the phase-space volume representing the "width" of the two delta functions; note that the one-point average $\langle g^2\rangle$ is a function only of $\boldsymbol{v}$ because the system is assumed to be statistically homogeneous in position space. I shall call (11.5) (perhaps a bit pompously) the *microgranulation ansatz*. A way to make it *almost* true by construction is to redefine our average as coarse-graining over phase-space macro-cells of at least the volume $\Delta\Gamma = \Delta r^3\Delta v^3$, where Δr and Δv are, respectively, the position- and velocity-space correlation scales of g. I said *"almost"* because the whole scheme depends on it being possible also to make both Δr and Δv sufficiently small:

$$\Delta r \ll V^{1/3},\quad \Delta v \ll \left(\frac{\mathscr{K}}{mN}\right)^{1/2} \sim v_{\mathrm{th}}, \tag{11.6}$$

where V is the system's volume and $\mathscr{K}$ its total energy. In other words, we are still assuming that, as a result of phase mixing, g would lose any system-scale correlations in either position or velocity space. Note that we are also assuming that Δr and Δv are sufficiently large for $\Delta\Gamma$ to contain so many particles that "true" Coulomb collisions are not a player—I will return later to what this means quantitatively.

If we assume that the correlation function (11.4) of δf is well-modelled by (11.5), viz.,

$$\begin{aligned} C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = \left\langle g_{\boldsymbol{k}}(\boldsymbol{v})g^*_{\boldsymbol{k}}(\boldsymbol{v}')\right\rangle &= \iint \frac{\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'}{V^2}\, e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}\left\langle g(\boldsymbol{r},\boldsymbol{v})g(\boldsymbol{r}',\boldsymbol{v}')\right\rangle \\ &= \langle g^2\rangle(\boldsymbol{v})\frac{\Delta\Gamma}{V}\,\delta(\boldsymbol{v}-\boldsymbol{v}'), \end{aligned} \tag{11.7}$$

a bitter disappointment awaits us: since (11.7) is manifestly real, the right-hand side of (11.3) is identically zero. This is a particular case of the general fact that, since, by the property of Fourier transforms of real functions, $C^*_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v})$, $\operatorname{Im} C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ contains only the antisymmetric part of the correlation function with respect to the transformation $\boldsymbol{v}\leftrightarrow\boldsymbol{v}'$. Thus, to get anything non-trivial, we must busy ourselves with the extraction of this antisymmetric part.

There are two ways in which $\operatorname{Im} C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ can be present. The first is that the microgranulation ansatz is plain wrong even approximately and the true correlation function of phase-space turbulence is not symmetric. The second (which is a mild variant

[105] This is the part of the evolution that Lynden-Bell (1967) calls "violent relaxation".

of the first) is that microgranulation ansatz is approximately fine, but there is a correction to it that gives rise to a non-zero collision integral. The first possibility will be examined in §12.2, but here and now I will explore the second one, which is, in a certain sense, a direct generalisation of the standard derivation of "true" collision integrals in the classic, textbook kinetic theory.

11.3. *Quasilinear Collision Integrals*

The idea is to treat the fully phase-mixed field $g(\boldsymbol{r},\boldsymbol{v})$ described by (11.5) as an initial condition for δf, then calculate the time evolution of the latter from that initial state over times long compared to streaming times ($\boldsymbol{k}\cdot\boldsymbol{v}\,t \gg 1$) but short compared to the times over which f_0 changes significantly, and then work out the contribution from that evolved δf to the evolution of f_0. A further approximation is to assume this evolution to be quasilinear, i.e., to segregate the linear part of the evolution of δf from the nonlinear part and make simplifying assumptions about the latter—as one does in QLT.

The perturbation $\delta f = f - f_0$ satisfies (2.12) exactly and its linear part (2.13) approximately (within the QL approximation), viz.,

$$\frac{\partial \delta f_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_{\boldsymbol{k}} = -i\frac{e}{m}\,\varphi_{\boldsymbol{k}}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{11.8}$$

Given an initial perturbation $g_{\boldsymbol{k}}(\boldsymbol{v})$, the Laplace-transformed solution of (11.8) is the same as (3.8) and (3.13), viz.,

$$\delta\hat{f}_{\boldsymbol{k}}(p) = -i\frac{e}{m}\frac{\hat{\varphi}_{\boldsymbol{k}}(p)}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + \hat{h}_{\boldsymbol{k}}(p), \tag{11.9}$$

$$\hat{h}_{\boldsymbol{k}}(p) = \frac{g_{\boldsymbol{k}}(\boldsymbol{v})}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}, \tag{11.10}$$

$$\hat{\varphi}_{\boldsymbol{k}}(p) = -\frac{4\pi e}{k^2\epsilon(p,\boldsymbol{k})}\int \mathrm{d}\boldsymbol{v}\,\hat{h}_{\boldsymbol{k}}(p), \tag{11.11}$$

$$\epsilon(p,\boldsymbol{k}) = 1 - i\frac{\omega_{\mathrm{pe}}^2}{n_0k^2}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{11.12}$$

Formally, this is linear theory, but we should keep in mind that if, instead of $\hat{h}_{\boldsymbol{k}}(p)$ being given by (11.10), we just treated it as an unknown function, then (11.9) could be interpreted as a decomposition of $\delta\hat{f}$ into a linear and nonlinear parts, with the latter to be determined. The calculations that follow will admit this possibility until the start of §11.3.1.

The time-dependent solution is recovered from (11.9) and (11.11) via the inverse Laplace transform (3.14), but I now wish to change the integration variable to $p = -i\omega + \sigma$:

$$\delta f_{\boldsymbol{k}}(t) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,e^{(-i\omega+\sigma)t}\delta f_{\boldsymbol{k}\omega}, \quad \varphi_{\boldsymbol{k}}(t) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,e^{(-i\omega+\sigma)t}\varphi_{\boldsymbol{k}\omega}. \tag{11.13}$$

The integration contour now runs along the real line and the Laplace-transformed

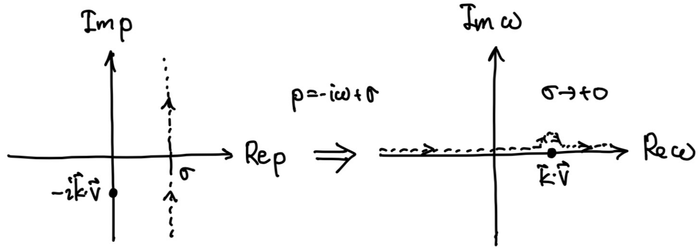


FIGURE 52. Change of variable in the inverse Laplace transform (11.13).

functions have been rebranded to look almost like Fourier-transformed ones:

$$\delta f_{\boldsymbol{k}\omega} = \delta\hat{f}_{\boldsymbol{k}}(-i\omega+\sigma) = \frac{e}{m}\frac{\varphi_{\boldsymbol{k}\omega}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} + h_{\boldsymbol{k}\omega}, \tag{11.14}$$

$$h_{\boldsymbol{k}\omega} = \hat{h}_{\boldsymbol{k}}(-i\omega+\sigma) = \frac{ig_{\boldsymbol{k}}(\boldsymbol{v})}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\sigma}, \tag{11.15}$$

$$\varphi_{\boldsymbol{k}\omega} = \hat{\varphi}_{\boldsymbol{k}}(-i\omega+\sigma) = -\frac{4\pi e}{k^2\epsilon_{\boldsymbol{k}\omega}}\int \mathrm{d}\boldsymbol{v}\, h_{\boldsymbol{k}\omega}, \tag{11.16}$$

$$\epsilon_{\boldsymbol{k}\omega} = \epsilon(-i\omega+\sigma,\boldsymbol{k}) = 1 + \frac{\omega_{\mathrm{pe}}^2}{n_0k^2}\int \mathrm{d}\boldsymbol{v}\,\frac{1}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}. \tag{11.17}$$

The reason for this rearrangement is that I now want to assume that $\hat{\varphi}_{\boldsymbol{k}}(p)$ and, therefore, $\delta\hat{f}_{\boldsymbol{k}}(p)$ have no poles at $\mathrm{Re}\,p>0$, i.e., f_0 supports no instabilities, so I may let $\sigma\to+0$, only needing it in the denominators to tell me how to circumvent the ballistic pole $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ (Fig. 52).

In the context of collisional relaxation of f_0, this assumption is justified as follows. The initial distribution may well be unstable, but we can always wait for its instabilities to get excited, saturate and change f_0 in such a way as to shut themselves down (e.g., quasilinearly, as they did in §6). After that, we have a distribution function that is stable or, at worst, marginally stable, i.e., $\epsilon_{\boldsymbol{k}\omega}$ might have zeros on the real-ω line and certainly at $\mathrm{Im}\,\omega=\mathrm{Re}\,p\leqslant 0$, but not at $\mathrm{Im}\,\omega=\mathrm{Re}\,p>0$. This assumption means that we should expect to derive an evolution equation for f_0 that conserves its kinetic energy (since we do not expect new electric fluctuations to be excited). This is a setting that is similar to what the Lynden-Bell-style calculation in §10 described [see discussion after (10.6)]—and thus (10.6) should be (and indeed will be) recoverable in this approach, under the right assumptions.

We are ready to roll. Substituting (11.13) and (11.14) into (11.1), we get[106]

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\boldsymbol{k}\,\mathrm{Im}\iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\,e^{-i(\omega-\omega')t}\left(-\frac{e^2}{m^2}\frac{\langle\varphi_{\boldsymbol{k}\omega}\varphi^*_{\boldsymbol{k}\omega'}\rangle}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} - \frac{e}{m}\langle h_{\boldsymbol{k}\omega}\varphi^*_{\boldsymbol{k}\omega'}\rangle\right). \tag{11.18}$$

The first term is the familiar QL diffusion (here written for a general-form φ) whereas the second term turns out to be a kind of drag [like in the Fokker–Planck collision

[106] A conscientious reader might at this point become worried about what exactly is meant by averages in (11.18). She will find some succour in §11.2.

integral (1.47)], whose form we shall now elaborate a little further. Using (11.16), we get

$$
\begin{aligned}
-\frac{e}{m}\langle h_{\boldsymbol{k}\omega}(\boldsymbol{v})\varphi^*_{\boldsymbol{k}\omega'}\rangle &= \frac{\omega_{\rm pe}^2}{n_0 k^2 \epsilon^*_{\boldsymbol{k}\omega'}}\int \mathrm{d}\boldsymbol{v}'\,\big\langle h_{\boldsymbol{k}\omega}(\boldsymbol{v})h^*_{\boldsymbol{k}\omega'}(\boldsymbol{v}')\big\rangle \\
&= \frac{\omega_{\rm pe}^2}{n_0 k^2 \epsilon_{\boldsymbol{k}\omega}\epsilon^*_{\boldsymbol{k}\omega'}}\int \mathrm{d}\boldsymbol{v}'\left[C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}') + \frac{\omega_{\rm pe}^2}{n_0 k^2}\int \mathrm{d}\boldsymbol{v}''\,\frac{C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}'' + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''}\right],
\end{aligned}
\tag{11.19}
$$

where $C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}') = \big\langle h_{\boldsymbol{k}\omega}(\boldsymbol{v})h^*_{\boldsymbol{k}\omega'}(\boldsymbol{v}')\big\rangle$ and the last expression was obtained by multiplying and dividing by $\epsilon_{\boldsymbol{k}\omega}$. The reason for this seemingly gratuitous manipulation is two-fold.

First, the first term in (11.19) vanishes after it is plugged into (11.18):

$$
\mathrm{Im}\iint \frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\,e^{-i(\omega-\omega')t}\frac{C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}')}{\epsilon_{\boldsymbol{k}\omega}\epsilon^*_{\boldsymbol{k}\omega'}} = \mathrm{Im}\big\langle \tilde{h}_{\boldsymbol{k}}(\boldsymbol{v})\tilde{h}^*_{\boldsymbol{k}}(\boldsymbol{v}')\big\rangle = 0,
\tag{11.20}
$$

provided the correlation function of the "dressed" distribution function $\tilde{h}_{\boldsymbol{k}} = (1/2\pi)\int \mathrm{d}\omega\, e^{-i\omega t}h_{\boldsymbol{k}\omega}/\epsilon_{\boldsymbol{k}\omega}$ is symmetric with respect to swapping velocities, $\boldsymbol{v} \leftrightarrow \boldsymbol{v}'$. If $h_{\boldsymbol{k}\omega}$ is given by (11.15) with the initial condition $g_{\boldsymbol{k}}(\boldsymbol{v})$ satisfying the microgranulation ansatz (11.5), this is indeed the case. Thus, here, effectively, is the place where the symmetric part of the correlation function (11.4) is got rid of and only the non-vanishing part of the collision integral (11.3) is kept.

Secondly, the first term in (11.18), upon insertion of (11.16), turns into an expression of a similar form to the second term in (11.19):

$$
-\frac{e^2}{m^2}\frac{\langle\varphi_{\boldsymbol{k}\omega}\varphi^*_{\boldsymbol{k}\omega'}\rangle}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = -\frac{\omega_{\rm pe}^4}{n_0^2 k^4 \epsilon_{\boldsymbol{k}\omega}\epsilon_{\boldsymbol{k}\omega'*}}\iint \mathrm{d}\boldsymbol{v}'\mathrm{d}\boldsymbol{v}''\,\frac{C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v}'',\boldsymbol{v}')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}}.
\tag{11.21}
$$

Putting all this together, we end up with (11.18) in the form that suggests that we might be on our way towards something like the particle-collision integral (7.37):

$$
\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}''\left[\mathsf{D}(\boldsymbol{v},\boldsymbol{v}'')\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} - \mathsf{D}(\boldsymbol{v}'',\boldsymbol{v})\cdot\frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''}\right]},
\tag{11.22}
$$

where the "diffusion kernel" is

$$
\mathsf{D}(\boldsymbol{v}'',\boldsymbol{v}) = -\frac{\omega_{\rm pe}^4}{n_0^2}\,\mathrm{Im}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\,\frac{e^{-i(\omega-\omega')t}}{\epsilon_{\boldsymbol{k}\omega}\epsilon^*_{\boldsymbol{k}\omega'}}\int \mathrm{d}\boldsymbol{v}'\,\frac{C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}'' + i\sigma}.
\tag{11.23}
$$

Note that the QL diffusion coefficient similar to the one appearing in (6.7) is then $\int \mathrm{d}\boldsymbol{v}''\,\mathsf{D}(\boldsymbol{v},\boldsymbol{v}'')$.

11.3.1. *Quasilinear Approximation*

Note that, except in justifying the vanishing of (11.20), I have not yet actually used the QL approximation, i.e., the explicit expression (11.15) for $h_{\boldsymbol{k}\omega}$ in terms of $g_{\boldsymbol{k}}$. Let me use it now:

$$
C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v},\boldsymbol{v}') = \frac{C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma)(\omega' - \boldsymbol{k}\cdot\boldsymbol{v}' - i\sigma)},\quad C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = \big\langle g_{\boldsymbol{k}}(\boldsymbol{v})g^*_{\boldsymbol{k}}(\boldsymbol{v}')\big\rangle,
\tag{11.24}
$$

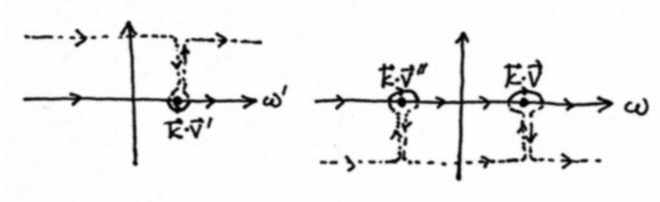


FIGURE 53. The contours for calculating the double integral (11.26).

and, therefore,

$$\mathsf{D}(\boldsymbol{v}'',\boldsymbol{v}) = -\frac{\omega_{\rm pe}^4}{n_0^2}\,\mathrm{Im}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\int\mathrm{d}\boldsymbol{v}'\,C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')I_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}''), \tag{11.25}$$

$$I_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}'') = \iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\,\frac{e^{-i(\omega-\omega')t}}{\epsilon_{\boldsymbol{k}\omega}\epsilon^*_{\boldsymbol{k}\omega'}(\omega-\boldsymbol{k}\cdot\boldsymbol{v}+i\sigma)(\omega'-\boldsymbol{k}\cdot\boldsymbol{v}'-i\sigma)(\omega-\boldsymbol{k}\cdot\boldsymbol{v}''+i\sigma)}. \tag{11.26}$$

The remaining work is now in calculating the double integral (11.26). If $\epsilon_{\boldsymbol{k}\omega}$ has no poles at real ω (i.e., f_0 is a stable distribution), the integral is done by shifting the ω' integration contour upwards to $\mathrm{Im}\,\omega'\to+\infty$, but snagging on the pole at $\omega'=\boldsymbol{k}\cdot\boldsymbol{v}'$, and the ω contour downwards to $\mathrm{Im}\,\omega\to-\infty$, snagging on the poles at $\omega=\boldsymbol{k}\cdot\boldsymbol{v}$ and $\omega=\boldsymbol{k}\cdot\boldsymbol{v}''$ (see Fig. 53). The result is

$$I_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}',\boldsymbol{v}'') = \frac{e^{-i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')t}}{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}\epsilon^*_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}'}}\underbrace{\frac{1}{\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')}\left(1-e^{i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')t}\frac{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}}{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}''}}\right)}_{\to\,-i\pi\delta(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}''))\ \text{as}\ t\to\infty}, \tag{11.27}$$

where the δ-function approximation is obtained in the same way as in (5.41). Thus, (11.25) becomes

$$\boxed{\mathsf{D}(\boldsymbol{v}'',\boldsymbol{v}) = \frac{\pi\omega_{\rm pe}^4}{n_0^2}\,\mathrm{Re}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\,\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)\int\mathrm{d}\boldsymbol{v}'\,\frac{C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')e^{-i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')t}}{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}\epsilon^*_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}'}}}\,. \tag{11.28}$$

Note that with the emergence of $\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)$, we have inched another step towards a result that looks like the particle-collision integral (7.37).

Exercise 11.1. Inclusion of plasma waves. What if $\epsilon_{\boldsymbol{k}\omega}$ does have poles at real ω, or close by? Try your hand at generalising this theory, from here and onwards to §11.5, to this case, which describes the evolution of a marginally stable or even weakly unstable f_0, and so of a plasma in communion with a population of waves. You will discover a bridge between the theory being developed here and the kind of QLT that was presented in §6. You may find some inspiration for this calculation in the classic paper by Rogister & Oberman (1968) (the sequel to it, Rogister & Oberman 1969, brings in the WT processes familiar from §7.2, but stops short of wading into strongly turbulent waters).

11.3.2. *Microgranulation Ansatz Re-applied*

In order to calculate the diffusion kernel (11.28) and, therefore, the collision integral (11.22), we yet again need to know the correlation function $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ of g. This is the moment to bring back, now with non-trivial pay-off, the microgranulation ansatz (11.7).

The result is that (11.22) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\pi\omega_{\rm pe}^4\Delta\Gamma}{n_0^2 V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}''\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2} \cdot\left[\langle g^2\rangle(\boldsymbol{v}'')\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} - \langle g^2\rangle(\boldsymbol{v})\frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''}\right]. \quad (11.29)$$

A reader who followed carefully the discussion and derivation of the quasiparticle-collision integrals in §§7 and 8.4.2 has realised that the contortions through which I am going here are an attempt to set up for a collisionless plasma something resembling the WT random-phase approximation or the usual Boltzmann's *Stosszahlansatz*. It is the microgranulation ansatz that breaks reversibility and is about to give us a "collision" integral with an H-theorem and a universal solution.

Exercise 11.2. Momentum and energy conservation for "collisionless collision integrals". Show that the general integral (11.29) conserves the momentum and kinetic energy of the particle distribution f_0.

This is as far as we can go without further closure assumptions: we still need to work out $\langle g^2\rangle$ in terms of f_0. Let us get on with it.

11.4. *Kadomtsev–Pogutse Collision Integral*

With the assumptions and the definition of averaging described above, f_0 is the same as f_0 in §10 and the macro-cells over which it is coarse-grained are the same randomly (and statistically independently) filled macro-cells as in the Lynden-Bell statistics (so, in the language of §10, $\Delta\Gamma = \mathcal{M}\delta\Gamma$). Unsurprisingly, the same result is about to pop out. Indeed, let us, for maximum simplicity, assume that the exact distribution is a single waterbag (10.1). Then[107]

$$\langle g^2\rangle = \langle (f-f_0)^2\rangle = \langle f^2\rangle - f_0^2 = (\eta - f_0)f_0, \quad (11.30)$$

because, for a single-waterbag distribution, $\langle f^2\rangle = \langle \eta f\rangle = \eta f_0$. With this additional closure, the "proto-collision integral" (11.29) turns into

$$\frac{\partial f_0}{\partial t} = \frac{\pi\omega_{\rm pe}^4\Delta\Gamma}{n_0^2 V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}''\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2} \cdot\left[\big(\eta - f_0(\boldsymbol{v}'')\big)f_0(\boldsymbol{v}'')\frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} - \big(\eta - f_0(\boldsymbol{v})\big)f_0(\boldsymbol{v})\frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''}\right]. \quad (11.31)$$

This is the "collisionless collision integral" of Kadomtsev & Pogutse (1970) (whose derivation I have more or less followed). It is not hard to show that it has an H-theorem

---

[107] It has not escaped my perceptive reader that a potential mathematical illegality is being perpetrated: is coarse-grain average of a coarse-grain average equal to itself, and is it true that $\langle ff_0\rangle = \langle f_0^2\rangle = f_0^2$? In general, no, not when the coarse-graining is done by convolution with some continuous shape function. However, it is fine if we formally do this as we did in §10, by grouping fixed discrete micro-cells into fixed discrete macro-cells. Alternatively, we may hope that the differences between f_0 and $\langle f_0\rangle$ and similar quantities are small enough to be ignored or, in a cleaner move, resort to an ensemble average over many possible realisations of f and assume that it will do the coarse-graining job automatically. Note that our time average (2.7) in fact suffered from the same problem. There too, we could instead have done an ensemble average and assumed that that would automatically have slow time evolution and no spatial dependence (in a statistically homogeneous system).

with the entropy (10.5) and that this entropy is maximised—and the right-hand-side of (11.31) is annihilated—by the Fermi–Dirac distribution (10.6).

Exercise 11.3. Show this.

Exercise 11.4. Generalise the above calculation to multiple species (electrons and ions).

The procedure that I have followed above to derive the collision integral (11.31) can be used in many different contexts. For one of the more recent and exotic such applications, see the derivation by Bar-Or *et al.* (2021) of quantum kinetic equation for "fuzzy dark matter". A more mundane, textbook example is the derivation of the "true" collision integrals for Fermi and Bose gases described in the appendix of Ewart *et al.* (2022).

11.5. *Lenard–Balescu Collision Integral*

Just as in §10, the non-degenerate limit is $f_0 \ll \eta$, turning (11.31) into

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{\pi\omega_{\mathrm{pe}}^4\eta\Delta\Gamma}{n_0^2 V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}'' \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2}\cdot\left[f_0(\boldsymbol{v}'')\frac{\partial f_0}{\partial \boldsymbol{v}} - f_0(\boldsymbol{v})\frac{\partial f_0}{\partial \boldsymbol{v}''}\right]}\,. \tag{11.32}$$

This has a functional form that is identical to the standard Lenard–Balescu collision integral (Lenard 1960; Balescu 1960) and its solution is, obviously, a Maxwellian.

How is it possible that collisionless and collisional behaviour turn out to be the same? Mathematically, this is not hard to understand. Let us recall what is meant by collisions in plasma physics. When a plasma (or, rather, a collection of individual particles) is described by its Klimontovich distribution function (1.19), the latter satisfies a "collisionless" Vlasov equation involving the microscopic electromagnetic field—the Klimontovich equation (1.35). Collisions acquire a specific mathematical meaning when this equation is averaged (coarse-grained) over the sub-Debye scales,[108] leading to the kinetic equation (1.41), where the collision integral is defined in terms of the correlation function of the differences between the averaged ("macroscopic") and the exact ("microscopic") fields and distribution. This is basically the same procedure as the one that led to (11.1) and thence to the "collisionless collision integral" (11.22) and its descendants (11.31) and (11.32). If we interpret the exact distribution f as the Klimontovich distribution,

$$f(\boldsymbol{r},\boldsymbol{v}) = \sum_{i=1}^{N}\delta(\boldsymbol{r}-\boldsymbol{r}_i)\,\delta(\boldsymbol{v}-\boldsymbol{v}_i)\,, \tag{11.33}$$

then the difference between it and f_0 will satisfy the microgranulation ansatz (11.5) automatically, because the Klimontovich distribution is only non-zero at the exact phase-space positions $(\boldsymbol{r}_i,\boldsymbol{v}_i)$ of the particles (assumed initially random), so the correlation function of $f - f_0$ must needs be a delta function. In this interpretation, $\Delta\Gamma$ is the effective width of the delta functions associated with individual particles, while η is their height (so the Klimontovich distribution is a kind of single waterbag). Clearly, they are related by

$$\eta\Delta\Gamma = 1, \tag{11.34}$$

which is all we need to know in (11.32), finally turning it into *the* Lenard–Balescu

---

[108]Our friends the galactic dynamicists do not make a distinction between "collisionless" and "collisional" dynamics because gravity is not shielded, so there is no natural coarse-graining analogous to (1.12) (Binney 2018; Fouvry 2021; Hamilton 2022; Hamilton & Fouvry 2024).

collision integral. It is not surprising that the solution is a Maxwellian because the Klimontovich distribution is extremely non-degenerate: all these delta functions always occupy a negligible fraction of the available phase space.

11.6. *Landau's Collision Integral*

While I am at it, let me show for completeness how the Landau (1936) collision integral is recovered from the Lenard–Balescu one. If we want to use (11.32) (with $\eta\Delta\Gamma = 1$) as an expression for *bona fide* collisions, i.e., if δf and φ are interpreted as fluctuating fields below the Debye scale, we must restrict the $\boldsymbol{k}$ summation to $k\lambda_{\rm D} \gg 1$. In this approximation, $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2 \approx 1$. The $\boldsymbol{k}$ sum then becomes tractable: denoting $\boldsymbol{v} - \boldsymbol{v}'' = \boldsymbol{w}$, we get

$$\frac{1}{V}\sum_{\boldsymbol{k}} \frac{\boldsymbol{kk}}{k^4}\,\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big) = \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}\,\frac{\boldsymbol{kk}}{k^4}\,\delta(\boldsymbol{k}\cdot\boldsymbol{w}) = \frac{1}{w}\int \frac{\mathrm{d}\boldsymbol{k}_\perp}{(2\pi)^3}\,\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k_\perp^4} = \frac{1}{8\pi^2 w}\left(\mathsf{I} - \frac{\boldsymbol{ww}}{w^2}\right)\int \frac{\mathrm{d}k_\perp}{k_\perp}, \tag{11.35}$$

where $\boldsymbol{k}_\perp = \boldsymbol{k}\cdot(\mathsf{I} - \boldsymbol{ww}/w^2)$. The divergent integral is the Coulomb logarithm $\Lambda = \ln(\lambda_{\rm D}/d)$ if the integration is cut off at the distance of closest approach d [see text after (1.13)] and the Debye scale $\lambda_{\rm D}$ (in fact, as explained in §1.11, it ought to be cut off well below $\lambda_{\rm D}$, but it matters little in a log). Consequently, (11.32) becomes

$$\boxed{\frac{\partial f_0}{\partial t} = \frac{\omega_{\rm pe}^4\Lambda}{8\pi n_0^2}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \frac{\mathrm{d}\boldsymbol{v}''}{w}\left(\mathsf{I} - \frac{\boldsymbol{ww}}{w^2}\right)\cdot\left[f_0(\boldsymbol{v}'')\frac{\partial f_0}{\partial \boldsymbol{v}} - f_0(\boldsymbol{v})\frac{\partial f_0}{\partial \boldsymbol{v}''}\right]}\,. \tag{11.36}$$

This is the Landau (1936) collision integral [cf. (1.47)]. Note that both it and the Lenard–Balescu integral (11.32) have the functional form anticipated in (7.37), which is as it should be—the calculations above are, thus, a way to work out the interaction probability $w(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{k})$.

11.7. *Hyperkinetic Collision Integral*

It is possible to generalise the Kadomtsev–Pogutse integral more or less straightforwardly to general distributions, not limited to a single waterbag or even a discrete set of them—alongside the more baroque scheme that you will have an opportunity to explore in Exercise 11.7, this is described in Ewart *et al.* (2022), but the derivation for discrete waterbags goes back to Severne & Luwel (1980) (in application to galactic dynamics). The idea is to construct a collision integral not for f_0 but for its hyperkinetic counterpart $P_0(\boldsymbol{v}, \eta)$ introduced in §10.3. This is a change of attitude with regard to the distribution function $f_0(\boldsymbol{v})$: it is now treated it as a "fluid" quanity—the mean density of the "phase fluid" that is fully described by the exact density $f(\boldsymbol{Q})$—and so $f_0(\boldsymbol{v})$ is just the first η moment of the "kinetic" quantity $P_0(\boldsymbol{v}, \eta)$, as per (10.18):

$$f_0 = \langle f \rangle = \int \mathrm{d}\eta\, \eta P_0(\boldsymbol{v}, \eta). \tag{11.37}$$

Higher-order correlation functions of f are, accordingly, higher η moments of $P_0(\boldsymbol{v}, \eta)$, e.g.,

$$\langle f^2 \rangle = \int \mathrm{d}\eta\, \eta^2 P_0(\boldsymbol{v}, \eta). \tag{11.38}$$

This is precisely the quantity that we needed in order to obtain closed expressions for the "collisionless collision integral" (11.29), enabling the step to (11.31) (but only for a single waterbag so far). Thus, just like in the derivation of fluid dynamics from kinetics, the closure problem for fluid (now "phase-fluid") moments is overcome by determining the kinetic (now hyperkinetic) distribution function over a higher-dimensional phase space.

Progress is possible because it is very easy to derive a closed equation for $P(\boldsymbol{Q}, \eta)$:

using Vlasov's equation for f, we get

$$\frac{\partial P}{\partial t} = \delta'(f-\eta)\frac{\partial f}{\partial t} = -\delta'(f-\eta)\left[-\boldsymbol{v}\cdot\boldsymbol{\nabla} f - \frac{e}{m}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right]$$
$$= -\boldsymbol{v}\cdot\boldsymbol{\nabla} P - \frac{e}{m}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial P}{\partial \boldsymbol{v}}. \tag{11.39}$$

Thus, P just satisfies the usual Vlasov's equation. Its average P_0, which we can assume to be spatially homogeneous, obeys, therefore,

$$\frac{\partial P_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\left[-\frac{e}{m}\left\langle(\boldsymbol{\nabla}\varphi)\delta P\right\rangle\right], \tag{11.40}$$

where $\delta P = P - P_0$ and, since $\delta f = \int \mathrm{d}\eta\,\eta\delta P$, Poisson's equation is

$$\nabla^2\varphi = -4\pi e\int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\eta\,\eta\delta P. \tag{11.41}$$

Working out the average in the right-hand side of (11.40) is a problem that can be solved entirely analogously to what has been done above to determine the rate of change of f_0, starting from (11.1). All the steps are analogous, except with $f_0 \to P_0$, $\delta f \to \delta P$, and wherever a velocity integral occurs, there must now also be an integral with respect to η, viz., $\int \mathrm{d}\boldsymbol{v} \to \int \mathrm{d}\boldsymbol{v}\int \mathrm{d}\eta\,\eta$, because all velocity integrals come from Poisson's law.

The first place where something mildly different happens is the microgranulation ansatz: assuming, as in §11.2, that different points in the (coarse-grained) phase space are uncorrelated, we get, instead of (11.5),

$$\left\langle g(\boldsymbol{Q},\eta)g(\boldsymbol{Q}',\eta')\right\rangle = \left\langle g(\boldsymbol{Q},\eta)g(\boldsymbol{Q},\eta')\right\rangle\Delta\Gamma\,\delta(\boldsymbol{Q}-\boldsymbol{Q}'). \tag{11.42}$$

The correlation function in the prefactor can now be worked out as follows:

$$\begin{aligned}\left\langle g(\boldsymbol{Q},\eta)g(\boldsymbol{Q},\eta')\right\rangle &= \left\langle\left[\delta\big(f(\boldsymbol{Q})-\eta\big) - P_0(\boldsymbol{v},\eta)\right]\left[\delta\big(f(\boldsymbol{Q})-\eta'\big) - P_0(\boldsymbol{v},\eta')\right]\right\rangle\\ &= \left\langle\delta\big(f(\boldsymbol{Q})-\eta\big)\delta\big(f(\boldsymbol{Q})-\eta'\big)\right\rangle - P_0(\boldsymbol{v},\eta)P_0(\boldsymbol{v},\eta')\\ &= \left[\delta(\eta-\eta') - P_0(\boldsymbol{v},\eta')\right]P_0(\boldsymbol{v},\eta).\end{aligned} \tag{11.43}$$

This is the generalisation of the single-waterbag formula (11.30). In what follows, we shall only need the first moment of this expression:

$$\int \mathrm{d}\eta'\,\eta'\left\langle g(\boldsymbol{Q},\eta)g(\boldsymbol{Q},\eta')\right\rangle = \big(\eta - f_0(\boldsymbol{v})\big)P_0(\boldsymbol{v},\eta). \tag{11.44}$$

With this, we are ready to write the hyperkinetic collision integral that generalises the Kadomtsev–Pogutse integral (11.31) to a continuum of waterbags:

$$\begin{aligned}\frac{\partial P_0}{\partial t} = \frac{\pi\omega_{\mathrm{pe}}^4\Delta\Gamma}{n_0^2 V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}''\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2}\\ \cdot\int \mathrm{d}\eta''\,\eta''\left[\big(\eta''-f_0(\boldsymbol{v}'')\big)P_0(\boldsymbol{v}'',\eta'')\frac{\partial P_0(\boldsymbol{v},\eta)}{\partial \boldsymbol{v}} - \big(\eta - f_0(\boldsymbol{v})\big)P_0(\boldsymbol{v},\eta)\frac{\partial P_0(\boldsymbol{v}'',\eta'')}{\partial \boldsymbol{v}''}\right].\end{aligned} \tag{11.45}$$

The last line of the above formula can be slightly simplified, sacrificing symmetry but eliciting the form of the diffusion and drag terms:

$$\left[\int \mathrm{d}\eta''\,\eta''^2 P_0(\boldsymbol{v}'',\eta'') - f_0^2(\boldsymbol{v}'')\right]\frac{\partial P_0(\boldsymbol{v},\eta)}{\partial \boldsymbol{v}} - \left[\big(\eta - f_0(\boldsymbol{v})\big)\frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''}\right]P_0(\boldsymbol{v},\eta). \tag{11.46}$$

The hyperkinetic collision integral (11.45) has an entropy—as anticipated in §10.3, it is (10.21)—and an H-theorem (Exercise 11.6), so it pushes the hyperkinetic distribution to the unique fixed point given by (10.24) (easily checked to be one by direct substitution). Thus, one might say that we have done for Lynden-Bell what Boltzmann did for Maxwell and Gibbs: shown that an equilibrium motivated by thermodynamics/statistical mechanics is reached dynamically, and how.

Exercise 11.5. Derivation of the hyperkinetic collision integral. (a) By retracing the steps that led from (11.1) to (11.31), but now keeping track of the η dependences, check that (11.45) is indeed duly recovered.

(b) Check that the integral (11.45) automatically preserves the properties (10.19) and (10.20), and that it conserves the total number of particles, their momentum, and energy.

Exercise 11.6. H-theorem for the hyperkinetic collision integral. State it and prove it.

Exercise 11.7. Chavanis–Ewart collision integral. Here I invite you to explore a clever alternative closure for a multi-waterbag collision integral. It can be traced back to Appendix C of the paper by Chavanis (2005) on geophysical turbulence; here we shall follow the derivation of it that was invented by Robbie Ewart in ignorance of that previous work (and published in Ewart *et al.* 2022). The applicability or validity of the Chavanis–Ewart integral are as yet unexplored, so this gives you some food for thought, as well nicely ties up some of the narrative threads that have lead us here.

(a) Rather than deriving an evolution equation for the hyperkinetic distribution $P_0(\boldsymbol{v},\eta)$, let us admit ignorance and adopt the kind of statistical-inference approach that is commonly used in statistical mechanics to find probability distributions that are guaranteed to reproduce certain quantities correctly—in this instance, we want $P_0(\boldsymbol{v},\eta)$ whose first η moment is $f_0(\boldsymbol{v})$, at every $\boldsymbol{v}$, as per (11.37). Obviously, $P_0(\boldsymbol{v},\eta)$ also needs to integrate to unity according to (10.19) and conserve the waterbag content (10.20). So, by maximising the Shannon entropy of the distribution $P_0(\boldsymbol{v},\eta)$ subject to these constraints, show that

$$P_0(\boldsymbol{v},\eta) = \frac{1}{Z\big(\psi(\boldsymbol{v})\big)}\,F(\eta)e^{-\psi(\boldsymbol{v})\eta}, \qquad Z\big(\psi(\boldsymbol{v})\big) = \int \mathrm{d}\eta\,F(\eta)e^{-\psi(\boldsymbol{v})\eta}, \tag{11.47}$$

where the function $F(\eta)$ is determined by the requirement that (10.20) be satisfied at every η and the function $\psi(\boldsymbol{v})$ is the solution of the implicit equation

$$f_0(\boldsymbol{v}) = -\frac{\partial \ln Z}{\partial \psi}(\boldsymbol{v}). \tag{11.48}$$

(b) Show further that[109]

$$\langle g^2\rangle = \langle f^2\rangle - f_0^2 = -\frac{\partial f_0}{\partial \psi}. \tag{11.49}$$

Derive therefore the Chavanis–Ewart collision integral:

$$\boxed{\frac{\partial f_0}{\partial t} = -\frac{\pi\omega_{\mathrm{pe}}^4\Delta\Gamma}{n_0^2 V}\frac{\partial}{\partial \boldsymbol{v}}\cdot\int \mathrm{d}\boldsymbol{v}''\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2}\cdot\left[\frac{\partial\psi(\boldsymbol{v})}{\partial\boldsymbol{v}} - \frac{\partial\psi(\boldsymbol{v}'')}{\partial\boldsymbol{v}''}\right]\frac{\partial f_0}{\partial\psi}(\boldsymbol{v})\frac{\partial f_0}{\partial\psi}(\boldsymbol{v}'')}\,. \tag{11.50}$$

(c) Check that the Kadomtsev–Pogutse integral (11.31) is readily recovered from (11.50) in the case of a single waterbag. What is $\psi(\boldsymbol{v})$ in terms of $f_0(\boldsymbol{v})$ in this case?

(d) Prove that the Lynden-Bell equilibria (10.24) are fixed points of (11.50). What are $\psi(\boldsymbol{v})$ and $F(\eta)$ in this case?

---

[109] Note that since the left-hand side of this equation is always positive, f_0 is always a monotonically decreasing function of ψ, which is good news from the point of view of solvability of (11.48).

(e) Show that the integral (11.50) has the entropy function

$$S = \int \mathrm{d}\boldsymbol{v}\ (\psi f_0 + \ln Z) - \int \mathrm{d}\eta\, \rho(\eta) \ln F(\eta), \tag{11.51}$$

which is never decreased by it, and that a stationary solution is achieved if and only if f_0 is the Lynden-Bell equilibrium (10.24).

(f) Observe that the closure underpinning the Chavanis–Ewart collision integral is equivalent to assuming that a collisionless system will always quickly maximise the entropy (10.21) *locally* in phase space (i.e., for each $\boldsymbol{v}$). The relaxation to the global Lynden-Bell equilibrium then happens more slowly, according to (11.50). This is somewhat analogous to standard collisional gases, which relax quickly, at the collision rate, to a local Maxwellian, and then slowly, at the diffusion rate, to the global one (see, e.g., my lectures on gas kinetics, Schekochihin 2024). If one could prove that the hyperkinetic collision integral (11.45) does indeed contain within itself such a two-time-scale evolution, that would promote the Chavanis–Ewart integral from a clever closure scheme to a valid approximation. I do not know how to do this, or whether it is true, so I give this problem to you as an exercise.

Exercise 11.8. Fluctuation energy in a Lynden-Bell plasma (cf. Ewart *et al.* 2022). (a) Use Poisson's equation and the microgranulation ansatz to show that, in a 3D plasma that is in a Lynden-Bell equilibrium (10.24), the electric-field fluctuation energy is

$$\mathscr{U} \equiv \frac{V}{8\pi} \sum_{\boldsymbol{k}} k^2 \langle |\varphi_{\boldsymbol{k}}|^2 \rangle = \frac{a e^2 \Delta\Gamma k_{\max} V}{\pi\beta} \int_0^\infty \frac{\mathrm{d}\varepsilon}{\varepsilon} \mathscr{N}(\varepsilon), \tag{11.52}$$

where β is the parameter of the Lynden-Bell equilibrium and $k_{\max}$ is some appropriate UV cutoff, needed to evaluate the wave-number integral (cf. §11.6), and $\mathscr{N}(\varepsilon) = g(\varepsilon) f_0$ is the particle-energy distribution [first used in (10.33)], with the density of states $g(\varepsilon)$ given by (10.30). In the derivation of (11.52), the formula (11.49) should prove handy, with $\psi = \beta\varepsilon$, as it is for the Lynden-Bell equilibria.

(b) Taking inspiration from the calculations in §10.4, show that, in the non-degenerate limit,

$$\mathscr{U} = \frac{e^2 \Delta\Gamma k_{\max} V}{\pi} \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, \eta^2 \rho(\eta) \sim \eta_{\max} \Delta\Gamma e^2 k_{\max} N. \tag{11.53}$$

What does this estimate imply for the relative size of $\mathscr{U}$ vs. the kinetic energy of the particles, and also vs. the energy of the "thermal noise" associated with Coulomb collisions? Note that the formula (11.53) could, in fact, be interpreted as an estimate for $\Delta\Gamma$ in terms of $\mathscr{U}$.

11.8. *Conclusions, Loose Ends, and Open Questions*

What can we conclude from the fact that the Lynden-Bell equilibria can be recovered from kinetic theory? The obvious observation so far is the correspondence between the assumptions of Lynden-Bell's statistical mechanics and those of the kinetic theory à la Kadomtsev & Pogutse (1974). In order to get the latter, we needed the stability of f_0 [see discussion just before (11.18)], the microgranulation ansatz (11.5), and phase-volume conservation in the form (10.20). The Lynden-Bell equilibria (10.24) and fixed energy (Exercise 11.2) followed. In the statistical-mechanical approach, we needed to fix the energy [see (10.16)] and assume thorough mixing of phase-density elements subject only to the exclusion principle dictated by phase-volume conservation [see discussion after (10.2)]. This gave us an entropy to maximise, and hence an equilibrium, whereas in kinetic theory, the existence of the Lynden-Bell entropy (10.21) was merely a humble property of the collision integral (Exercise 11.6).

Suspending for a moment all the legitimate doubts about the assumptions that have gone into this, let us observe that the progress achieved in §11 in comparison with the statistical-mechanical calculations of §10 is that we can now follow the relaxation of f_0 in time, at least after some initial period during which all the instabilities of an initial state sort themselves out and f_0 becomes stable. The rate at which f_0 relaxes towards equilibrium can be read off from (11.31) or (11.45): ignoring any factors of order unity and using (11.35) to estimate the size of the $\boldsymbol{k}$ sum,[110] we get the effective collision frequency

$$\boxed{\nu_{\text{eff}} \sim \nu\eta\Delta\Gamma}\,, \tag{11.54}$$

where ν is the "true" (Coulomb) collision frequency. In the "true" collisional limit, $\eta\Delta\Gamma = 1$ [see (11.34)], but in the collisionless regime, $\eta\Delta\Gamma$ (the number of particles that fit into a fully occupied macro-cell whose size is the correlation volume of g) must be large, so the "collisionless collision frequency" ν_{eff} is, generally speaking, much larger than ν. Can we estimate ν_{eff} in terms of some physical (measurable) parameters?

Since, generally, there are many waterbags, η in (11.54) must be interpreted as some characteristic value: perhaps, in view of (11.29),

$$\eta_{\text{eff}} = \frac{\langle g^2\rangle}{f_0}, \tag{11.55}$$

i.e., it is the typical deviation of the phase-space density from its mean. Using (11.49) with $\psi = \beta\varepsilon$ (as is appropriate for the Lynden-Bell equilibria), this gives

$$\eta_{\text{eff}} = -\frac{1}{\beta}\frac{\partial \ln f_0}{\partial\varepsilon} \sim \eta_{\max}, \tag{11.56}$$

assuming that it is the energy dependence in the core of the distribution that matters here [see (10.49)]. This assumption is not, perhaps, obviously correct or inevitable, and indeed it is possible that different parts of the distribution relax at significantly different rates—an open question for research. If the estimate (11.56) is correct, then η_{eff} can be related to the density of the particles and the underlying Gardner ground-state energy via (10.48).

The correlation volume $\Delta\Gamma = \Delta r^3\Delta v^3$ is an (even) trickier quantity to make sense of. As the evolution of a collisionless system proceeds, one might argue that g would get ever more fine-scaled ("phase-mixed"; cf. §5.3), i.e., that Δv (and, due to nonlinear coupling of $\boldsymbol{k}$'s, also Δr) would decrease with time. Under this scenario, $\Delta\Gamma(t)$ will decrease with time and so the convergence of f_0 towards equilibrium will slow down as time goes on.[111] Eventually, both Δr and Δv will be limited by collisions, however small they are—with regard to Δv, we saw that in §5.5, and I will show in §12.4.3 that there is a collisional cutoff also in position space.

As anticipated at the end of §9.1 (right after Exercise 9.1), and as will be demonstrated quantitatively in §12.4.4, it is a very bad assumption that purely collisionless dynamics can continue for as long a time as $\sim \nu^{-1}$. This means that, while the evolution of f_0 might still be collisional, that of δf will not be—and, therefore, conservation of $\rho(\eta)$ can no longer be used as an exact, immutable constraint. This might seem to render the whole theory presented in §§9–11 quite useless, but we could hope to rescue it if the effect of

[110] This includes the $1/|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2$ factor. For $k\lambda_{\text{D}} \gg 1$, $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}| \approx 1$ and the contribution to the $\boldsymbol{k}$ sum from these wave numbers is $\sim \Lambda$; for $k\lambda_{\text{D}} \ll 1$, $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}| \sim 1/(k\lambda_{\text{D}})^2$ and the contribution from these wave numbers is ~ 1.

[111] Since $\Delta\Gamma$ had to be chosen in such a way that the perturbed distribution g were decorrelated over it as per the microgranulation ansatz (11.5), one could perhaps make do with any $\Delta\Gamma$ that is larger than the true correlation volume $\Delta r^3\Delta v^3$. This suggests that the relaxation to equilibrium of a distribution f_0 coarse-grained on larger $\Delta\Gamma$ might occur faster, perhaps not an entirely crazy proposition. Then, as time goes on, the equilibrium would be reached with "greater precision", i.e., by f_0 coarse-grained on smaller $\Delta\Gamma$.

collisions turned out to be to evolve $\rho(\eta)$ in some universal way—perhaps towards the universal asymptotic $\rho(\eta) \sim \eta^{-1}$ intuited by Ewart [see (10.34)]? The latter question was asked, somewhat in the spirit of wishful thinking, in the discussional part of the paper by Ewart *et al.* (2023) and shortly thereafter proved to be a good hunch: 1D simulations of the relaxation of the streaming instability do indeed appear to follow such a scenario (Ewart *et al.* 2024). Exactly why this is the case remains an open question: a proper theory would have to append to the collision integrals derived above another equation, evolving the waterbag content:

$$\frac{\partial \rho}{\partial t} = \dots? \tag{11.57}$$

Yet another open question is whether the QL approximation (§11.3.1) coupled with the microgranulation ansatz (§11.3.2) is at all an acceptable closure. As I intimated in §11.3, one could reinterpret $h_{k\omega}$ in (11.14) as containing not just the ballistic evolution (11.15) but also the "nonlinear part" of $\delta f_{k\omega}$. The question is what is its correlation function $C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v}, \boldsymbol{v}')$, which then goes into the general collision integral (11.22)—and indeed whether (11.22) is still valid, i.e., if the correlation function has the symmetry required to achieve (11.20) (in §12.4, I will alight on a scenario in which this is not the case); in the absence of such a symmetry, one has to go back to the protypical collision integral (11.3) and calculate $C_{\boldsymbol{k}}(\boldsymbol{v}, \boldsymbol{v}')$.

You will find some further work around these issues in the classic papers by Kadomtsev & Pogutse (1971) and Dupree (1972), although I mention them with some hesitation: neither is particularly transparent (and both are almost certainly technically wrong even if suggestive of ways forward), so you might end up with less clarity rather than more.[112] The key contention of these papers is that the correlation scales Δr and Δv will decrease in time much less quickly than you might imagine based on naïve phase-mixing estimates—because long-time correlated phase-space "clumps" (or "granulations") will allegedly form. The "collisionless collision integral" (11.22), or something resembling it, with an appropriate correlation function $C_{\boldsymbol{k}\omega\omega'}(\boldsymbol{v}, \boldsymbol{v}')$, then represents some effective scattering of particles by these clumps. The status of all this is rather uncertain: there was a lot of analytical work done in the 1970s and 1980s along and around these lines, but most of it has remained both incomplete and hypothetical (as well as, I am sorry to say, quite impenetrably written), due to the difficulty of nonlinear theory and impossibility at the time of kinetic numerical simulations capable of resolving anything. The latter impediment to progress is now rapidly being lifted, so it is a good time to revisit the old theories and attempt new ones. This is what I shall do in §12.

12. Phase-Space Turbulence I: Echoes, Cascades, and Nonlinear Phase Mixing

Since the question of the phase-space, and, in particular, velocity-space, structure of the perturbed distribution functions has acquired new urgency in §11, especially in what concerns the *nonlinear* evolution of fluctuations in plasmas, it is time to revisit phase mixing, but in the nonlinear regime.

[112]The textbook by Diamond *et al.* (2010, Chapter 8) is on an ideological continuum from Dupree (1972), and, while also not a transparent read (imho), is probably the most up-to-date exposition of where that line of thinking had led before the current generation of Oxford students got interested.

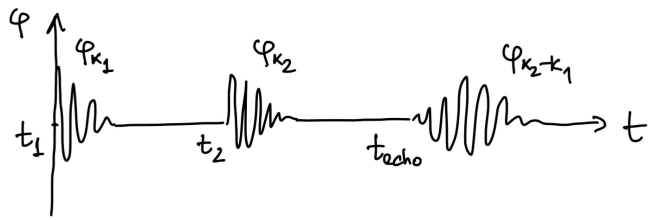


FIGURE 54. Two subsequent pulses, then an echo at time (12.4).

12.1. *Plasma Echo*

In §5.3, we saw that the perturbed distribution function (5.26) associated with a Landau-damped electric perturbation contained an undamped ballistic term proportional to $e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$. As time went on, this term became ever more oscillatory in $\boldsymbol{v}$ and so no longer contributed to φ, but the information contained in it was not destroyed until it hit collisions. It turns out that this information can be recovered.

Let us think in 1D and imagine sending an electric pulse into a plasma (i.e., setting up some initial perturbation) at time t_1, with wave number k_1. It will be Landau damped, but the undamped ballistic part of the resulting perturbed distribution function will be

$$\delta f_{k_1} = a_1 e^{-ik_1 v(t-t_1)}, \tag{12.1}$$

where a_1 is the contents of the curly bracket in (5.26) and v the component of the velocity parallel to the direction of the wave vector. Now let us wait for a while and send in another pulse, with $k_2 > k_1$ at $t_2 > t_1$. This too will be Landau damped, leaving behind

$$\delta f_{k_2} = a_2 e^{-ik_2 v(t-t_2)}. \tag{12.2}$$

In the linear approximation, these are independent solutions, which simply add up. But if nonlinearity is allowed, they can couple to produce

$$\delta f_{k_2-k_1} \propto \delta f_{k_2}\delta f^*_{k_1} = a_1^* a_2 e^{-ik_2 v(t-t_2)+ik_1 v(t-t_1)} = a_1^* a_2 e^{-i[(k_2-k_1)t-(k_2t_2-k_1t_1)]v}. \tag{12.3}$$

Around the time

$$\boxed{t_{\rm echo} = \frac{k_2t_2 - k_1t_1}{k_2-k_1}}\,, \tag{12.4}$$

the distribution function (12.3) does not oscillate strongly in v and so can have an associated electric perturbation $\varphi_{k_2-k_1} \propto \int \mathrm{d}v\,\delta f_{k_2-k_1}$, which will pop out of nowhere some time after the second pulse has gone in (e.g., if $k_2 = 2k_1$, $t_{\rm echo} = 2t_2 - t_1 > t_2$; Fig. 54). Physically, what happens is that the second pulse "catches up" with the first pulse in phase space, couples to it, and effectively reverses the direction of phase mixing, producing "phase unmixing" and bringing information back into the electric field. This is called *plasma echo* and is a strange but real thing: having been derived by Gould *et al.* (1967), it was promptly produced in a laboratory by Malmberg *et al.* (1968).

The original derivation is a nice exercise in perturbation theory, which I will reproduce in §12.1.1. But this is not a mere curiosity: for nonlinear plasma systems with many interacting modes (i.e., for plasma turbulence), the implication is that Landau damping and phase mixing are suppressed at larger spatial scales, or, more precisely, that the velocity-space (linear) mixing is intertwined with position-space (nonlinear) one, thus rendering the nonlinear plasma in a certain sense more "fluid-like" than the linear one is. That in turn may mean that the microgranulation ansatz (11.5) is suspect and the phase-space correlations are a lot longer than linear, quasilinear, or weakly nonlinear theories might lead one to believe. This is a research frontier—I will develop this topic starting from §12.2.

12.1.1. *Textbook Derivation of Plasma Echo*

Let us consider a situation in which only the electron distribution function is perturbed (i.e., frequencies are high) and its perturbation satisfies (2.12), as usual:

$$\frac{\partial \delta f_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{v}\delta f_{\boldsymbol{k}} + i\frac{e}{m}\varphi_{\boldsymbol{k}}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} = -i\frac{e}{m}\sum_{\boldsymbol{k}'}\varphi_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial \delta f_{\boldsymbol{k}-\boldsymbol{k}'}}{\partial \boldsymbol{v}} \equiv N_{\boldsymbol{k}}(\boldsymbol{v},t), \tag{12.5}$$

where $\varphi_{\boldsymbol{k}}$ now contains both the self-consistent field arising from $\delta f_{\boldsymbol{k}}$ via Poisson's equation (2.9) and an externally applied field $\chi_{\boldsymbol{k}}$ (cf. Q-13):

$$\varphi_{\boldsymbol{k}} = \chi_{\boldsymbol{k}} - \frac{4\pi e}{k^2}\int \mathrm{d}\boldsymbol{v}\,\delta f_{\boldsymbol{k}}. \tag{12.6}$$

We shall solve an initial-value problem with $\delta f_{\boldsymbol{k}}(t=0)=0$ and use $\chi_{\boldsymbol{k}}$ to set up our two pulses:

$$\chi(t,\boldsymbol{r}) = \chi_1\delta(t-t_1)\cos k_1 z + \chi_2\delta(t-t_2)\cos k_2 z, \tag{12.7}$$

where z is the axis along which they propagate and $t_2 > t_1 > 0$, $k_2 > k_1$ (the case of $\chi_2 = 0$, $t_1 = 0$, and $N_{\boldsymbol{k}} = 0$ is exactly equivalent to the standard Landau-damping problem solved in §3).

Laplace transforming everything, we get

$$\delta\hat{f}_{\boldsymbol{k}}(p) = \frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\left[-i\frac{e}{m}\hat{\varphi}_{\boldsymbol{k}}(p)\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} + \hat{N}_{\boldsymbol{k}}(p)\right]. \tag{12.8}$$

This is not really a solution, just a rewriting of (12.5), because the Laplace-transformed nonlinear term $\hat{N}_{\boldsymbol{k}}(p)$ contains $\delta f_{\boldsymbol{k}}$:

$$\begin{aligned}\hat{N}_{\boldsymbol{k}}(p) &= -i\frac{e}{m}\sum_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial}{\partial\boldsymbol{v}}\int_0^\infty \mathrm{d}t\, e^{-pt}\int_{-i\infty+\sigma'}^{+i\infty+\sigma'}\int_{-i\infty+\sigma''}^{+i\infty+\sigma''}\frac{\mathrm{d}p'\mathrm{d}p''}{(2\pi i)^2}\, e^{(p'+p'')t}\hat{\varphi}_{\boldsymbol{k}'}(p')\delta\hat{f}_{\boldsymbol{k}-\boldsymbol{k}'}(p'')\\ &= i\frac{e}{m}\sum_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}p'\mathrm{d}p''}{(2\pi i)^2}\frac{\hat{\varphi}_{\boldsymbol{k}'}(p')\delta\hat{f}_{\boldsymbol{k}-\boldsymbol{k}'}(p'')}{p'+p''-p}, \qquad \mathrm{Re}\, p > \sigma' + \sigma'',\end{aligned} \tag{12.9}$$

where I have dropped, but will remember, the integration limits. Keeping this expression in mind, let us combine (12.6) and (12.8):

$$\hat{\varphi}_{\boldsymbol{k}}(p) = \frac{\hat{\chi}_{\boldsymbol{k}}(p)}{\epsilon(p,\boldsymbol{k})} - \frac{4\pi e}{k^2\epsilon(p,\boldsymbol{k})}\int\mathrm{d}\boldsymbol{v}\,\frac{\hat{N}_{\boldsymbol{k}}(p)}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}, \tag{12.10}$$

where $\epsilon(p,\boldsymbol{k})$ is the usual dielectric function, most recently written out in (11.12). The first term in (12.10) is the linear response [cf. (6.59) in Q-7], while the second one is the nonlinear one. Assuming the latter to be small, we can calculate it perturbatively, i.e., by using the linear approximation to $\hat{\varphi}$ and $\delta\hat{f}$ in (12.9). Namely,

$$\begin{aligned}\hat{N}_{\boldsymbol{k}}(p) = \frac{e^2}{m^2}\sum_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial}{\partial\boldsymbol{v}}\iint\frac{\mathrm{d}p'\mathrm{d}p''}{(2\pi i)^2}&\frac{\hat{\chi}_{\boldsymbol{k}'}(p')\hat{\chi}_{\boldsymbol{k}-\boldsymbol{k}'}(p'')}{\epsilon(p',\boldsymbol{k}')\epsilon(p'',\boldsymbol{k}-\boldsymbol{k}')\left[p''+i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{v}\right](p'+p''-p)}\\ &\times(\boldsymbol{k}-\boldsymbol{k}')\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}.\end{aligned} \tag{12.11}$$

This is as far as we can get without inputting specific information about $\hat{\chi}_{\boldsymbol{k}}(p)$, which we shall need in order to do the p' and p'' integrals. From (12.7),

$$\hat{\chi}_{\boldsymbol{k}}(p) = \frac{\chi_1}{2}e^{-pt_1}\left(\delta_{\boldsymbol{k},\boldsymbol{k}_1} + \delta_{\boldsymbol{k},-\boldsymbol{k}_1}\right) + \frac{\chi_2}{2}e^{-pt_2}\left(\delta_{\boldsymbol{k},\boldsymbol{k}_2} + \delta_{\boldsymbol{k},-\boldsymbol{k}_2}\right). \tag{12.12}$$

There will be a lot of terms in $\hat{\chi}_{\boldsymbol{k}'}(p')\hat{\chi}_{\boldsymbol{k}-\boldsymbol{k}'}(p'')$, but we only want the ones that are proportional to $\delta_{\boldsymbol{k},\boldsymbol{k}_2-\boldsymbol{k}_1}$ (it is a tedious, but character-building **exercise** to confirm that there are no echoes

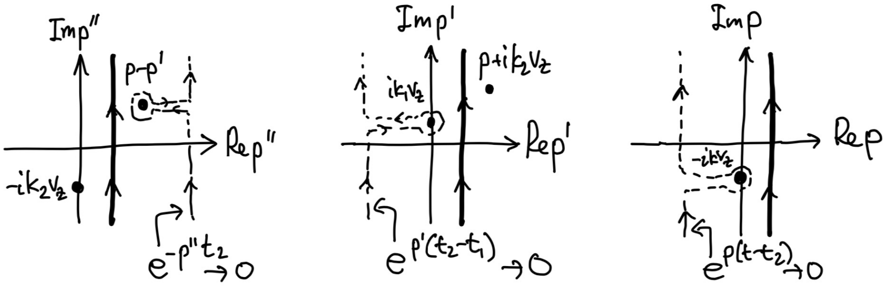


FIGURE 55. Integration countours for the integrals in (12.14) and (12.15).

in the rest of them). They are

$$\frac{\chi_1\chi_2}{4}\,\delta_{\boldsymbol{k},\boldsymbol{k}_2-\boldsymbol{k}_1}\left(e^{-p't_1-p''t_2}\delta_{\boldsymbol{k}',-\boldsymbol{k}_1}+e^{-p't_2-p''t_1}\delta_{\boldsymbol{k}',\boldsymbol{k}_2}\right). \tag{12.13}$$

Putting this into (12.11) and swapping integration variables $p'\leftrightarrow p''$ in the second term, we get, for $\boldsymbol{k}=\boldsymbol{k}_2-\boldsymbol{k}_1$,

$$\begin{aligned}\hat{N}_{\boldsymbol{k}}(p) &= -\frac{e^2\chi_1\chi_2k_1k_2}{4m^2}\frac{\partial}{\partial v_z}\iint\frac{\mathrm{d}p'\mathrm{d}p''}{(2\pi i)^2}\frac{e^{-p't_1-p''t_2}}{\epsilon(p',-k_1)\epsilon(p'',k_2)(p'+p''-p)}\\ &\qquad\times\left(\frac{1}{p''+ik_2v_z}+\frac{1}{p'-ik_1v_z}\right)\frac{\partial f_0}{\partial v_z}\\ &= \frac{e^2\chi_1\chi_2k_1k_2}{4m^2}\frac{\partial}{\partial v_z}\int\frac{\mathrm{d}p'}{2\pi i}\frac{e^{p'(t_2-t_1)-pt_2}(p+ikv_z)}{\epsilon(p',-k_1)\epsilon(p-p',k_2)(p-p'+ik_2v_z)(p'-ik_1v_z)}\frac{\partial f_0}{\partial v_z}\\ &= \frac{e^2\chi_1\chi_2k_1k_2}{4m^2}\frac{\partial}{\partial v_z}\frac{e^{ik_1v_z(t_2-t_1)-pt_2}}{\epsilon(ik_1v_z,-k_1)\epsilon(p-ik_1v_z,k_2)}\frac{\partial f_0}{\partial v_z}.\end{aligned} \tag{12.14}$$

In the last two lines, I did the the p'' and p' integrals, in that order. The p'' integral is done by pushing the integration contour rightwards to $\mathrm{Re}\,p''\to\infty$, but circumventing the pole at $p''=p-p'$ clockwise; only the pole's contribution survives because $e^{-p''t_2}\to 0$ at $\mathrm{Re}\,p''\to\infty$ (Fig. 55). The p' integral is done by pushing the contour leftwards to $\mathrm{Re}\,p'\to-\infty$, but circumventing the ballistic pole at $p'=ik_1v_z$ anticlockwise; again only the pole's contribution survives because $e^{p'(t_2-t_1)}\to 0$ at $\mathrm{Re}\,p'\to-\infty$ (Fig. 55). The dielectric function is assumed to have no poles at $\mathrm{Re}\,p'>0$ (i.e., there is only damping, no instabilities) and the Landau-damping poles it might have at $\mathrm{Re}\,p'<0$ are ignored because they will give rise to damped solutions, not echoes.

Finally, let us put (12.14) into (12.10), integrate out v_x and v_y dependence, so $f_0(\boldsymbol{v})\to F(v_z)$ [cf. (3.18)], and inverse Laplace transform, to get, for the echo (with $\boldsymbol{k}=\boldsymbol{k}_2-\boldsymbol{k}_1$),

$$\begin{aligned}\varphi_{\boldsymbol{k}}^{\mathrm{echo}}(t) &= -\frac{4\pi e}{k^2}\int\mathrm{d}v_z\int\frac{\mathrm{d}p}{2\pi i}\frac{e^{pt}\hat{N}_{\boldsymbol{k}}(p)}{\epsilon(p,k)(p+ikv_z)}\\ &= -i\frac{\pi e^3\chi_1\chi_2k_1k_2}{m^2k}\int\mathrm{d}v_z\int\frac{\mathrm{d}p}{2\pi i}\frac{e^{p(t-t_2)+ik_1v_z(t_2-t_1)}}{\epsilon(p,k)\epsilon(ik_1v_z,-k_1)\epsilon(p-ik_1v_z,k_2)(p+ikv_z)^2}\frac{\partial F}{\partial v_z}\\ &= -i(t-t_2)\frac{\pi e^3\chi_1\chi_2k_1k_2}{m^2k}\int\mathrm{d}v_z\frac{e^{-i[k(t-t_2)-k_1(t_2-t_1)]v_z}}{\epsilon(-ikv_z,k)\epsilon(ik_1v_z,-k_1)\epsilon(-ik_2v_z,k_2)}\frac{\partial F}{\partial v_z}.\end{aligned} \tag{12.15}$$

In the second line, I integrated by parts with respect to v_z, producing a second-order ballistic pole at $p=-ikv_z$, and in the third line, I profited from this pole by pushing the p-integration contour leftwards to $\mathrm{Re}\,p\to-\infty$, using the fact that $e^{p(t-t_2)}\to 0$ there, for $t>t_2$, and thus being left only with the contribution from the pole, circumnavigated anticlockwise (Fig. 55). Note that, since the pole is second order, I had to differentiate the expression under the integral with respect to p to obtain the residue—in doing so, I ignored the derivatives of the dielectric functions in comparison with the derivative of $e^{p(t-t_2)}$, because the latter was large in $t-t_2$.

We see that the exponential in (12.15) that oscillates in v_z does not in fact oscillate very much around precisely the time $t = t_{\rm echo}$ given by (12.4)—this is the effect that we were after!

There is a nice, simple answer in the limit of $k\lambda_{\rm De} \gg 1$, when all dielectric functions in (12.15) can be approximated by unity (and Landau damping is strong). Assuming $F(v_z)$ is a Maxwellian, we get

$$\varphi_{\boldsymbol{k}}^{\rm echo}(t) \approx (t - t_{\rm echo})(t - t_2)\frac{\pi e^3 \chi_1\chi_2 k_1 k_2}{m^2} e^{-k^2 v_{\rm th}^2 (t-t_{\rm echo})^2/4}. \tag{12.16}$$

Thus, the echo pulse arises at $t = t_{\rm echo}$, and then phase-mixes away again.

Exercise 12.1. Plasma echo for weakly damped waves. Work out $\varphi_{\boldsymbol{k}}^{\rm echo}(t)$ in the opposite limit, $k\lambda_{\rm De} \ll 1$, when the Landau-damping rates are exponentially small and the v_z integral in (12.15) can be done by picking up the poles associated with the zeros of the dielectric functions in the denominator. Show in particular that the echo pulse first grows exponentially at the rate $\gamma_{\boldsymbol{k}_1}(k_2 - k_1)/k_1$ and then decays at the rate $\gamma_{\boldsymbol{k}_2-\boldsymbol{k}_1}$, where $\gamma_{\boldsymbol{k}}$ is the Landau-damping rate at the wave number $\boldsymbol{k}$. You will find the solution in Gould *et al.* (1967).

12.2. *Phase-Space Turbulence and Stochastic Echo*

I shall now show how perturbations being dragged back from phase space by nonlinear interactions can be described formally in a fairly transparent fashion. In the process, we will learn a new way of looking at phase mixing—and unmixing—and gain some insight into the nature of "phase-space turbulence".

At the beginning of §11.1, we saw that a crucial role in the "collisionless collision integrals" was played by the correlation function (11.4) of the perturbed distribution function, $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = \left\langle \delta f_{\boldsymbol{k}}(\boldsymbol{v})\delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')\right\rangle$. Here I shall be interested in this function both with a view of deriving a better "collisionless collision integral" than I managed in §11 and because it is interesting in itself, as a descriptor of phase-space turbulence. Phase mixing should manifest itself as $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ developing ever smaller-scale dependence on $\boldsymbol{v}-\boldsymbol{v}'$, and phase unmixing the opposite.

Since

$$\frac{\partial C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')}{\partial t} = \left\langle \delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')\frac{\partial \delta f_{\boldsymbol{k}}(\boldsymbol{v})}{\partial t} + \delta f_{\boldsymbol{k}}(\boldsymbol{v})\frac{\partial \delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')}{\partial t}\right\rangle, \tag{12.17}$$

to derive the evolution equation for $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$, one must multiply (12.5) by $\delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')$ and add to the resulting equation its complex conjugate with $\boldsymbol{v}$ and $\boldsymbol{v}'$ swapped:

$$\begin{aligned}\frac{\partial C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')}{\partial t} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = -i\frac{e}{m}\Bigg[&\left\langle\varphi_{\boldsymbol{k}}\delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')\right\rangle\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} \\ &+ \sum_{\boldsymbol{k}'}\boldsymbol{k}'\cdot\frac{\partial}{\partial\boldsymbol{v}}\left\langle\varphi_{\boldsymbol{k}'}\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{v})\delta f_{\boldsymbol{k}}^*(\boldsymbol{v}')\right\rangle\Bigg] + (\boldsymbol{v}\leftrightarrow\boldsymbol{v}')^* \\ \equiv S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') + N_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}'),&\end{aligned} \tag{12.18}$$

where $S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ is the "source" term containing the second-order correlators and $N_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ is the nonlinear term containing the third-order ones.

12.2.1. *Hacking the Closure Problem*

With third-order correlators appearing in the evolution equation for the second-order one, we are faced with the usual "closure problem" of nonlinear theory (cf. §8.4.1). In order to make the points I wish to make, I am going to have to ram through some rather brutal simplifications.

First, I will treat $\varphi_{\boldsymbol{k}}$ as a known field, i.e., ignore any self-consistent contribution that $\delta f_{\boldsymbol{k}}$ makes to it. This can be interpreted as adopting a "forced Poisson law" [cf. (12.6) and (6.109)]

$$\varphi_{\boldsymbol{k}} = \chi_{\boldsymbol{k}} + \delta\varphi_{\boldsymbol{k}}, \qquad \frac{e}{m}\delta\varphi_{\boldsymbol{k}} = -\frac{\omega_{\rm pe}^2}{k^2 n_0}\int \mathrm{d}\boldsymbol{v}\,\delta f_{\boldsymbol{k}}, \tag{12.19}$$

and ignoring the self-consistent field $\delta\varphi$, or hoping, non-rigorously, that the effect of correlations between $\delta f_{\boldsymbol{k}}(\boldsymbol{v})$ under the velocity integral in (12.19) and the δf's appearing in the right-hand side of (12.18) can be ignored. This *passive-advection approximation* will turn out not too bad in certain regimes (§12.5).

Secondly, I will assume that $\varphi_{\boldsymbol{k}}(t)$ is a random and rapidly decorrelating field—more rapidly than δf evolves either linearly or nonlinearly. This is obviously not a great assumption unless we are literally imposing a white-noise field $\chi_{\boldsymbol{k}}$ onto our system (as we did in Q-13), but it will have to do if we want some kind of closure: just like with the random-phase approximation in §8.4.2, the microgranulation ansatz in §11.3.2, or the *Stosszahlansatz* in the standard particle-collision theory (Parra 2019*a*; Kunz 2021), one needs to break long-term/long-range correlations in some way to get closed equations. The proposed assumption, known in different contexts (see reviews by Falkovich *et al.* 2001, Rincon 2019, and §15.2 of these Lectures) as the Kazantsev–Kraichnan model, or the "short-sudden approximation", is [cf. (6.62) in Q-7 or (6.110) in Q-13]

$$\boxed{\frac{e^2}{m^2}\left\langle\varphi_{\boldsymbol{k}}(t)\varphi^*_{\boldsymbol{k}'}(t')\right\rangle = 2A_k\delta_{\boldsymbol{k},\boldsymbol{k}'}\delta(t-t')}\,, \tag{12.20}$$

where A_k is a known function. Here $k = |\boldsymbol{k}|$, so statistical isotropy is assumed, as well as statistical spatial homogeneity: $\delta_{\boldsymbol{k},\boldsymbol{k}'}$ in (12.20) enforces the latter, viz., the requirement that $\langle\varphi(\boldsymbol{r},t)\varphi(\boldsymbol{r}',t')\rangle$ is a function only of $\boldsymbol{r}-\boldsymbol{r}'$ (it is a simple **exercise** to confirm this).

The correlators in the right-hand side of the $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$-evolution equation (12.18) can now be computed as follows. Let us formally integrate the kinetic equation (12.5) up to time t:

$$\delta f_{\boldsymbol{k}}(t) = -i\int_{\rm past}^t \mathrm{d}t'\left\{\boldsymbol{k}\cdot\boldsymbol{v}\,\delta f_{\boldsymbol{k}}(t') + \frac{e}{m}\left[\varphi_{\boldsymbol{k}}(t')\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} + \sum_{\boldsymbol{k}'}\varphi_{\boldsymbol{k}'}(t')\boldsymbol{k}'\cdot\frac{\partial\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t')}{\partial\boldsymbol{v}}\right]\right\}. \tag{12.21}$$

It does not matter from when in the past we integrate, because only a very short period immediately preceding t will contribute to correlations between $\delta f(t)$ and $\varphi(t)$. Let us now substitute (12.21) into the right-hand side of (12.18). Since $\delta f(t)$ can only depend on the values of $\varphi(t')$ in the past, $t' < t$, we may split any correlators between $\delta f(t')$ and $\varphi(t')$ or $\varphi(t)$. Namely:

$$\begin{aligned}
-i\frac{e}{m}\left\langle\varphi_{\boldsymbol{k}}(t)\delta f^*_{\boldsymbol{k}}(t,\boldsymbol{v}')\right\rangle = \frac{e}{m}\int_{\rm past}^t \mathrm{d}t'\Bigg\{&\boldsymbol{k}\cdot\boldsymbol{v}\cancel{\left\langle\varphi_{\boldsymbol{k}}(t)\right\rangle}\,\cancel{\left\langle\delta f^*_{\boldsymbol{k}}(t',\boldsymbol{v}')\right\rangle}\\
&+\frac{e}{m}\Bigg[\underbrace{\left\langle\varphi_{\boldsymbol{k}}(t)\varphi^*_{\boldsymbol{k}}(t')\right\rangle}_{\text{use (12.20)}}\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v}')}{\partial\boldsymbol{v}'}\\
&+\sum_{\boldsymbol{k}'}\left\langle\varphi_{\boldsymbol{k}}(t)\varphi^*_{\boldsymbol{k}'}(t')\right\rangle\boldsymbol{k}'\cdot\frac{\partial\cancel{\left\langle\delta f^*_{\boldsymbol{k}-\boldsymbol{k}'}(t',\boldsymbol{v}')\right\rangle}}{\partial\boldsymbol{v}'}\Bigg]\Bigg\}\\
= A_k\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v}')}{\partial\boldsymbol{v}'},&
\end{aligned} \tag{12.22}$$

so the source term in the $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$-evolution equation (12.18) is

$$S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = 2A_k\boldsymbol{k}\boldsymbol{k} : \frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}}\frac{\partial f_0(\boldsymbol{v}')}{\partial \boldsymbol{v}'}. \tag{12.23}$$

In the same vein, to calculate $N_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$, we may construct from (12.5) an evolution equation for $\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}}(t,\boldsymbol{v}')$ and formally integrate it:

$$\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}}(t,\boldsymbol{v}') = -i\int_{\text{past}}^t \mathrm{d}t' \left\{\frac{e}{m}\sum_{\boldsymbol{k}''}\left[\varphi_{\boldsymbol{k}''}(t')\boldsymbol{k}''\cdot\frac{\partial \delta f_{\boldsymbol{k}-\boldsymbol{k}'-\boldsymbol{k}''}(t',\boldsymbol{v})}{\partial \boldsymbol{v}}\delta f^*_{\boldsymbol{k}}(t',\boldsymbol{v}')\right.\right.$$
$$\left.\left. - \varphi^*_{\boldsymbol{k}''}(t')\boldsymbol{k}''\cdot\frac{\partial \delta f^*_{\boldsymbol{k}-\boldsymbol{k}''}(t',\boldsymbol{v}')}{\partial \boldsymbol{v}'}\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t',\boldsymbol{v})\right] + \dots\right\}. \tag{12.24}$$

The terms abbreviated by "..." will not matter because all odd-order correlators of φ vanish when we calculate

$$-i\frac{e}{m}\langle\varphi_{\boldsymbol{k}'}(t)\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}}(t,\boldsymbol{v}')\rangle$$
$$= A_{k'}\left[\boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{v}}\langle\delta f_{\boldsymbol{k}}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}}(t,\boldsymbol{v}')\rangle + \boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{v}'}\langle\delta f^*_{\boldsymbol{k}-\boldsymbol{k}'}(t,\boldsymbol{v}')\delta f_{\boldsymbol{k}-\boldsymbol{k}'}(t,\boldsymbol{v})\rangle\right]$$
$$= A_{k'}\boldsymbol{k}'\cdot\left[\frac{\partial C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')}{\partial \boldsymbol{v}} + \frac{\partial C_{\boldsymbol{k}-\boldsymbol{k}'}(\boldsymbol{v},\boldsymbol{v}')}{\partial \boldsymbol{v}'}\right]. \tag{12.25}$$

This has worked out this way because, when the model $\varphi_{\boldsymbol{k}}$ correlator (12.20) was used, $\boldsymbol{k}''$ in the first term in (12.24) was set to $-\boldsymbol{k}'$ and $\boldsymbol{k}''$ in the second term to $\boldsymbol{k}'$. Finally, since $C^*_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v}) = C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$, the nonlinear term in the $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$-evolution equation (12.18) becomes

$$N_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = \sum_{\boldsymbol{k}'}A_{k'}\boldsymbol{k}'\boldsymbol{k}' : \left(2\frac{\partial^2 C_{\boldsymbol{k}-\boldsymbol{k}'}}{\partial \boldsymbol{v}\partial \boldsymbol{v}'} + \frac{\partial^2 C_{\boldsymbol{k}}}{\partial \boldsymbol{v}\partial \boldsymbol{v}} + \frac{\partial^2 C_{\boldsymbol{k}}}{\partial \boldsymbol{v}'\partial \boldsymbol{v}'}\right)$$
$$= D\left(\frac{\partial}{\partial \boldsymbol{v}} + \frac{\partial}{\partial \boldsymbol{v}'}\right)^2 C_{\boldsymbol{k}} + 2\sum_{\boldsymbol{k}'}A_{k'}\boldsymbol{k}'\boldsymbol{k}' : \frac{\partial^2}{\partial \boldsymbol{v}\partial \boldsymbol{v}'}(C_{\boldsymbol{k}-\boldsymbol{k}'} - C_{\boldsymbol{k}}), \tag{12.26}$$

where $D = \sum_{\boldsymbol{k}'} k'^2 A_{k'}/d$ is the QL diffusion coefficient (cf. Q-12), d being the dimension of space (usually $= 3$, but it will be useful to have results for $d = 1$ and 2).

We have paid a heavy price both in dodgy assumptions and in algebra, but the evolution equation (12.18) for the phase-space correlation function $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ is now closed.

Exercise 12.2. Furutsu–Novikov formula. At least one element of the dodgyness can, in fact, be removed. The average-splitting scheme that I employed above is, more formally, based on the following mathematical formula, due to Furutsu (1963) and Novikov (1965). If $\varphi(q)$ is a Gaussian random field that depends on variable(s) q (which can include time, space, wave number, vector index, etc.) and $F[\varphi]$ is a functional of φ, then their correlation function can be expressed as a convolution of the second-order correlator of φ and the functional derivative of F with respect to φ:

$$\boxed{\langle\varphi(q)F[\varphi]\rangle = \int \mathrm{d}q'\langle\varphi(q)\varphi(q')\rangle\left\langle\frac{\delta F[\varphi]}{\delta\varphi(q')}\right\rangle}. \tag{12.27}$$

Convince yourself that the results (12.23) and (12.26) can be derived from (12.21) using this formula. If you are unsure how to handle functional derivatives, here are three basic rules:

$$\frac{\delta\varphi(q)}{\delta\varphi(q')} = \delta(q-q'), \qquad \frac{\delta}{\delta\varphi(q)}\int \mathrm{d}q'\varphi(q')K(q') = K(q), \qquad \frac{\delta}{\delta\varphi}FG = \frac{\delta F}{\delta\varphi}G + F\frac{\delta G}{\delta\varphi}. \tag{12.28}$$

Note also that $\delta F/\delta\varphi(q) = 0$ if F does not depend on the value of φ at q: e.g., taking q to be time, this is the case if F is some function that only depends on the past history of φ (as δf does in the calculation above), and we are trying to compute a functional derivative of it with respect to φ taken at a future moment.

Exercise 12.3. Two-time correlation function. Consider the two-time correlation function

$$\tilde{C}_{\boldsymbol{k}}(t-t',\boldsymbol{v},\boldsymbol{v}') = \langle \delta f_{\boldsymbol{k}}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}}(t',\boldsymbol{v}')\rangle, \tag{12.29}$$

where, without loss of generality, $t > t'$, and, in a statistical steady state, $\tilde{C}_{\boldsymbol{k}}$ depends only on the time delay $\tau = t - t'$, rather than separately on t and t'. Using the same methods, assumptions, and approximations as above, derive the following equation:

$$\frac{\partial \tilde{C}_{\boldsymbol{k}}}{\partial \tau} + i\boldsymbol{k}\cdot\boldsymbol{v}\tilde{C}_{\boldsymbol{k}} = D\frac{\partial^2 \tilde{C}_{\boldsymbol{k}}}{\partial \boldsymbol{v}^2}. \tag{12.30}$$

Solve this equation to show that

$$\boxed{\tilde{C}_{\boldsymbol{k}}(\tau,\boldsymbol{v},\boldsymbol{v}') = \int \mathrm{d}\boldsymbol{v}'' \frac{1}{(4\pi D\tau)^{d/2}} \exp\left(-\frac{|\boldsymbol{v}-\boldsymbol{v}''|^2}{4D\tau} - i\boldsymbol{k}\cdot\frac{\boldsymbol{v}+\boldsymbol{v}''}{2}\tau - \frac{Dk^2\tau^3}{12}\right) C_{\boldsymbol{k}}(\boldsymbol{v}'',\boldsymbol{v}')}\,, \tag{12.31}$$

where $C_{\boldsymbol{k}}$ is the one-time correlation function (11.4). What is, therefore, the correlation time of $\delta f_{\boldsymbol{k}}$?

Exercise 12.4. Heating rate in forced plasma. Go back to Q-13, where the electrostatic potential (6.109) was assumed to be the sum of its self-consistent part and an external white noise (6.110). Use the averaging method presented in this section to prove that the heating rate in such a plasma will always be given by (6.114), irrespective of the form of A_k. This is a manifestation of the general fact that systems subject to white-noise forcing receive a fixed, externally pre-determined amount of power from it, which no internal dynamics are able to modify because the forcing is entirely unpredictable and the system cannot adopt a defensive posture—the undergraduate-textbook version of this result is (6.73), for the Langevin problem.

12.2.2. *Phase Mixing and Unmixing*

Let us now discuss what the $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$-evolution equation (12.18), with (12.23) for $S_{\boldsymbol{k}}$ and (12.26) for $N_{\boldsymbol{k}}$, tells us. For a moment, let us ignore $N_{\boldsymbol{k}}$ and solve the linear equation

$$\frac{\partial C_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')C_{\boldsymbol{k}} = S_{\boldsymbol{k}}. \tag{12.32}$$

The solution is

$$C_{\boldsymbol{k}} = S_{\boldsymbol{k}}\frac{1-e^{-i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')t}}{i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')} \to S_{\boldsymbol{k}}\pi\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}')\big) \quad \text{as} \quad t\to\infty, \tag{12.33}$$

with the δ-function limit derived in the same way as in (5.41). This tells us that linear physics wants to make $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$ very fine scaled in phase space. This is a manifestation of a familiar phenomenon—phase mixing.

It is then natural to change variables as follows:

$$(\boldsymbol{v},\boldsymbol{v}') \to (\boldsymbol{u},\boldsymbol{w}), \quad \text{where} \quad \boldsymbol{u} = \frac{\boldsymbol{v}+\boldsymbol{v}'}{2}, \quad \boldsymbol{w} = \boldsymbol{v}-\boldsymbol{v}', \tag{12.34}$$

$$\frac{\partial}{\partial \boldsymbol{v}} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{u}} + \frac{\partial}{\partial \boldsymbol{w}}, \qquad \frac{\partial}{\partial \boldsymbol{v}'} = \frac{1}{2}\frac{\partial}{\partial \boldsymbol{u}} - \frac{\partial}{\partial \boldsymbol{w}}, \tag{12.35}$$

so the derivatives appearing in (12.26) become

$$\left(\frac{\partial}{\partial \boldsymbol{v}} + \frac{\partial}{\partial \boldsymbol{v}'}\right)^2 = \frac{\partial^2}{\partial \boldsymbol{u}^2}, \qquad \boldsymbol{k}'\boldsymbol{k}' : \frac{\partial^2}{\partial \boldsymbol{v}\partial \boldsymbol{v}'} = \frac{1}{4}\left(\boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{u}}\right)^2 - \left(\boldsymbol{k}'\cdot\frac{\partial}{\partial \boldsymbol{w}}\right)^2. \tag{12.36}$$

In these new variables, phase mixing will look like growth of $\partial/\partial \boldsymbol{w}$, whereas the $\boldsymbol{u}$

dependence of the correlation function will just capture the large-scale, inhomogeneous structure of the velocity space—the fact that f_0 has a scale $\sim v_{\mathrm{th}}$. As a matter of fact, this scale will tend to increase with time via what is recognisably QL diffusion: for f_0, the version of QLT with an external field was the subject of Q-12 and Q-13, while for $C_{\boldsymbol{k}}$, the first term in (12.26) is just diffusion in $\boldsymbol{u}$ [see (12.36)]. Thus, let us assume that $\partial/\partial\boldsymbol{w} \gg \partial/\partial\boldsymbol{u}$ and write out (12.18) with (12.23) and (12.26) in this approximation, using (12.36):

$$\frac{\partial C_{\boldsymbol{k}}}{\partial t} + i\boldsymbol{k}\cdot\boldsymbol{w}C_{\boldsymbol{k}} \approx S_{\boldsymbol{k}} + D\frac{\partial^2 C_{\boldsymbol{k}}}{\partial \boldsymbol{u}^2} - 2\sum_{\boldsymbol{k}'} A_{k'}\left(\boldsymbol{k}'\cdot\frac{\partial}{\partial\boldsymbol{w}}\right)^2 (C_{\boldsymbol{k}-\boldsymbol{k}'} - C_{\boldsymbol{k}}). \tag{12.37}$$

Since our objective is to characterise the emergence of the small-scale structure in $\boldsymbol{w}$, let us make another formal step and Fourier transform in this variable:

$$C_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{u}) = \int \frac{\mathrm{d}\boldsymbol{w}}{(2\pi)^3}\, C_{\boldsymbol{k}}(\boldsymbol{u},\boldsymbol{w})e^{i\boldsymbol{s}\cdot\boldsymbol{w}}. \tag{12.38}$$

This "mixed-variable" object, containing information both about the large-scale real-velocity-space inhomogeneity and a Fourier-space representation of the small-scale structure (in both velocities and positions), is known in quantum mechanics (and in signal processing) as the *Wigner function* (it already appeared, in a different context, in §8.5). It satisfies

$$\boxed{\frac{\partial C_{\boldsymbol{k},\boldsymbol{s}}}{\partial t} + \boldsymbol{k}\cdot\frac{\partial C_{\boldsymbol{k},\boldsymbol{s}}}{\partial \boldsymbol{s}} \approx S_{\boldsymbol{k}}(\boldsymbol{u},\boldsymbol{u})\delta(\boldsymbol{s}) + D\frac{\partial^2 C_{\boldsymbol{k},\boldsymbol{s}}}{\partial \boldsymbol{u}^2} + 2\sum_{\boldsymbol{k}'} A_{k'}(\boldsymbol{k}'\cdot\boldsymbol{s})^2\big(C_{\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{s}} - C_{\boldsymbol{k},\boldsymbol{s}}\big)}\,. \tag{12.39}$$

A further approximation deployed above is

$$S_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') \approx S_{\boldsymbol{k}}(\boldsymbol{u},\boldsymbol{u}) = 2A_k\left[\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{u})}{\partial\boldsymbol{u}}\right]^2, \tag{12.40}$$

because $\boldsymbol{w}$ is assumed small while $S_{\boldsymbol{k}}$ only contains large-scale velocity dependence, via f_0.

The new equation (12.39) gives us a transparent way to see phase mixing: the second term on the left-hand side is just a propagation term in $\boldsymbol{s}$ space, at "velocity" $\boldsymbol{k}$, while the source ($S_{\boldsymbol{k}}$) term on the right is a "boundary condition" at small s. Perturbations seeded by this source travel to larger values of s provided $\boldsymbol{k}\cdot\boldsymbol{s} > 0$—this is phase mixing. In principle, $\boldsymbol{k}\cdot\boldsymbol{s} < 0$ would take perturbations from larger to smaller s—phase unmixing!—but there are no sources at large s within linear theory. The nonlinear game changer is the last, mode-coupling term on the right-hand side of (12.39): it can easily couple a phase-mixing perturbation with $(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{s} > 0$ to a phase-unmixing one with $\boldsymbol{k}\cdot\boldsymbol{s} < 0$ and thus send a pulse back to low s. This is a stochastic version of the echo effect: perturbations chase each other into phase space and can turn around if they couple in the right way.[113]

Will they couple in the right way? Even though (12.39) is a closed equation, it is not so easy to solve, but we do know some solutions of it, in 1D and with further simplifying assumptions (Adkins & Schekochihin 2018; Nastac *et al.* 2024, Fig. 50a,c)—on this basis,

[113]In Schekochihin *et al.* (2016) and Adkins & Schekochihin (2018), you will find two versions of this argument set up in terms of Hermite, rather than Fourier, transforms, which are, of course, equivalent at large Hermite order and large s (see Q-8 for a quick tutorial). In the Hermite formalism, there exists a neat decomposition of the distribution function into phase-mixing and phase-unmixing parts, with the nonlinear term represented as their coupling, but I now nevertheless prefer the Fourier language because it does not shackle one to a Maxwellian f_0.

the tentative answer is yes, phase unmixing will indeed change things in an interesting way (the same conclusion emerges from numerical simulations: see Nastac *et al.* 2025). I will show the Adkins–Nastac solution in §12.4 and use it as a vehicle for explaining how phase-space turbulence works.

Exercise 12.5. Multispecies phase-space turbulence. Generalise the above developments to the case of multiple species, i.e., derive closed evolution equations for

$$C_{\boldsymbol{k}\alpha\alpha'}(\boldsymbol{v},\boldsymbol{v}') = \left\langle \delta f_{\boldsymbol{k}\alpha}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}\alpha'}(t,\boldsymbol{v}')\right\rangle, \tag{12.41}$$

$$\tilde{C}_{\boldsymbol{k}\alpha\alpha'}(t-t',\boldsymbol{v},\boldsymbol{v}') = \left\langle \delta f_{\boldsymbol{k}\alpha}(t,\boldsymbol{v})\delta f^*_{\boldsymbol{k}\alpha'}(t',\boldsymbol{v}')\right\rangle, \quad t>t'. \tag{12.42}$$

What is the new version of the $(\boldsymbol{u},\boldsymbol{w})$ variables?

The equation that you will have derived for $\tilde{C}_{\boldsymbol{k}\alpha\alpha'}(\tau)$ is identical to (12.30) and is solved in the same way. The equation for the one-time function $C_{\boldsymbol{k}\alpha\alpha'}$ has not been properly analysed, so it is not known what interesting effects might lurk there, if any. If you investigate, you might find something new.

12.3. *f-strophy*

Before solving (12.39), however, I would like to make an important observation: the Wigner function $C_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{u})$ is evidently the spectral density of a quadratic integral quantity that is conserved except by the source term. This conserved quantity is

$$\sum_{\boldsymbol{k}}\int \mathrm{d}\boldsymbol{s}\int \mathrm{d}\boldsymbol{u}\, C_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{u}) = \sum_{\boldsymbol{k}}\int \mathrm{d}\boldsymbol{u}\, C_{\boldsymbol{k}}(\boldsymbol{u},\boldsymbol{w}=0) = \frac{1}{V}\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\left\langle \delta f(\boldsymbol{v},\boldsymbol{r})^2\right\rangle \equiv \delta\mathscr{C}_2, \tag{12.43}$$

the total variance of the "kinetic scalar", which I shall call, for the love of coining a new term, *f-strophy*,[114] and which satisfies

$$\frac{\mathrm{d}\delta\mathscr{C}_2}{\mathrm{d}t} = \sum_{\boldsymbol{k}}\int \mathrm{d}\boldsymbol{u}\, S_{\boldsymbol{k}}(\boldsymbol{u},\boldsymbol{u}) = \sum_{\boldsymbol{k}} 2A_k\int \mathrm{d}\boldsymbol{u}\left[\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{u})}{\partial\boldsymbol{u}}\right]^2 = 2D\int \mathrm{d}\boldsymbol{u}\left|\frac{\partial f_0(\boldsymbol{u})}{\partial\boldsymbol{u}}\right|^2 \equiv \varepsilon. \tag{12.44}$$

I shall discuss the physical meaning and role of this invariant in §12.3.3, but let us first see what its conservation implies for us.

The most immediate thing that it implies is that, since $\delta\mathscr{C}_2$ has a positive-definite source but no sink, (12.39) cannot have a steady-state solution. In fact, of course it can, but only if we do what is now manifestly the right thing to do and restore collisions, to provide the necessary sink.

Why is it legitimate to do that? In §5.5, my mantra was that, however infrequent particle collisions were, phase mixing in a linear system would always (and fairly quickly) produce sufficiently small velocity-space scales in order for collisions to switch on, thermalise perturbations, and make the system's evolution irreversible. We are going to see that in the nonlinear, echoing/phase-unmixing regime, this happens in an even more efficient way.

12.3.1. *Budget of f-strophy*

The easiest way to model the effect of collisions is to add to (12.39) an extra damping term proportional to $-2\nu v_{\mathrm{th}}^2 s^2$, where ν is the collision frequency, to reflect the fact that

---

[114] Knorr (1977) and Servidio *et al.* (2017) call a similar quantity "enstrophy" (by analogy with the second invariant, additional to energy, in 2D hydrodynamics), Diamond *et al.* (2010) call it "phaseostrophy", and for Nastac *et al.* (2024, 2025), it is just "δC_2", but I cannot resist a new name.

the collision operator (1.47) is a velocity-space diffusion operator and so, in the crudest approximation, it should look like $\nu v_{\rm th}^2 \partial^2 \delta f/\partial v^2$:

$$\frac{\partial C_{\boldsymbol{k},\boldsymbol{s}}}{\partial t} = \text{all terms from (12.39)} - 2\nu v_{\rm th}^2 s^2 C_{\boldsymbol{k},\boldsymbol{s}}. \tag{12.45}$$

Then (12.44) turns into the f-strophy budget equation:

$$\frac{\mathrm{d}\delta\mathscr{C}_2}{\mathrm{d}t} = \varepsilon - \mathscr{D}, \qquad \mathscr{D} = 2\nu v_{\rm th}^2 \sum_{\boldsymbol{k}} \int \mathrm{d}\boldsymbol{s}\, s^2 \int \mathrm{d}\boldsymbol{u}\, C_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{u}). \tag{12.46}$$

The steady state, as usual, will involve a balance between injection and dissipation, and the latter, despite being collisional, will be independent of the collision rate, however small the latter is, $\mathscr{D} = \varepsilon$. The rest of the terms in (12.39) must then provide the means by which the system gets the f-strophy, injected at large scales, to flow to small scales in velocity (large s), so it can be dissipated efficiently.

12.3.2. *Flux of f-strophy*

We have ended up with yet another turbulent-cascade problem, involving, inevitably, a constant flux (cf. §§7.4, 8.4.6, 16.4, and 23.5.3). This is actually quite straightforward to see. Let us consider (12.39) at values of s that are small enough for collisions not to be important, integrate over $\boldsymbol{u}$ and $\boldsymbol{k}$ and demand a steady state:

$$\frac{\partial}{\partial \boldsymbol{s}} \cdot \boldsymbol{\Gamma}_{\boldsymbol{s}} = \varepsilon\delta(\boldsymbol{s}), \qquad \boldsymbol{\Gamma}_{\boldsymbol{s}} \equiv \sum_{\boldsymbol{k}} \boldsymbol{k} \int \mathrm{d}\boldsymbol{u}\, C_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{u}). \tag{12.47}$$

This is an equation for a flux in $\boldsymbol{s}$ space, $\boldsymbol{\Gamma}_{\boldsymbol{s}}$. If we integrate it further over all but one (does not matter which one) components of $\boldsymbol{s}$, the solution is a constant:

$$\Gamma_{s_i} = \mathrm{sgn}(s_i)\,\frac{\varepsilon}{2}, \tag{12.48}$$

an outward flow of f-strophy. I could, of course, have integrated (12.47) over angles instead and found a constant radial flux out towards large s, $\Gamma_s = \varepsilon$.

This result is, obviously, exactly the same in the linear regime: all effects of nonlinearity have been integrated out. This does not mean, however, that nonlinearity, echoes, phase unmixing, etc., are unimportant to the structure, and nature, of the phase-space cascade. In §12.4, I shall consider the promised simple 1D case to see how it all plays out. In particular, I will show that there is a constant flux of f-strophy not only in $\boldsymbol{s}$ space but also in $\boldsymbol{k}$ space, the generation of fine velocity and spatial scales being interwtined in a distinctive phase-space cascade.

12.3.3. *Meaning of f-strophy*

Although I am speaking in such terms as "dissipation", "flux", "cascade", etc., f-strophy is *not* the same thing as the entropic part of the free energy (5.20) or its Kruskal–Oberman generalisation (9.32)—a different kind of entropy turns out to be the relevant one. Since the exact f satisfies the collisionless Vlasov equation, it has an infinite number of Casimir invariants (9.8). One way of enumerating them is

$$\mathscr{C}_n = \frac{1}{V}\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, f^n = \int \mathrm{d}\eta\, \eta^n \rho(\eta), \tag{12.49}$$

where $\rho(\eta)$ is the waterbag content of f, defined in (10.20). Apart from the trivial case of $\mathscr{C}_1 = n_0$, only one of these invariants is additive in terms of the mean and fluctuating parts of the phase-space density:

$$\mathscr{C}_2 = \int \mathrm{d}\boldsymbol{v}\, f_0^2 + \frac{1}{V}\iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\,\delta f^2 \equiv \mathscr{C}_{2,0} + \delta\mathscr{C}_2, \tag{12.50}$$

where $\delta\mathscr{C}_2$ is our f-strophy.

This puts the anticipated cascade of $\delta\mathscr{C}_2$ into the right physical context. First, let us ask how $\mathscr{C}_{2,0}$ evolves in time. For that, we need the evolution of f_0—if you have done Q-12 and worked out the QLT of stochastic acceleration, you know that f_0 will be diffusively heated. For simplicity, let us use the zero-correlation-time version of that calculation, which can also be arrived at by direct application of the already-derived closure (12.22) (after replacing $\boldsymbol{k} \to -\boldsymbol{k}$), viz., from (11.1),

$$\frac{\partial f_0}{\partial t} = D\frac{\partial^2 f_0}{\partial \boldsymbol{v}^2}, \qquad D = \frac{1}{d}\sum_{\boldsymbol{k}} k^2 A_k, \tag{12.51}$$

where d is the number of spatial dimensions. Solving this equation and finding that the kinetic energy of the distribution grows as $\mathscr{K} = \mathscr{K}(t=0) + mn_0 Dt$ is not a great challenge,[115] but here I am interested in a different consequence of (12.51), viz., that the part of the invariant $\mathscr{C}_2$ that is associated with f_0 is manifestly decreased by the heating process:

$$\frac{\mathrm{d}\mathscr{C}_{2,0}}{\mathrm{d}t} = -2D\int \mathrm{d}\boldsymbol{v}\left|\frac{\partial f_0}{\partial \boldsymbol{v}}\right|^2 = -\varepsilon, \tag{12.52}$$

where ε is the same f-strophy flux that was defined in (12.44) and figured as the source of $\delta\mathscr{C}_2$ in the f-strophy budget (12.46). Thus, $\mathscr{C}_2$ plays the role of minus-entropy in the problem at hand: a collisionless Casimir invariant whose large-scale part $\mathscr{C}_{2,0}$ is minimised by the heating of f_0,[116] transferring the lost $\mathscr{C}_2$ into $\delta\mathscr{C}_2$, which, via a phase-space cascade (to be worked out: §12.4), is eventually drained by collisions, however infrequent. This is the means by which irreversibility of heating is sealed.

Let me stress that these considerations are not restricted to the Kazantsev–Kraichnan model (12.20) of the electric field: $\mathscr{C}_2$ is a collisionless invariant no matter what, $\partial f_0/\partial \boldsymbol{v}$ will always source δf, so f-strophy will flow from f_0 to δf (from $\mathscr{C}_{0,2}$ to $\delta\mathscr{C}_2$), δf will always access small phase-space scales, and thus collisions, fairly quickly, via a nonlinear f-strophy cascade, and, it turns out, even the scalings associated with this cascade, to be derived in §12.4, are quite insensitive to modelling assumptions (§12.5).

Exercise 12.6. f-strophy budget. Starting from the Vlasov–Landau equation (1.30), show, with no approximations, that

$$\boxed{\frac{\mathrm{d}\delta\mathscr{C}_2}{\mathrm{d}t} = \varepsilon - \mathscr{D}, \quad \text{where} \quad \varepsilon = -\frac{\mathrm{d}\mathscr{C}_{2,0}}{\mathrm{d}t}} \tag{12.53}$$

and $\mathscr{D} \geqslant 0$ is the collisional dissipation.

12.4. *Adkins–Nastac Theory of Phase-Space Cascade in 1D*

Finally, let us tackle the much-anticipated phase-space cascade. To turn (12.39) into something solvable, we make three further simplifications:

[115] And the result is of somewhat limited usefulness, as we will discover in §12.5.3.

[116] Unlike the maximisation of the $-f\log f$ entropy by Coulomb collisions (converging to a Maxwellian), this process can never be completed because $f_0 = \text{const}$ is not normalisable.

(i) restrict everything to 1D;

(ii) ignore the u-space diffusion term on the grounds that $\partial/\partial u \ll s$—mathematically, it is convenient just to integrate out the u dependence;

(iii) assume that the spectrum of the electric perturbations, $k'^2 A_{k'}$ (and, therefore, also the source S_k), is concentrated at larger scales than $C_{k,s}$, i.e., $k' \ll k$ (in the context of passive-scalar advection, this is known as the Batchelor 1959 regime).

The last of these allows us to approximate

$$C_{k-k',s} - C_{k,s} \approx -k' \frac{\partial C_{k,s}}{\partial k} + \frac{k'^2}{2} \frac{\partial^2 C_{k,s}}{\partial k^2}, \tag{12.54}$$

i.e., the coupling between k's now happens in small local steps, diffusively, rather than in finite leaps. The first term in (12.54) vanishes under the k' integration in (12.39), and one arrives at the following rather cute equation:

$$\boxed{\frac{\partial C_{k,s}}{\partial t} + k\frac{\partial C_{k,s}}{\partial s} = S_k \delta(s) + \gamma^3 s^2 \frac{\partial^2 C_{k,s}}{\partial k^2} - 2\nu v_{\rm th}^2 s^2 C_{k,s}}\,, \qquad \gamma^3 = \sum_{k'} k'^4 A_{k'}, \tag{12.55}$$

where I have kept the collisional bolt-on introduced in (12.45).

What this equation does is quite easy to see qualitatively. The source at low s and k (low k because of the Batchelor assumption) and the collisional sink at large s are responsible for the injection and dissipation of f-strophy, as discussed in §12.3. I shall pin down the collisional cutoff in §12.4.3, but first let us consider what happens at such s and k that neither the source nor the collisions play a role.

This is an "inertial range", through which a constant flux of f-strophy, mandated by (12.48), must somehow be set up. There are two competing effects: phase (un)mixing, i.e., propagation in s at velocity k, and mode coupling, which in the Batchelor regime takes the form of k-space diffusion, with diffusivity $\gamma^3 s^2$. At lower s, phase (un)mixing dominates, but at higher s, the wave-number diffusion takes over. This change of behaviour takes place when the characteristic time scales associated with the two processes become equal:

$$\frac{k}{s} \sim \frac{\gamma^3 s^2}{k^2} \quad \Rightarrow \quad \boxed{s \sim \frac{k}{\gamma}}\,. \tag{12.56}$$

At $s \ll k/\gamma$ but away from $s = 0$, (12.55) reduces to

$$\frac{\partial C_{k,s}}{\partial s} = 0 \quad \Rightarrow \quad C_{k,s} = f(k), \tag{12.57}$$

a flat s spectrum with an as yet not obvious k dependence.

Exercise 12.7. This is just the linear phase-mixing spectrum. Show that it is equivalent to the Hermite spectrum (6.81) derived in Q-8.

At $s \gg k/\gamma$, (12.55) reduces to

$$\frac{\partial^2 C_{k,s}}{\partial k^2} = 0 \quad \Rightarrow \quad C_{k,s} = g(s), \tag{12.58}$$

a flat k spectrum with a non-obvious s dependence—the result of wave-number diffusion taking over. This diffusion can take perturbations from positive to negative k, or vice versa, and thus allow them to go from phase mixing to phase unmixing.

The dynamics that give rise to these stationary spectra can be schematically described

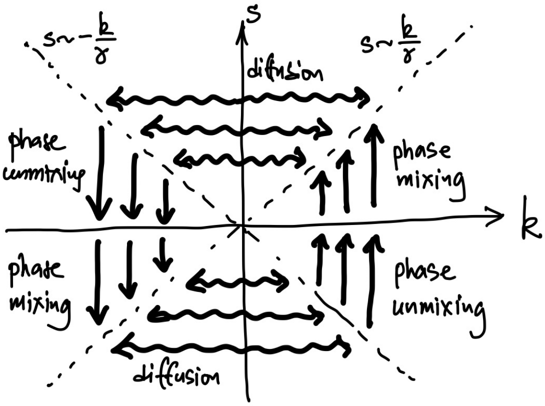


FIGURE 56. Schematic of the circulation in phase space: phase mixing, k-space diffusion, phase unmixing, etc. The circulation is, in fact, not perfectly symmetric as depicted here, but supports a constant net flux of f-strophy outwards to large k and s [see (12.65) and Exercise 12.9].

as follows (Fig. 56). Suppose a perturbation is launched at $k > 0$ and low s. Phase mixing will move it "vertically" in the (k, s) plane to $s \sim k/\gamma$. Then diffusion will spread it "horizontally" from $k > 0$ to $k < 0$ until it reaches $s \sim -k/\gamma$ (which is positive at negative k). There, phase unmixing will take over and push the perturbation to lower s, towards $s = 0$, and then to larger negative s, until it reaches $s \sim k/\gamma$, again becomes subject to diffusion, spreads back to positive k, reaches $s \sim -k/\gamma$, and is again phase-unmixed towards $s = 0$. The cycle is complete.

In fact, it is, of course, not quite a cycle but an outward unwinding spiral—a property of Adkins' exact solution spotted by Nastac *et al.* (2024), and indeed inevitable because of the requirement for a constant f-strophy flux to be pushed through phase space towards larger s. Let us derive that exact solution (read §12.4.2 instead if you want to stick to pure twiddle algebra).

12.4.1. *Self-Similar Phase-Space Spectrum*

We look for a self-similar solution of

$$k\frac{\partial C_{k,s}}{\partial s} = \gamma^3 s^2 \frac{\partial^2 C_{k,s}}{\partial k^2} \tag{12.59}$$

in the form

$$C_{k,s} = \frac{1}{|s|^\alpha}\,\Phi(\xi), \qquad \xi = \frac{k}{\gamma s}. \tag{12.60}$$

We can fix α by noticing that (12.59) implies

$$\frac{\partial}{\partial s}\int_{-\infty}^{+\infty} \mathrm{d}k\, kC_{k,s} = 0, \qquad \int_{-\infty}^{+\infty} \mathrm{d}k\, kC_{k,s} = \gamma^2|s|^{2-\alpha}\mathrm{sgn}(s)\int_{-\infty}^{+\infty} \mathrm{d}\xi\,\xi\Phi(\xi) \quad \Rightarrow \quad \alpha = 2. \tag{12.61}$$

Note that this condition is the 1D version of the constant-flux requirement (12.47) at $s > 0$. Substituting (12.60) with $\alpha = 2$ into (12.59), we get

$$\Phi'' = -\xi^2\Phi' - 2\xi\Phi = -(\xi^2\Phi)' \quad \Rightarrow \quad \Phi' + \xi^2\Phi = c_1 = \text{const}. \tag{12.62}$$

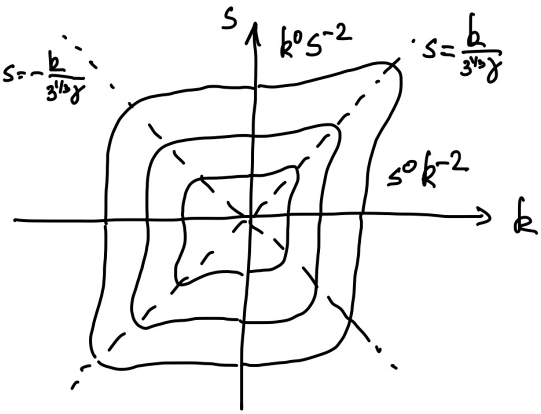


FIGURE 57. Sketch of the contour lines of the solution (12.64). The asymmetric skew of this solution towards large k and s of the same sign is a manifestation of it supporting a constant f-strophy flux.

Integrating once again, we get

$$\varPhi(\xi) = e^{-\xi^3/3}\left(c_1\int_{-\infty}^{\xi} \mathrm{d}\xi' e^{\xi'^3/3} + c_2\right) = c_1\int_0^\infty \frac{\mathrm{d}z\, e^{-z}}{|\xi^3 - 3z|^{2/3}}. \tag{12.63}$$

The second expression was obtained by setting the integration constant $c_2 = 0$ to prevent blow-up at $\xi \to -\infty$ and by changing the integration variable to $z = (\xi^3 - \xi'^3)/3$. Finally, putting this back into (12.60), we get

$$\boxed{C_{k,s} = C_0\int_0^\infty \frac{\mathrm{d}z\, e^{-z}}{|k^3 - 3\gamma^3 s^3 z|^{2/3}}} \to \begin{cases} C_0 k^{-2}, & s \ll \dfrac{k}{3^{1/3}\gamma}, \\[2ex] \dfrac{C_0\varGamma(1/3)}{3^{2/3}\gamma^2}\, s^{-2}, & s \gg \dfrac{k}{3^{1/3}\gamma}, \end{cases} \tag{12.64}$$

where $C_0 = c_1\gamma^2$. This solution is sketched in Fig. 57. Asymptotically, it is just the solution anticipated in (12.57) and (12.58), but we now also know how the "phase-space spectrum" scales with k and s in all relevant limits (we shall recover these scalings from a more hand-waving, but also more easily generalisable, argument in §12.4.2).

It is not hard to show that the solution (12.64) carries a constant flux, i.e., that it can be made to satisfy (12.48). Indeed, using (12.61) and (12.63), we get

$$\begin{aligned}\frac{2\pi}{L}\mathrm{sgn}(s)\varGamma_s &= \mathrm{sgn}(s)\int_{-\infty}^{+\infty}\mathrm{d}k\, kC_{k,s} = \gamma^2\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\xi\varPhi(\xi)\\ &= C_0\int_0^\infty \mathrm{d}z\, e^{-z}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x\, x}{|x^3-1|^{2/3}} = C_0\int_{-\infty}^{+\infty}\frac{\mathrm{d}y}{y|1-y^3|^{2/3}}\\ &= C_0\int_0^\infty \mathrm{d}y\left(\frac{1}{y|1-y^3|^{2/3}} - \frac{1}{y|1+y^3|^{2/3}}\right) \approx 1.81C_0,\end{aligned} \tag{12.65}$$

via three changes of integration variable $x = \xi/(3z)^{1/3}$, $y = 1/x$, and then $y \to -y$ in the negative half of the integration interval. Therefore, from (12.48),

$$\boxed{C_0 \approx \frac{2\pi}{L}0.28\,\varepsilon}\,. \tag{12.66}$$

12.4.2. *Critically Balanced Phase-Space Cascade*

If we assume constant flux, we can recover the asymptotics (12.64) of the phase-space spectrum at both small s/large k and small k/large s from a "twiddle" argument reminiscent of the theories of turbulent cascades in media where linear and nonlinear physics coexist, e.g., MHD (§16.4), rotating and stratified fluids (Nazarenko & Schekochihin 2011), or drift-kinetic (§23.5.4) and gyrokinetic (Schekochihin *et al.* 2009) plasmas. The specific argument that I outline below was proposed by Nastac *et al.* (2024, Fig. 50c).

Let us associate with each value of s a velocity scale $w = 1/s$, and call δf_w the typical size of the increment of the distribution function between two velocity points separated by w. Mathematically, one might think of δf_w^2 as the structure function of the perturbed phase-space density, understood to be averaged over the position space (or independent of it by statistical homogeneity):

$$v_{\rm th}\delta f_w^2 \sim \int \mathrm{d}u \left\langle |\delta f(x, u + w/2) - \delta f(x, u - w/2)|^2 \right\rangle \sim \int_{s=1/w}^{\infty} \mathrm{d}s' \sum_k C_{k,s'}. \tag{12.67}$$

Physically, $v_{\rm th}\delta f_w^2$ is "the amount of f-strophy at scale w". Since this is transferred to smaller w with a constant flux ε, we have

$$\frac{v_{\rm th}\delta f_w^2}{\tau_w} \sim \varepsilon, \qquad \tau_w \sim \frac{s}{k}, \tag{12.68}$$

where τ_w is the transfer time, which is just the linear-mixing time. But the linear-mixing time corresponding to which wave number k? Here we bring in an assumption that the relevant k can be found by assuming that the linear- and nonlinear-mixing times are comparable, i.e., that this k satisfies (12.56):

$$k \sim \gamma s. \tag{12.69}$$

This is the phase-space analog of the famous *critical-balance conjecture*, which, as a prescription to equate linear-propagation and nonlinear-advection time scales at each spatial scale, originates in the theory of MHD turbulence (§16.4) and has been increasingly widely used elsewhere (see the papers cited at the beginning of this section and many papers cited by those papers). Here, the simple justification for it is that, according to (12.58), $C_{k,s}$ is independent of k up to $k \sim \gamma s$ and, assuming that it decays with k sufficiently quickly above $k \sim \gamma s$ (we can check *a posteriori* that it does), $\sum_k C_{k,s}$ is dominated by $k \sim \gamma s$ at any given s, and so is (12.67) (assuming $C_{k,s}$ decays quickly enough with s). In words, qualitatively, we assume that the f-strophy transfer is dominated by the region of phase space where linear and nonlinear mixing compete as equals. With this assumption, (12.68) gives us

$$\tau_w \sim \gamma^{-1} \quad \Rightarrow \quad v_{\rm th}\delta f_w^2 \sim \varepsilon\gamma^{-1} = \mathrm{const}. \tag{12.70}$$

On the other hand, if we assume that $g(s) \propto s^{-a}$ in (12.58) and $f(k) \propto k^{-b}$ in (12.57), i.e., that

$$\boxed{C_{k,s} \propto \left\{ \begin{array}{ll} k^0 s^{-a}, & k \ll \gamma s, \\ s^0 k^{-b}, & k \gg \gamma s, \end{array} \right.} \tag{12.71}$$

then, assuming $b > 1$,

$$\sum_k C_{k,s} \propto s^{-a+1} \quad \Rightarrow \quad \delta f_w^2 \propto w^{a-2}. \tag{12.72}$$

Comparing this with (12.70) immediately implies $\boxed{a = 2}$. We have recovered the second asymptotic in (12.64). From (12.72), or (12.70), the 1D s spectrum of f-strophy is

$$\boxed{C_s \equiv \sum_k C_{k,s} \sim \varepsilon\gamma^{-1}s^{-1}}\,. \tag{12.73}$$

To recover the first asymptotic in (12.64), notice that, in the Batchelor approximation, the term in (12.55) associated with the f-strophy flow through k space has assumed an explicit

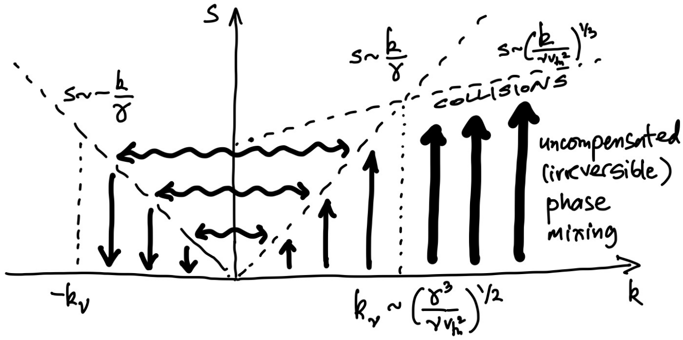


FIGURE 58. Same as the upper half-plane of Fig. 56 but with collisions.

divergence form, and so, in steady state and away from the collisional scales, integrating (12.55) gives us a constant flux through k space:

$$-\gamma^3\frac{\partial^2}{\partial k^2}\int \mathrm{d}s\, s^2 C_{k,s} = S_k \approx \varepsilon\frac{2\pi}{L}\delta(k) \quad\Rightarrow\quad \Gamma_k \equiv -\frac{L}{2\pi}\gamma^3\frac{\partial}{\partial k}\int \mathrm{d}s\, s^2 C_{k,s} = \mathrm{sgn}(k)\,\frac{\varepsilon}{2}, \tag{12.74}$$

where I have put all the "source power" at the origin in k space. Note that this is different from a linear plasma, where no f-strophy exchange between different k's is possible. The scaling argument that led us from (12.67) to $a = 2$ can now be cloned, but for spatial scales $\ell = 1/k$ instead of velocity scales $w = 1/s$. The outcome is $\boxed{b = 2}$, just as it must be, according to (12.64).[117]

Exercise 12.8. Construct this argument and get this result. Prove also that the 1D k spectrum of f-strophy is

$$\boxed{C_k \equiv \int \mathrm{d}s\, C_{k,s} \sim \frac{2\pi}{L}\varepsilon\gamma^{-1}k^{-1}}\,. \tag{12.75}$$

Exercise 12.9. Confirm explicitly that, for the spectrum (12.64) with C_0 given by (12.66), the k-flux is what (12.74) says it is.

Exercise 12.10. Work out what the dimensional prefactors should be in (12.71).

All of the above is, in fact, almost obvious: in the Batchelor approximation, the "kinetic passive scalar" δf is mixed by an external, single-spatial-scale electric field, and there is a single assosiated time scale, γ^{-1}. The spectrum (12.75) is a close analogue of the Batchelor (1959) k^{-1} spectrum of a passive scalar field. I have gone through the argument featuring critical balance and matching asymptotics primarily for pedagogical reasons, hoping to impart a "deeper understanding" of how all this comes about in a kinetic system.

12.4.3. *Collisional Cutoff*

The constant f-strophy flux through phase space has the collisional cutoff as its destination. Let us work out where this cutoff is.

The stationary ($\partial/\partial t = 0$) solution of (12.55) in the linear ($\gamma = 0$) approximation is

$$C_{k,s} = f(k)\, e^{-(s/s_\nu)^3}, \qquad s_\nu(k) = \left(\frac{3k}{2\nu v_{\mathrm{th}}^2}\right)^{1/3}, \tag{12.76}$$

[117] The same result is, in fact, obtainable instantly by matching the two asymptotics in (12.71) along the critical-balance line $k \sim \gamma s$.

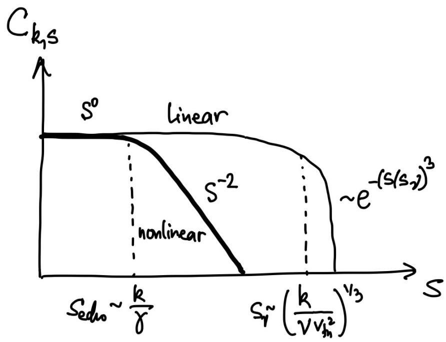


FIGURE 59. Velocity-space spectra $C_{k,s}$ in the linear regime with a collisional cutoff, $k \gtrsim k_\nu$ [see (12.76)] and in the nonlinear regime, $k \ll k_\nu$ [see (12.64)].

reflecting the fact, obvious from comparing time scales (k/s vs. $\nu v_{\rm th}^2 s^2$), that collisions become competitive with phase mixing when $s \gtrsim s_\nu$. If perturbations that phase-mix, moving "vertically" in the (k, s) plane, hit s_ν before the echo threshold $s \sim k/\gamma$, they will get dissipated and lose their chance to diffuse to negative k's and phase unmix (Fig. 58). These are the perturbations with

$$s_\nu(k) \sim \left(\frac{k}{\nu v_{\rm th}^2}\right)^{1/3} \lesssim \frac{k}{\gamma} \quad \Leftrightarrow \quad k \gtrsim \left(\frac{\gamma^3}{\nu v_{\rm th}^2}\right)^{1/2} \equiv k_\nu. \tag{12.77}$$

Another way to present this argument is just to say that the f-strophy cascade is terminated at the scale where the collision rate becomes comparable to the cascade rate, which, by the critical-balance conjecture (§12.4.2), is the same as both the linear- and the nonlinear-mixing rates:

$$\frac{k}{s} \sim \frac{\gamma^3 s^2}{k^2} \sim \nu v_{\rm th}^2 s^2 \quad \Rightarrow \quad \boxed{k_\nu \sim \left(\frac{\gamma^3}{\nu v_{\rm th}^2}\right)^{1/2}, \quad s_\nu(k_\nu) \sim \left(\frac{\gamma}{\nu}\right)^{1/2} v_{\rm th}^{-1}}\,. \tag{12.78}$$

Thus, there is a k-space cutoff k_ν, a kind of "Kolmogorov scale" for kinetic plasmas, beyond which the nonlinearity does not matter and perturbations just "Landau-damp" as they would in a linear system. Indeed, k_ν is amplitude-dependent, via γ, so in a linearised system, $k_\nu \to 0$. This means, by the way, that the limits of small nonlinearity and small collisionality do not commute: the linear approximation is only valid if the amplitudes are so small that the collisional cutoff k_ν is below the smallest wavenumber, or, equivalently, $s_\nu v_{\rm th} \ll 1$.[118]

Let me give advance warning that a certain shadow will be cast in §12.6 on the literal validity of treating collisions, and the associated cutoffs, in this way. This will not destroy the overall story, but raise an important caveat, in view of which the calculation that is about to be given in §12.4.4 will need a slight (non-lethal) tweak.

12.4.4. *Efficiency of Nonlinear Phase Mixing*

Being in possession both of the scalings of the phase-space spectrum and of the collisional cutoff for it allows us to work out the amount of f-strophy stored in phase

[118] This was spotted a long time ago, in the context of Landau (or not so Landau) damping of small-amplitude waves, by Zakharov & Karpman (1963), and resurrected for the modern times by Catto (2025).

space in a statistically stationary state: from (12.73),

$$\delta\mathscr{C}_2 \sim \int_{v_{\rm th}^{-1}}^{s_\nu(k_\nu)} \mathrm{d}s\, C_s \sim \varepsilon\gamma^{-1}\ln\frac{\gamma}{\nu}, \tag{12.79}$$

where $s_\nu(k_\nu)$ is given by (12.78). This gives us a physically reasonable estimate for the characteristic time scale on which perturbations of δf are processed by the phase-space cascade and erased by collisions:

$$\boxed{t_{\rm c} \sim \frac{\delta\mathscr{C}_2}{\varepsilon} \sim \gamma^{-1}\ln\frac{\gamma}{\nu}}\,. \tag{12.80}$$

Obviously, one gets the same answer by using the k spectrum (12.75) or by positing that, like in the fluid-dynamical Batchelor (1959) passive-scalar cascade, k increases exponentially at the rate γ until it reaches the collisional cutoff k_ν, given by (12.77) (you will find a detailed argument to this effect in Nastac *et al.* 2025).

The nonlinear phase-mixing time (12.80) should be contrasted with the linear one (5.33), viz., $t_{\rm c} \sim \nu^{-1/3}(kv_{\rm th})^{-2/3}$, which can also be obtained by the same method as (12.80), except using the linear spectrum (12.76) for a single wavenumber k (cf. Fig. 59). While $t_{\rm c} \propto \nu^{-1/3}$ is not a very long time, $t_{\rm c} \propto |\ln\nu|$ is even shorter—the nonlinear phase mixing is quickened compared to the linear one because the phase-space cascade increases k simultaneously with s.

Thus, as I promised repeatedly above, δf reaches collisional scales quite quickly. This means that purely collisionless, phase-volume/Casimir-conserving dynamics cannot last long—but perhaps it is still possible to argue that they last a little longer than the dynamical time (here γ^{-1}), enabling a formal (if in practice never very good) asymptotic separation of scales between the latter and the evolution of $\rho(\eta)$, as hoped in §11.8.

12.5. *Nastac's Theory of Phase-Space Turbulence with Self-Consistent Electric Field*

The model of phase-space turbulence developed in §§12.2–12.4 gave us a lot of good physical insight accompanied by the emotional comfort of exact solvability. We paid for that comfort with some very dramatic assumptions, most of them undoubtedly wrong quantitatively and maybe even qualitatively. Let me now inquire to what extent those assumptions can be lifted while preserving (or modifying) the key physical conclusions. In doing so, I shall mostly follow the paper by Nastac *et al.* (2025).

The most dramatic simplification, which allowed us to derive a closed equation for the correlation function $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$, was decoupling the electric field from the perturbed phase-space density δf, assigning to it the convenient statistics (12.20), and assuming that it was concentrated in some range of k bounded from above [assumption (iii) at the beginning of §12.4]—I shall call these scale(s) the *outer scale(s)* and denote by k_0 the characteristic wavenumber of the electric fluctuations. Formally, this amounted to imagining the electric field to be externally forced according to (12.19) and ignoring the self-consistent bit $\delta\varphi_{\boldsymbol{k}}$. Since, as a reward for this sharp practice, we got to know $C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')$, we can now calculate the spectrum of the self-consistent field and ask whether the assumption that it would not matter was a good one. We have, for $\delta\boldsymbol{E}_{\boldsymbol{k}} = -i\boldsymbol{k}\delta\varphi_{\boldsymbol{k}}$,

$$C_{\boldsymbol{k}}^{(E)} \equiv \frac{e^2}{m^2}\left\langle|\delta\boldsymbol{E}_{\boldsymbol{k}}|^2\right\rangle = \frac{\omega_{\rm pe}^4}{k^2n_0^2}\underbrace{\iint\mathrm{d}\boldsymbol{v}\mathrm{d}\boldsymbol{v}'\,C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}')}_{=(2\pi)^3\int\mathrm{d}\boldsymbol{u}\,C_{\boldsymbol{k},\boldsymbol{s}=0}(\boldsymbol{u})}\,. \tag{12.81}$$

In 1D, using the asymptotic scaling (12.64) that we derived for our model,

$$\boxed{C_k^{(E)} = \frac{\omega_{\rm pe}^4}{k^2n_0^2}\,2\pi\int\mathrm{d}u\,C_{k,s=0}(u) \approx \frac{4\pi^2\omega_{\rm pe}^4}{Ln_0^2}\,0.28\,\varepsilon k^{-4}}\,, \tag{12.82}$$

formally valid at $k \gg k_0$. I shall show below that this is too steep to interfere with the advection of δf by the outer-scale field. To do that, I need to establish a quick way to work out the phase-space spectra given an electric field whose spectrum is known.

Let us go back to the (1D) equation for δf:

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} + E\frac{\partial \delta f}{\partial v} = -E\frac{\partial f_0}{\partial v}, \tag{12.83}$$

where the factor $-e/m$ has been absorbed into E for brevity. The second and third terms on the left-hand side represent the linear and nonlinear phase mixing, respectively, with the rates, at the scales w in velocity space and ℓ in position space,

$$t_{\rm lin}^{-1} \sim \frac{w}{\ell}, \qquad t_{\rm nl}^{-1} \sim \frac{E_\ell}{w}, \tag{12.84}$$

where E_ℓ is a typical size of the electric fluctuations at scale ℓ. As in §12.4.2, let us assume *critical balance* between these time scales, $t_{\rm lin} \sim t_{\rm nl}$, either of which will then also be the transfer time τ_w for the f-strophy δf_w^2 at the velocity scale

$$w \sim (E_\ell \ell)^{1/2} \tag{12.85}$$

to be passed to the next smaller scale according to (12.68). How E_ℓ depends on ℓ and, therefore, how $\tau_w \sim t_{\rm lin} \sim t_{\rm nl}$ depends on w, is determined by the spectrum of the electric field. If $C_k^{(E)} \propto k^{-\alpha}$ with $1 < \alpha < 3$, then $E_\ell \propto \ell^{(\alpha-1)/2}$, and the nonlinear phase mixing of δf structures of scale ℓ is done "locally" by the electric-field structures of the same scale. We expect, however, from (12.82), $\alpha = 4$. This means that E_ℓ is dominated by the Taylor expansion of the electric field in small distance:

$$E_\ell \sim \gamma^2 \ell \quad \Rightarrow \quad w \sim \gamma\ell \quad \Rightarrow \quad \tau_w \sim t_{\rm lin} \sim t_{\rm nl} \sim \gamma^{-1}, \tag{12.86}$$

where γ^2 is the typical gradient of the electric field at the outer scale, which then determines a single, scale-independent transfer time that controls the f-strophy cascade. This is what I referred to in §12.4 as the Batchelor (1959) regime of passive-scale advection, and this new γ is physically the same as the old γ introduced in (12.55), but the scaling theory of §12.4.2 (and, therefore, also of §§12.4.3–12.4.4) can now be recovered without recourse to the modelling assumptions of §§12.2–12.4, leading still to the phase-space spectrum (12.71) with $a = b = 2$ and the electric-field spectrum (12.82) (modulo numerical constants). To get (12.86), we needed to assume $\alpha > 3$, and indeed we got a k^{-4} spectrum.

Exercise 12.11. Imagine that, in fact, $1 < \alpha < 3$ and work out, analogously to what was done in §12.4.2, what the phase-space spectrum of f-strophy would be and what that would imply for the spectrum of the electric field. You should arrive at a contradiction, confirming that the Batchelor regime is the only possible regime (Nastac *et al.* 2025).

Note that in §12.4.2, I justified the independence of $C_{k,s}$ of s at $s \ll \gamma^{-1}k$ by (12.57) and the independence of $C_{k,s}$ of k at $s \gg \gamma^{-1}k$ by (12.58), which were terms in the model equation (12.55)—and at least the latter term was certainly the result of my closure assumptions. These flat dependences on s and k in the linear- and nonlinear-mixing limits, respectively, can instead be deduced on very general grounds. When the linear phase mixing dominates, it is straightforward that the s dependence of the spectrum should be wiped out—no approximations here. When the nonlinearity is dominant, one may argue that, in the absence of the linear term, there is little to preserve the coherence of δf fluctuations across these (large) scales, so they become white in space—and so have a flat k spectrum.

All of these results have been confirmed numerically by Nastac *et al.* (2025), who simulated a 1D Vlasov–Poisson turbulence forced by an external potential χ as in (12.19), injected at the outer scale $k_0^{-1} \sim \lambda_{{\rm D}e}$. Unlike in Q-13 or in §§12.2–12.4, their forcing was not white in time though, so they were able to see the transition to slower, v-dependent diffusion and a "flat-top" f_0 that you worked out in Q-12(d) (QL theory turned out to work for f_0 in such a system, even though a phase-space cascade was present, so δf clearly could not satisfy the linear approximation perfectly—but, at the Debye scale, for the purposes of diffusing f_0, it seemed to satisfy it well enough). Another numerical experiment where phase-space spectra predicted above have been seen was the simulation by Ewart *et al.* (2024), also in 1D, of the relaxation of two-stream instability (cf. Exercise 3.5). The headline result there was the emergence of a universal collisionless Lynden-Bell equilibrium with a ε^{-2} tail (§10.4), but under the hood of this dynamics of f_0, there was a phase-space cascade with a k^{-4} spectrum of the electric field and the phase-space distribution of f-strophy described above. So these things that I am telling you about are real (at least in the sense of virtual reality).

12.5.1. *f-strophy and Electric-Field Spectra in d Dimensions*

The restriction to 1D was required in §12.4 to get an exactly solvable equation, but we have now fully shed the requirements of exactitude and embraced "twiddle" algebra, so why not generalise everything to the obviously more relevant 3D case—or indeed d-D. As far as we know at this point, nothing qualitatively dramatic happens, and this generalisation is a purely technical exercise.

As seems obvious from §12.2, or indeed on general grounds, the phase-space f-strophy spectrum in a statistically isotropic, homogeneous system will depend on k, s, and the angle $\cos\theta = \boldsymbol{k}\cdot\boldsymbol{s}/ks$. It is clear that it will be asymmetric with respect the changing of the sign of $\cos\theta$ as this corresponds to phase-mixing vs. phase-unmixing modes, and the asymmetry will be needed, as it was in the 1D solution (§12.4.1), to accommodate a constant f-strophy flux out to large k and s. It is also likely that, as the linear phase mixing in each direction occurs at the "speed" equal to the projection of $\boldsymbol{k}$ on that direction, f-strophy should concentrate preferentially closer to $\cos\theta \sim \pm 1$ and less so around $\cos\theta \sim 0$.

In what follows, I will assume, however, that these are all order-unity effects that can be integrated out by shell-averaging in $\boldsymbol{k}$ and $\boldsymbol{s}$, and that the remaining dependence on k and s can still be described by power laws in two regions of the (k, s) space separated by the critical-balance line $k \sim \gamma s$. The argument that the spectrum should be independent of $\boldsymbol{s}$ at $s \ll \gamma^{-1}k$ and of $\boldsymbol{k}$ at $k \ll \gamma s$ remains sound, so, taking into account the "density-of-states" factors ($\sim Vk^{d-1}$ and s^{d-1}) in the double-shell-integrated spectrum, we expect the latter to have the scalings

$$\bar{\bar{C}}_{k,s} \propto \begin{cases} k^{d-1}s^{-a}, & k \ll \gamma s, \\ s^{d-1}k^{-b}, & k \gg \gamma s. \end{cases} \tag{12.87}$$

Following the same logic as in §12.4.2 and in the argument following (12.83),

$$v_{\rm th}^d \delta f_w^2 \sim v_{\rm th}^d \delta f_\ell^2 \sim \varepsilon\gamma^{-1} = \text{const}, \tag{12.88}$$

whereas, on the other hand,

$$\int \mathrm{d}k\, \bar{\bar{C}}_{k,s} \propto \int_0^{\gamma s} \mathrm{d}k\, k^{d-1}s^{-a} \propto s^{-a+d} \quad\Rightarrow\quad \delta f_w^2 \propto w^{a-d-1} \quad\Rightarrow\quad a = d+1, \tag{12.89}$$

$$\int \mathrm{d}s\, \bar{\bar{C}}_{k,s} \propto \int_0^{\gamma^{-1}k} \mathrm{d}s\, s^{d-1}k^{-b} \propto k^{-b+d} \quad\Rightarrow\quad \delta f_\ell^2 \propto \ell^{b-d-1} \quad\Rightarrow\quad b = d+1. \tag{12.90}$$

Either of these exponents is also obtainable from the other by matching the two asymptotics of $\bar{\bar{C}}_{k,s}$ along $k \sim \gamma s$. Using (12.88) to work out dimensional prefactors, we have

$$\boxed{\bar{\bar{C}}_{k,s} \sim \begin{cases} \varepsilon k^{d-1}(\gamma s)^{-d-1}, & k \ll \gamma s, \\ \varepsilon(\gamma s)^{d-1}k^{-d-1}, & k \gg \gamma s. \end{cases}} \tag{12.91}$$

Note that the 1D (i.e., integrated over k or s) spectra are still the Batchelor spectra s^{-1} and k^{-1}, same as (12.73) and (12.75)—in the latter case, without the factor of $2\pi/L$, eliminated by shell averaging, which includes a factor of $V/(2\pi)^d$, where, obviously, $V = L^d$.

The electric-field spectrum (12.81) is obtained from $C_{\boldsymbol{k},\boldsymbol{s}=0}$, shell-averaged only in $\boldsymbol{k}$, so the s^{d-1} factor should be removed from (12.91):

$$\boxed{\bar{C}_k^{(E)} \sim \frac{\omega_{\rm pe}^4}{k^2 n_e^2}\bar{C}_{k,\boldsymbol{s}=0} \sim \frac{\omega_{\rm pe}^4\varepsilon\gamma^{d-1}}{n_e^2}\,k^{-d-3}}\,, \tag{12.92}$$

even steeper than (12.82) was in 1D, so the advection of f-strophy is still safely in the Batchelor regime, as assumed *a priori*. Voilà, here we have *the universal fluctuation spectrum of the turbulent Vlasov–Poisson plasma* (Nastac *et al.* 2025).

12.5.2. *Validity of Nastac's Theory*

How universal is Nastac's "universal spectrum" (12.92)? Let me explore under what conditions the passive-advection approximation applies.

At the beginning of §12.5, I introduced k_0 as the characteristic wavenumber of the electric fluctuations, a.k.a. the outer scale—Nastac's universal spectrum then emerged at $k > k_0$, where the passive-advection (Batchelor) approximation held. By the definition of γ in (12.86), the size of the characteristic electric field (the larger of the external and self-consistent fields) that controls the f-strophy cascade is

$$\frac{e^2}{m^2}\langle E^2\rangle \sim \gamma^4 k_0^{-2} \quad \Rightarrow \quad W \equiv \frac{\langle E^2\rangle}{8\pi n_e T_e} \sim \left(\frac{\gamma}{\omega_{\rm pe}}\right)^4 (k_0\lambda_{\rm D})^{-2}. \tag{12.93}$$

In what follows, it will be convenient, at times, to express γ in terms of the "fluctuation parameter" W:[119]

$$\frac{\gamma}{\omega_{\rm pe}} \sim (k_0\lambda_{\rm D})^{1/2} W^{1/4}. \tag{12.94}$$

Thanks to (12.92), we can also estimate the size of the self-consistent field at k_0:

$$\frac{e^2}{m^2}\langle \delta E^2\rangle \sim \int_{k_0}^{\infty} \mathrm{d}k\, \bar{C}_k^{(E)} \sim \frac{\omega_{\rm pe}^4 \varepsilon \gamma^{d-1}}{n_e^2} k_0^{-d-2}. \tag{12.95}$$

In order for the passive-advection approximation not to lead to a contradiction, $\langle\delta E^2\rangle$ clearly cannot be allowed to exceed $\langle E^2\rangle$, so a consistency constraint emerges:

$$\frac{\langle\delta E^2\rangle}{\langle E^2\rangle} \sim \left(\frac{\omega_{\rm pe}}{\gamma}\right)^{4-d} \frac{\varepsilon\gamma^{-1} v_{\rm th}^d}{n_e^2} (k_0\lambda_{\rm D})^{-d} \lesssim 1. \tag{12.96}$$

The physics content of this inequality becomes more transparent if we estimate the f-strophy flux from (12.52):

$$\varepsilon = -\frac{\mathrm{d}\mathscr{C}_{2,0}}{\mathrm{d}t} \sim \frac{v_{\rm th}^d f_0^2}{t} \sim \frac{n_e^2}{v_{\rm th}^d} t^{-1}. \tag{12.97}$$

The characteristic time scale of the f-strophy change is just t because $\mathscr{C}_{2,0}$ will generally decay with time as a power law (see §12.5.3). Using also (12.94), the constraint (12.96) turns into a lower bound on the electric fluctuation level:

$$\boxed{W \gtrsim (\omega_{\rm pe} t)^{-4/(5-d)} (k_0\lambda_{\rm D})^{-2(5+d)/(5-d)}}\,. \tag{12.98}$$

Another constraint is that, if a critically balanced cascade starts at k_0, the corresponding velocity scale (which I will denote s_0^{-1}) cannot, obviously, be larger than $v_{\rm th}$:

$$\boxed{(s_0 v_{\rm th})^{-1} \sim \frac{\gamma}{k_0 v_{\rm th}} \sim W^{1/4} (k_0\lambda_{\rm D})^{-1/2} \lesssim 1}\,, \tag{12.99}$$

which gives us an upper bound on W. Another upper bound is simply $W \lesssim 1$, expressing the reasonable expectation that the electric-energy density (12.93) should not to exceed the thermal one. Note that the consistency of these bounds with (12.98) is not a problem as long as $\omega_{\rm pe} t \gg 1$.

The two inequalities (12.98) and (12.99) can both be viewed as telling us that the outer scale of our turbulence must not be too large in comparison with $\lambda_{\rm D}$. That is just as well, as Nastac's theory is unlikely to apply at $k\lambda_{\rm D} \ll 1$. Indeed, in this limit, (12.94) implies $\gamma \ll \omega_{\rm pe}$—such slow dynamics must be subject to Debye screening, set up on fast time scales $\sim \omega_{\rm pe}^{-1}$ (for "externally" introduced fields, you worked this out in Q-7); very crudely, this should cut the low-frequency field by a factor of $\epsilon(0, k_0) \sim (k_0\lambda_{\rm D})^{-2}$ [see, e.g., (3.90)], but no account was taken of any such effect in Nastac's theory. Furthermore, in the event that the injection mechanism at the outer scale is high-frequency but low-amplitude, i.e., the nonlinear rate $\gamma \ll \omega_{\rm pe}$ but plasma-frequency

[119] We have encountered this parameter before, when we dealt with Lamngmuir turbulence in §8. It first appeared in (8.23) (and then again in §8.4.7), where too it was a measure of the nonlinear rate in terms of $\omega_{\rm pe}$. Note how slow the Langmuir turbulence was compared to what we are dealing with here.

waves are exited, we are in the realm of Langmuir turbulence (§8), which, in either weak or strong regime, does not, as far as we know, look like Nastac's critically balanced cascade.

Keeping all this in mind and tentatively letting $k_0\lambda_{\rm D} \sim 1$, (12.98) and (12.99) reduce to

$$(\omega_{\rm pe}t)^{-4/(5-d)} \lesssim W \lesssim 1. \tag{12.100}$$

Although this looks reasonable, and perhaps not particularly constraining in the long-time limit, complications do arise—I will explain them in §12.5.3, but do feel free to skip to §12.6.

12.5.3. *Heating and Decay Laws of Phase-Space Turbulence*

When plasma turbulence is solved completely, we will be able to check the constraints derived in §12.5.2 by working out the time evolution of W, i.e., the plasma-heating law (T vs. t) and the fluctuation-decay law ($\langle E^2\rangle$ vs. t), as well as the evolution of the outer scale ($k_0\lambda_{\rm D}$ vs. t). In general, these will depend on how the system is set up: how (and if) the energy is injected, what happens at large scales, etc.

The simplest example of how such a calculation might proceed is furnished by the Kazantsev–Kraichnan model of advection, in which f_0 satisfies the diffusion equation (12.51), whence

$$v_{\rm th}^2 \sim v_{\rm th,0}^2 + Dt, \quad D \sim \sum_{\boldsymbol{k}} k^2 A_k \sim \frac{e^2}{m^2}\langle E^2\rangle\,\tau_{\rm corr} \sim v_{\rm th}^2 W\omega_{\rm pe}^2\tau_{\rm corr}$$
$$\Rightarrow \quad W \sim \left(1 - \frac{v_{\rm th,0}^2}{v_{\rm th}^2}\right)\left(\omega_{\rm pe}^2\tau_{\rm corr}t\right)^{-1}, \tag{12.101}$$

where $n_e m v_{\rm th,0}^2$ is some reference initial thermal energy and $\tau_{\rm corr}$ is the correlation time of the field (of course, in the Kazantsev–Kraichnan model, formally, $\tau_{\rm corr} = 0$ and $\langle E^2\rangle = \infty$, but we shall treat it as an approximation or even just an order-unity-wrong closure, and see that the above estimate gives us what we need). If we think of D (equivalently, $\langle E^2\rangle$), $\tau_{\rm corr}$, and k_0 as fixed parameters of some external forcing, the decay law (12.101) and the evolution of $k_0\lambda_{\rm D} \propto v_{\rm th}$ appear to be compatible with (12.98) and (12.99) at long times, with the self-consistent field decaying and the fixed outer scale becoming ever more sub-Debye. Problems begin, however, if we ask what value it would be physically sensible to assign to $\tau_{\rm corr}$. From (12.55), still in the Kazatsev–Kraichnan model,

$$\gamma^3 \sim \sum_{\boldsymbol{k}} k^4 A_k \sim k_0^2 D \quad \Rightarrow \quad \frac{e^2}{m^2}\langle E^2\rangle \sim \frac{D}{\tau_{\rm corr}} \sim \tau_{\rm corr}^{-1}\gamma^3 k_0^{-2}, \tag{12.102}$$

assuming the wavenumber integral is dominated by the outer scale. Comparing (12.102) with (12.93) gives $\tau_{\rm corr} \sim \gamma^{-1}$, which makes sense: the correlation time is the nonlinear time. But this immediately implies that the white-noise (diffusion) approximation cannot, in fact be used at the Debye scale: indeed, as we saw in Q-12, it requires the correlation time to be short compared to the particles' transit time across the electric fluctuations (after which they start averaging out the heating), $\tau_{\rm corr}k_0 v_{\rm th} \ll 1$, but this is the opposite of (12.99) if $\tau_{\rm corr} \sim \gamma^{-1}$ (and even if $\tau_{\rm corr}$ were small, it could not remain small enough forever with $v_{\rm th}$ growing).

So, instead of the diffusive heating law (12.101), let us use the result of Q-12(d), the subdiffusive heating:

$$v_{\rm th}^2 \sim v_{\rm th,0}^2 + (\alpha t)^{1/2}, \quad \alpha \sim \sum_{\boldsymbol{k}} \frac{A_k}{\tau_{\rm corr}^2} \sim \frac{D}{k_0^2\tau_{\rm corr}^2} \sim \frac{e^2}{m^2}\langle E^2\rangle\frac{1}{k_0^2\tau_{\rm corr}}. \tag{12.103}$$

First considering fixed $\langle E^2\rangle$, $\tau_{\rm corr}$, k_0, and ignoring $v_{\rm th,0}^2$ in the long-time limit, we get $W \propto t^{-1/2}$ and $k_0\lambda_{\rm D} \propto t^{1/4}$, again consistent with (12.98) and (12.99). So a numerical experiment with such an external forcing should vindicate Nastac's theory—and indeed does (Nastac *et al.* 2025).

Let us now see if, instead of applying an external force, we can imagine a more organic variety of turbulence: letting $\langle E^2\rangle \sim W n_e m v_{\rm th}^2$ and $\tau_{\rm corr}^{-1} \sim \gamma \sim \omega_{\rm pe}\,(k_0\lambda_{\rm D})^{1/2}\,W^{1/4}$ in (12.103), we get

$$\alpha \sim \frac{v_{\rm th}^4 W^{5/4}\omega_{\rm pe}}{(k_0\lambda_{\rm D})^{3/2}} \quad \Rightarrow \quad W \sim \left(1 - \frac{v_{\rm th,0}^2}{v_{\rm th}^2}\right)^{8/5}(\omega_{\rm pe}t)^{-4/5}\,(k_0\lambda_{\rm D})^{6/5}. \tag{12.104}$$

By energy conservation in a system without external inputs, $v_{\rm th}^2(1+W) = v_{\rm th,0}^2(1+W_0)$, so

$1 - v_{\rm th}^2/v_{\rm th,0}^2 \to W_0/(1+W_0)$ as $W \to 0$. In a system where there is no external forcing and everything is generated (or decaying) in a self-contained way, it is the definition of the outer scale that $\langle \delta E^2\rangle \sim \langle E^2\rangle$ at it, i.e., the inequality (12.98) turns into equality, determining k_0. Comparing it with (12.104), we find $k_0\lambda_{\rm D} \propto (\omega_{\rm pe}t)^{-d/(20+d)}$. Thus, the outer scale wants to get (quite slowly) larger—but we cannot trust this result asymptotically because it implies $k_0\lambda_{\rm D} \ll 1$, so a theory of turbulence at such scales, with Debye shielding, is needed to work out the heating and decay laws.

Such a theory is currently lacking (you will find some speculations as to what it might be in Nastac *et al.* 2025). For now, the key conclusion of the above argument is that in a self-contained system, the outer scale will be pushed to the Debye scale or larger. This implies that Nastac's theory is certainly safe below the Debye scale, where the advection of δf is "passive", done by the electric field from larger scales. In this sense, his sub-Debye spectrum is indeed universal.

12.6. *Phase-Space Turbulence vs. Particle Noise*

When, at the end of §12.5.1, I uttered the words "universal fluctuation spectrum", those who had read old papers (Hubbard 1961; Rostoker 1961) or textbooks (e.g., Klimontovich 1967; Krall & Trivelpiece 1973; Lifshitz & Pitaevskii 1981) might have been triggered to start thinking about the "thermal" fluctuation spectrum of a plasma due to particle noise and wondering how that fits with Nastac's spectrum (12.92).

I have so far avoided doing the theory of Coulomb collisions in these Lectures, but in fact, much of it is already here as a simple consequence of working out the "collisionless collision integrals". As I explained in §11.5, the Klimontovich distribution, which describes the exact positions of point particles in phase space (§1.7), can be viewed as a single-waterbag phase-space density, with $\eta \gg f_0$. The microgranulation ansatz (11.5), or, in $\boldsymbol{k}$ space, (11.7), then holds true as a consequence of the Klimontovich distribution being a sum of δ functions, with $\langle g^2\rangle = \eta f_0$ [see (11.30)] and $\mathit{\Delta}\Gamma = \eta^{-1}$ [see (11.34)]:

$$C_{\boldsymbol{k}}(\boldsymbol{v},\boldsymbol{v}') = \frac{f_0(\boldsymbol{v})}{V}\,\delta(\boldsymbol{v}-\boldsymbol{v}'). \tag{12.105}$$

We can use this formula in (12.81) to calculate the electric-field spectrum arising from particle noise—there is no need to worry about "dressing" the fluctuations (dividing by $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2$) because we shall only be interested in short wavelengths, $k\lambda_{\rm D} \gg 1$, where $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2 \approx 1$. This gives

$$C_{\boldsymbol{k}}^{E,{\rm th}} = \frac{\omega_{\rm pe}^4}{N}\,k^{-2}, \tag{12.106}$$

where $N = n_0 V$. The shell-integrated version of this is, in d dimensions,

$$\boxed{\bar{C}_k^{E,{\rm th}} = \frac{S_d}{(2\pi)^d}\frac{\omega_{\rm pe}^4}{n_e}\,k^{d-3}}\,, \tag{12.107}$$

where S_d is the area of a unit sphere in d dimensions ($=2$ in 1D, 2π in 2D, and 4π in 3D).[120]

This spectrum is much shallower than the turbulent one (12.92), so it provides a kind of floor below which the latter disappears into the noise. The wavenumber at which they meet is obtained by comparing them:

$$k_{\rm th} \sim \left(\frac{\varepsilon\gamma^{d-1}}{n_e}\right)^{1/2d} \sim \left(\frac{n_e\gamma^{d-1}}{v_{\rm th}^d t}\right)^{1/2d}, \tag{12.108}$$

[120] You might wonder why this spectrum is called "thermal" if it does not depend on the temperature. In fact, both its IR and UV cutoffs do. At the long-wavelength side, it is valid for $k \gg \lambda_{\rm D}^{-1} \sim \omega_{\rm pe}/v_{\rm th}$ (otherwise it needs "dressing"), which obviously does depend on temperature. At the short-wavelength side, $k \ll d^{-1} \sim T/e^2$ (§1.4), where the distance of closest approach d also depends on T. Integrating (12.107), in 3D, up to this cutoff gives the total electric-energy density $\langle E^2\rangle \sim nT$, as it ought to do, as it represents Coulomb collisions.

where I used (12.97). Non-dimensionalising in a physically relevant way, this thermal cutoff is

$$\boxed{k_{\rm th}\lambda_{\rm D} \sim \left(\frac{n_e\lambda_{\rm D}^d}{\gamma t}\right)^{1/2d}\left(\frac{\gamma}{\omega_{\rm pe}}\right)^{1/2} \sim \left(\frac{N_{\rm D}}{\omega_{\rm pe}t}\right)^{1/2d} W^{(d-1)/8d}}\,, \tag{12.109}$$

where the last expression in (12.109) was obtained by using (12.94) (with $k_0\lambda_{\rm D}\sim 1$, to simplify the algebra a bit) and (re)introducing $N_{\rm D} = n_e\lambda_{\rm D}^d$, the number of particles in a Debye sphere, which is the fundamental large parameter in a weakly interacting plasma (§1.3). In order for the turbulent spectrum (12.92) to show itself above the noise, we must have $k_{\rm th}\lambda_{\rm D}\gg 1$, which translates into

$$\boxed{\omega_{\rm pe}t \ll N_{\rm D}W^{(d-1)/4}}\,. \tag{12.110}$$

This is either an expiration date for the turbulent spectrum at some late time or a condition that gets easier to satisfy as time goes on—which of these it is will depend on the macroscopic arrangements in a particular system that set its heating and decay laws (§12.5.3) and, therefore, the time evolution of $N_{\rm D}\propto T^{d/2}$ and $W\propto \langle E^2\rangle T^{-1}$.

Technically, if we are to stick with the notion that there is such a thing as "true" collisions that dissipate the f-strophy cascade, there is a more stringent condition required to be satisfied in order for the collisionless (turbulence) and collisional (noise) dynamics to be well separated: the wavenumber $k_{\rm th}$ must not be allowed to be reached at all, i.e., the turbulent spectrum's collisional cutoff, given by (12.77),

$$k_\nu\lambda_{\rm D} \sim \frac{\gamma^{3/2}\lambda_{\rm D}}{\nu^{1/2}v_{\rm th}} \sim \left(\frac{\gamma}{\omega_{\rm pe}}\right)^{3/2}\left(\frac{\nu}{\omega_{\rm pe}}\right)^{-1/2} \sim W^{3/8}N_{\rm D}^{1/2}, \tag{12.111}$$

must be smaller than (12.109). This seems to be very difficult to satisfy, ever.[121] Thus, one should not take the collisional cutoff derived in §12.4.3 too literally—in reality, the cutoff is $\sim k_{\rm th}$, and the exact way in which f-strophy is destroyed at small scales is not necessarily controlled by an operator that resembles Landau's (11.36). This means also that, in estimating the characteristic timescale of the destruction of f-strophy, we ought to replace $s_\nu(k_\nu)$ in (12.79) by $s_{\rm th}\sim k_{\rm th}/\gamma$, giving

$$t_{\rm c} \sim \frac{\delta\mathscr{C}_2}{\varepsilon} \sim \gamma^{-1}\ln\left(k_{\rm th}\lambda_{\rm D}\right), \tag{12.112}$$

give or take factors of order unity. This is not very different from (12.80), so the conclusion that nonlinear phase mixing is efficient (§12.4.4) should survive the onslaught of the noise.

12.7. *Moral of the Story*

What is the moral of this rather long story? It has been known since the beginning of times that the evolution of any coarse-grained distribution is controlled by the fluctuations around it (e.g., Klimontovich 1967). When the coarse-graining is over deeply sub-Debye scales (§1.4) and the fluctuations are particle noise, this evolution is, in plasma physics, deemed "collisional". When the coarse-graining includes fluctuations on scales around $\lambda_{\rm D}$ and above, the evolution is "collisionless" (§11)—assuming that it turns out to be faster than collisions. On the level of fluctuation spectra, this new physics is manifested in the emergence of Nastac's spectra of f-strophy and of the electric field, which are non-equilibrium turbulence spectra, set by the f-strophy cascade (§12.5.1), rather than thermal equipartition spectra usually associated with particle noise (§12.6). It

[121] The last estimate in (12.111) followed from (1.14): $\nu/\omega_{\rm pe}\sim N_{\rm D}^{-1}$ in 3D. Note that in 1D, the δ-function inside the collision integral (11.32) sets $v''=v$ and annihilates the integral, so one has to go to higher order in $N_{\rm D}^{-1}$, leading to a $\nu/\omega_{\rm pe}\sim N_{\rm D}^{-2}$ scaling—but this makes k_ν even larger (for the 1D scaling, take a look at Fouvry 2022 if you want to be awed by a truly heroic, if perhaps quixotic, calculation—and learn how to use MATHEMATICA to make yourself all-powerful).

remains an open problem how to convert this insight, and the calculations above, into the correct "collisionless collision integral" *à la* (11.45) but with a better prescription than the microgranulation ansatz (11.42) and hence with a method to calculate $\Delta\Gamma$—this might then tell us how the system is dynamically pushed towards Ewart's universal collisionless equilibrium (§10.4), including by evolving the waterbag content $\rho(\eta)$, presumably on the nonlinear-mixing timescale (12.80), or (12.112). Watch this space.

13. Phase-Space Turbulence II: Particle Trapping and Long-Lived Structures

13.1. *Nonlinear Landau Damping*

Coming soon. See O'Neil (1965); Mazitov (1965).

13.2. *BGK Solutions*

Coming soon. See Bernstein *et al.* (1957).

13.3. *Phase-Space Holes*

PART III

Magnetohydrodynamics

14. MHD Equations

Like hydrodynamics from gas kinetics, MHD can be derived systematically from the Vlasov–Maxwell–Landau equations for a plasma in the limit of large collisionality + a number of additional assumptions (see, e.g., Goedbloed & Poedts 2004; Parra 2019*a*; Kunz 2021). Here I will adopt a purely fluid approach—partly to make this Part III of the Lectures self-consistent and partly because there is a certain beauty in it: we need to know relatively little about the properties of the constituent substance in order to spin out a very sophisticated and complete theory about the way in which it flows. This approach is also more generally applicable because the substance that we will be dealing with need not be gaseous, like plasma—you may also think of liquid metals, various conducting solutions, etc. For plasmas, I shall re-derive all this from kinetic theory in Part IV.

So, let us declare an interest in the flow of a conducting fluid and attempt to be guided in our description of it by the very basic things: conservation laws of mass, momentum and energy plus Maxwell's equations for the electric and magnetic fields. This will prove sufficient for most of our purposes. So we shall consider a fluid characterised by the following quantities:

ρ—mass density,
$\boldsymbol{u}$—flow velocity,
p—pressure,
σ—charge density,
$\boldsymbol{j}$—current density,
$\boldsymbol{E}$—electric field,
$\boldsymbol{B}$—magnetic field.

Our immediate objective is to find a set of closed equations that would allow us to determine all of these quantities as functions of time and space within the fluid.

14.1. *Conservation of Mass*

This is the most standard of all arguments in fluid dynamics (Fig. 60):

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \mathrm{d}\boldsymbol{r}\,\rho}_{\substack{\text{mass inside a}\\\text{volume of}\\\text{fluid}}} = -\underbrace{\int_{\partial V}(\rho\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S}}_{\substack{\text{mass flux}\\\text{in/out}}} = -\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}). \tag{14.1}$$

As this equation holds for any V, however small, it can be converted into a differential relation

$$\boxed{\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0}\,. \tag{14.2}$$

This is the *continuity equation*.

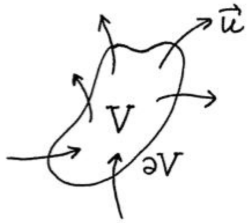


FIGURE 60. Stuff flowing in and out of a volume V enclosed by surface ∂V.

14.2. *Conservation of Momentum*

A similar approach:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \mathrm{d}\boldsymbol{r}\,\rho\boldsymbol{u}}_{\substack{\text{momentum}\\\text{inside a}\\\text{volume of}\\\text{fluid}}} = \underbrace{-\int_{\partial V}\underbrace{(\rho\boldsymbol{u}\boldsymbol{u})}_{\substack{\text{Reynolds}\\\text{stress}}}\cdot\,\mathrm{d}\boldsymbol{S}}_{\substack{\text{momentum flux}\\\text{through boundary}\\\text{(fluid flow carrying}\\\text{its own momentum)}}} - \underbrace{\int_{\partial V} p\,\mathrm{d}\boldsymbol{S}}_{\substack{\text{pressure on}\\\text{boundary}\\\text{(momentum}\\\text{flux due to}\\\text{internal}\\\text{motion)}}} - \underbrace{\int_{\partial V}\boldsymbol{\mathit{\Pi}}\cdot\mathrm{d}\boldsymbol{S}}_{\text{viscous stress}} + \underbrace{\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{F}}_{\substack{\text{all other}\\\text{forces (E\&M}\\\text{will come in}\\\text{here)}}}$$

$$= \int_V \mathrm{d}\boldsymbol{r}\,\left[-\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}) - \boldsymbol{\nabla}p - \boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}} + \boldsymbol{F}\right]. \tag{14.3}$$

In differential form, this becomes

$$\underbrace{\frac{\partial}{\partial t}\rho\boldsymbol{u}}_{\substack{=\rho\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\frac{\partial\rho}{\partial t}\\ =\rho\frac{\partial\boldsymbol{u}}{\partial t}-\cancel{\boldsymbol{u}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})}}} = \underbrace{-\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u})}_{\substack{-\rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\\ -\cancel{\boldsymbol{u}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u})}}} - \boldsymbol{\nabla}p - \boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}} + \boldsymbol{F}, \tag{14.4}$$

and so, finally,

$$\boxed{\rho\underbrace{\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)}_{\substack{\equiv\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\\\text{convective}\\\text{derivative}}} = -\boldsymbol{\nabla}p - \boldsymbol{\nabla}\cdot\boldsymbol{\mathit{\Pi}} + \boldsymbol{F}}\,. \tag{14.5}$$

This is the *momentum equation.*

One part of this equation does have to be calculated from some knowledge of the microscopic properties of the constituent fluid or gas—the viscous stress. For a gas, it is done in kinetic theory (e.g., Lifshitz & Pitaevskii 1981; Dellar 2015; Schekochihin 2024, §6.8):

$$\boldsymbol{\mathit{\Pi}} = -\rho\nu\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T - \frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}\,\mathsf{I}\right], \tag{14.6}$$

where ν is the kinematic (Newtonian) viscosity.

In a magnetised plasma (i.e., such that its collision frequency $\ll$ Larmor frequency of the gyrating charges), the viscous stress is much more complicated and anisotropic with respect to the direction of the magnetic field: because of their Larmor motion, charged particles diffuse differently across and along the field. This gives rise to the so-called *Braginskii (1965) stress* (see §22.2.4 onwards, or texts by Helander & Sigmar 2005; Parra 2019*a*; Kunz 2021).

In what follows (within this part of the Lectures, Part III), we will never require the explicit form of $\boldsymbol{\Pi}$.

14.3. *Electromagnetic Fields and Forces*

The fact that the fluid is conducting means that it can have distributed charges (σ) and currents ($\boldsymbol{j}$) and so the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields will exert body forces on the fluid. Indeed, for one particle of charge q, the Lorentz force is

$$\boldsymbol{f}_{\mathrm{L}} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right), \tag{14.7}$$

and if we sum this over all particles (or, to be precise, average over their distribution and sum over species), we will get

$$\boldsymbol{F} = \sigma\boldsymbol{E} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c}. \tag{14.8}$$

This body force (force density) goes into (14.5) and so we must know $\boldsymbol{E}$, $\boldsymbol{B}$, σ and $\boldsymbol{j}$ in order to compute the fluid flow $\boldsymbol{u}$.

Clearly it is a good idea to bring in Maxwell's equations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\sigma \quad \text{(Gauss)}, \tag{14.9}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \tag{14.10}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E} \quad \text{(Faraday)}, \tag{14.11}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} \quad \text{(Ampère–Maxwell)}. \tag{14.12}$$

To these, we append Ohm's law in its simplest form: The electric field in the frame of a fluid element moving with velocity $\boldsymbol{u}$ is

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c} = \eta\boldsymbol{j}, \tag{14.13}$$

where $\boldsymbol{E}$ is the electric field in the laboratory frame and η is the *Ohmic resistivity.*

Normally, the resistivity, like viscosity, has to be computed from kinetic theory (see, e.g., Helander & Sigmar 2005; Parra 2019*a*; Kunz 2021) or tabulated by assidiuous experimentalists. In a magnetised plasma, the simple form (14.13) of Ohm's law is only valid at spatial scales longer than the Larmor radii and time scales longer than the Larmor periods of the particles (see, e.g., Goedbloed & Poedts 2004; Parra 2019*b*).

Equations (14.9–14.13) can be reduced somewhat if we assume (quite reasonably for most applications) that our fluid flow is *non-relativistic.* Let us stipulate that all fields evolve on time scales $\sim\tau$, have spatial scales $\sim\ell$ and that the flow velocity is

$$u \sim \frac{\ell}{\tau} \ll c. \tag{14.14}$$

Then, from Ohm's law (14.13),

$$E \sim \frac{u}{c}B \ll B, \tag{14.15}$$

so *electric fields are small compared to magnetic fields.*

In Ampère–Maxwell's law (14.12),

$$\frac{\left|\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right|}{|\boldsymbol{\nabla}\times\boldsymbol{B}|} \sim \frac{\frac{1}{c}\frac{1}{\tau}\frac{u}{c}B}{\frac{1}{\ell}B} \sim \frac{u^2}{c^2} \ll 1, \tag{14.16}$$

so *the displacement current is negligible* (note that at this point we have ordered out light waves; see Q-2 in Kinetic Theory). This allows us to revert to the pre-Maxwell form of Ampère's law:

$$\boxed{\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}}\,. \tag{14.17}$$

Thus, the current is no longer an independent field, there is a one-to-one correspondence $\boldsymbol{j} \leftrightarrow \boldsymbol{B}$.

Finally, comparing the electric and magnetic parts of the Lorentz force (14.8), and using Gauss's law (14.9) to estimate $\sigma \sim E/\ell$, we get

$$\frac{|\sigma\boldsymbol{E}|}{\left|\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B}\right|} \sim \frac{\frac{1}{\ell}E^2}{\frac{1}{c}\frac{c}{\ell}B^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{c^2} \ll 1. \tag{14.18}$$

Thus, the MHD body force is

$$\boxed{\boldsymbol{F} = \frac{\boldsymbol{j}\times\boldsymbol{B}}{c} = \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}}\,. \tag{14.19}$$

This goes into (14.5) and we note with relief that σ, $\boldsymbol{j}$ and $\boldsymbol{E}$ have all fallen out of the momentum equation—we only need to know $\boldsymbol{B}$.

14.4. *Maxwell Stress and Magnetic Forces*

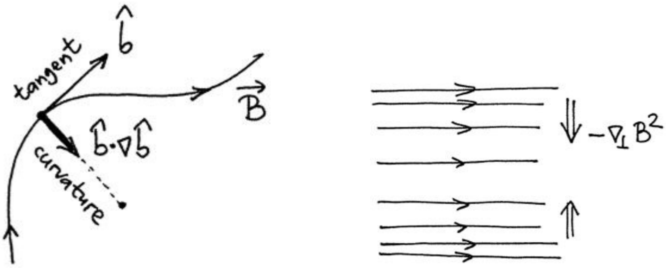
Let us take a break from formal derivations to consider what (14.19) teaches us about the sort of new dynamics that our fluid will experience as a result of being conducting. To see this, it is useful to play with the expression (14.19) in a few different ways.

By simple vector algebra,

$$\boldsymbol{F} = \underbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}}_{\substack{\text{“magnetic}\\ \text{tension”}}} - \underbrace{\boldsymbol{\nabla}\frac{B^2}{8\pi}}_{\substack{\text{“magnetic}\\ \text{pressure”}}} = -\boldsymbol{\nabla}\cdot\underbrace{\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right)}_{\substack{\text{“Maxwell}\\ \text{stress”}}}, \tag{14.20}$$

where the last expression was obtained with the aid of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Thus, the action of the Lorentz force in a conducting fluid amounts to a new form of stress. Mathematically, this *“Maxwell stress”* is somewhat similar to the kind of stress that would arise from a suspension in the fluid of elongated molecules—e.g., polymer chains, or other kinds of “balls on springs” (see, e.g., Dellar 2017; the analogy can be made rigorous: see Ogilvie & Proctor 2003). Thus, we expect that the magnetic field threading the fluid will impart to it a degree of “elasticity” (you will have an opportunity of a practical engagement with this analogy in Exercise 16.4).

Exactly what this means dynamically becomes obvious if we rewrite the magnetic tension and pressure forces in (14.20) in the following way. Let $\boldsymbol{b} = \boldsymbol{B}/B$ be the unit



(a) Curvature force (b) Magnetic pressure

FIGURE 61. Magnetic forces.

vector in the direction of $\boldsymbol{B}$ (the unit tangent to the field line). Then

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} = B\boldsymbol{b}\cdot\boldsymbol{\nabla}(B\boldsymbol{b}) = B^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{B^2}{2} \tag{14.21}$$

and, putting this back into (14.20), we get

$$\boldsymbol{F} = \underbrace{\frac{B^2}{4\pi}\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}}_{\substack{\text{“curvature}\\ \text{force”}}} \underbrace{-\underbrace{(\mathsf{I}-\boldsymbol{b}\boldsymbol{b})\cdot\boldsymbol{\nabla}}_{\equiv\boldsymbol{\nabla}_\perp}\frac{B^2}{8\pi}}_{\substack{\text{magnetic}\\ \text{pressure force}}}. \tag{14.22}$$

Thus, we learn that the Lorentz force consists of two distinct parts (Fig. 61):

- *curvature force*, so called because $\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}$ is the vector curvature of the magnetic field line—the implication being that field lines, if bent, will want to straighten up;
- *magnetic pressure*, whose presence implies that field lines will resist compression or rarefication (the field wants to be uniform in strength).

Note that both forces act perpendicularly to $\boldsymbol{B}$, as they must, since magnetic field never exerts a force along itself on a charged particle [see (14.7)].

So this is the effect of the field on the fluid. What is the effect of the fluid on the field?

14.5. *Evolution of Magnetic Field*

Returning to deriving MHD equations, we use Ohm's law (14.13) to express $\boldsymbol{E}$ in terms of $\boldsymbol{u}$, $\boldsymbol{B}$ and $\boldsymbol{j}$ in the right-hand side of Faraday's law (14.11). We then use Ampere's law (14.17) to express $\boldsymbol{j}$ in terms of $\boldsymbol{B}$. The result is

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{u}\times\boldsymbol{B} - \frac{c^2\eta}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}\right). \tag{14.23}$$

After using also $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ to get $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = -\nabla^2\boldsymbol{B}$ and renaming $c^2\eta/4\pi\to\eta$, the *magnetic diffusivity*, we arrive at the *magnetic induction equation* (due to Hertz):

$$\boxed{\frac{\partial\boldsymbol{B}}{\partial t} = \underbrace{\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})}_{\text{advection}} + \underbrace{\eta\nabla^2\boldsymbol{B}}_{\text{diffusion}}}. \tag{14.24}$$

Note that if $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ is satisfied initially, any solution of (14.24) will remain divergence-free at all times.

14.6. *Magnetic Reynolds Number*

The relative importance of the diffusion term (it is obvious what this does) and the advection term (to be discussed in the next few sections) in (14.24) is measured by a dimensionless number:

$$\frac{|\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})|}{|\eta\nabla^2\boldsymbol{B}|}\sim\frac{\dfrac{u}{\ell}B}{\dfrac{\eta}{\ell^2}B}=\frac{u\ell}{\eta}\equiv\mathrm{Rm},\tag{14.25}$$

called the *magnetic Reynolds number*. In nature, it can take a very broad range of values:

$$\begin{aligned}\text{liquid metals in idustrial contexts (metallurgy):}&\quad \mathrm{Rm}\sim10^{-3}\dots10^{-1},\\ \text{planet interiors:}&\quad \mathrm{Rm}\sim100\dots300,\\ \text{solar convective zone:}&\quad \mathrm{Rm}\sim10^{6}\dots10^{9},\\ \text{interstellar medium (“warm” phase):}&\quad \mathrm{Rm}\sim10^{18},\\ \text{intergalactic medium (cores of galaxy clusters):}&\quad \mathrm{Rm}\sim10^{29},\\ \text{laboratory “dynamo” experiments:}&\quad \mathrm{Rm}\sim1\dots10^{3}.\end{aligned}$$

Generally speaking, when flow velocities are large/distances are large/resistivities are low, $\mathrm{Rm}\gg1$ and it makes sense to consider *"ideal MHD,"* i.e., the limit $\eta\to0$. In fact, η often needs to be brought back in to deal with instances of large $\boldsymbol{\nabla}\boldsymbol{B}$, which arise naturally from solutions of ideal MHD equations (see §15.1, Q-5, §19 and Parra 2019*a*), but let us consider the ideal case for now to understand what the advective part of the induction equation does to $\boldsymbol{B}$.

14.7. *Lundquist Theorem*

The ideal ($\eta=0$) version of the induction equation (14.24),

$$\frac{\partial\boldsymbol{B}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}),\tag{14.26}$$

implies that *fluid elements that lie on a field line initially will remain on this field line*, i.e., "the magnetic field moves with the flow."

Proof. Unpacking the double vector product in (14.26),

$$\frac{\partial\boldsymbol{B}}{\partial t}=-\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\boldsymbol{u}\cancel{\boldsymbol{\nabla}\cdot\boldsymbol{B}},\tag{14.27}$$

or, using the notation for the "convective derivative" [see (14.5)],

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}\equiv\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}.\tag{14.28}$$

The continuity equation (14.2) can be rewritten in a somewhat similar-looking form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\equiv\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\rho=-\rho\boldsymbol{\nabla}\cdot\boldsymbol{u}\quad\Rightarrow\quad\boldsymbol{\nabla}\cdot\boldsymbol{u}=-\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}.\tag{14.29}$$

The last expression is now used for $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ in (14.28):

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}=\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}+\frac{\boldsymbol{B}}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}.\tag{14.30}$$

Multiplying this equation by $1/\rho$ and noting that

$$\frac{1}{\rho}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t}-\frac{\boldsymbol{B}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{B}}{\rho},\tag{14.31}$$

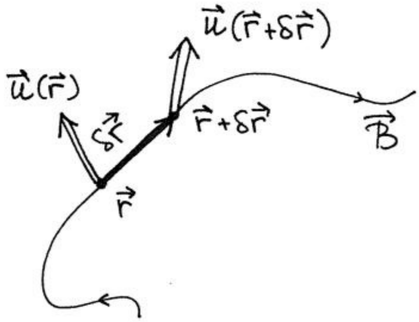


FIGURE 62. Pressure-balanced perturbations.

we arrive at

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{B}}{\rho} = \frac{\boldsymbol{B}}{\rho}\cdot\boldsymbol{\nabla}\boldsymbol{u}}\,. \tag{14.32}$$

Let us compare the evolution of the vector $\boldsymbol{B}/\rho$ with the evolution of an infinitesimal Lagrangian separation vector in a moving fluid: the convective derivative is the Lagrangian time derivative, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{r}(t) = \boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r}) \approx \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{u}. \tag{14.33}$$

Thus, $\delta\boldsymbol{r}$ and $\boldsymbol{B}/\rho$ satisfy the same equation. This means that if two fluid elements are initially on the same field line,

$$\delta\boldsymbol{r} = \mathrm{const}\,\frac{\boldsymbol{B}}{\rho}, \tag{14.34}$$

then they will stay on the same field line, q.e.d.[122]

This means that in MHD, the fluid flow will be entraining the magnetic-field lines with it—and, as we saw in §14.4, the field lines will react back on the fluid:
—when the fluid tries to bend the field, the field will want to spring back,
—when the fluid tries to compress or rarefy the field, the field will resist as if it possessed (perpendicular) pressure.

This is the sense in which MHD fluid is "elastic": it is threaded by magnetic-field lines, which move with it and act as elastic bands.

14.8. *Flux Freezing*

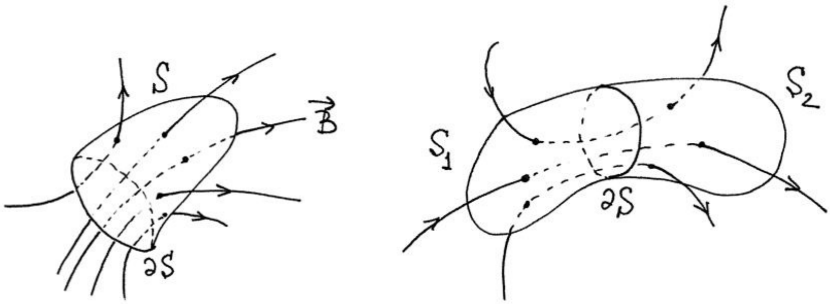
There is an essentially equivalent formulation of the result of §14.7 that highlights the fact that the ideal induction equation (14.26) is a *conservation law*—conservation of *magnetic flux*.

The magnetic flux through a surface S (Fig. 63a) is, by definition,

$$\Phi = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \tag{14.35}$$

($\mathrm{d}\boldsymbol{S} \equiv \hat{\boldsymbol{n}}\,\mathrm{d}S$, where $\hat{\boldsymbol{n}}$ is a unit normal pointing out of the surface). *The flux Φ depends on the loop ∂S, but not on the choice of the surface spanning it.* Indeed, if we consider two

[122]Lundquist theorem opens the door to a Lagrangian description of MHD fluid that contains some mathematical and physical delights: I will pick up this thread in §14.13.



(a) (b)

FIGURE 63. Magnetic flux through a loop ∂S (a) is independent of the surface spanning the loop (b).

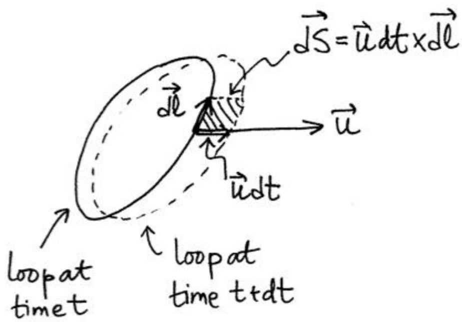


FIGURE 64. A loop moving with the fluid.

surfaces, S_1 and S_2, spanning the same loop ∂S (Fig. 63b) and define $\Phi_{1,2} = \int_{S_{1,2}} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$, then the flux out of the volume V enclosed by $S_1 \cup S_2 = \partial V$ is

$$\Phi_2 - \Phi_1 = \int_{\partial V} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \int_V \mathrm{d}\boldsymbol{r}\, \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \text{q.e.d.} \tag{14.36}$$

Alfvén's Theorem. *Flux through any loop moving with the fluid is conserved.*

Proof. Let $S(t)$ be a surface spanning the loop at time t. If the loop moves with the fluid (Fig. 64), at the slightly later time $t + \mathrm{d}t$ it is spanned (for example) by the surface

$$S(t+\mathrm{d}t) = S(t) \;\cup\; \begin{array}{l}\text{ribbon traced by the loop}\\ \text{as it moves over time } \mathrm{d}t.\end{array} \tag{14.37}$$

Then the flux at time t is

$$\Phi(t) = \int_{S(t)} \boldsymbol{B}(t)\cdot\mathrm{d}\boldsymbol{S} \tag{14.38}$$

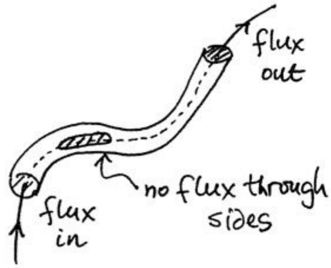


FIGURE 65. Flux tube.

and at the later time,

$$
\begin{aligned}
\Phi(t+\mathrm{d}t) &= \int_{S(t+\mathrm{d}t)} \boldsymbol{B}(t+\mathrm{d}t)\cdot\mathrm{d}\boldsymbol{S} \\
&= \underbrace{\int_{S(t)} \boldsymbol{B}(t+\mathrm{d}t)\cdot\mathrm{d}\boldsymbol{S}}_{= \underbrace{\int_{S(t)} \boldsymbol{B}(t)\cdot\mathrm{d}\boldsymbol{S}}_{=\Phi(t)} + \mathrm{d}t\int_{S(t)}\frac{\partial \boldsymbol{B}}{\partial t}\cdot\mathrm{d}\boldsymbol{S}} + \underbrace{\int_{\text{ribbon}} \boldsymbol{B}(t+\mathrm{d}t)\cdot\underbrace{\mathrm{d}\boldsymbol{S}}_{=\boldsymbol{u}\mathrm{d}t\times\mathrm{d}\boldsymbol{l}}}_{=\mathrm{d}t\int_{\partial S(t)}\boldsymbol{B}(t)\cdot(\boldsymbol{u}\times\mathrm{d}\boldsymbol{l})} \\
&\qquad = -\mathrm{d}t\int_{\partial S(t)}(\boldsymbol{u}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l} \\
&\qquad = -\mathrm{d}t\int_{S(t)}[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})]\cdot\mathrm{d}\boldsymbol{S}.
\end{aligned}
\tag{14.39}
$$

Therefore,

$$
\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\Phi(t+\mathrm{d}t)-\Phi(t)}{\mathrm{d}t} = \int_{S(t)} \underbrace{\left[\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\right]}_{\substack{\text{ideal induction}\\ \text{equation (14.26)}}}\cdot\mathrm{d}\boldsymbol{S} = 0, \quad \text{q.e.d.} \tag{14.40}
$$

This result means that *field lines are frozen into the flow.* Indeed, consider a flux tube enclosing a field line (Fig. 65). As the tube deforms, the field line stays inside it because fluxes through the ends and sides of the tube cannot change.

Note that *Ohmic diffusion breaks flux freezing*, as is obvious from (14.40) if in the integrand one uses the induction equation (14.24) keeping the resistive term.

14.9. *Amplification of Magnetic Field by Fluid Flow*

An interesting physical consequence of these results is that *flows of conducting fluid can amplify magnetic fields.* For example, consider a flow that stretches an initial cylindrical tube of length l_1 and cross section S_1 into a long thin spaghetto of length l_2 and cross section S_2 (Fig. 66). By conservation of flux,

$$B_1S_1 = B_2S_2. \tag{14.41}$$

By conservation of mass,

$$\rho_1 l_1 S_1 = \rho_2 l_2 S_2. \tag{14.42}$$

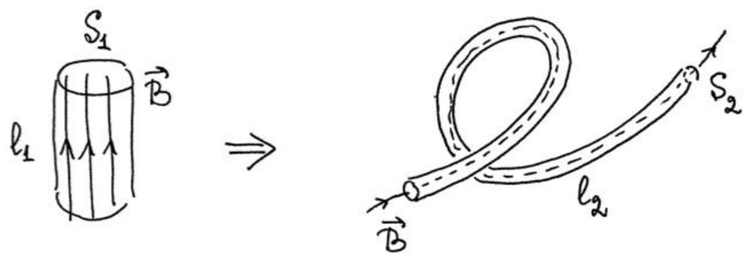


FIGURE 66. Amplification of magnetic field by stretching.

Therefore,

$$\frac{B_2}{\rho_2 l_2} = \frac{B_1}{\rho_1 l_1} \quad \Rightarrow \quad \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}. \tag{14.43}$$

In an incompressible fluid, $\rho_2 = \rho_1$, and the field is amplified by a factor l_2/l_1. In a compressible fluid, the field can also be amplified by compression.

Going back to the induction equation in the form (14.27),

$$\frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B}}_{\text{advection}} = \underbrace{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}}_{\text{stretching}} - \underbrace{\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}}_{\text{compression}}, \tag{14.44}$$

the three terms in it are responsible for, in order, advection of the field by the flow (i.e., the flow carrying the field around with it), "stretching" (amplification) of the field by velocity gradients that make fluid elements longer and, finally, compression or rarefication of the field by convergent or divergent flows (unless $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, as it is in an incompressible fluid).

Hence arises the famous problem of *MHD dynamo:* are there fluid flows that lead to sustained amplification of the magnetic fields? The answer is yes—but the flow must be 3D (the absence of dynamo action in 2D is a theorem, the simplest version of which is due to Zeldovich 1956; see Q-4). Magnetic fields of planets, stars, galaxies, etc. are all believed to owe their origin and persistence to this effect. This topic requires (and merits) a more detailed treatment (see reading suggestions below), but for now let us flag two important aspects:

• resistivity, however small, turns out to be impossible to neglect because large gradients of $\boldsymbol{B}$ appear as the field is advected by the flow (see §15.1);

• the amplification of the field is checked by the Lorentz force once the field is strong enough that it can act back on the flow, viz., when their energy densities become comparable:

$$\frac{B^2}{8\pi} \sim \frac{\rho u^2}{2}. \tag{14.45}$$

If you wish to educate yourself on the topic of MHD dynamos (as everyone should), a classic (and mostly timeless) text is Moffatt (1978). The new classic is the review by Rincon (2019), which I recommend strongly. In these Lectures, some elements of dynamo theory are introduced in §15.

14.10. *Conservation of Energy*

Let us summarise the equations that we have derived so far, namely (14.2), (14.5) and (14.24), expressing conservation of

$$\text{mass}\quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{14.46}$$

$$\begin{aligned}\text{momentum}\quad \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) &= -\boldsymbol{\nabla}p - \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} + \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}\\ &= -\boldsymbol{\nabla}\cdot\underbrace{\left[\left(p + \frac{B^2}{8\pi}\right)\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + \boldsymbol{\Pi}\right]}_{\text{total stress}},\end{aligned} \tag{14.47}$$

$$\text{and flux}\quad \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) + \eta\nabla^2\boldsymbol{B}. \tag{14.48}$$

To complete the system, we need an equation for p, which has to come from the one conservation law that we have not yet utilised: *conservation of energy.*

The total *energy density* is

$$\varepsilon = \underbrace{\frac{\rho u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{internal}} + \underbrace{\cancel{\frac{E^2}{8\pi}}}_{\text{electric}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}}, \tag{14.49}$$

where the electric energy can (and, for consistency with §14.3, must) be neglected because $E^2/B^2 \sim u^2/c^2 \ll 1$. We follow the same logic as we did in §§14.1 and 14.2:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_V \mathrm{d}\boldsymbol{r}\,\varepsilon}_{\substack{\text{energy inside}\\ \text{a volume of}\\ \text{fluid}}} = &-\underbrace{\int_{\partial V}\left(\frac{\rho u^2}{2} + \frac{p}{\gamma-1}\right)\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{S}}_{\substack{\text{fluid flow carrying its}\\ \text{kinetic and internal energy}\\ \text{through boundary}}} - \underbrace{\int_{\partial V}[(p\mathsf{I} + \boldsymbol{\Pi})\cdot\boldsymbol{u}]\cdot\mathrm{d}\boldsymbol{S}}_{\substack{\text{work done on boundary}\\ \text{by pressure and viscous}\\ \text{stress}}}\\ &- \underbrace{\int_{\partial V}\boldsymbol{q}\cdot\mathrm{d}\boldsymbol{S}}_{\text{heat flux}} - \underbrace{\int_{\partial V}\frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{S}}_{\text{Poynting flux}}.\end{aligned} \tag{14.50}$$

Like the viscous stress $\boldsymbol{\Pi}$, the heat flux $\boldsymbol{q}$ must be calculated kinetically (in a plasma) or tabulated (in an arbitrary complicated substance). In a gas, $\boldsymbol{q} = -\kappa\boldsymbol{\nabla}T$, but it is more complicated in a magnetised plasma (see §22.3 and Braginskii 1965; Helander & Sigmar 2005; Parra 2019*a*; Kunz 2021).

Note that the magnetic energy and the work done by the Lorentz force are not included in the first two terms on the right-hand side of (14.50) because all of that must already be correctly acounted for by the Poynting flux. Indeed, since $c\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B} + \eta\boldsymbol{\nabla}\times\boldsymbol{B}$

[this is (14.13), with η renamed as in (14.24)], we have

$$\int_{\partial V} \frac{c}{4\pi}\left(\boldsymbol{E}\times\boldsymbol{B}\right)\cdot \mathrm{d}\boldsymbol{S} = \underbrace{\int_{\partial V} \frac{B^2}{8\pi}\,\boldsymbol{u}\cdot \mathrm{d}\boldsymbol{S}}_{\text{magnetic energy flow}} + \underbrace{\int_{\partial V} \left[\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}\right)\cdot\boldsymbol{u}\right]\cdot \mathrm{d}\boldsymbol{S}}_{\text{work done by Maxwell stress}}$$
$$+ \underbrace{\int_{\partial V} \eta\,\frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}\cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{resistive slippage accounting}\\ \text{for field not being precisely}\\ \text{frozen into flow}}}\,. \tag{14.51}$$

After application of Gauss's theorem and shrinking of the volume V to infinitesimality, we get the differential form of (14.50):

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{8\pi}\right) = -\,\boldsymbol{\nabla}\cdot\left[\frac{\rho u^2}{2}\boldsymbol{u} + \frac{\gamma}{\gamma-1}\,p\boldsymbol{u} + \boldsymbol{\Pi}\cdot\boldsymbol{u} + \boldsymbol{q}\right.$$
$$\left. + \frac{B^2\mathsf{I} - \boldsymbol{BB}}{4\pi}\cdot\boldsymbol{u} + \eta\,\frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}\right]. \tag{14.52}$$

It remains to separate the evolution equation for p by using the fact that we know the equations for ρ, $\boldsymbol{u}$ and $\boldsymbol{B}$ and so can deduce the rates of change of the kinetic and magnetic energies.

14.10.1. *Kinetic Energy*

Using (14.46) and (14.47),

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\rho u^2}{2} &= \frac{u^2}{2}\frac{\partial \rho}{\partial t} + \rho\boldsymbol{u}\cdot\frac{\partial \boldsymbol{u}}{\partial t}\\
&= -\frac{u^2}{2}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\frac{u^2}{2} - \boldsymbol{u}\cdot\left\{\boldsymbol{\nabla}\cdot\left[\left(p + \frac{B^2}{8\pi}\right)\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi} + \boldsymbol{\Pi}\right]\right\}\\
&= -\boldsymbol{\nabla}\cdot\left[\cancel{\frac{\rho u^2}{2}}\boldsymbol{u} + \cancel{p\boldsymbol{u}} + \left(\frac{B^2}{\cancel{8\pi}}\cancel{\mathsf{I} - \boldsymbol{BB}}\over{4\pi}\right)\cdot\boldsymbol{u} + \cancel{\boldsymbol{\Pi}\cdot\boldsymbol{u}}\right]\\
&\quad + \underbrace{p\boldsymbol{\nabla}\cdot\boldsymbol{u}}_{\substack{\text{compressional}\\ \text{work}}} + \underbrace{\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}\right):\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{energy exchange}\\ \text{with magnetic field}}} + \underbrace{\boldsymbol{\Pi}:\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{viscous}\\ \text{dissipation}}}\,.
\end{aligned} \tag{14.53}$$

The flux terms (energy flows and work by stresses on boundaries) that have been crossed out cancel with corresponding terms in (14.52) once (14.53) is subtracted from it.

14.10.2. *Magnetic Energy*

Using the induction equation (14.48),

$$\begin{aligned}\frac{\partial}{\partial t}\frac{B^2}{8\pi} &= \frac{\boldsymbol{B}}{4\pi}\cdot\left[-\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}+\eta\nabla^2\boldsymbol{B}\right]\\ &= -\boldsymbol{\nabla}\cdot\left[\cancel{\frac{B^2}{8\pi}\boldsymbol{u}}+\cancel{\eta\frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}}\right]-\underbrace{\left(\frac{B^2}{8\pi}\mathsf{I}-\frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right):\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{energy exchange}\\ \text{with velocity field}}}-\underbrace{\eta\frac{|\boldsymbol{\nabla}\times\boldsymbol{B}|^2}{4\pi}}_{\substack{\text{Ohmic}\\ \text{dissipation}}}.\end{aligned}\tag{14.54}$$

Again, the crossed out flux terms will cancel with corresponding terms in (14.52). The metamorphosis of the resistive term into a flux term and an Ohmic dissipation term is a piece of vector algebra best checked by expanding the divergence of the flux term. Finally, the $\boldsymbol{u}$-to-$\boldsymbol{B}$ energy exchange term (penultimate on the right-hand side) corresponds precisely to the $\boldsymbol{B}$-to-$\boldsymbol{u}$ exchange term in (14.53) and cancels with it if we add (14.53) and (14.54).

14.10.3. *Thermal Energy*

Subtracting (14.53) and (14.54) from (14.52), consummating the promised cancellations, and mopping up the remaining $\boldsymbol{\nabla}\cdot(p\boldsymbol{u})$ and $p\boldsymbol{\nabla}\cdot\boldsymbol{u}$ terms, we end up with the desired evolution equation for the thermal (internal) energy:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{p}{\gamma-1}}_{\substack{\text{advection of}\\ \text{internal}\\ \text{energy}}} = \underbrace{-\boldsymbol{\nabla}\cdot\boldsymbol{q}}_{\text{heat flux}}\underbrace{-\frac{\gamma}{\gamma-1}p\boldsymbol{\nabla}\cdot\boldsymbol{u}}_{\substack{\text{compressional}\\ \text{heating}}}\underbrace{-\boldsymbol{\mathit{\Pi}}:\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{viscous}\\ \text{heating}}}+\underbrace{\eta\frac{|\boldsymbol{\nabla}\times\boldsymbol{B}|^2}{4\pi}}_{\substack{\text{Ohmic}\\ \text{heating}}}.\tag{14.55}$$

A further rearrangement and the use of the continuity equation (14.46) to express $\boldsymbol{\nabla}\cdot\boldsymbol{u} = -\mathrm{d}\ln\rho/\mathrm{d}t$ turn (14.55) into

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p}{\rho^\gamma} = \frac{\gamma-1}{p}\left(-\boldsymbol{\nabla}\cdot\boldsymbol{q}-\boldsymbol{\mathit{\Pi}}:\boldsymbol{\nabla}\boldsymbol{u}+\eta\frac{|\boldsymbol{\nabla}\times\boldsymbol{B}|^2}{4\pi}\right)}.\tag{14.56}$$

This form of the thermal-energy equation has very clear physical content: the left-hand side represents advection of the entropy of the MHD fluid by the flow—each fluid element behaves adiabatically, except for the sundry non-adiabatic effects on the right-hand side. The latter are the heat flux in/out of the fluid element and the dissipative (viscous and resistive) heating, leading to entropy production. Note that the form of the viscous stress $\boldsymbol{\mathit{\Pi}}$ ensures that the viscous heating is always positive [see, e.g., (14.6)]. In these Lectures, I will, for the most part, focus on ideal MHD and so use the adiabatic version of (14.56), with the right-hand side set to zero.

Let me reiterate the equations of *ideal MHD*, now complete:

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{14.57}$$

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) = -\boldsymbol{\nabla}p + \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}, \tag{14.58}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{14.59}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\frac{p}{\rho^\gamma} = 0. \tag{14.60}$$

In what follows, we shall study various solutions and symptotic regimes of these rather nice equations.

Exercise 14.1. MHD with self-gravity. Consider an MHD system subject to gravity with acceleration $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$, where the gravitational potential Φ satisfies Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho, \tag{14.61}$$

and G is the gravitational constant. There will then be an additional body force in the momentum equation (14.5), equal to $\rho\boldsymbol{g}$. Show that, like the magnetic force (14.20), the gravitational force can be written as a divergence of a stress tensor:

$$\rho\boldsymbol{g} = \boldsymbol{\nabla}\cdot\left(\frac{g^2}{8\pi G}\mathsf{I} - \frac{\boldsymbol{g}\boldsymbol{g}}{4\pi G}\right). \tag{14.62}$$

Show also that the total energy of this system is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{r}\left(\frac{\rho u^2}{2} + \frac{p}{\gamma-1} + \frac{B^2}{8\pi} + \frac{\rho\Phi}{2}\right) = 0, \tag{14.63}$$

where the integration is over the entire system, surface integrals over its boundary are assumed to vanish and

$$\mathscr{E}_G \equiv \int\mathrm{d}\boldsymbol{r}\,\frac{\rho\Phi}{2} = -\int\mathrm{d}\boldsymbol{r}\,\frac{g^2}{8\pi G} < 0 \tag{14.64}$$

is the gravitational energy of the system.

14.11. *Beyond MHD Approximation*

In the context of plasma physics, the ideal MHD equations (14.57–14.60) are a *collisional* approximation: the key place where that comes in is the isotropy of pressure. A more general situation is one where pressure is a tensor, P, with $\boldsymbol{\nabla}p$ in (14.58) replaced by $\boldsymbol{\nabla}\cdot\mathsf{P}$. To calculate P, one then needs the kinetic equation or some appropriate closure approximation (or model). In the presence of a magnetic field that is strong enough to make the ion Larmor frequency much greater than the collision rate, the pressure tensor on scales that are longer than the Larmor radius becomes diagonal with respect to the local direction of $\boldsymbol{B}$:

$$\mathsf{P} = p_\perp(\mathsf{I} - \boldsymbol{b}\boldsymbol{b}) + p_\parallel\boldsymbol{b}\boldsymbol{b}. \tag{14.65}$$

A particularly popular and well-known closure model for calculating the pependicular and parallel pressures is the so-called *double-adiabatic*, or *CGL equations* (after the orginal authors Chew *et al.* 1956):

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_\perp}{\rho B} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{p_\parallel B^2}{\rho^3} = 0. \tag{14.66}$$

These equations, which replace the adiabatic law (14.60), express the property of particles on Larmor orbits in magnetic field to conserve certain adiabatic invariants of motion.

Plasma dynamics become very different (and often quite counterintuitive) in this and similar pressure-anisotropic approximations than for regular ideal MHD (e.g., it is difficult to amplify, or, generally, change the magnetic field's strength in such a plasma: see Helander *et al.* 2016 and Squire *et al.* 2019; the latter paper gave the effect a sexy name, *magneto-immutability*). I shall come back to this topic in Exercise 16.8 and from §22.2.1 (where the CGL equations will be derived) onwards (see, in particular, the short discussion at the end of §22.2.4).

A feature that makes this foray beyond MHD particularly difficult is that pressure anisotropies in certain types of plasmas can trigger small-scale instabilities—the simplest of these is the *firehose instability*, which will make a cameo appearance in Exercise 16.9, to be treated properly in §20.8, followed by its friend the *mirror instability* (§22.1). These break the fluid approximation and leave us without a good mean-field theory for the description of macroscopic motions in such environments. What to do about that is a subject of a rich vein of current research in plasma astrophysics, for which you will be well prepared by the part of these Lectures where MHD and kinetic theory join hands (Part IV).

Another important route away from MHD is one that leads to "two-fluid" approximations of plasma dynamics. These deal with the question of what happens at scales where different species (ions and electrons) cannot be considered to move together (the most popular such models are *Hall MHD* and *Electron MHD*; you will meet the latter in Q-7). This too is a result (and manifestation) of various plasma microphysics (e.g., Larmor motion) kicking in because collisions are not large enough to suppress them—I will return to the subject in Part IV of these Lectures (§24.3).

14.12. *Virial Theorem*

If you have done Exercise 14.1, you are ready for the following rather nice, exact result that follows directly from MHD equations and helps one decide whether an MHD system can "self-confine", i.e., whether a blob of plasma (or, more generally, a conducting fluid) can exist without blowing itself apart.

Consider an MHD system whose volume is V. Its moment of inertia is

$$I = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\rho r^2. \tag{14.67}$$

Let us see how it evolves with time: using the continuity equation (14.2), we get

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,r^2\frac{\partial\rho}{\partial t} = -\frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,r^2\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = \int \mathrm{d}\boldsymbol{r}\,\rho\boldsymbol{u}\cdot\boldsymbol{r}, \tag{14.68}$$

after integration by parts and assuming $\boldsymbol{u}\perp\partial V$ (no in/outflows). Let us take another time derivative of this, this time using the momentum equation in the form (14.4),

$$\frac{\partial}{\partial t}\rho\boldsymbol{u} = -\boldsymbol{\nabla}\cdot\mathsf{T}, \tag{14.69}$$

where the total stress tensor

$$\mathsf{T} = \rho\boldsymbol{u}\boldsymbol{u} + p\mathsf{I} + \boldsymbol{\Pi} + \frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} - \left(\frac{g^2}{8\pi G}\mathsf{I} - \frac{\boldsymbol{g}\boldsymbol{g}}{4\pi G}\right) \tag{14.70}$$

includes the Maxwell stress (14.20) and the gravitational stress (14.62). This gives us

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = -\int \mathrm{d}\boldsymbol{r}\,(\boldsymbol{\nabla}\cdot\mathsf{T})\cdot\boldsymbol{r} = -\int \mathrm{d}\boldsymbol{r}\,\left[\boldsymbol{\nabla}\cdot(\mathsf{T}\cdot\boldsymbol{r}) - \mathsf{T}:\boldsymbol{\nabla}\boldsymbol{r}\right] = -\int \mathrm{d}\boldsymbol{S}\cdot\mathsf{T}\cdot\boldsymbol{r} + \int \mathrm{d}\boldsymbol{r}\,\mathrm{tr}\,\mathsf{T}. \tag{14.71}$$

Therefore, in steady state, the stresses on the boundary satisfy

$$\boxed{\int \mathrm{d}\boldsymbol{S}\cdot\mathsf{T}\cdot\boldsymbol{r} = \int \mathrm{d}\boldsymbol{r}\,\mathrm{tr}\,\mathsf{T} = 2\mathscr{E}_{\mathrm{kin}} + 3(\gamma-1)\mathscr{E}_{\mathrm{th}} + \mathscr{E}_{\mathrm{mag}} + \mathscr{E}_G}\,, \tag{14.72}$$

where $\mathscr{E}_{\mathrm{kin}}$, $\mathscr{E}_{\mathrm{th}}$, $\mathscr{E}_{\mathrm{mag}}$ and $\mathscr{E}_G$ are the total kinetic, thermal, magnetic and gravitational energies of the system—the four terms in (14.63); note that $\mathrm{tr}\,\boldsymbol{\Pi} = 0$ (any trace is part of p). In the absence of gravity, the weighted sum of energies in (14.72) is strictly positive and so it is not possible to have zero stress on the boundary of the system—the system cannot be self-confined. With gravity, since the gravitational energy (14.64) is negative, the right-hand side of (14.72) can be–and, in a steady, self-confined state, is—zero, telling us gravity can confine MHD systems, e.g., stars. The total energy in this case is, as you might have guessed, negative:

$$\mathscr{E} = \mathscr{E}_{\mathrm{kin}} + \mathscr{E}_{\mathrm{th}} + \mathscr{E}_{\mathrm{mag}} + \mathscr{E}_G = -\mathscr{E}_{\mathrm{kin}} - (3\gamma-4)\mathscr{E}_{\mathrm{th}} < 0. \tag{14.73}$$

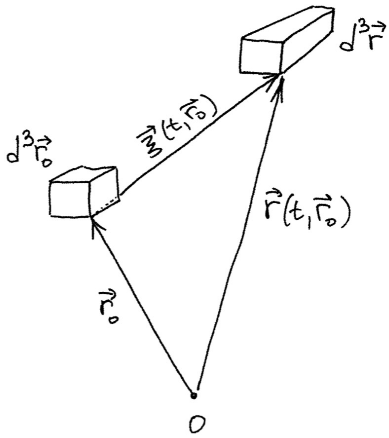


FIGURE 67. Lagrangian dispalcement of an infinitesimal fluid element.

A perhaps unexpected corollary of this is that if a star radiates and, therefore, loses energy, then $\mathscr{E}_{\rm th}$ must increase (assuming $\mathscr{E}_{\rm kin} \ll \mathscr{E}_{\rm th}$, i.e., subsonic motions inside)—i.e., "as a star cools, it heats up".

14.13. *Lagrangian MHD*

There is a Lagrangian formulation of ideal MHD, due to Newcomb (1962, a classic and elegant paper, which I recommend for your reading pleasure). This is both mathematically attractive and sheds some physical light.

Let us label each fluid element's position at $t = 0$ by the *Lagrangian coordinate* $\boldsymbol{r}_0$. Then the *Eulerian coordinate* $\boldsymbol{r}$ at any given time t is the position of the same fluid element at that time:

$$\boldsymbol{r}(t,\boldsymbol{r}_0) = \boldsymbol{r}_0 + \boldsymbol{\xi}(t,\boldsymbol{r}_0), \tag{14.74}$$

where $\boldsymbol{\xi}$ is the *displacement*. Formally, we shall treat (14.74) as just a coordinate transformation, which can also be inverted: given the Eulerian coordinate $\boldsymbol{r}$ of a fluid element at time t, the Lagrangian coordinate $\boldsymbol{r}_0(t,\boldsymbol{r})$ is where this fluid element was at $t = 0$. The coordinate transformation is determined by the history of the fluid flow:

$$\frac{\partial \boldsymbol{r}(t,\boldsymbol{r}_0)}{\partial t} = \frac{\partial \boldsymbol{\xi}(t,\boldsymbol{r}_0)}{\partial t} = \boldsymbol{u}(t,\boldsymbol{r}(t,\boldsymbol{r}_0)) \equiv \boldsymbol{u}_{\rm L}(t,\boldsymbol{r}_0). \tag{14.75}$$

I shall use the subscript "L" to designate Lagrangian fields, i.e., MHD fields as functions of the Lagrangian coordinate and time.

Let us learn how to transform derivatives between Eulerian and Lagrangian variables:

$$\frac{\partial}{\partial r_{0i}} = \frac{\partial r_j}{\partial r_{0i}}\frac{\partial}{\partial r_j} = \left(\delta_{ji} + \frac{\partial \xi_j}{\partial r_{0i}}\right)\frac{\partial}{\partial r_j}, \quad \text{or} \quad \boldsymbol{\nabla}_0 = (\boldsymbol{\nabla}_0\boldsymbol{r})\cdot\boldsymbol{\nabla} = (\mathsf{I} + \boldsymbol{\nabla}_0\boldsymbol{\xi})\cdot\boldsymbol{\nabla}, \tag{14.76}$$

$$\frac{\partial}{\partial t_{\rm L}} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}_0} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{r}} + \left(\frac{\partial r_i}{\partial t}\right)_{\boldsymbol{r}_0}\frac{\partial}{\partial r_i} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} = \frac{\mathrm{d}}{\mathrm{d}t}. \tag{14.77}$$

Thus, the Lagrangian time derivative is the convective derivative—you knew that! We are now ready to convert the MHD equations (14.57–14.60) to the Lagrangian frame.

14.13.1. *Density*

The continuity equation is an expression of conservation of mass (§14.1). Let us write that for an infinitesimal volume element moving with the fluid:

$$\rho_0(\boldsymbol{r}_0)\mathrm{d}\boldsymbol{r}_0 = \rho_{\rm L}(t,\boldsymbol{r}_0)\mathrm{d}\boldsymbol{r}, \tag{14.78}$$

where ρ_0 and $\mathrm{d}\boldsymbol{r}_0$ are the density and the volume of the fluid element at $t = 0$, and $\rho_{\rm L}$ and $\mathrm{d}\boldsymbol{r}$ are its density and volume at time t (Fig. 67). So we need to know how volumes change under the coordinate transformation (14.74), i.e., we need the Jacobian of the *strain matrix* $\partial r_i/\partial r_{0j}$:

$$\mathrm{d}\boldsymbol{r} = J(t,\boldsymbol{r}_0)\mathrm{d}\boldsymbol{r}_0, \quad \text{where} \quad J = |\det \boldsymbol{\nabla}_0\boldsymbol{r}| = \frac{1}{6}\epsilon_{ijk}\epsilon_{lmn}\frac{\partial r_i}{\partial r_{0l}}\frac{\partial r_j}{\partial r_{0m}}\frac{\partial r_k}{\partial r_{0n}}. \tag{14.79}$$

The continuity equation in the Lagrangian frame is, therefore,

$$\boxed{\rho_{\mathrm{L}}(t,\boldsymbol{r}_0) = \frac{\rho_0(\boldsymbol{r}_0)}{J(t,\boldsymbol{r}_0)}}. \tag{14.80}$$

14.13.2. *Pressure*

The result (14.80), together with (14.77), makes it really easy to work out the Lagrangian pressure. Indeed, the adiabatic law (14.60) implies that the entropy density of Lagrangian fluid elements never changes:

$$\frac{\partial}{\partial t}\frac{p_{\mathrm{L}}}{\rho_{\mathrm{L}}^\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{p}{\rho^\gamma} = 0 \quad \Rightarrow \quad \frac{p_{\mathrm{L}}}{\rho_{\mathrm{L}}^\gamma} = \frac{p_0}{\rho_0^\gamma}. \tag{14.81}$$

Using (14.80), we get

$$\boxed{p_{\mathrm{L}}(t,\boldsymbol{r}_0) = \frac{p_0(\boldsymbol{r}_0)}{J^\gamma(t,\boldsymbol{r}_0)}}. \tag{14.82}$$

14.13.3. *Magnetic Field*

To work out the magnetic field, consider the induction equation in the form (14.32). In Lagrangian variables,

$$\frac{\partial}{\partial t}\frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}} = \frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}}\cdot\boldsymbol{\nabla}\boldsymbol{u} = \frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}}\cdot(\boldsymbol{\nabla}\boldsymbol{r}_0)\cdot\boldsymbol{\nabla}_0\boldsymbol{u}_{\mathrm{L}} = \frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}}\cdot(\boldsymbol{\nabla}\boldsymbol{r}_0)\cdot\boldsymbol{\nabla}_0\frac{\partial\boldsymbol{r}}{\partial t}. \tag{14.83}$$

The solution to this that satisfies $\boldsymbol{B}_{\mathrm{L}}/\rho_{\mathrm{L}} = \boldsymbol{B}_0/\rho_0$ at $t=0$ is

$$\frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}} = \frac{\boldsymbol{B}_0}{\rho_0}\cdot\boldsymbol{\nabla}_0\boldsymbol{r}. \tag{14.84}$$

Indeed:

$$\frac{\partial}{\partial t}\frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}} = \frac{\boldsymbol{B}_0}{\rho_0}\cdot\boldsymbol{\nabla}_0\frac{\partial\boldsymbol{r}}{\partial t} = \frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}}\cdot(\boldsymbol{\nabla}_0\boldsymbol{r})^{-1}\cdot\boldsymbol{\nabla}_0\frac{\partial\boldsymbol{r}}{\partial t} = \frac{\boldsymbol{B}_{\mathrm{L}}}{\rho_{\mathrm{L}}}\cdot(\boldsymbol{\nabla}\boldsymbol{r}_0)\cdot\boldsymbol{\nabla}_0\frac{\partial\boldsymbol{r}}{\partial t}, \quad \text{q.e.d.} \tag{14.85}$$

Thus, the magnetic field in the Lagrangian frame satisfies (14.84), or, using (14.80),

$$\boxed{\boldsymbol{B}_{\mathrm{L}}(t,\boldsymbol{r}_0) = \frac{\boldsymbol{B}_0(\boldsymbol{r}_0)\cdot\boldsymbol{\nabla}_0\boldsymbol{r}(t,\boldsymbol{r}_0)}{J(t,\boldsymbol{r}_0)}}. \tag{14.86}$$

This solution is really just a restatement of the Lundquist theorem: (14.84) says that $\boldsymbol{B}/\rho$ transforms in the same way as a vector connecting two infinitesimally close material points in the fluid [see (14.33)].

14.13.4. *Fluid Flow*

Finally, let us deal with the momentum equation (14.58), which in this context is the equation that actually defines the Lagrangian variable transformation (14.74). Using the decomposition (14.20) of the Lorentz force into magnetic pressure and tension, and substituting for ρ, p and $\boldsymbol{B}$ from (14.80), (14.82) and (14.86), respectively, we get, noting that $(\boldsymbol{\nabla}_0\boldsymbol{r})\cdot\boldsymbol{\nabla} = \boldsymbol{\nabla}_0$,

$$\boxed{\frac{\rho_0}{J}\frac{\partial^2\boldsymbol{r}}{\partial t^2} = -(\boldsymbol{\nabla}_0\boldsymbol{r})^{-1}\cdot\boldsymbol{\nabla}_0\left(\frac{p_0}{J^\gamma} + \frac{|\boldsymbol{B}_0\cdot\boldsymbol{\nabla}_0\boldsymbol{r}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\frac{\boldsymbol{B}_0}{J}\cdot\boldsymbol{\nabla}_0\frac{\boldsymbol{B}_0}{J}\cdot\boldsymbol{\nabla}_0\boldsymbol{r}}. \tag{14.87}$$

Coupled with the initial condition $\boldsymbol{r}(0) = \boldsymbol{r}_0$ and the formula (14.79) for J, this equation determines the trajectory $\boldsymbol{r}(t,\boldsymbol{r}_0)$ of each fluid element and hence its Lagrangian displacement $\boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{r}_0$, its velocity $\boldsymbol{u}_{\mathrm{L}} = \partial\boldsymbol{\xi}/\partial t$, and the associated density, pressure and magnetic field via (14.80), (14.82) and (14.86). It is not a particularly pretty equation—the price we have paid for being able to integrate explicitly all the other MHD equations.

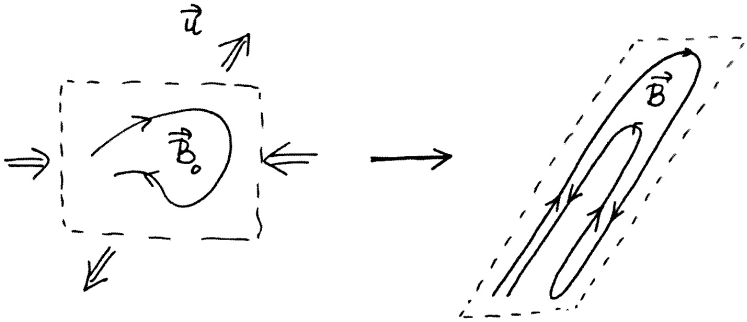


FIGURE 68. Effect of stretching/shearing/compression on magnetic field: "folded" field with small-scale direction reversals.

Exercise 14.2. Energy in Lagrangian MHD. Show that, in Lagrangian variables, the total energy of a volume of MHD fluid is

$$\mathscr{E} = \int \mathrm{d}\boldsymbol{r}_0 \left[\frac{1}{2}\rho_0 \left(\frac{\partial \boldsymbol{r}}{\partial t}\right)^2 + \frac{p_0 J^{-(\gamma-1)}}{\gamma - 1} + \frac{|\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}_0 \boldsymbol{r}|^2}{8\pi J}\right]. \tag{14.88}$$

Exercise 14.3. Action principle for Lagrangian MHD. Show that (14.87) can be derived from an action principle, $\delta S = 0$, with

$$S = \int_0^t \mathrm{d}t \int \mathrm{d}\boldsymbol{r}_0 \, \mathscr{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}), \tag{14.89}$$

where the Lagrangian density is exactly what you would have guessed from (14.88):

$$\mathscr{L}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{\rho_0 |\dot{\boldsymbol{r}}|^2}{2} - \frac{p_0 J^{-(\gamma-1)}}{\gamma - 1} - \frac{|\boldsymbol{B}_0 \cdot \boldsymbol{\nabla}_0 \boldsymbol{r}|^2}{8\pi J} \tag{14.90}$$

(cf. Kulsrud 2005, §4.7).

The Lagrangian formalism, besides shedding conceptual light, turns out to give one some useful analytical tools, e.g., for the treatment of explosive MHD instabilities (Pfirsch & Sudan 1993; Cowley & Artun 1997).

An obvious problem with using this approach to describe the MHD fluid over any significant intervals of time is that it only works for ideal MHD. Even if we restrict ourselves to nice flows that do not have small scales and are thus immune to viscosity, (14.86) tells us that magnetic field can quickly develop small scales and thus access resistivity. An example of how that could happen starting with a fairly generic magnetic configuration is in Q-5. An even simpler example is an application of a simple, linear stretching/shearing/compressing transformation (which can be achieved by an arbitrarily large-scale flow) to a curved field line, producing a "folded" field with direction reversals on ever smaller scales (Fig. 68). It is clear that resistivity must kick in, at which point the Lagrangian solution (14.86) can no longer be used. I shall offer two examples (§§14.14 and 15.1) of how one might deal with such a situation mathematically and in the process get an idea of what happens physically.

14.14. *Eyink's Stochastic Lundquist Theorem*

Coming soon...

15. Small-Scale Dynamo

One day, I shall add here a general introduction to small-scale dynamo. For now, you will find a (somewhat outdated) review in my handwritten 2007 Les Houches lectures available here: Schekochihin (2007*b*). A very short printed review is Schekochihin & Cowley (2007, §3). One of the calculations from those notes and this review is reproduced in §15.1. An up-to-date treatment focussing on the nonlinear stages of the dynamo and on the yet-unresolved issues is Schekochihin (2022, §13).

15.1. *Zeldovich's Dynamo in a Linear Flow*

Let us consider the evolution of magnetic field in a linear flow, i.e.,

$$\boldsymbol{u}(t,\boldsymbol{r}) = \mathsf{A}(t)\cdot\boldsymbol{r}, \tag{15.1}$$

where $\mathsf{A}(t)$ is some matrix that may be a function of time, but not of space. The key simplification that comes with this assumption (admittedly at a heavy price!) is that the strain matrix $\boldsymbol{\nabla}_0\boldsymbol{r}$ is independent of position: indeed, it is the solution of

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}_0\boldsymbol{r} = \boldsymbol{\nabla}_0\boldsymbol{u}_{\rm L} = (\boldsymbol{\nabla}_0\boldsymbol{r})\cdot\mathsf{A}^T(t), \quad \boldsymbol{\nabla}_0\boldsymbol{r}(t=0) = \mathsf{I}. \tag{15.2}$$

Let us further assume, for simplicity, that the flow is incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{u} = \mathrm{tr}\,\mathsf{A} = 0$). Then $\rho = \rho_0 = \mathrm{const}$ and so $J = 1$.

15.1.1. *"Lagrangian" Solution of Induction Equation with Resistivity*

The (Lagrangian) magnetic field satisfies the induction equation with resistivity:

$$\frac{\partial \boldsymbol{B}_{\rm L}}{\partial t} = \underbrace{\boldsymbol{B}_{\rm L}\cdot\boldsymbol{\nabla}\boldsymbol{u}}_{=\mathsf{A}\cdot\boldsymbol{B}_{\rm L}} + \eta\nabla^2\boldsymbol{B}_{\rm L}, \tag{15.3}$$

where the gradients in the right-hand side are still Eulerian. Now let us seek a solution of this equation in the form

$$\boldsymbol{B}_{\rm L}(t,\boldsymbol{r}_0) = \hat{\boldsymbol{B}}(t)e^{i\boldsymbol{k}(t)\cdot\boldsymbol{r}(t,\boldsymbol{r}_0)}. \tag{15.4}$$

At $t=0$, this is simply a single-Fourier-mode initial field: $\boldsymbol{B}_0(\boldsymbol{r}_0) = \hat{\boldsymbol{B}}_0 e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}_0}$, where $\boldsymbol{k}_0 = \boldsymbol{k}(0)$ and $\hat{\boldsymbol{B}}_0 = \hat{\boldsymbol{B}}(0)$. Let us see if such functions $\hat{\boldsymbol{B}}(t)$ and $\boldsymbol{k}(t)$ can be found that (15.4) works. Substituting (15.4) into (15.3), one gets

$$\frac{\partial\hat{\boldsymbol{B}}}{\partial t} + i\left(\frac{\partial\boldsymbol{k}}{\partial t}\cdot\boldsymbol{r} + \boldsymbol{k}\cdot\frac{\partial\boldsymbol{r}}{\partial t}\right)\hat{\boldsymbol{B}} = \mathsf{A}\cdot\hat{\boldsymbol{B}} - \eta k^2(t)\hat{\boldsymbol{B}}. \tag{15.5}$$

Since $\dot{\boldsymbol{r}} = \boldsymbol{u} = \mathsf{A}\cdot\boldsymbol{r}$, the second term on the left-hand side is annihilated if

$$\frac{\partial\boldsymbol{k}}{\partial t} = -\boldsymbol{k}\cdot\mathsf{A} \quad\Rightarrow\quad \boldsymbol{k}(t) = (\boldsymbol{\nabla}\boldsymbol{r}_0)(t)\cdot\boldsymbol{k}_0. \tag{15.6}$$

The last expression follows from (15.2) if one works out the time derivative of $\boldsymbol{\nabla}\boldsymbol{r}_0$ by differentiating $(\boldsymbol{\nabla}\boldsymbol{r}_0)\cdot(\boldsymbol{\nabla}_0\boldsymbol{r}) = \mathsf{I}$. We are left with

$$\frac{\partial\hat{\boldsymbol{B}}}{\partial t} = \mathsf{A}\cdot\hat{\boldsymbol{B}} - \eta k^2(t)\hat{\boldsymbol{B}} \quad\Rightarrow\quad \boxed{\hat{\boldsymbol{B}}(t) = \hat{\boldsymbol{B}}_0\cdot(\boldsymbol{\nabla}_0\boldsymbol{r})(t)\exp\left[-\eta\int_0^t \mathrm{d}t' k^2(t')\right]}. \tag{15.7}$$

The solution has been obtained by eliminating the resistive term via an integrating factor and taking care of the rest by using (15.2). The Lagrangian solution (14.86) has re-emerged as a prefactor, now attenuated by resistive decay.

Since the induction equation is linear in $\boldsymbol{B}$, any linear combination of solutions of the form (15.4) is also a solution. Therefore, we can accommodate any initial field $\boldsymbol{B}_0$: it will evolve according to

$$\begin{aligned}\boldsymbol{B}_{\rm L}(t,\boldsymbol{r}_0) &= \sum_{\boldsymbol{k}_0}\hat{\boldsymbol{B}}(t,\boldsymbol{k}_0)e^{i\boldsymbol{k}(t,\boldsymbol{k}_0)\cdot\boldsymbol{r}(t,\boldsymbol{r}_0)}\\ &= \sum_{\boldsymbol{k}_0}\hat{\boldsymbol{B}}_0(\boldsymbol{k}_0)\cdot(\boldsymbol{\nabla}_0\boldsymbol{r})(t)\exp\left[i\boldsymbol{k}(t,\boldsymbol{k}_0)\cdot\boldsymbol{r}(t,\boldsymbol{r}_0) - \eta\int_0^t \mathrm{d}t' k^2(t',\boldsymbol{k}_0)\right],\end{aligned} \tag{15.8}$$

where $\hat{\boldsymbol{B}}_0(\boldsymbol{k}_0)$ is the Fourier coefficient of the initial field $\boldsymbol{B}_0(\boldsymbol{r}_0) = \sum_{\boldsymbol{k}_0}\hat{\boldsymbol{B}}(\boldsymbol{k}_0)e^{i\boldsymbol{k}_0\cdot\boldsymbol{r}_0}$ and $\boldsymbol{k}(t,\boldsymbol{k}_0)$ satisfies (15.6).

15.1.2. *Is There Dynamo?*

An interesting question now is whether the energy of the solution (15.8) grows with time—is the velocity field (15.1) a dynamo? It is not hard to prove a version of the Parseval theorem: the volume average of the magnetic energy is

$$\langle B^2\rangle(t) \equiv \int \frac{\mathrm{d}\boldsymbol{r}}{V}|\boldsymbol{B}(t,\boldsymbol{r})|^2 = \sum_{\boldsymbol{k}_0}|\hat{\boldsymbol{B}}(t,\boldsymbol{k}_0)|^2. \tag{15.9}$$

Exercise 15.1. Prove (15.9).

Using (15.7),

$$|\hat{\boldsymbol{B}}(t,\boldsymbol{k}_0)|^2 = \hat{\boldsymbol{B}}_0(\boldsymbol{k}_0)\cdot\mathbf{g}(t)\cdot\hat{\boldsymbol{B}}_0^*(\boldsymbol{k}_0)\exp\left[-2\eta\int_0^t \mathrm{d}t'\boldsymbol{k}_0\cdot\mathbf{g}^{-1}(t')\cdot\boldsymbol{k}_0\right], \tag{15.10}$$

where $\mathbf{g}$ is the (covariant) metric tensor associated with the Largangian transformation of variables:

$$g_{ij} = \frac{\partial r_k}{\partial r_{0i}}\frac{\partial r_k}{\partial r_{0j}}, \qquad (g^{-1})_{ij} = \frac{\partial r_{0i}}{\partial r_k}\frac{\partial r_{0j}}{\partial r_k}. \tag{15.11}$$

These matrices can (in principle) be calculated via (15.2) for any given $\mathsf{A}(t)$.

It is possible to do this with a degree of generality, but a lot of formalism is needed along the way. I will instead opt for an extremely simple case:

$$\mathsf{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \quad\Rightarrow\quad \mathbf{g} = \begin{pmatrix} e^{2\lambda_1 t} & & 0 \\ & e^{2\lambda_2 t} & \\ 0 & & e^{2\lambda_3 t} \end{pmatrix}, \tag{15.12}$$

where $\lambda_1 > \lambda_2 > 0 > \lambda_3$ are constants and $\lambda_1+\lambda_2+\lambda_3 = 0$ (incompressibility). These three rates are rates of stretching and compression by the flow. Ignoring exponentially small terms, we get

$$|\hat{\boldsymbol{B}}(t,\boldsymbol{k}_0)|^2 \approx |\hat{B}_{01}|^2\exp\left[2\lambda_1 t - \eta\left(\frac{k_{01}^2}{\lambda_1}+\frac{k_{02}^2}{\lambda_2}+\frac{k_{03}^2}{|\lambda_3|}e^{2|\lambda_3|t}\right)\right]. \tag{15.13}$$

This says that for most $\boldsymbol{k}_0$, the corresponding modes will decay superexponentially (after initial transient growth at the "ideal" rate $2\lambda_1$). At any given time $t \gg \lambda_1^{-1}$, the domain in the $\boldsymbol{k}_0$ space that will dominate the integral (15.9) is one containing modes for which the resistivity has not yet managed to cut the initial growth rate:

$$\eta\left(\frac{k_{01}^2}{\lambda_1}+\frac{k_{02}^2}{\lambda_2}+\frac{k_{03}^2}{|\lambda_3|}e^{2|\lambda_3|t}\right) \lesssim \text{const}. \tag{15.14}$$

This is the interior of an ellipsoid whose volume is $\propto e^{-|\lambda_3|t}$. Within this volume, $|\hat{\boldsymbol{B}}(t,\boldsymbol{k}_0)|^2 \sim |\hat{B}_{01}|^2e^{2\lambda_1 t}$. Therefore, the integral (15.9) is, roughly,

$$\boxed{\langle B^2\rangle(t) \propto e^{(2\lambda_1-|\lambda_3|)t} = e^{(\lambda_1-\lambda_2)t}}, \tag{15.15}$$

because $\lambda_3 = -\lambda_1-\lambda_2$. Since $\lambda_1 > \lambda_2$, the conclusion is that magnetic energy will grow, albeit thanks to an ever shrinking subset of initial modes.

This calculation is due to Zeldovich *et al.* (1984) (but was cleverly anticipated 20 years previously in a one-page note by Moffatt & Saffman 1964). The physical interpretation of it was some time in coming—let me explain it the way I understand it (Schekochihin *et al.* 2004).

15.1.3. *Folded Fields*

The three rates λ_1, λ_2 and λ_3 represent the flow's action along the three (Lyapunov) directions locally associated with it: stretching at the rate λ_1 along the first direction ($\hat{\boldsymbol{e}}_1$), compression at the rate $|\lambda_3|$ along the third ($\hat{\boldsymbol{e}}_3$) and something along the second, "null" direction ($\hat{\boldsymbol{e}}_2$)—it can indeed be null ($\lambda_2 = 0$), but it can also be sretching or compression at a smaller rate than the other two. It follows from (15.7) and (15.6) that the magnetic field will align with the stretching

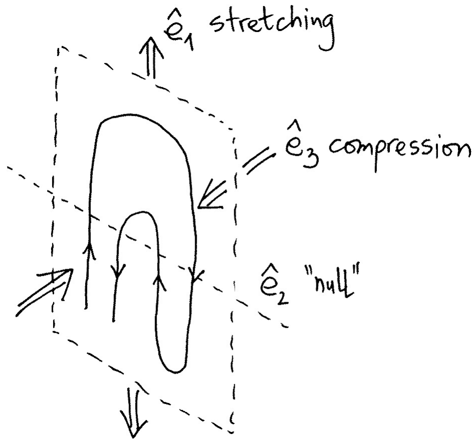


FIGURE 69. A "winning" fold in Zeldovich-type dynamo.

direction whereas is wave vector will align with the compression direction, both exponentially fast:

$$\boldsymbol{B} \sim B_{01}\hat{\boldsymbol{e}}_1 e^{\lambda_1 t}, \qquad \boldsymbol{k} \sim k_{03}\hat{\boldsymbol{e}}_3 e^{|\lambda_3| t}. \tag{15.16}$$

The latter alignment is what makes most modes decay superexponentially—in physical terms, this is the tendency, illustrated by Fig. 68, for the fields to become folded and reverse direction at ever smaller scales. The only modes that survive are those for which the initial wave vector $\boldsymbol{k}_0$ was nearly perpendicular to $\hat{\boldsymbol{e}}_3$, with its permitted angular deviation from 90° decaying $\sim e^{-|\lambda_3|t}$. Since $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, $\boldsymbol{k}_0 \cdot \hat{\boldsymbol{B}}_0 = 0$ must be satisfied for the initial field. Thus, the modes that get amplified most are ones for which $\hat{\boldsymbol{B}}_0 \parallel \hat{\boldsymbol{e}}_1$ and $\boldsymbol{k}_0 \parallel \hat{\boldsymbol{e}}_2$. Such a "winning" fold is sketched in Fig. 69.

Exercise 15.2. What happens, mathematically and physically, if (a) $\lambda_2 = 0$, (b) $\lambda_2 < 0$?

Exercise 15.3. Work through an analogous argument in 2D and show that magnetic field always decays (no 2D dynamo; cf. Q-4). Does this make sense physically?

15.1.4. *Further Reading*

Considering that the above calculation was done for a preposterously simplistic flow, is it at all useful in understanding real dynamos? It turns out to be surprisingly so. One does see folded fields in generic fluctuation dynamos when they are driven by smooth velocity fields, even random ones (Schekochihin *et al.* 2004; Rincon 2019). Locally such fields can be viewed as linear (15.1), but perhaps not constant time. This is OK as the above construction can be generalised to time-dependent fields. Indeed, since the matrix $\mathsf{g}(t)$ is symmetric, it can at any time be diagonalised by an appropriate rotation:

$$\mathsf{g}(t) = \mathsf{R}^T(t)\cdot\begin{pmatrix} e^{\Lambda_1(t)} & & 0 \\ & e^{\Lambda_2(t)} & \\ 0 & & e^{\Lambda_3(t)} \end{pmatrix}\cdot\mathsf{R}(t), \tag{15.17}$$

where the quantities $\Lambda_i(t)/2t$ are known as the *finite-time Lyapunov exponents (FTLEs)*. It is possible to prove (Goldhirsch *et al.* 1987) that $\mathsf{R}(t)$ converges to a constant matrix $(\hat{\boldsymbol{e}}_1\ \hat{\boldsymbol{e}}_2\ \hat{\boldsymbol{e}}_3)$ (*Lyapunov basis*) exponentially fast in time and that (more slowly) $\Lambda_i(t)/2t \to \lambda_i$ (*Lyapunov exponents*).

When the flow is random, the FTLEs are random functions and then one can prove that (15.15) survives in the form

$$\overline{\langle B^2\rangle}(t) \propto \overline{e^{[\Lambda_1(t)-\Lambda_2(t)]/2}}, \tag{15.18}$$

where the overline means averaging over the distribution of the Λ_i's. This distribution is usually quite hard to calculate for any real flow, so it has only been done for a few very special examples.

If you are intrigued by this line of inquiry, you might find further enlightenment in Zeldovich *et al.* (1984) and then, e.g., in Ott (1998) and Chertkov *et al.* (1999).

15.2. *Kazantsev Spectrum of Dynamo-Generated Field*

Coming soon. In the meanwhile, you may read about this topic in my hand-written notes available here: `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec23_Spectrum.pdf`.

---

16. MHD in a Straight Magnetic Field

Equations (14.57–14.60) have a very simple static, uniform *equilibrium solution:*

$$\rho_0 = \text{const}, \quad p_0 = \text{const}, \quad \boldsymbol{u}_0 = 0, \quad \boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}} = \text{const}. \tag{16.1}$$

We will turn to more nontrivial equilibria in due course, but first we shall study this one carefully—because it is very generic in the sense that many other, more complicated, equilibria *locally* look just like this.

16.1. *MHD Waves*

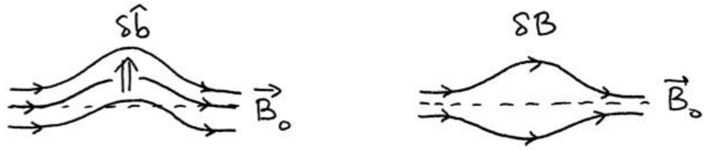
If you have an equilibrium solution of any set of equations, your first reflex ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly interesting nonlinear behaviour of small perturbations (e.g., §§6–8).

So we now seek solutions to the MHD equations (14.57–14.60) in the form of

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \boldsymbol{u} = \frac{\partial\boldsymbol{\xi}}{\partial t}, \quad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}, \tag{16.2}$$

where we have introduced the fluid *displacement* field $\boldsymbol{\xi}$ (cf. §14.13).[123] To start with, we consider all perturbations to be infinitesimal and so linearise the MHD equations

[123]Thinking in terms of displacements makes sense in MHD but not so much in (homogeneous) hydrodynamics because in the latter case, just displacing a fluid element produces no back reaction, whereas in MHD, since magnetic fields are frozen into the fluid and are elastic, displacing fluid elements causes magnetic restoring forces to switch on. In other words, an (ideal) MHD fluid "remembers" the state from which it has been displaced, whereas neutral (Newtonian) fluids only "know" about velocities at which they flow.



(a) Perpendicular perturbations, $\delta\boldsymbol{b} = \delta\boldsymbol{B}_\perp/B_0$ (b) Parallel perturbations, $\delta B_\parallel = \delta B$

FIGURE 70. Magnetic perturbations.

(14.57–14.60) as follows.

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad\Rightarrow\quad \frac{\partial\delta\rho}{\partial t} = -\rho_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}$$
$$\Rightarrow\quad \boxed{\frac{\delta\rho}{\rho_0} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}}\,, \tag{16.3}$$
$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\frac{p}{\rho^\gamma} = 0 \quad\Rightarrow\quad \frac{\partial}{\partial t}\frac{\delta p}{p_0} = \gamma\,\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0}$$
$$\Rightarrow\quad \boxed{\frac{\delta p}{p_0} = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi}}\,, \tag{16.4}$$
$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad\Rightarrow\quad \frac{\partial\delta\boldsymbol{B}}{\partial t} = B_0\nabla_\parallel\frac{\partial\boldsymbol{\xi}}{\partial t} - \hat{\boldsymbol{z}}B_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}$$
$$\Rightarrow\quad \frac{\delta\boldsymbol{B}}{B_0} = \nabla_\parallel\boldsymbol{\xi} - \hat{\boldsymbol{z}}\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \nabla_\parallel\boldsymbol{\xi}_\perp - \hat{\boldsymbol{z}}\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp$$
$$\Rightarrow\quad \boxed{\frac{\delta\boldsymbol{B}_\perp}{B_0} = \nabla_\parallel\boldsymbol{\xi}_\perp}\,,\quad \boxed{\frac{\delta B_\parallel}{B_0} = -\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp}\,, \tag{16.5}$$

where $\parallel$ and $\perp$ denote projections onto the direction (z) of $\boldsymbol{B}_0$ and onto the plane (x, y) perpendicular to it, respectively. Equations (16.5) tell us that parallel displacements produce no perturbation of the magnetic field—obviously not, because the magnetic field is carried with the fluid flow and nothing will happen if you displace a straight uniform field parallel to itself.

The physics of magnetic-field perturbations becomes clearer if we observe that

$$\frac{\delta\boldsymbol{B}}{B_0} = \frac{\delta(B\boldsymbol{b})}{B_0} = \delta\boldsymbol{b} + \hat{\boldsymbol{z}}\,\frac{\delta B}{B_0}. \tag{16.6}$$

The perturbed field-direction vector $\delta\boldsymbol{b}$ must be perpendicular to $\hat{\boldsymbol{z}}$ (otherwise the field direction is unperturbed; formally this is shown by perturbing the equation $\boldsymbol{b}^2 = 1$). Therefore, the perpendicular and parallel perturbations of the magnetic field are the perturbations of its direction and strength, respectively (Fig. 70):

$$\frac{\delta\boldsymbol{B}_\perp}{B_0} = \delta\boldsymbol{b},\qquad \frac{\delta B_\parallel}{B_0} = \frac{\delta B}{B_0}. \tag{16.7}$$

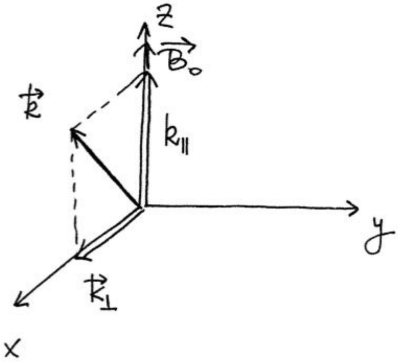


FIGURE 71. Coordinate system for the treatment of MHD waves.

Finally, linearising (14.58) gives us

$$\underbrace{\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)}_{=\rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}} = \underbrace{-\boldsymbol{\nabla}p}_{\substack{=-\boldsymbol{\nabla}\delta p\\ =\gamma p_0\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\xi},\\ \text{from (16.4)}}} \underbrace{\underbrace{-\boldsymbol{\nabla}\frac{B^2}{8\pi}}_{=-\frac{B_0^2}{4\pi}\boldsymbol{\nabla}\frac{\delta B}{B_0}} + \underbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}}_{=\frac{B_0^2}{4\pi}\nabla_\parallel\left(\delta\boldsymbol{b}+\hat{\boldsymbol{z}}\frac{\delta B}{B_0}\right)}}_{\substack{=\frac{B_0^2}{4\pi}\left(-\boldsymbol{\nabla}_\perp\frac{\delta B}{B_0}+\nabla_\parallel\delta\boldsymbol{b}\right)\\ =\frac{B_0^2}{4\pi}\left(\boldsymbol{\nabla}_\perp\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp+\nabla_\parallel^2\boldsymbol{\xi}_\perp\right),\\ \text{from (16.7) and (16.5)}}}. \tag{16.8}$$

Assembling all this, we get

$$\boxed{\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = c_{\rm s}^2\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + v_{\rm A}^2\left(\boldsymbol{\nabla}_\perp\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp + \nabla_\parallel^2\boldsymbol{\xi}_\perp\right)}\,, \tag{16.9}$$

where two special velocities have emerged:

$$c_{\rm s} = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad v_{\rm A} = \frac{B_0}{\sqrt{4\pi\rho_0}}, \tag{16.10}$$

the *sound speed* and the *Alfvén speed*, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

Exercise 16.1. Derive (16.3–16.5) and (16.9) directly from Lagrangian MHD equations (14.80), (14.82), (14.86) and (14.87). A useful starting point is that, for an infinitesimal displacement, $J = 1 + \boldsymbol{\nabla}_0\cdot\boldsymbol{\xi}$ [follows from (14.79)].

Let us seek wave-like solutions of (16.9), $\boldsymbol{\xi} \propto \exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{r})$. For such perturbations,

$$\omega^2\boldsymbol{\xi} = c_{\rm s}^2\boldsymbol{k}\boldsymbol{k}\cdot\boldsymbol{\xi} + v_{\rm A}^2\left(\boldsymbol{k}_\perp\boldsymbol{k}_\perp\cdot\boldsymbol{\xi}_\perp + k_\parallel^2\boldsymbol{\xi}_\perp\right). \tag{16.11}$$

Without loss of generality, let $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ (i.e., by definition, x is the direction of $\boldsymbol{k}_\perp$;

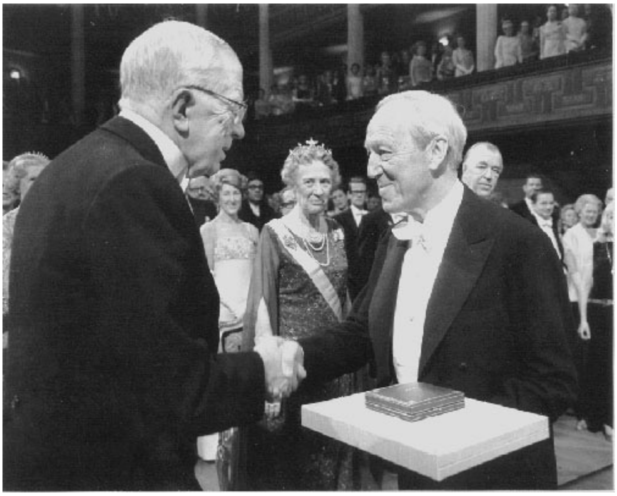


FIGURE 72. Hannes Olof Gösta Alfvén (1908-1995), Swedish electrical engineer and plasma physicist. He was the father of MHD, distrusted religion, computers and Big Bang theory, and got a Nobel Prize "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics" (1970). In this picture, he is receiving it from King Gustaf VI Adolf of Sweden.

see Fig. 71). Then (16.11) becomes

$$\omega^2\xi_x = c_s^2 k_\perp(k_\perp\xi_x + k_\parallel\xi_\parallel) + v_A^2 k^2\xi_x, \tag{16.12}$$

$$\omega^2\xi_y = v_A^2 k_\parallel^2\xi_y, \tag{16.13}$$

$$\omega^2\xi_\parallel = c_s^2 k_\parallel(k_\perp\xi_x + k_\parallel\xi_\parallel). \tag{16.14}$$

The perturbations of the rest of the fields are

$$\frac{\delta\rho}{\rho_0} = -i\boldsymbol{k}\cdot\boldsymbol{\xi} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel), \tag{16.15}$$

$$\frac{\delta p}{p_0} = \gamma\frac{\delta\rho}{\rho_0}, \tag{16.16}$$

$$\delta\boldsymbol{b} = ik_\parallel\boldsymbol{\xi}_\perp = ik_\parallel\begin{pmatrix}\xi_x\\ \xi_y\\ 0\end{pmatrix}, \tag{16.17}$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x. \tag{16.18}$$

16.1.1. *Alfvén Waves*

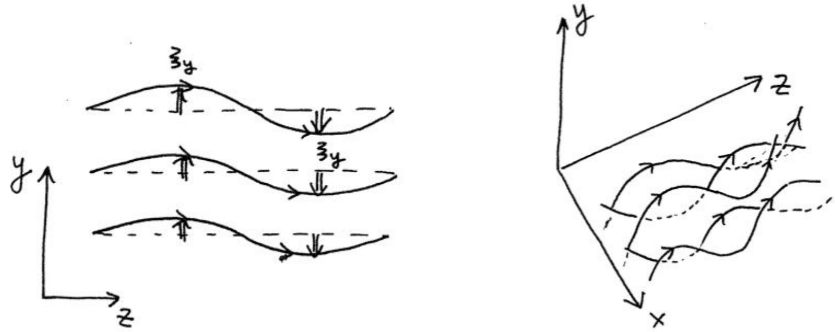
We start by spotting, instantly, that (16.13) decouples from the rest of the system. Therefore, $\boldsymbol{\xi} = (0, \xi_y, 0)$ is an eigenvector, with two associated eigenvalues

$$\boxed{\omega = \pm k_\parallel v_A}, \tag{16.19}$$

representing *Alfvén waves* propagating parallel and antiparallel to $\boldsymbol{B}_0$. An Alfvénic perturbation is (Fig. 73a)

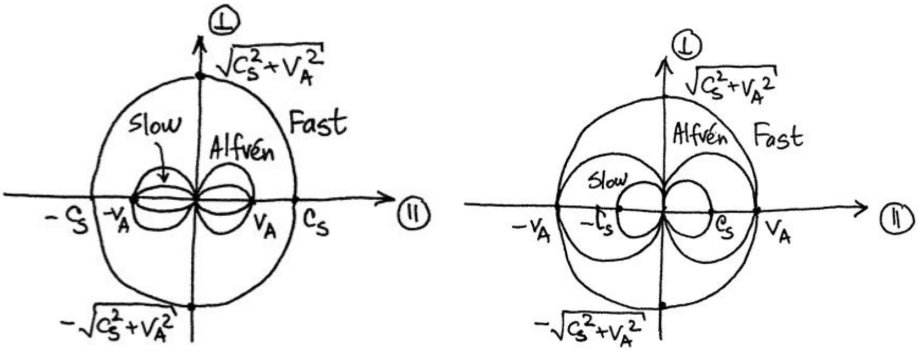
$$\boldsymbol{\xi} = \xi_y\hat{\boldsymbol{y}}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta\boldsymbol{b} = ik_\parallel\xi_y\hat{\boldsymbol{y}}, \tag{16.20}$$

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force. Note



(a) (b) $k_\perp \neq 0$

FIGURE 73. Alfvén waves.



(a) $c_s > v_A$ ("high β") (b) $c_s < v_A$ ("low β")

FIGURE 74. Friedricks diagram: radius ω/k, angle $\cos\theta = k_\parallel/k$.

that these waves can have $k_\perp \neq 0$ even though their dispersion relation (16.19) does not depend on $k_\perp$ (Fig. 73b).

16.1.2. *Magnetosonic Waves*

Equations (16.12) and (16.14) form a closed 2D system:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix} = \begin{pmatrix} c_s^2 k_\perp^2 + v_A^2 k^2 & c_s^2 k_\parallel k_\perp \\ c_s^2 k_\parallel k_\perp & c_s^2 k_\parallel^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix}. \tag{16.21}$$

The resulting dispersion relation is

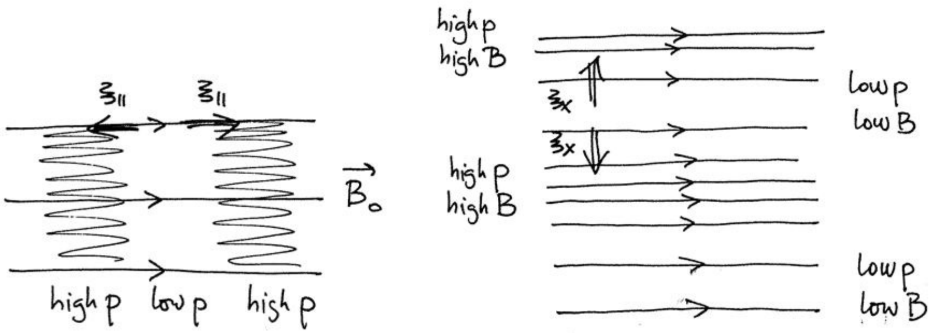
$$\omega^4 - k^2(c_s^2 + v_A^2)\omega^2 + c_s^2 v_A^2 k^2 k_\parallel^2 = 0. \tag{16.22}$$

This has four solutions:

$$\boxed{\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \sqrt{(c_s^2 + v_A^2)^2 - 4 c_s^2 v_A^2 \cos^2\theta} \right]}, \quad \cos^2\theta = \frac{k_\parallel^2}{k^2}. \tag{16.23}$$

The two "+" solutions are the *"fast magnetosonic waves"* and the two "−" ones are the *"slow magnetosonic waves"*.

Since both sound and Alfvén speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of



(a) Parallel propagation, $k_\perp = 0$ (b) Perpendicular propagation, $k_\parallel = 0$

FIGURE 75. Sound waves.

the thermal to magnetic energies in the MHD medium, known as the *plasma beta:*

$$\beta = \frac{p_0}{B_0^2/8\pi} = \frac{2}{\gamma}\frac{c_{\rm s}^2}{v_{\rm A}^2}. \tag{16.24}$$

The magnetosonic waves can be conveniently summarised by the so-called *Friedricks diagram*, a graph of (16.23) in polar coordinates where the radius is the phase speed ω/k and the angle is θ, the direction of propagation with respect to $\boldsymbol{B}_0$ (Fig. 74).

Clearly, magnetosonic waves contain perturbations of both the magnetic field and of the "hydrodynamic" quantities ρ, p, $\boldsymbol{u}$, but working them all out for the case of general oblique propagation ($\theta \sim 1$) is a bit messy. The physics of what is going on is best understood via a few particular cases.

16.1.3. *Parallel Propagation*

Consider $\boxed{k_\perp = 0}$ ($\theta = 0$). Then $(\xi_x, 0, 0)$ and $(0, 0, \xi_\parallel)$ are eigenvectors of the matrix in (16.21) and the two corresponding waves are

- another *Alfvén wave*, this time with perturbation in the x direction (which, however, is not physically different from the y direction when $k_\perp = 0$):

$$\omega^2\xi_x = k_\parallel^2 v_{\rm A}^2 \xi_x \quad \Rightarrow \quad \boxed{\omega = \pm k_\parallel v_{\rm A}}\,, \tag{16.25}$$

$$\boldsymbol{\xi} = \xi_x\hat{\boldsymbol{x}}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta\boldsymbol{b} = ik_\parallel \xi_x \hat{\boldsymbol{x}} \tag{16.26}$$

(at high β, this is the slow wave, at low β, this is the fast wave);

- the parallel-propagating *sound wave* (Fig. 75a):

$$\omega^2\xi_\parallel = k_\parallel^2 c_{\rm s}^2 \xi_\parallel \quad \Rightarrow \quad \boxed{\omega = \pm k_\parallel c_{\rm s}}\,, \tag{16.27}$$

$$\boldsymbol{\xi} = \xi_\parallel\hat{\boldsymbol{z}}, \quad \frac{\delta\rho}{\rho_0} = -ik_\parallel\xi_\parallel, \quad \frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0}, \quad \delta B = 0, \quad \delta\boldsymbol{b} = 0 \tag{16.28}$$

(at high β, this is the fast wave, at low β, this is the slow wave); the magnetic field does not participate here at all.

Note that this wave is hydrodynamically very similar to the ion-acoustic wave in unmagnetised plasma (cf. Exercise 3.6), but its ability to propagate undamped does hinge on the validity of the fluid approximation, i.e., on the plasma being collisional (except at high β and finite amplitudes, when it can "create" its own "collisionality"—see Kunz *et al.* 2020).

16.1.4. *Perpendicular Propagation*

Now consider $\boxed{k_\parallel = 0}$ ($\theta = 90^\circ$). Then $(\xi_x, 0, 0)$ is again an eigenvector of the matrix in (16.21).[124] The resulting fast magnetosonic wave is again a *sound wave*, but because it is perpendicular-propagating, both thermal and magnetic pressures get involved, the perturbations are compressions/rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfvén speeds (with the latter now representing the magnetic pressure response):

$$\omega^2 \xi_x = k_\perp^2 (c_\mathrm{s}^2 + v_\mathrm{A}^2) \xi_x \quad \Rightarrow \quad \boxed{\omega = \pm k_\perp \sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}}\,, \tag{16.29}$$

$$\boldsymbol{\xi} = \xi_x \hat{\boldsymbol{x}}, \quad \frac{\delta\rho}{\rho_0} = -ik_\perp \xi_x, \quad \frac{\delta p}{p_0} = \gamma\, \frac{\delta\rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -ik_\perp \xi_x, \quad \delta\boldsymbol{b} = 0. \tag{16.30}$$

Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field (Fig. 75b).

16.1.5. *Anisotropic Perturbations: $k_\parallel \ll k_\perp$*

Taking $k_\parallel = 0$ in §16.1.4 was perhaps a little radical as we lost all waves apart from the fast one. As we are about to see, a lot of babies were thrown out with this particular bathwater.

So let us consider MHD waves in the limit $\boxed{k_\parallel \ll k_\perp}$. This turns out to be an extremely relevant regime, because, in a strong magnetic field, realistically excitable perturbations, both linear and nonlinear, tend to be highly elongated in the direction of the field. Going back to (16.23) and enforcing this limit, we get

$$\begin{aligned}
\omega^2 &= \frac{1}{2}\, k^2 (c_\mathrm{s}^2 + v_\mathrm{A}^2) \left[1 \pm \sqrt{1 - \frac{4c_\mathrm{s}^2 v_\mathrm{A}^2}{(c_\mathrm{s}^2 + v_\mathrm{A}^2)^2} \frac{k_\parallel^2}{k^2}} \right] \\
&\approx \frac{1}{2}\, k^2 (c_\mathrm{s}^2 + v_\mathrm{A}^2) \left[1 \pm 1 \mp \frac{2c_\mathrm{s}^2 v_\mathrm{A}^2}{(c_\mathrm{s}^2 + v_\mathrm{A}^2)^2} \frac{k_\parallel^2}{k^2} \right].
\end{aligned} \tag{16.31}$$

The upper sign gives the familiar *fast wave*

$$\boxed{\omega = \pm k \sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}}\,. \tag{16.32}$$

This is just the magnetically enhanced sound wave that was considered in §16.1.4. The small corrections to it due to $k_\parallel / k$ are not particularly interesting.

The lower sign in (16.31) gives the *slow wave*

$$\boxed{\omega = \pm k_\parallel \frac{c_\mathrm{s} v_\mathrm{A}}{\sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}}}\,, \tag{16.33}$$

[124] As is $(0, 0, \xi_\parallel)$, but with $\omega = 0$; we will deal with this mode in §16.3.4.

which is more interesting. Let us find the corresponding eigenvector: from (16.14),

$$\underbrace{\left(\omega^2 - k_\parallel^2 c_{\rm s}^2\right)}_{\substack{= -k_\parallel^2 \dfrac{c_{\rm s}^4}{c_{\rm s}^2 + v_{\rm A}^2}, \\ \text{from (16.33)}}} \xi_\parallel = k_\parallel k_\perp c_{\rm s}^2 \xi_x. \tag{16.34}$$

Therefore, the displacements are mostly parallel:

$$\boxed{\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel}{k_\perp}\frac{c_{\rm s}^2}{c_{\rm s}^2 + v_{\rm A}^2} \ll 1}\,. \tag{16.35}$$

Using this equation together with (16.15–16.18), we find that the perturbations in the remaining fields are

$$\frac{\delta\rho}{\rho_0} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel) = -i\,\frac{v_{\rm A}^2}{c_{\rm s}^2 + v_{\rm A}^2}\,k_\parallel\xi_\parallel, \tag{16.36}$$

$$\frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0}, \tag{16.37}$$

$$\delta\boldsymbol{b} = ik_\parallel\xi_x\hat{\boldsymbol{x}} = -i\,\frac{k_\parallel}{k_\perp}\frac{c_{\rm s}^2}{c_{\rm s}^2 + v_{\rm A}^2}\,k_\parallel\xi_\parallel\hat{\boldsymbol{x}} \to 0, \tag{16.38}$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x = i\,\frac{c_{\rm s}^2}{c_{\rm s}^2 + v_{\rm A}^2}\,k_\parallel\xi_\parallel. \tag{16.39}$$

Thus, to lowest order in $k_\parallel/k_\perp$, this wave involves no bending of the magnetic field, but has a pressure/density perturbation and a magnetic-field-strength perturbation—the latter in counter-phase to the former (Fig. 76). To be precise, *the slow-wave perturbations are pressure balanced:*

$$\delta\left(p + \frac{B^2}{8\pi}\right) = p_0\,\frac{\delta p}{p_0} + \frac{B_0^2}{4\pi}\frac{\delta B}{B_0} = \rho_0\left(c_{\rm s}^2\,\frac{\delta\rho}{\rho_0} + v_{\rm A}^2\,\frac{\delta B}{B_0}\right) = 0. \tag{16.40}$$

The same is, of course, already obvious from the momentum equation (16.8), where, in the limit $k_\parallel \ll k_\perp$ and $\omega \ll kc_{\rm s}$ ("incompressible" perturbations; see §16.2), the dominant balance is

$$\boldsymbol{\nabla}_\perp\left(p + \frac{B^2}{8\pi}\right) = 0. \tag{16.41}$$

Finally, the Alfvén waves in the limit of anisotropic propagation are just the same as ever (§16.1.1)—they are unaffected by $k_\perp$, while being perfectly capable of having perpendicular variation (Fig. 73b).

16.1.6. *High-β Limit: $c_{\rm s} \gg v_{\rm A}$*

Another limit in which high-frequency acoustic response (fast waves) and low-frequency, pressure-balanced Alfvénic response (slow and Alfvén waves) are separated is $\beta \gg 1 \Leftrightarrow c_{\rm s} \gg v_{\rm A}$.[125] In this limit, the approxmate expression (16.31) for the magnetosonic frequencies is still valid, but because $v_{\rm A}/c_{\rm s}$, rather than $k_\parallel/k$, is small.

[125]This limit is astrophysically very interesting because magnetic fields locally produced by plasma motions in various astrophysical environments (e.g., interstellar and intergalactic media) can only be as strong energetically as the motions that make them [see (14.45)] and so, the latter being subsonic, $v_{\rm A} \sim u \ll c_{\rm s}$.

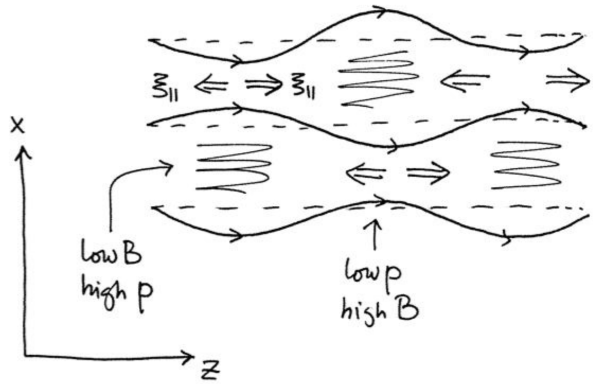


FIGURE 76. Slow wave in the anistropic limit $k_\parallel \ll k_\perp$: pressure balanced, $\xi_x \ll \xi_\parallel$.

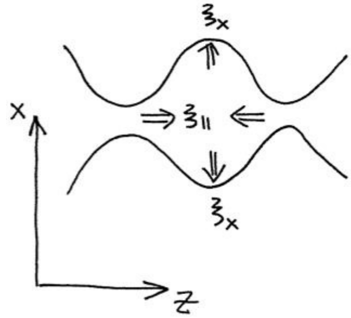


FIGURE 77. Slow wave in the high-β limit: pressure balanced, $\xi_x \sim \xi_\parallel$.

The rest of the calculations in §16.1.5 are also valid, with the following simplifications arising from v_A being negligible compared to c_s.

The upper sign in (16.31) again gives us the *fast wave*, which, this time, is a pure *sound wave*:

$$\boxed{\omega = \pm k c_s}\,. \tag{16.42}$$

This is natural because, at high β, the magnetic pressure is negligible compared to thermal pressure and sound can propagate oblivious of the magnetic field.

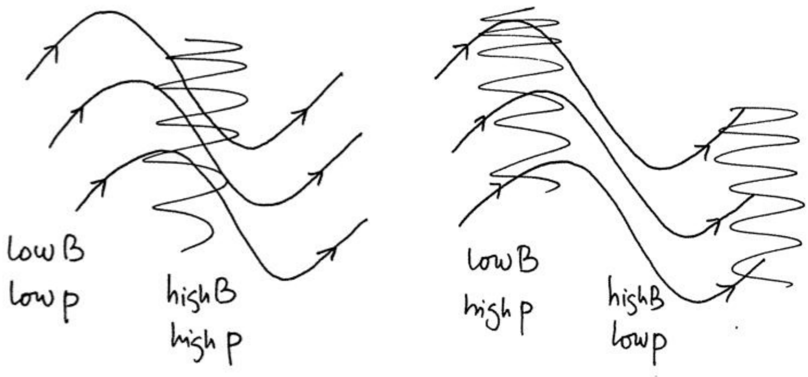
The lower sign in (16.31) yields the *slow wave*: (16.33) is still valid and becomes, for $v_A \ll c_s$,

$$\boxed{\omega = \pm k_\parallel v_A}\,. \tag{16.43}$$

Because the slow wave's dispersion relation in this limit looks exactly like the dispersion relation (16.19) of an Alfvén wave, it is called the *pseudo-Alfvén wave.* The similarity is deceptive as the nature of the perturbation (the eigenvector) is completely different. Substituting $\omega^2 = k_\parallel^2 v_A^2$ into (16.14), we find

$$k_\perp \xi_x + k_\parallel \xi_\parallel = \frac{v_A^2}{c_s^2} k_\parallel \xi_\parallel \ll k_\parallel \xi_\parallel. \tag{16.44}$$

This just says that, to lowest order in $1/\beta$, $\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0$, i.e., the perturbations are incompressible. In contrast to the anisotropic case (16.35), the perpendicular and parallel



(a) Fast waves. (b) Slow waves.

FIGURE 78. General oblique magnetosonic waves.

displacements are now comparable (assuming, in general, $k_\parallel \sim k_\perp$):

$$\boxed{\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel}{k_\perp}}. \tag{16.45}$$

Also in contrast to the anisotropic case, the density and pressure perturbations are now vanishingly small, but the field can be bent as well as compressed:

$$\frac{\delta\rho}{\rho_0} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel) = -i\,\frac{v_A^2}{c_s^2}\,k_\parallel\xi_\parallel \to 0, \tag{16.46}$$

$$\frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0} \to 0, \tag{16.47}$$

$$\delta\boldsymbol{b} = ik_\parallel\xi_x\hat{\boldsymbol{x}} = -i\,\frac{k_\parallel}{k_\perp}\,k_\parallel\xi_\parallel\hat{\boldsymbol{x}}, \tag{16.48}$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x = ik_\parallel\xi_\parallel. \tag{16.49}$$

The δB and $\delta\boldsymbol{b}$ perturbations are in counter-phase, as are $\xi_\parallel$ and ξ_x (Fig. 77). It is easy to check that pressure balance (16.40) is again maintained by these perturbations.

In the more general case of oblique propagation ($k_\parallel \sim k_\perp$) and finite beta ($\beta \sim 1$), the fast and slow magnetosonic waves generally have comparable frequencies and contain perturbations of all relevant fields, with the fast waves tending to have the perturbations of the thermal and magnetic pressure in phase and slow waves in counter-phase (Fig. 78).

16.2. *Subsonic Ordering*

Enough linear theory! We shall now occupy ourselves with the behaviour of *finite* (although still small) perturbations of a straight-field equilibrium. While we abandon linearisation (i.e., the neglect of nonlinear terms), much of what the linear theory has taught us about the basic responses of an MHD fluid remains true and useful. In particular, the linear relations between the perturbation amplitudes of various fields provide us with a guidance as to the *relative size* of finite perturbations of these fields. This makes sense if, while allowing the nonlinearities back in, we do not assume the linear physics to be completely negligible, i.e., if we allow the linear and nonlinear time scales to compete (§16.2.3). We shall see that solutions for which this is the case satisfy self-consistent equations, so can be expected to be realisable (and, as we know from experimental, observational and numerical evidence, *are* realised).

I shall start by constructing nonlinear equations that describe the incompressible limit, i.e., fields and motions that are subsonic: both their phase speeds and flow velocities will be assumed small compared to the speed of sound:

$$\frac{\omega/k}{\sqrt{c_{\rm s}^2+v_{\rm A}^2}} \ll 1, \quad \mathrm{Ma} \equiv \frac{u}{\sqrt{c_{\rm s}^2+v_{\rm A}^2}} \ll 1. \tag{16.50}$$

In this limit, we expect all fast-wave-like perturbations to disappear (in a similar way to the sound waves disappearing in the incompressible Navier–Stokes hydrodynamics) and for the MHD dynamics to contain only Alfvénic and slow-wave-like perturbations. We saw in §§16.1.5 and 16.1.6 that, linearly, fast and slow waves are well separated either in the limit of $k_\parallel/k_\perp \ll 1$ or in the limit of $\beta \gg 1$. Indeed, comparing the Alfvén frequency (16.19) and slow-wave frequency (16.33) to the sound (fast-wave) frequency (16.32), we get

$$\frac{\omega_{\text{Alfvén}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel v_{\rm A}}{k\sqrt{c_{\rm s}^2+v_{\rm A}^2}} \sim \frac{k_\parallel}{k}\frac{1}{\sqrt{1+\beta}}, \quad \frac{\omega_{\text{slow}}}{\omega_{\text{fast}}} \sim \frac{k_\parallel c_{\rm s} v_{\rm A}}{k(c_{\rm s}^2+v_{\rm A}^2)} \sim \frac{k_\parallel}{k}\frac{\sqrt{\beta}}{1+\beta}, \tag{16.51}$$

both of which are small in either of the two limits, satisfying the first of the conditions (16.50).

The second condition (16.50) involves the *"magnetic Mach number"* Ma (generalised to compare the flow velocity to the speed of sound in a magnetised fluid), which measures the size of the perturbations themselves—in the linear theory, this was arbitrarily small, but now we will need to relate it to our other small parameter(s), $k_\parallel/k$ or $1/\beta$. This means that we would like to construct an asymptotic *ordering* in which there will be some prescription as to how small, or otherwise, various (fractional) perturbations and small parameters are—not by themselves, i.e., compared to 1, but compared to each other (compared to 1, the small parameters can all formally be taken to be as small as we desire).

The general strategy for ordering perturbations with respect to each other will be to use the linear relations obtained in the two incompressible limits ($k_\parallel/k \ll 1$ or $\beta \gg 1$). If we do not specifically expect one perturbation to be larger or smaller than another on some physical grounds (like the properties of the linear response), we must order them the same; this does not stop us later from constructing subsidiary expansions in which they might be different. For example, MHD equations themselves were an expansion in a number of small parameters, in particular u/c [see (14.14)]. However, at the time of deriving them, I did not want to rule out sonic or supersonic motions and so, effectively, I ordered $\mathrm{Ma} \sim 1$, $k_\parallel/k \sim 1$ and $\beta \sim 1$, as far as the u/c expansion was concerned, i.e., Ma, $k_\parallel/k$, $1/\beta \gg u/c$. Now we are constructing a subsidiary expansion in these other parameters, keeping in mind that they are allowed to be small but not as small as the small parameter already used in the derivation of the MHD equations.[126]

[126]In principle, you should always feel a little paranoid about the question of whether such "nested" asymptotic expansions commute, i.e., whether it matters in which order they are done. They usually do commute, but this is not guaranteed and you ought to check if you want to be sure. Another formally justified mathematical worry is whether asymptotic solutions of exact equations are the same as exact solutions of asymptotic equations. This will lead you on a journey to the world of proofs of existence and uniqueness—where I wish you an enjoyable stay.

16.2.1. *Ordering of Alfvénic Perturbations*

Since the Alfvénic perturbations decouple completely from the rest (§16.1.1), linear theory does not give us a way to relate u_y to $u_\parallel$, so we shall exercise the no-prejudice principle stated above and assume

$$u_y \sim u_\parallel, \tag{16.52}$$

i.e., the Mach numbers for the Alfvénic and slow-wave-like motions are comparable. We can, however, relate u_y to $\delta\boldsymbol{b}$, via the curvature-force response (16.20):

$$|\delta\boldsymbol{b}| \sim k_\parallel \xi_y \sim \frac{k_\parallel u_y}{\omega} \sim \frac{u_y}{v_A} \sim \mathrm{Ma}\sqrt{1+\beta}. \tag{16.53}$$

16.2.2. *Ordering of Slow-Wave-Like Perturbations*

For slow-wave-like perturbations, in either the anisotropic or the high-β limit, from (16.14) and (16.33),

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} \sim \omega(k_\perp\xi_x + k_\parallel\xi_\parallel) \sim \frac{\omega^2}{k_\parallel c_\mathrm{s}^2}\,\omega\xi_\parallel \sim \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2+v_\mathrm{A}^2}\,k_\parallel u_\parallel \sim \frac{k_\parallel u_\parallel}{1+\beta}. \tag{16.54}$$

Thus, the divergence of the flow velocity is small (the dynamics are incompressible) in all three of our (potentially) small parameters:

$$\frac{\boldsymbol{\nabla}\cdot\boldsymbol{u}}{k\sqrt{c_\mathrm{s}^2+v_\mathrm{A}^2}} \sim \frac{k_\parallel}{k}\frac{1}{1+\beta}\,\mathrm{Ma}. \tag{16.55}$$

From this, we can immediately obtain an ordering for the density and pressure perturbations: using (16.3), (16.4), (16.33) and (16.54) [cf. (16.36) and (16.46)],

$$\frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \boldsymbol{\nabla}\cdot\boldsymbol{\xi} \sim \frac{\boldsymbol{\nabla}\cdot\boldsymbol{u}}{\omega} \sim \frac{\mathrm{Ma}}{\sqrt{\beta}}. \tag{16.56}$$

The magnetic-field-strength (magnetic-pressure) perturbation is, using (16.39) and (16.33) [cf. (16.49)],

$$\frac{\delta B}{B_0} \sim k_\perp\xi_x \sim \frac{c_\mathrm{s}^2}{c_\mathrm{s}^2+v_\mathrm{A}^2}\frac{k_\parallel u_\parallel}{\omega} \sim \sqrt{\beta}\,\mathrm{Ma}, \tag{16.57}$$

or, perhaps more straightforwardly, from pressure balance (16.40) and using (16.56),

$$\frac{\delta B}{B_0} = -\frac{\beta}{2}\frac{\delta p}{p_0} \sim \sqrt{\beta}\,\mathrm{Ma}. \tag{16.58}$$

Finally, in a similar fashion, using (16.17) and (16.57) [cf. (16.38) and (16.48)], we find

$$|\delta\boldsymbol{b}| \sim k_\parallel\xi_x \sim \frac{k_\parallel}{k_\perp}\sqrt{\beta}\,\mathrm{Ma} \tag{16.59}$$

for slow-wave-like perturbations. Note that in all interesting limits this is superceded by the Alfvénic ordering (16.53).

16.2.3. *Ordering of Time Scales*

Let us recall that our motivation for using linear relations between perturbations to determine their relative sizes in a nonlinear regime was that linear response will lose its exclusive sway but remain non-negligible. In formal terms, this means that we must order the linear and nonlinear time scales to be comparable.[127] The nonlinearities in

[127] In the theory of MHD turbulence, this principle, applied at each scale, is known as the *critical balance* (see §16.4).

MHD equations are advective, i.e., they are of the form $\boldsymbol{u}\cdot\boldsymbol{\nabla}$(stuff) and similar, so the rate of nonlinear interaction is $\sim ku$ (in the case of anisotropic perturbations, $\sim k_\perp u_\perp$). Ordering this to be comparable to the frequencies of the Alfvén and slow waves [see (16.51)] gives us

$$\omega_{\text{Alfvén}} \sim ku \quad\Rightarrow\quad \text{Ma} \sim \frac{k_\parallel}{k}\frac{1}{\sqrt{1+\beta}}, \tag{16.60}$$

$$\omega_{\text{slow}} \sim ku \quad\Rightarrow\quad \text{Ma} \sim \frac{k_\parallel}{k}\frac{\sqrt{\beta}}{1+\beta}. \tag{16.61}$$

Note that the first of these relations supersedes the second in all interesting limits.

16.2.4. *Summary of Subsonic Ordering*

Thus, the ordering of the time scales determines the size of the perturbations via (16.60). Using this restriction on Ma, we may summarise our subsonic ordering as follows[128]

$$\boxed{\text{Ma} \equiv \frac{u}{\sqrt{c_{\text{s}}^2+v_{\text{A}}^2}} \sim \frac{|\delta\boldsymbol{b}|}{\sqrt{1+\beta}} \sim \frac{1}{\sqrt{\beta}}\frac{\delta B}{B} \sim \sqrt{\beta}\,\frac{\delta\rho}{\rho_0} \sim \sqrt{\beta}\,\frac{\delta p}{p_0} \sim \frac{k_\parallel}{k}\frac{1}{\sqrt{1+\beta}} \ll 1} \tag{16.62}$$

and $\omega \sim ku$. The ordering can be achieved either in the limit of $k_\parallel/k \ll 1$ or $1/\beta \ll 1$, or both. Note that if one of these parameters is small, the other can be order unity or even large (as long as it is not larger than the inverse of the small one).

The case of anisotropic perturbations and arbitrary β applies in a broad range of plasmas, from magnetically confined fusion ones (tokamaks, stellarators) to space (e.g., the solar corona or the solar wind). We shall consider the implications of this ordering in §16.3.

The case of high β applies, e.g., to high-energy galactic and extragalactic plasmas. It is the direct generalisation to MHD of incompressible Navier–Stokes hydrodynamics, i.e., in this case, all one needs to do is solve MHD equations assuming $\rho = \text{const}$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$. We shall consider this case now.

16.2.5. *Incompressible MHD Equations*

Assuming $\beta \gg 1$, our ordering becomes

$$\boxed{\frac{u}{c_{\text{s}}} \sim \frac{\omega}{kc_{\text{s}}} \sim \frac{1}{\sqrt{\beta}} \sim \text{Ma}, \quad |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim \sqrt{\beta}\,\text{Ma} \sim 1, \quad \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\text{Ma}}{\sqrt{\beta}} \sim \text{Ma}^2}\,. \tag{16.63}$$

Thus, the density and pressure perturbations are minuscule, while magnetic perturbations are order unity—magnetic fields are relatively easy to bend (i.e., subsonic motions can tangle the field substantially in this regime). Because of this, it will not make sense to split $\boldsymbol{B}$ into $\boldsymbol{B}_0$ and $\delta\boldsymbol{B}$ explicitly, we will treat the magnetic field as a single field, with no need for a strong mean component.

Let us examine the MHD equations (14.57–14.60) under the ordering (16.63).

Since $\omega \sim ku$, the convective derivative $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ survives intact in all equations, allowing the advective nonlinearity to enter.

[128]Note that it is not absolutely necessary to work out the detailed linear theory of a set of equations in order to be able to construct such orderings: it is often enough to know roughly where you are going and simply balance terms representing the physics that you wish to keep (or expect to have to keep). An example of this approach is given in §16.2.8.

The continuity equation (14.57) simply reiterates our earlier statement that the velocity field is divergenceless to lowest order:

$$\nabla\cdot\boldsymbol{u} = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \sim \omega\,\frac{\delta\rho}{\rho_0} \sim \mathrm{Ma}^3 k c_\mathrm{s} \to 0. \tag{16.64}$$

The momentum equation (14.58) becomes

$$\left(1+\cancel{\frac{\delta\rho}{\rho_0}}\right)\left(\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla\underbrace{\left(\frac{c_\mathrm{s}^2}{\gamma}\frac{\delta p}{p_0}+\frac{B^2}{8\pi\rho_0}\right)}_{\equiv\,\tilde{p}}+\frac{\boldsymbol{B}\cdot\nabla\boldsymbol{B}}{4\pi\rho_0}. \tag{16.65}$$

The density perturbation in the left-hand side is $\sim \mathrm{Ma}^2$ and so negligible compared to unity. The remaining terms in this equation are all the same order ($\sim \mathrm{Ma}^2 k c_\mathrm{s}^2$) and so they must all be kept. The total "pressure" $\tilde{p}$ is determined by enforcing $\nabla\cdot\boldsymbol{u}=0$ [see (16.64)]. Namely, our equations are

$$\boxed{\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla\tilde{p}+\boldsymbol{B}\cdot\nabla\boldsymbol{B}}\,, \tag{16.66}$$

where

$$\nabla^2\tilde{p} = -\nabla\nabla : (\boldsymbol{u}\boldsymbol{u}-\boldsymbol{B}\boldsymbol{B}) \tag{16.67}$$

and the magnetic field has been rescaled to velocity units, $\boldsymbol{B}/\sqrt{4\pi\rho_0}\to\boldsymbol{B}$.

In the induction equation, best written in the form (14.27), all terms are the same order $\sim kuB \sim \mathrm{Ma}\,kc_\mathrm{s}B$ except the one containing $\nabla\cdot\boldsymbol{u}$, which is $\sim \mathrm{Ma}^3 kc_\mathrm{s}B$ and so must be neglected. We are left with

$$\boxed{\frac{\partial\boldsymbol{B}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}}\,. \tag{16.68}$$

Finally, the internal-energy equation (14.60), which, keeping only the lowest-order terms, becomes

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right)\left(\frac{\delta p}{p_0}-\gamma\,\frac{\delta\rho}{\rho_0}\right) = 0, \tag{16.69}$$

can be used to find $\delta\rho/\rho_0$, once $\delta p/p_0 = \gamma(\tilde{p}-B^2/2)/c_\mathrm{s}^2$ is calculated from the solution of (16.66–16.68). Note that $\delta\rho/\rho_0$ is merely a spectator quantity, not required to solve (16.66–16.68), which form a closed set.

Equations (16.66–16.68) are the equations of *incompressible MHD* (let us call it *iMHD*). Note that while they have been obtained in the limit of $\beta\gg 1$, all β dependence has disappeared from them—basically, they describe subsonic dynamics on top of an infinite heat bath. This is how it should be: formally, in any good asymptotic theory, it must be possible to make the small parameter artbitrarily small without changing anything in the equations.

Exercise 16.2. Show that iMHD conserves the sum of kinetic and magnetic energies,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{r}\left(\frac{u^2}{2}+\frac{B^2}{2}\right) = 0. \tag{16.70}$$

Exercise 16.3. Check that you can obtain the right waves, viz., Alfvén (§16.1.1) and pseudo-Alfvén (§16.1.6), directly from iMHD.

Exercise 16.4. Magnetoelastic waves.[129] (a) Show that the iMHD equations can be rewritten as the following closed set describing the evolution of the velocity field $\boldsymbol{u}$ and the Maxwell tensor $M_{ij} = B_i B_j$:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial r_j} = -\frac{\partial \tilde{p}}{\partial r_i} + \frac{\partial M_{ij}}{\partial r_j}, \tag{16.71}$$

$$\frac{\partial M_{ij}}{\partial t} + u_n \frac{\partial M_{ij}}{\partial r_n} = M_{nj} \frac{\partial u_i}{\partial r_n} + M_{in} \frac{\partial u_j}{\partial r_n} \tag{16.72}$$

(summation over repeated indices is implied).

(b) Imagine that there is no mean magnetic field, the MHD medium is static, and it is filled with chaotically tangled magnetic fields that are constant in time. Denote their Maxwell tensor $M_{ij}^{(0)}$. Assume that these fields have a characteristic scale that is no larger than ℓ and are statistically isotropic, so if we introduce an average (denoted by angle brackets) over scales of order ℓ, then

$$\langle M_{ij}^{(0)} \rangle = v_{\mathrm{A}}^2 \delta_{ij}, \tag{16.73}$$

where (obviously) $v_{\mathrm{A}}^2 = \langle B^2 \rangle / 3 = \mathrm{const}$. This is clearly a static ($u_i = 0$) equilibrium solution of (16.71) and (16.72). Consider infinitesimal perturbations δu_i and δM_{ij} around this equilibrium and assume that they vary in space on scales much longer than ℓ, viz.,

$$\langle u_i \rangle = 0 + \delta u_i \ll v_{\mathrm{A}}, \quad \langle M_{ij} \rangle = \langle M_{ij}^{(0)} \rangle + \delta M_{ij}, \quad \delta M_{ij} \ll v_{\mathrm{A}}^2. \tag{16.74}$$

Ignore any possible perturbations of u_i and M_{ij} on scales ℓ or smaller. Show that δu_i and δM_{ij} will describe propagating waves, derive their dispersion relation and also the relationship between δM_{ij} and the displacement vector ξ_i associated with $\delta u_i = \partial \xi_i / \partial t$. These are called *magnetoelastic waves*. Think about their physical nature, their similarities with, or differences from, Alfvén waves.

As I already intimated in §14.4, the iMHD equations written in the form of (16.71) and (16.72) are mathematically similar to the equations describing certain kinds of polymer-laden fluids. The intrinsic "elasticity" of the Maxwell stress leads to a kind of isotropic Alfvénic response that gives rise to the magnetoelastic waves. Note, however, that a significant difference between polymer chains and magnetic fields is that the latter have a sign, so there is a distinction between parallel and antiparallel fields, while polymers do not have that. Consequently, the analogy between MHD and polymer fluids becomes very imperfect if dissipation of M_{ij} is included: for polymers, there is a relaxation term in (16.72) of the form $-(M_{ij} - v_{\mathrm{A}}^2 \delta_{ij})/\tau$, describing the polymers' desire to curl up due to entropic forces; whereas in MHD, the resistive term $\eta (B_i \nabla^2 B_j + B_j \nabla^2 B_i)$ (§14.5) cannot be converted into anything that depends only on M_{ij}—indeed, it would, e.g., heavily damp antiparralel fields that reverse direction on small scales, an effect invisible to M_{ij}, where the field's sign cancels out.

16.2.6. *Elsasser MHD*

The iMHD equations possess a remarkable symmetry. Let us introduce *Elsasser (1950) fields*

$$\boldsymbol{Z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B} \tag{16.75}$$

and rewrite (16.66) and (16.68) as evolution equations for $\boldsymbol{Z}^{\pm}$: after trivial algebra,

$$\boxed{\frac{\partial \boldsymbol{Z}^{\pm}}{\partial t} + \boldsymbol{Z}^{\mp} \cdot \boldsymbol{\nabla} \boldsymbol{Z}^{\pm} = -\boldsymbol{\nabla} \tilde{p}} \tag{16.76}$$

[129] This is based on the 2019 exam question. You will find a much more sophisticated version of this calculation in Hosking *et al.* (2020), where you will learn how to deal with situations in which the dodgy assumption that perturbations of u_i and M_{ij} on scales ℓ can be ignored is actually wrong, and also when this assumption is OK.

FIGURE 79. Walter Elsasser (1904-1991), born in Germany, ended up in America, worked on MHD and, apparently, theoretical biology. Photo taken from the AIP Emilio Segrè Visual Archives, Physics Today Collection, https://repository.aip.org/islandora/object/nbla:296432.

and, since $\boldsymbol{\nabla}\cdot\boldsymbol{Z}^{\pm}=0$,

$$\nabla^2\tilde{p} = -\boldsymbol{\nabla}\boldsymbol{\nabla} : \boldsymbol{Z}^{+}\boldsymbol{Z}^{-}. \tag{16.77}$$

Thus, one can think of iMHD as representing the evolution of two incompressible "velocity fields" advecting each other.

Let us restore the separation of the magnetic field into its mean and perturbed parts, $\boldsymbol{B}=\boldsymbol{B}_0+\delta\boldsymbol{B}=v_{\rm A}\hat{\boldsymbol{z}}+\delta\boldsymbol{B}$ (recall that $\boldsymbol{B}$ is in velocity units). Then

$$\boldsymbol{Z}^{\pm} = \pm v_{\rm A}\hat{\boldsymbol{z}} + \delta\boldsymbol{Z}^{\pm} \tag{16.78}$$

and (16.76) becomes

$$\boxed{\frac{\partial\delta\boldsymbol{Z}^{\pm}}{\partial t} \mp v_{\rm A}\nabla_{\parallel}\delta\boldsymbol{Z}^{\pm} + \delta\boldsymbol{Z}^{\mp}\cdot\boldsymbol{\nabla}\delta\boldsymbol{Z}^{\pm} = -\boldsymbol{\nabla}\tilde{p}}\,. \tag{16.79}$$

Thus, $\delta\boldsymbol{Z}^{\pm}$ are finite, counter-propagating (at the Alfvén speed $v_{\rm A}$) perturbations—and they interact nonlinearly only with each other, not with themselves. If we let, say, $\delta\boldsymbol{Z}^{-}=0 \Leftrightarrow \boldsymbol{u}=\delta\boldsymbol{B}$, then $\delta\boldsymbol{Z}^{+}$ satisfies

$$\frac{\partial\delta\boldsymbol{Z}^{+}}{\partial t} - v_{\rm A}\nabla_{\parallel}\delta\boldsymbol{Z}^{+} = 0, \tag{16.80}$$

and similarly for $\delta\boldsymbol{Z}^{-}$ (propagting at $-v_{\rm A}$) if $\delta\boldsymbol{Z}^{+}=0$. Therefore,

$$\delta\boldsymbol{Z}^{\pm} = \boldsymbol{f}(\boldsymbol{r}\pm v_{\rm A}t\hat{\boldsymbol{z}}), \quad \delta\boldsymbol{Z}^{\mp}=0, \tag{16.81}$$

where $\boldsymbol{f}$ is an arbitrary function, are *exact nonlinear solutions* of iMHD. They are called *Elsasser states*. Physically, they are isolated Alfvén-wave packets that propagate along the guide field and never interact (because they all travel at the same speed and so can never catch up with or overtake one another). In order to have any interesting *nonlinear* dynamics, the system must have *counter-propagating* Alfvén-wave packets (see §16.4).

16.2.7. *Cross-Helicity*

Equations (16.76) manifestly support *two* conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}\boldsymbol{r}\,\frac{|\boldsymbol{Z}^{\pm}|^2}{2} = 0, \tag{16.82}$$

i.e., the energy of each Elsasser field is individually conserved. This can be reformulated as conservation of the total energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,\frac{1}{2}\left(\frac{|\boldsymbol{Z}^+|^2}{2}+\frac{|\boldsymbol{Z}^-|^2}{2}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,\left(\frac{u^2}{2}+\frac{B^2}{2}\right)=0, \tag{16.83}$$

and of a new quantity, known as the *cross-helicity:*

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,\frac{1}{2}\left(\frac{|\boldsymbol{Z}^+|^2}{2}-\frac{|\boldsymbol{Z}^-|^2}{2}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{u}\cdot\boldsymbol{B}=0}\,. \tag{16.84}$$

In the Elsasser formulation, the cross-helicity is a measure of energy *imbalace* between the two Elssasser fields[130]—this is observed, for example, in the solar wind, where there is significantly more energy in the Alfvénic fluctuations propagating away from the Sun than towards it (see, e.g., Wicks *et al.* 2013).

Exercise 16.5. To see why we needed incompressibility to get this new conservation law, work out the time evolution equation for $\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{u}\cdot\boldsymbol{B}$ from the general (compressible) MHD equations and hence the condition under which the cross-helicity is conserved.

16.2.8. *Stratified MHD*

It is quite instructive to consider a very simple example of non-uniform MHD equilibrium: the case of a *stratified atmosphere*. Let us introduce gravity into MHD equations, viz., the momentum equation (14.58) becomes

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=-\boldsymbol{\nabla}\left(p+\frac{B^2}{8\pi}\right)+\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}-\rho g\hat{\boldsymbol{z}} \tag{16.85}$$

(uniform gravitational acceleration pointing downward, against the z direction). We wish to consider a static equilibrium inhomogeneous in the z direction and threaded by a uniform magnetic field (which may be zero):

$$\rho_0=\rho_0(z),\quad p_0=p_0(z),\quad \boldsymbol{u}_0=0,\quad \boldsymbol{B}_0=B_0\boldsymbol{b}_0=\text{const}, \tag{16.86}$$

where $\boldsymbol{b}_0$ is at some general angle to $\hat{\boldsymbol{z}}$ and $p_0(z)$ and $\rho_0(z)$ are constrained by the vertical force balance:

$$\boxed{\frac{\mathrm{d}p_0}{\mathrm{d}z}=-\rho_0 g}\quad\Rightarrow\quad g=-\frac{p_0}{\rho_0}\frac{\mathrm{d}\ln p_0}{\mathrm{d}z}=\frac{c_{\mathrm{s}}^2}{\gamma}\frac{1}{L_p}, \tag{16.87}$$

where it has been opportune to define the pressure scale height L_p. We shall now seek time-dependent solutions of the MHD equations for which

$$\rho=\rho_0(z)+\delta\rho,\quad \frac{\delta\rho}{\rho_0}\ll 1,\quad p=p_0(z)+\delta p,\quad \frac{\delta p}{p_0}\ll 1, \tag{16.88}$$

and the spatial variation of all pertubations occurs on scales that are small compared to the pressure scale height L_p or the analogously defined density scale height $L_\rho=-(\mathrm{d}\ln\rho_0/\mathrm{d}z)^{-1}$ (for ordering purposes, we denote them both L):

$$\boxed{kL\gg 1}\,. \tag{16.89}$$

After the equilibrium pressure balance is subtracted from (16.85), this equation becomes, under any ordering in which $\delta\rho\ll\rho_0$,

$$\boxed{\rho_0\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}=-\boldsymbol{\nabla}\left(\delta p+\frac{B^2}{8\pi}\right)+\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}-\delta\rho g\hat{\boldsymbol{z}}}\,. \tag{16.90}$$

[130] Cross-helicity can also be interpreted as a topological invariant, counting the linkages between flux tubes and vortex tubes analogously to what magnetic helicity does for the flux tubes alone (see §17.2).

The last term is the *buoyancy (Archimedes) force*. In order for this new feature to give rise to any nontrivial new physics, it must be ordered comparable to all the other terms in the equation: using (16.87) to express $g \sim p_0/\rho_0 L$, we find

$$\delta\rho g \sim k\delta p \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim kL\,\frac{\delta p}{p_0} \gg \frac{\delta p}{p_0}, \tag{16.91}$$

$$\delta\rho g \sim \frac{kB^2}{4\pi} \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim \frac{kL}{\beta} \ll 1 \quad \Rightarrow \quad \beta \gg kL \gg 1. \tag{16.92}$$

So we learn that the density perturbations must now be much larger than the pressure perturbations, but, in order for the former to remain small and for the magnetic field to be in the game, β must be high (it is in anticipation of this that we did not split $\boldsymbol{B}$ into $\boldsymbol{B}_0$ and $\delta\boldsymbol{B}$, expecting them to be of the same order).

Let us now expand the internal-energy equation (14.60) in small density and pressure perturbations. Denoting $s = p/\rho^\gamma = s_0(z) + \delta s$ (entropy density) and introducing the entropy scale height

$$\frac{1}{L_s} \equiv \frac{\mathrm{d}\ln s_0}{\mathrm{d}z} = -\frac{1}{L_p} + \frac{\gamma}{L_\rho} \tag{16.93}$$

(assumed positive), we find[131]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta s}{s_0} = -\frac{u_z}{L_s}, \qquad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \gamma\,\frac{\delta\rho}{\rho_0} \approx -\gamma\,\frac{\delta\rho}{\rho_0}. \tag{16.94}$$

The last, approximate, expression follows from the smallness of pressure perturbations [see (16.91)]. This then gives us

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\rho}{\rho_0} = \frac{u_z}{\gamma L_s}}\,. \tag{16.95}$$

But, on the other hand, the continuity equation (14.57) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\rho}{\rho_0} = -\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{u_z}{L_\rho} \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = u_z\left(\frac{1}{L_\rho} - \frac{1}{\gamma L_s}\right) = \frac{u_z}{\gamma L_p} \quad \Rightarrow \quad \frac{\boldsymbol{\nabla}\cdot\boldsymbol{u}}{ku} \sim \frac{1}{kL} \ll 1. \tag{16.96}$$

Thus, the dynamics are incompressible again and the role of the continuity equation is to tell us that we must find δp from the momentum equation (16.90) by enforcing $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ to lowest order. The difference with iMHD (§16.2.5) is that $\delta\rho/\rho$ now participates in the dynamics via the buoyancy force and must be found self-consistently from (16.95).

Finally, we rewrite our newly found simplified system of equations for a stratified, high-β atmosphere, in the following neat way:

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}\tilde{p} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} + a\hat{\boldsymbol{z}}, \tag{16.97}$$

$$\nabla^2\tilde{p} = -\boldsymbol{\nabla}\boldsymbol{\nabla} : (\boldsymbol{u}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{B}) + \frac{\partial a}{\partial z}, \tag{16.98}$$

$$\frac{\partial a}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}a = -N^2 u_z, \qquad N = \frac{c_{\mathrm{s}}}{\gamma\sqrt{L_s L_p}}, \tag{16.99}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tag{16.100}$$

where we have rescaled $\boldsymbol{B}/\sqrt{4\pi\rho_0} \to \boldsymbol{B}$ and denoted the Archimedes acceleration

$$a = -\frac{\delta\rho}{\rho_0}\,g = -\frac{\delta\rho}{\rho_0}\frac{c_{\mathrm{s}}^2}{\gamma L_p}, \tag{16.101}$$

a quantity also known as *the buoyancy* of the fluid. We shall call (16.97–16.100) the equations of *stratified MHD (SMHD)*.

---

[131] We are able to take equilibrium quantities in and out of spatial derivatives because $kH \gg 1$ and the perturbations are small.

A new frequency N, known as the *Brunt–Väisälä frequency*, has appeared in our equations.[132] In order for all the linear and nonlinear time scales that are present in our equations to coexist legitimately within our ordering, we must demand that the Alfvén, Brunt–Väisälä and nonlinear time scales all be comparable:

$$kv_{\mathrm{A}} \sim N \sim ku \quad \Rightarrow \quad \boxed{\frac{1}{\sqrt{\beta}} \sim \frac{1}{kL} \sim \mathrm{Ma}}\,. \tag{16.102}$$

This gives us a relative ordering between all the small parameters that have appeared so far, including the new one, $1/kL$. Using (16.95) and recalling (16.91), let us summarise the ordering of the perturbations:

$$\frac{u}{c_{\mathrm{s}}} \sim \frac{\delta\rho}{\rho_0} \sim \mathrm{Ma}, \quad \frac{\delta p}{p_0} \sim \mathrm{Ma}^2, \quad |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim 1. \tag{16.103}$$

The difference with the iMHD high-β ordering (16.63) is that the density perturbations have now been promoted to dynamical relevance, thankfully without jeopardising incompressibility (i.e., still ordering out the sonic perturbations). The ordering (16.103) can be thought of as a generalisation to MHD of the *Boussinesq approximation* in hydrodynamics.

Further investigations of the SMHD equations are undertaken in Q-6.

16.3. *Reduced MHD*

We now turn to the anisotropic ordering, $k_\parallel/k \ll 1$ (while $\beta \sim 1$, in general), for which we studied the linear theory in §16.1.5. Specialising to this case from our general ordering (16.62), we have

$$\boxed{\mathrm{Ma} \sim \frac{u_\perp}{c_{\mathrm{s}}} \sim \frac{u_\parallel}{c_{\mathrm{s}}} \sim |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_\perp c_{\mathrm{s}}} \sim \frac{k_\parallel}{k_\perp} \ll 1}\,. \tag{16.104}$$

Starting again with the continuity equation (14.57), dividing through by ρ_0 and ordering all terms, we get

$$\underbrace{\Bigg(\underbrace{\frac{\partial}{\partial t}}_{\mathrm{Ma}} + \underbrace{\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}_\perp}_{\mathrm{Ma}} + \underbrace{\cancel{u_\parallel \nabla_\parallel}}_{\mathrm{Ma}^2}\Bigg)\underbrace{\frac{\delta\rho}{\rho_0}}_{\mathrm{Ma}}}_{\mathrm{Ma}^2} = -\underbrace{\Bigg(1 + \underbrace{\cancel{\frac{\delta\rho}{\rho_0}}}_{\mathrm{Ma}}\Bigg)\Big(\underbrace{\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}_\perp}_{\mathrm{Ma}} + \underbrace{\nabla_\parallel u_\parallel}_{\mathrm{Ma}^2}\Big)}_{\mathrm{Ma}}. \tag{16.105}$$

Thus, to lowest order, the perpendicular velocity field is 2D-incompressible:

$$\mathcal{O}(\mathrm{Ma}): \quad \boxed{\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}_\perp = 0}\,. \tag{16.106}$$

In the next order (which we will need in §16.3.2),

$$\mathcal{O}(\mathrm{Ma}^2): \quad (\boldsymbol{\nabla}\cdot\boldsymbol{u})_2 = -\left(\frac{\partial}{\partial t} + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}_\perp\right)\frac{\delta\rho}{\rho_0} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\rho}{\rho_0}, \tag{16.107}$$

where, to leading order, the convective derivative now involves only perpendicular advection.

Equation (16.106) implies that $\boldsymbol{u}_\perp$ can be written in terms of a *stream function:*

$$\boxed{\boldsymbol{u}_\perp = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Phi}\,. \tag{16.108}$$

---

[132] N is real because we assumed $L_s > 0$ (a "stably stratified atmosphere"), otherwise the atmosphere becomes convectively unstable—this happens when the equilibrium entropy decreases upwards (cf. §18.3, Q-10 and Q-6c).

Similarly, for the magnetic field, we have

$$0 = \boldsymbol{\nabla}\cdot\boldsymbol{B} = \underbrace{\boldsymbol{\nabla}_\perp\cdot\delta\boldsymbol{B}_\perp}_{\mathrm{Ma}} + \underbrace{\cancel{\nabla_\parallel\delta B_\parallel}}_{\mathrm{Ma}^2} \approx \boldsymbol{\nabla}_\perp\cdot\delta\boldsymbol{B}_\perp, \tag{16.109}$$

so $\delta\boldsymbol{B}_\perp$ is also 2D-solenoidal and can be written in terms of a *flux function:*

$$\boxed{\frac{\delta\boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\Psi}\,. \tag{16.110}$$

Note that $\Psi = -A_\parallel/\sqrt{4\pi\rho_0}$, the parallel component of the vector potential.

Thus, Alfvénically polarised perturbations, $\boldsymbol{u}_\perp$ and $\delta\boldsymbol{B}_\perp$ (see §16.1.1), can be described by two scalar functions, Φ and Ψ. Let us work out the evolution equations for them.

16.3.1. *Alfvénic Perturbations*

We start with the induction equation, again most useful in the form (14.27). Dividing through by B_0, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\boldsymbol{B}}{B_0} = \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{b}\boldsymbol{\nabla}\cdot\boldsymbol{u}. \tag{16.111}$$

Throwing out the obviously subdominant $\delta\boldsymbol{b}$ contribution in the last term on the right-hand side (i.e., approximating $\boldsymbol{b}\approx\hat{\boldsymbol{z}}$ in that term), then taking the perpendicular part of the remaining equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\boldsymbol{B}_\perp}{B_0} = \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\perp. \tag{16.112}$$

As we saw above, the convective derivative is with respect to the perpendicular velocity only and, in view of the stream-function representation (16.108) of the latter, for any function f, we have, to leading order,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}_\perp f = \frac{\partial f}{\partial t} + \hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}_\perp\Phi\times\boldsymbol{\nabla}_\perp f) = \frac{\partial f}{\partial t} + \{\Phi, f\}, \tag{16.113}$$

where the "Poisson bracket" is

$$\{\Phi, f\} = \frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial f}{\partial x}. \tag{16.114}$$

Similarly, to leading order,

$$\boldsymbol{b}\cdot\boldsymbol{\nabla}f = \frac{\partial f}{\partial z} + \delta\boldsymbol{b}\cdot\boldsymbol{\nabla}_\perp f = \frac{\partial f}{\partial z} + \frac{1}{v_\mathrm{A}}\,\hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}_\perp\Psi\times\boldsymbol{\nabla}_\perp f) = \frac{\partial f}{\partial z} + \frac{1}{v_\mathrm{A}}\{\Psi, f\}. \tag{16.115}$$

Finally, using (16.113) and (16.115) in (16.112) and expressing $\delta\boldsymbol{B}_\perp$ in terms of Ψ [see (16.110)] and $\boldsymbol{u}_\perp$ in terms of Φ [see (16.108)], it is a straightforward exercise to show, after "uncurling" (16.112), that

$$\boxed{\frac{\partial\Psi}{\partial t} + \{\Phi, \Psi\} = v_\mathrm{A}\frac{\partial\Phi}{\partial z}}\,. \tag{16.116}$$

Turning now to the momentum equation (14.58), taking its perpendicular part and dividing through by $\rho\approx\rho_0$, we get

$$\underbrace{\frac{\mathrm{d}\boldsymbol{u}_\perp}{\mathrm{d}t}}_{\mathrm{Ma}^2} = \frac{1}{\rho_0}\left[-\boldsymbol{\nabla}_\perp\left(p + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\boldsymbol{B}_\perp}{4\pi}\right] = -\underbrace{\boldsymbol{\nabla}_\perp\left(\frac{c_\mathrm{s}^2}{\gamma}\frac{\delta p}{p_0} + v_\mathrm{A}^2\frac{\delta B}{B_0}\right)}_{\mathrm{Ma}} + \underbrace{v_\mathrm{A}^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta\boldsymbol{B}_\perp}{B_0}}_{\mathrm{Ma}^2}. \tag{16.117}$$

To lowest order,

$$\mathcal{O}(\mathrm{Ma}):\quad \nabla_\perp\left(\frac{c_\mathrm{s}^2}{\gamma}\frac{\delta p}{p_0}+v_\mathrm{A}^2\frac{\delta B}{B_0}\right)=0\quad\Rightarrow\quad \boxed{\frac{\delta p}{p_0}=-\gamma\frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\frac{\delta B}{B_0}}\,. \tag{16.118}$$

This is a statement of pressure balance, which is physically what has been expected [see (16.41)] and which will be useful in §16.3.2. In the next order, (16.117) contains the perpendicular gradient of the second-order contribution to the total pressure. To avoid having to calculate it, we take the curl of (16.117) and thus obtain

$$\mathcal{O}(\mathrm{Ma}^2):\quad \nabla_\perp\times\frac{\mathrm{d}\boldsymbol{u}_\perp}{\mathrm{d}t}=v_\mathrm{A}^2\nabla_\perp\times\left(\boldsymbol{b}\cdot\nabla\frac{\delta\boldsymbol{B}_\perp}{B_0}\right). \tag{16.119}$$

Finally, using again (16.108), (16.110), (16.113) and (16.115) in (16.119), some slightly tedious algebra leads us to

$$\boxed{\frac{\partial}{\partial t}\nabla_\perp^2\Phi+\{\Phi,\nabla_\perp^2\Phi\}=v_\mathrm{A}\frac{\partial}{\partial z}\nabla_\perp^2\Psi+\{\Psi,\nabla_\perp^2\Psi\}}\,. \tag{16.120}$$

Note that $\nabla_\perp^2\Phi$ is the vorticity of the flow $\boldsymbol{u}_\perp$ and so the above equation is the MHD generalisation of the 2D Euler equation.

To summarise the equations (16.120) and (16.116) in their most compact form, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2\Phi=v_\mathrm{A}\boldsymbol{b}\cdot\nabla\nabla_\perp^2\Psi, \tag{16.121}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}=v_\mathrm{A}\frac{\partial\Phi}{\partial z}, \tag{16.122}$$

where the convective time derivative $\mathrm{d}/\mathrm{d}t$ and the parallel spatial derivative $\boldsymbol{b}\cdot\nabla$ are given by (16.113) and (16.115), respectively. Beautifully, these *nonlinear* equations describing Alfvénic perturbations have decoupled completely from everything else: we do not need to know $\delta\rho$, δp, $u_\parallel$ or δB in order to solve for $\boldsymbol{u}_\perp$ and $\delta\boldsymbol{B}_\perp$. *Alfvénic dynamics are self-contained.*

Equations (16.121) and (16.122) are called the Equations of *Reduced MHD (RMHD)*. They were originally derived in the context of tokamak plasmas (Kadomtsev & Pogutse 1974; Strauss 1976) and are extremely popular as a simple paradigm for MHD is a strong guide field—not just in tokamaks, but also in space.[133]

16.3.2. *Compressive Perturbations*

What about the rest of our fields—in the linear language, the slow-wave-like perturbations (§16.1.5)? While we do not need them to compute the Alfvénic perturbations, we might still wish to know them for their own sake.

Returning to the induction equation (16.111) and taking its z component, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_\parallel}{B_0}=\boldsymbol{b}\cdot\nabla u_\parallel-\nabla\cdot\boldsymbol{u}\quad\Rightarrow\quad\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta B}{B_0}-\frac{\delta\rho}{\rho_0}\right)=\boldsymbol{b}\cdot\nabla u_\parallel}\,, \tag{16.123}$$

where all terms are $\mathcal{O}(\mathrm{Ma}^2)$, $\delta B_\parallel\approx\delta B$ to leading order and we used (16.107) to express

[133] In the latter context, they are used most prominently as a description of Alfvénic turbulence at small scales (see §16.4), for which the RMHD equations can be shown to be the correct description even if the plasma is collisionless and in general requires kinetic treatment (Schekochihin *et al.* 2009; Kunz *et al.* 2015, 2018).

$\boldsymbol{\nabla}\cdot\boldsymbol{u}$. The derivatives $\mathrm{d}/\mathrm{d}t$ and $\boldsymbol{b}\cdot\boldsymbol{\nabla}$ contain the nonlinearities involving Φ and Ψ, which we already know from (16.121) and (16.122).

To find an equation for $u_\parallel$, we take the z component of the momentum equation (14.58):

$$\underbrace{\frac{\mathrm{d}u_\parallel}{\mathrm{d}t}}_{\mathrm{Ma}^2} = \frac{1}{\rho_0}\left[-\underbrace{\cancel{\frac{\partial}{\partial z}\left(p+\frac{B^2}{8\pi}\right)}}_{\mathrm{Ma}^3} + \underbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\delta B_\parallel}{4\pi}}_{\mathrm{Ma}^2}\right] \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}u_\parallel}{\mathrm{d}t} = v_\mathrm{A}^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta B}{B_0}}\,. \tag{16.124}$$

The parallel pressure gradient is $\mathcal{O}(\mathrm{Ma}^3)$ because there is pressure balance (16.118) to lowest order.

Finally, let us bring in the energy equation (14.60), as yet unused. To leading order, it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta s}{s_0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0 \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\rho}{\rho_0} + \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\frac{\delta B}{B_0}\right) = 0}\,, \tag{16.125}$$

where, to obtain the final version of the equation, we substituted (16.118) for $\delta p/p_0$.

Equations (16.123–16.125) are a complete set of equations for δB, $u_\parallel$ and $\delta\rho$, given Φ and Ψ. These equations are linear in the Lagrangian frame associated with the Alfvénic perturbations, provided the parallel distances are measured along perturbed field lines. Physically, they tell us that *slow waves propagate along perturbed field lines and are passively (i.e., without acting back) advected by the perpendicular Alfvénic flows.*

In what follows, when we refer to RMHD, we will mean all five equations (16.121–16.122) and (16.123–16.125).

Exercise 16.6. Check that the linear relationships between various perturbations in a slow wave derived in §16.1.5 are manifest in (16.123–16.125).

Exercise 16.7. Show that RMHD equations possess the following exact symmetry: $\forall\,\epsilon$ and a, one can simultaneously scale all perturbation amplitudes by ϵ, perpendicular distances by a, parallel distances and times by a/ϵ. This means that parallel and perpendicular distances in RMHD are effectively measured in different units. It also means that the small parameter Ma in RMHD can be made arbitrarily small, without any change in the form of the equations, so RMHD is a *bona fide* asymptotic theory (see remark at the end of §16.2.5).

Exercise 16.8. RMHD and waves in double-adiabatic plasmas.[134] (a) Recall that in plasmas where collisions are not sufficiently strong to isotropise pressure with respect to the local magnetic-field direction, the scalar pressure is replaced by the diagonal tensor (14.65). Consider a static equilibrium with a constant, uniform magnetic field $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, density ρ_0, and perpendicular and parallel pressures $p_{\perp 0} = p_{\parallel 0} = p_0$. Start from the usual MHD equations, but with $-\boldsymbol{\nabla}p$ replaced with $-\boldsymbol{\nabla}\cdot\mathsf{P}$, and adopt the RMHD ordering. Show that the reduced equations for the Alfvénic perturbations are unaffected by the introduction of the anisotropic pressure, and that the compressive perturbations still satisfy (16.123), but (16.124) is replaced by

$$\rho_0\frac{\mathrm{d}u_\parallel}{\mathrm{d}t} = -\boldsymbol{b}\cdot\boldsymbol{\nabla}\delta p_\parallel. \tag{16.126}$$

Instead of (16.125), work out the reduced version of the CGL closure (14.66).

(b) Use the linearised version of these equations to find the dispersion relation for the slow waves in a double-adiabatic plasma and to show that, in the limit of $\beta \gg 1$, these "slow waves" in fact

---

[134] This was the 2021 exam question. If do not do it now, it will come back to haunt you as Exercise 22.5.

propagate much faster than the Alfvén waves, viz.,

$$\boxed{\omega = \pm k_\parallel \sqrt{\frac{5+6/\beta}{2+2/\beta}\frac{p_0}{\rho_0}}}\,. \tag{16.127}$$

In what way is the physics of these CGL slow waves different from the physics of the slow waves in standard MHD with isotropic pressure and why, therefore, are they able to propagate faster?

Exercise 16.9. Firehose instability.[135] (a) Consider again a plasma with anisotropic pressure (14.65) in a static, uniform equilibrium with a straight magnetic field, but this time allow the equilibrium pressure, not just its perturbations, to be anisotropic: $p_{\perp 0} \neq p_{\parallel 0}$. Working in RMHD ordering, show that the Alfvénic perturbations still decouple from the compressive ones and do not depend on $\delta p_\perp$ and $\delta p_\parallel$.

(b) Derive their dispersion relation and determine the condition under which they become unstable. This is called the *firehose instability*. Can you explain the physical mechanism of this instability? What changes in the feedback to fluid displacements that makes Alfvénic perturbations in a pressure-anisotropic plasma unstable, while when $p_{\perp 0} = p_{\parallel 0}$, these perturbations behave as propagating waves?

(c) In the intergalactic medium, the typical pressure anisotropy is $|p_\perp - p_\parallel|/p_\parallel \sim 10^{-2}$ and plasma beta is $\beta \sim 10^2$ (or larger). In (certain parts of) fusion devices, $|p_\perp - p_\parallel|/p_\parallel \sim 10^{-1}$ and $\beta \sim 10^{-2}$. Which of these plasmas is likely to suffer from the firehose instability?

16.3.3. *Elsasser Fields and the Energetics of RMHD*

The Elsasser approach (§16.2.6) can be adapted to the RMHD system. Defining *Elsasser potentials*

$$\boxed{\zeta^\pm = \Phi \pm \Psi} \quad \Leftrightarrow \quad \delta\boldsymbol{Z}_\perp^\pm = \boldsymbol{u}_\perp \pm \frac{\delta\boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \zeta^\pm, \tag{16.128}$$

it is a straighforward exercise to show that the "vorticities" of the the two Elsasser fields,

$$\omega^\pm = \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_\perp \times \delta\boldsymbol{Z}_\perp^\pm) = \nabla_\perp^2 \zeta^\pm \tag{16.129}$$

(fluid vorticities $\pm$ electric currents), satisfy the following evolution equation

$$\boxed{\frac{\partial \omega^\pm}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \omega^\pm}{\partial z} + \{\zeta^\mp, \omega^\pm\} = \{\partial_j \zeta^\pm, \partial_j \zeta^\mp\}}\,, \tag{16.130}$$

where summation over the repeated index j is implied. The main corrolary of this equation is the same as in §16.2.6, although here it applies to perpendicular perturbations only: *only counter-propagting Alfvénic perturbations can interact* and *any finite-amplitude perturbation composed of just one Elsasser field is a nonlinear solution.*

Some light is perhaps shed on the nature of the interaction between Elasser fields if we notice that the left-hand side of (16.130) tells us that the Elsasser vorticity $\omega^\pm$ is propagated along the mean field at the speed v_{A} and advected across the field by the Elsasser field $\delta\boldsymbol{Z}_\perp^\mp$. The right-hand side of (16.130) is a kind of vortex-stretching term, implying a tendency for vortices and current layers to be produced in the (x, y) plane. There is a preference for current layers, as it turns out. The term in the right-hand side of (16.130) has opposite signs for the two Elsasser fields. Therefore, arguably, nonlinear dynamics favour $\omega^+\omega^- < 0$, i.e., $|\nabla_\perp^2 \Psi|^2 > |\nabla_\perp^2 \Phi|^2$ (larger currents than vorticities). This is, indeed, what is seen in numerical simulations of MHD turbulence (see Zhdankin *et al.* 2016 and §16.4).

---

[135] The 2016 exam question was based on this. While this is a useful exercise that can be done at this point in our story, I will come back to the firehose instability in §20.8, where you will find most of the answers.

The energies of the two Elsasser fields are individually conserved (cf. §16.2.7),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,|\boldsymbol{\nabla}_\perp\zeta^\pm|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,|\delta\boldsymbol{Z}_\perp^\pm|^2 = 0, \tag{16.131}$$

i.e., when the two fields do interact, they scatter each other nonlinearly, but do *not* exchange energy.

There is an Elsasser-like formulation for the slow waves as well:[136]

$$\boxed{\delta Z_\parallel^\pm = u_\parallel \pm \frac{\delta B}{\sqrt{4\pi\rho_0}}\sqrt{1+\frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}}}\,. \tag{16.132}$$

Then, from (16.123–16.125), one gets, after more algebra,

$$\frac{\partial\delta Z_\parallel^\pm}{\partial t} \mp \frac{c_\mathrm{s}v_\mathrm{A}}{\sqrt{c_\mathrm{s}^2+v_\mathrm{A}^2}}\frac{\partial\delta Z_\parallel^\pm}{\partial z} = -\frac{1}{2}\left[\left(1\mp\frac{1}{\sqrt{1+v_\mathrm{A}^2/c_\mathrm{s}^2}}\right)\{\zeta^+,\delta Z_\parallel^\pm\} + \left(1\pm\frac{1}{\sqrt{1+v_\mathrm{A}^2/c_\mathrm{s}^2}}\right)\{\zeta^-,\delta Z_\parallel^\pm\}\right]. \tag{16.133}$$

Note the (expected) appearance of the slow-wave phase speed [cf. (16.33)] in the left-hand side. Thus, slow waves interact only with Alfvénic perturbations—when $v_\mathrm{A} \ll c_\mathrm{s}$, only with the counterpropagating ones, but at finite β, because the slow waves are slower, a co-propagating Alfvénic perturbation can catch up with a slow one, have its way with it in passing and speed on (it's a tough world).

There is no energy exchange in these interactions: the "+" and "−" slow-wave energies are individually conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,|\delta Z_\parallel^\pm|^2 = 0. \tag{16.134}$$

16.3.4. *Entropy Mode*

There are only two equations in (16.133), whereas we had three equations (16.123–16.125) for our three compressive fields δB, $u_\parallel$ and $\delta\rho$. The third equation, (16.125), was in fact for the entropy perturbation:

$$\boxed{\frac{\mathrm{d}\delta s}{\mathrm{d}t} = 0}\,, \quad \frac{\delta s}{s_0} = -\gamma\left(\frac{\delta\rho}{\rho_0} + \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\frac{\delta B}{B_0}\right). \tag{16.135}$$

We see that δs is a decoupled variable, independent from $\zeta^\pm$ or $\delta Z_\parallel^\pm$ (because it is the only one that involves $\delta\rho/\rho_0$). Equation (16.135) says that δs is a *passive scalar field*, simply carried around by the Alfvénic velocity $\boldsymbol{u}_\perp$ (via $\mathrm{d}/\mathrm{d}t$). At high β, this is just a density perturbation.

The associated linear mode is not a wave: its dispersion relation is

$$\boxed{\omega = 0}\,. \tag{16.136}$$

This is the (famously often forgotten) 7th MHD mode, known as the *entropy mode* (there are 7 equations in MHD, so there must be 7 linear modes: two fast waves, two Alfvén waves, two slow waves and one entropy mode).

[136] At high β, $v_\mathrm{A} \ll c_\mathrm{s}$, so we recover from (16.132) and (16.128) the Elsasser fields as defined for iMHD in (16.75).

Exercise 16.10. Go back to §16.1 and find where this mode was overlooked.

Since the entropy mode is decoupled, its "energy" (variance) is individually conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,|\delta s|^2 = 0. \tag{16.137}$$

Thus, *in RMHD, the (nonlinear) evolution of all perturbations is constrained by 5 separate conservation laws:* $\int \mathrm{d}\boldsymbol{r}\,|\delta\boldsymbol{Z}_\perp^\pm|^2$, $\int \mathrm{d}\boldsymbol{r}\,|\delta Z_\parallel^\pm|^2$ *and* $\int \mathrm{d}\boldsymbol{r}\,|\delta s|^2$ *are all invariants.*

16.3.5. *Discussion*

Such are the simplifications allowed by anisotropy. Besides greater mathematical simplicity, what is the moral of this story, physically? Let me leave you with two observations.

• In a strong magnetic field, linear propagation is a parallel effect, whilst nonlinearity is a perpendicular effect (advection by $\boldsymbol{u}_\perp$, adjustment of propagation direction by $\delta\boldsymbol{B}_\perp$). RMHD equations express the idea that linear and nonlinear physics play equally important role—this becomes the fundamental guiding principle in the theory of MHD turbulence (§16.4). The idea is that complicated nonlinear dynamics that emerge in the perpendicular plane get teased out along the field because propagating waves enforce a degree of parallel spatial coherence. The distances over which this happens are determined by equating linear and nonlinear time scales, $k_\parallel v_\mathrm{A} \sim k_\perp u_\perp$. Dynamics cannot stay coherent over distances longer than $\sim k_\parallel^{-1}$ determined by this balance because of causality: points separated by longer parallel distances cannot exchange information quickly enough to catch up with perpendicular nonlinearities acting locally at each of these points. This principle is called *critical balance.*

• Restricting the size of perturbations to be small made the RMHD system, in a certain sense, "less nonlinear" than the full MHD (or than iMHD, where $\delta B/B_0 \sim 1$ was allowed). This led to the system's dynamics being constrained by more invariants: the MHD energy invariant got split into 5 individually conserved quadratic quantities.

Exercise 16.11. You might find it an interesting excercise to think about properties of the RMHD system in 2D, in the light of the two observations above. How many invariants are there? In what kind of physical circumstances *can* we use 2D RMHD without necessarily expecting parallel coherence of the system to break down by the causality argument?

16.4. *MHD Turbulence*

RMHD is a good starting point for developing the theory of *MHD turbulence*—a phenomenon observed with great precision in the solar wind and believed ubiquitous in the Universe. Everything that I have to say on this subject can be found in Schekochihin (2022).

Exercise 16.12. Weak RMHD turbulence. If you followed §§7.2 and 8.4, try your hand at constructing a WT theory for Alfvén waves in RMHD. A good starting point is the RMHD equations in the form (16.130) as the underlying dynamical equations for the waves. You can check your theory by consulting Appendix A of Schekochihin (2022).

17. MHD Relaxation

So far, we have only considered MHD in a straight field against the background of constant density and pressure (except in §16.2.8, where this was generalised slightly). As any more complicated (static) equilibrium will locally look like this, what we have done has considerable universal significance. Now we shall occupy ourselves with a somewhat less universal (i.e., dependent on the circumstances of a particular problem) and more "large-scale" (compared to the dynamics of wavy perturbations) question: *what kind of (static) equilibrium states are there and into which of those states will an MHD fluid normally relax?*

17.1. *Static MHD Equilibria*

Let us go back to the MHD equations (14.57–14.60) and seek static equilibria, i.e., set $\boldsymbol{u} = 0$ and $\partial/\partial t = 0$. The remaining equations are

$$\boxed{-\boldsymbol{\nabla} p + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c} = 0, \quad \boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}, \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0} \tag{17.1}$$

(the force balance, Ampère's law and the solenoidality-of-$\boldsymbol{B}$ constraint). These are 7 equations for 7 unknowns (p, $\boldsymbol{B}$, $\boldsymbol{j}$), so a complete set. Density is irrelevant because nothing moves and so inertia does not matter.

The force-balance equation has two immediate general consequences:

$$\boldsymbol{B}\cdot\boldsymbol{\nabla} p = 0, \tag{17.2}$$

so *magnetic surfaces are surfaces of constant pressure*, and

$$\boldsymbol{j}\cdot\boldsymbol{\nabla} p = 0, \tag{17.3}$$

so *currents flow along those surfaces.*

Equation (17.2) implies that if magnetic field lines are stochastic and fill the volume of the system, then $p = \text{const}$ across the system and so the force balance becomes

$$\boldsymbol{j}\times\boldsymbol{B} = 0. \tag{17.4}$$

Such equilibria are called *force-free* and turn out to be very interesting, as we shall discover soon (from §17.1.2 onwards).

17.1.1. *MHD Equilibria in Cylindrical Geometry*

As the simplest example of an inhomogeneous equilibrium, let us consider the case of axial and cylindrical symmetry:

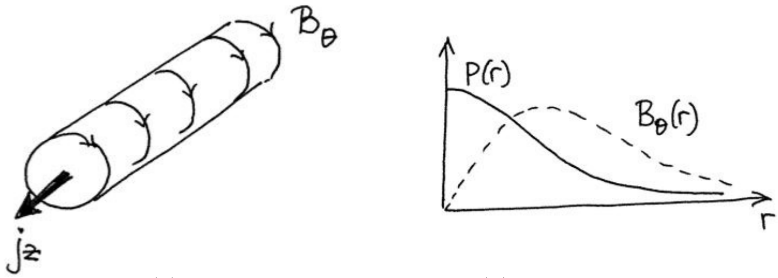
$$\frac{\partial}{\partial\theta} = 0, \quad \frac{\partial}{\partial z} = 0. \tag{17.5}$$

Solenoidality of the magnetic field then rules out it having a radial component:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r} rB_r = 0 \quad\Rightarrow\quad rB_r = \text{const} \quad\Rightarrow\quad B_r = 0. \tag{17.6}$$

Ampère's law tells us that currents do not flow radially either:

$$\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B} \quad\Rightarrow\quad \begin{cases} j_r = 0, \\[2ex] j_\theta = -\dfrac{c}{4\pi}\dfrac{\partial B_z}{\partial r}, \\[2ex] j_z = \dfrac{c}{4\pi}\dfrac{1}{r}\dfrac{\partial}{\partial r} rB_\theta. \end{cases} \tag{17.7}$$



(a) (b) Plasma is confined.

FIGURE 80. z pinch.

Finally, the radial pressure balance gives us

$$\frac{\partial p}{\partial r} = \frac{(\boldsymbol{j}\times\boldsymbol{B})_r}{c} = \frac{j_\theta B_z - j_z B_\theta}{c} = \frac{1}{4\pi}\left(-B_z\frac{\partial B_z}{\partial r} - \frac{B_\theta}{r}\frac{\partial}{\partial r}rB_\theta\right)$$
$$= -\frac{\partial}{\partial r}\frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r}\frac{B_\theta^2}{8\pi} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}}\,. \tag{17.8}$$

This simply says that the total pressure gradient is balanced by the tension force. A general equilibrium for which this is satisfied is called a *screw pinch*.

One simple particular case of this is the *z pinch* (Fig. 80a). This is achieved by letting a current flow along the z axis, giving rise to an azimuthal field:

$$j_\theta = 0, \quad j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}rB_\theta \quad \Rightarrow \quad B_\theta = \frac{4\pi}{c}\frac{1}{r}\int_0^r \mathrm{d}r'r'j_z(r'), \quad B_z = 0. \tag{17.9}$$

Equation (17.8) becomes

$$\boxed{\frac{\partial p}{\partial r} = -\frac{1}{c}j_z B_\theta}\,. \tag{17.10}$$

The "pinch" comes from magnetic loops and is due to the curvature force: the loops want to contract inwards, the pressure gradient opposes this and so *plasma is confined* (Fig. 80b). This configuration will, however, prove to be very badly unstable (§18.4)—which does not stop it from being a popular laboratory set up for short-term confinement experiments (see, e.g., review by Haines 2011).

Another simple particular case is the *θ pinch* (Fig. 81a). This is achieved by imposing a straight but radially non-uniform magnetic field in the z direction and, therefore, azimuthal currents:

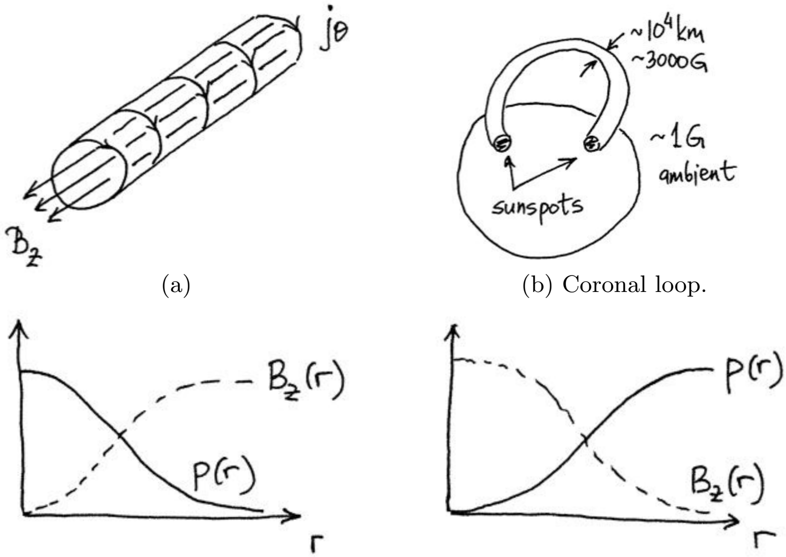
$$B_\theta = 0, \quad j_z = 0, \quad j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}. \tag{17.11}$$

Equation (17.8) is then just a pressure balance, pure and simple:

$$\boxed{\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{8\pi}\right) = 0}\,. \tag{17.12}$$

In this configuration, we can confine the plasma (Fig. 81c) or the magnetic flux (Fig. 81d). The latter is what happens, for example, in flux tubes that link sunspots (Fig. 81b). The θ pinch is a stable configuration (Q-11).

The more general case of a screw pinch (17.8) is a superposition of z and θ pinches, with both magnetic fields and currents wrapping themselves around cylindrical flux surfaces.



(a) (b) Coronal loop.

(c) Plasma is confined. (d) Magnetic field is confined.

FIGURE 81. θ pinch.

The next step in complexity is to assume axial, but not cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z \neq 0$). This is explored in Q-9.

For a much more thorough treatment of MHD equilibria, the classic textbook is Freidberg (2014).

17.1.2. *Force-Free Equilibria*

Another interesting and elegant class of equilibria arises if we consider situations in which $\boldsymbol{\nabla} p$ is negligible and can be completely omitted from the force balance. This can happen in two possible sets of circumstances:

—pressure is the same across the system, e.g., because the field lines are stochastic [a previously mentioned consequence of (17.2)];

—$\beta = p/(B^2/8\pi) \ll 1$, so thermal energy is negligible compared to magnetic energy and so p is irrelevant.

A good example of the latter situation is the *solar corona*, where $\beta \sim 1$–10^{-6} (assuming $n \sim 10^9$ cm^{-3}, $T \sim 10^2$ eV and $B \sim 1 - 10^3$ G, the lower value applying in the photosphere, the upper one in the coronal loops; see Fig. 81b)

In such situations, the equilibrium is purely magnetic, i.e., the magnetic field is *"force-free,"* which implies that the current must be parallel to the magnetic field:

$$\boxed{\boldsymbol{j}\times\boldsymbol{B} = 0} \quad \Rightarrow \quad \boldsymbol{j} \parallel \boldsymbol{B} \quad \Rightarrow \quad \frac{4\pi}{c}\boldsymbol{j} = \boldsymbol{\nabla}\times\boldsymbol{B} = \alpha(\boldsymbol{r})\boldsymbol{B}, \tag{17.13}$$

where $\alpha(\boldsymbol{r})$ is an arbitrary scalar function. Taking the divergence of the last equation tells us that

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\alpha = 0, \tag{17.14}$$

so the function $\alpha(\boldsymbol{r})$ is constant on magnetic surfaces. If $\boldsymbol{B}$ is chaotic and volume-filling, then $\alpha = \text{const}$ across the system.

The case of $\alpha = \text{const}$ is called the *linear force-free field*. In this case, taking the curl of (17.13) and then iterating it once gives us

$$-\nabla^2 \boldsymbol{B} = \alpha \boldsymbol{\nabla} \times \boldsymbol{B} = \alpha^2 \boldsymbol{B} \quad \Rightarrow \quad \boxed{\left(\nabla^2 + \alpha^2\right) \boldsymbol{B} = 0}, \tag{17.15}$$

so the magnetic field satisfies a Helmholtz equation (to solve which, one must, of course, specify some boundary conditions).

Thus, there is, potentially, a large zoo of MHD equilibria. Some of them are stable, some are not, and, therefore, some are more interesting and/or more relevant than others. How does one tell? A good question to ask is as follows. Suppose we set up some initial configuration of magnetic field (by, say, switching on some current-carrying coils, driving currents inside plasma, etc.)—*to what (stable) equilibrium will this system eventually relax?*

In general, any initially arranged magnetic configuration will exert forces on the plasma, these will drive flows, which in turn will move the magnetic fields around; eventually, everything will settle into some static equilibrium. We expect that, normally, some amount of the energy contained in the initial field will be lost in such a relaxation process because the flows will be dissipating, the fields diffusing and/or reconnecting, etc.—the losses occur due to the resistive and viscous terms in the non-ideal MHD equations derived in §14. Thus, one expects that *the final relaxed static state will be a minimum-energy state* and so we must be able to find it by minimising magnetic energy:

$$\int \mathrm{d}\boldsymbol{r}\, \frac{B^2}{8\pi} \to \min. \tag{17.16}$$

Clearly, if the relaxation occured without any constraints, the solution would just be $\boldsymbol{B} = 0$. In fact, there are constraints. These constraints are *topological:* if you think of magnetic field lines as a tangled mess, you will realise that, while you can change this tangle by moving field lines around, you cannot easily undo linkages, knots, etc.—anything that, to be undone, would require the field lines to have "ends". This intuition can be turned into a quantitative theory once we discover that the induction equation (14.59) has an invariant that involves the magnetic field only and is, in a certain sense, "better conserved" than energy.

17.2. *Helicity*

Magnetic helicity in a volume V is defined as

$$\boxed{H = \int_V \mathrm{d}\boldsymbol{r}\, \boldsymbol{A} \cdot \boldsymbol{B}}, \tag{17.17}$$

where $\boldsymbol{A}$ is the vector potential, $\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}$.

17.2.1. *Helicity Is Well-Defined*

This is not obvious because $\boldsymbol{A}$ is not unique: a gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi, \tag{17.18}$$

with χ an arbitrary scalar function, leaves $\boldsymbol{B}$ unchanged and so does not affect physics. Under this transformation, helicity stays invariant:

$$H \to H + \int_V \mathrm{d}\boldsymbol{r}\, \boldsymbol{B} \cdot \boldsymbol{\nabla}\chi = H + \int_{\partial V} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B}\, \chi = H, \tag{17.19}$$

provided $\boldsymbol{B}$ at the boundary is parallel to the boundary, i.e., provided the volume V encloses the field (nothing sticks out).

17.2.2. *Helicity Is Conserved*

Let us go back to the induction equation (14.23) (in which we retain resistivity to keep track of non-ideal effects, i.e., of the breaking of flux conservation):

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B}). \tag{17.20}$$

"Uncurling" this equation, we get

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B} + \boldsymbol{\nabla}\chi. \tag{17.21}$$

Using (17.20) and (17.21), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\boldsymbol{A}\cdot\boldsymbol{B} &= \boldsymbol{B}\cdot(\cancel{\boldsymbol{u}\times\boldsymbol{B}} - \eta\boldsymbol{\nabla}\times\boldsymbol{B} + \boldsymbol{\nabla}\chi) + \boldsymbol{A}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B} - \eta\boldsymbol{\nabla}\times\boldsymbol{B})] \\
&= -\eta\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{\nabla}\cdot(\boldsymbol{B}\chi) \\
&\quad - \boldsymbol{\nabla}\cdot[\boldsymbol{A}\times(\boldsymbol{u}\times\boldsymbol{B} - \eta\boldsymbol{\nabla}\times\boldsymbol{B})] + (\cancel{\boldsymbol{u}\times\boldsymbol{B}} - \eta\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\underbrace{(\boldsymbol{\nabla}\times\boldsymbol{A})}_{=\boldsymbol{B}} \\
&= \boldsymbol{\nabla}\cdot[\boldsymbol{B}\chi - \boldsymbol{u}\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}\cdot\boldsymbol{u} + \eta\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})] - 2\eta\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}).
\end{aligned} \tag{17.22}$$

Integrating this and using Gauss's theorem, we get

$$\begin{aligned}
\frac{\partial}{\partial t}\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{A}\cdot\boldsymbol{B} = \int_{\partial V} &\mathrm{d}\boldsymbol{S}\cdot[\boldsymbol{B}\chi - \boldsymbol{u}\boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}\cdot\boldsymbol{u} + \eta\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B})] \\
&- 2\eta\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}).
\end{aligned} \tag{17.23}$$

The surface integral vanishes provided both $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel to the boundary (no fields stick out and no flows cross). The resistive term in the surface integral can also be ignored either by arranging V appropriately or simply by taking it large enough so $\boldsymbol{B} \to 0$ on ∂V, or, indeed, by taking $\eta \to +0$. Thus,

$$\boxed{\frac{\mathrm{d}H}{\mathrm{d}t} = -2\eta\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})}\,, \tag{17.24}$$

magnetic helicity is conserved in ideal MHD.[137]

Furthermore, it turns out that even in resistive MHD, helicity is "better conserved" than energy, in the following sense. As we saw in §14.10.2, the magnetic energy evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\boldsymbol{r}\,B^2 = \left(\begin{array}{c}\text{energy exchange terms}\\ \text{and fluxes}\end{array}\right) - 2\eta\int \mathrm{d}\boldsymbol{r}\,|\boldsymbol{\nabla}\times\boldsymbol{B}|^2. \tag{17.25}$$

The first term on the right-hand side contains various fluxes and energy exchanges with the velocity field [see (14.54)], all of which eventually decay as the system relaxes (flows decay by viscosity). The second term represents Ohmic heating. If η is small but the Ohmic heating is finite, it is finite because magnetic field develops fine-scale gradients: $\nabla \sim \eta^{-1/2}$, so

$$-2\eta\int \mathrm{d}\boldsymbol{r}\,|\boldsymbol{\nabla}\times\boldsymbol{B}|^2 \to \text{const} \quad \text{as} \quad \eta \to +0. \tag{17.26}$$

[137]The resistive term in the right-hand side of (17.24) is $\propto \int \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot\boldsymbol{j}$, a quantity known as the *current helicity*.

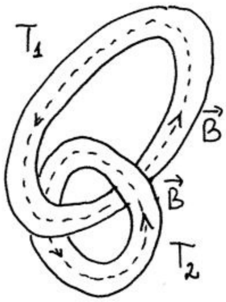


FIGURE 82. Linked flux tubes.

But then the right-hand side of (17.24) is

$$-2\eta \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mathcal{O}(\eta^{1/2}) \to 0 \quad \text{as} \quad \eta \to +0. \tag{17.27}$$

Thus, as an initial magnetic configuration relaxes, while its energy can change quickly (on dynamical times), its helicity changes only very slowly in the limit of small η. *The constancy of H (as $\eta \to +0$) provides us with the constraint subject to which the energy will need to be minimised.*

Before we use this idea, let us discuss what the conservation of helicity means physically, or, rather, topologically.

17.2.3. *Helicity Is a Topological Invariant*

Consider two linked flux tubes, T_1 and T_2 (Fig. 82). The helicity of T_1 is the product of the fluxes through T_1 and T_2:

$$\begin{aligned} H_1 &= \int_{T_1} \mathrm{d}\boldsymbol{r}\, \boldsymbol{A}\cdot\boldsymbol{B} = \int_{T_1} \underbrace{\mathrm{d}\boldsymbol{l}}_{\boldsymbol{b}\mathrm{d}l} \cdot \underbrace{\mathrm{d}\boldsymbol{S}}_{\boldsymbol{b}\mathrm{d}S}\, \underbrace{\boldsymbol{A}\cdot\boldsymbol{B}}_{\boldsymbol{A}\cdot\boldsymbol{b}B} \\ &= \int_{T_1} \boldsymbol{A}\cdot\boldsymbol{b}\mathrm{d}l\, B\boldsymbol{b}\cdot\boldsymbol{b}\mathrm{d}S = \int_{T_1} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}\, \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \Phi_1 \underbrace{\int_{T_1} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}_{\substack{= \int_{S_1} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \\ = \Phi_{\text{through hole in } T_1}}} = \Phi_1\Phi_2. \end{aligned} \tag{17.28}$$

By the same token, in general, in a system of many linked tubes, the helicity of tube i is

$$H_i = \Phi_i \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}, \tag{17.29}$$

where N_{ij} is the number of times tube j passes through the hole in tube i. The total helicity of the this entire assemblage of flux tubes is then

$$\boxed{H = \sum_{ij} \Phi_i \Phi_j N_{ij}}\,. \tag{17.30}$$

Thus, H is the number of linkages of the flux tubes weighted by the field strength in them. It is in this sense that helicity is a topological invariant.

Note that the cross-helicity $\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{u}\cdot\boldsymbol{B}$ (§16.2.7) can similarly be interpreted as counting the linkages between flux tubes ($\boldsymbol{B}$) and vortex tubes ($\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}$). The current helicity $\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{B}\cdot\boldsymbol{j}$

[appearing in the right-hand side of (17.24)] counts the number of linkages between current loops. The latter is not an MHD invariant though.

17.3. *J. B. Taylor Relaxation*

Let us now work out the equilibrium to which an MHD system will relax by *minimising magnetic energy subject to constant helicity:*

$$\delta \int_V \mathrm{d}\boldsymbol{r}\ \left(B^2 - \alpha\,\boldsymbol{A}\cdot\boldsymbol{B}\right) = 0, \tag{17.31}$$

where α is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Let us work out the two terms:

$$\begin{aligned}
\delta \int_V \mathrm{d}\boldsymbol{r}\, B^2 &= 2\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot\delta\boldsymbol{B} = 2\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\delta\boldsymbol{A})\\
&= 2\int_V \mathrm{d}\boldsymbol{r}\,\left[-\boldsymbol{\nabla}\cdot(\boldsymbol{B}\times\delta\boldsymbol{A}) + (\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\delta\boldsymbol{A}\right]\\
&= -2\int_{\partial V}\mathrm{d}\boldsymbol{S}\cdot(\boldsymbol{B}\times\delta\boldsymbol{A}) + 2\int_V \mathrm{d}\boldsymbol{r}\,(\boldsymbol{\nabla}\times\boldsymbol{B})\cdot\delta\boldsymbol{A},
\end{aligned} \tag{17.32}$$

$$\begin{aligned}
\delta H = \delta\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{A}\cdot\boldsymbol{B} &= \int_V \mathrm{d}\boldsymbol{r}\,\left(\boldsymbol{B}\cdot\delta\boldsymbol{A} + \boldsymbol{A}\cdot\delta\boldsymbol{B}\right) = \int_V \mathrm{d}\boldsymbol{r}\,\left[\boldsymbol{B}\cdot\delta\boldsymbol{A} + \boldsymbol{A}\cdot(\boldsymbol{\nabla}\times\delta\boldsymbol{A})\right]\\
&= \int_V \mathrm{d}\boldsymbol{r}\,\Big[\boldsymbol{B}\cdot\delta\boldsymbol{A} - \boldsymbol{\nabla}\cdot(\boldsymbol{A}\times\delta\boldsymbol{A}) + \underbrace{(\boldsymbol{\nabla}\times\boldsymbol{A})}_{=\,\boldsymbol{B}}\cdot\delta\boldsymbol{A}\Big]\\
&= -\int_{\partial V}\mathrm{d}\boldsymbol{S}\cdot(\boldsymbol{A}\times\delta\boldsymbol{A}) + 2\int_V \mathrm{d}\boldsymbol{r}\,\boldsymbol{B}\cdot\delta\boldsymbol{A}.
\end{aligned} \tag{17.33}$$

Now, since

$$\frac{\partial\delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = \boldsymbol{\nabla}\times\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\times\boldsymbol{B}\right) \tag{17.34}$$

for small displacements, we have $\delta\boldsymbol{A} = \boldsymbol{\xi}\times\boldsymbol{B}$, whence

$$\boldsymbol{B}\times\delta\boldsymbol{A} = B^2\boldsymbol{\xi} - \boldsymbol{B}\cdot\boldsymbol{\xi}\,\boldsymbol{B}, \tag{17.35}$$

$$\boldsymbol{A}\times\delta\boldsymbol{A} = \boldsymbol{A}\cdot\boldsymbol{B}\,\boldsymbol{\xi} - \boldsymbol{A}\cdot\boldsymbol{\xi}\,\boldsymbol{B}. \tag{17.36}$$

Therefore, the surface terms in (17.32) and (17.33) vanish if $\boldsymbol{B}$ and $\boldsymbol{\xi}$ are parallel to the boundary ∂V, i.e., if the volume V encloses both $\boldsymbol{B}$ and the plasma—there are no displacements through the boundary.

This leaves us with

$$\delta\int_V \mathrm{d}\boldsymbol{r}\ \left(B^2 - \alpha\,\boldsymbol{A}\cdot\boldsymbol{B}\right) = 2\int_V \mathrm{d}\boldsymbol{r}\ (\boldsymbol{\nabla}\times\boldsymbol{B} - \alpha\boldsymbol{B})\cdot\delta\boldsymbol{A} = 0, \tag{17.37}$$

which instantly implies that $\boldsymbol{B}$ *is a linear force-free field:*

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \alpha\boldsymbol{B} \quad\Rightarrow\quad \boxed{\nabla^2\boldsymbol{B} = -\alpha^2\boldsymbol{B}}\,. \tag{17.38}$$

Thus, our system will relax to a linear force-free state determined by (17.38) and system-specific boundary conditions. Here $\alpha = \alpha(H)$ depends on the (fixed by initial conditions) value of H via the equation

$$H(\alpha) = \int \mathrm{d}\boldsymbol{r}\,\boldsymbol{A}\cdot\boldsymbol{B} = \frac{1}{\alpha}\int \mathrm{d}\boldsymbol{r}\,B^2, \tag{17.39}$$

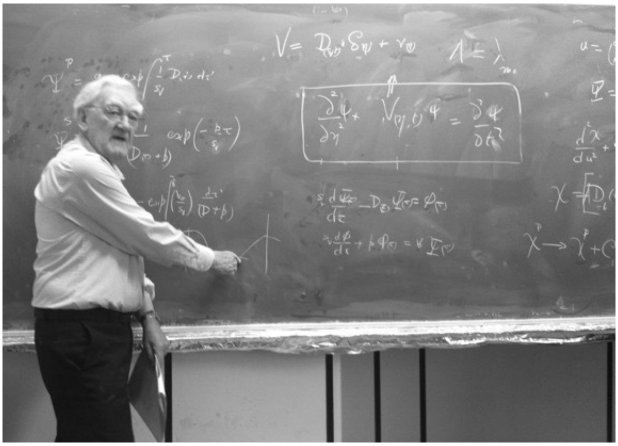


FIGURE 83. John Bryan Taylor (born 1928), one of the founding fathers of modern plasma physics, author of the Taylor relaxation (§17.3), Taylor constraint (in dynamo theory), Chirikov–Taylor map (in chaos theory), the ballooning theory (in tokamak MHD), and many other clever things, including a key role in the design of the UK's first hydrogen bomb (1957). This picture was taken in 2012 at the Wolfgang Pauli Institute in Vienna.

where $\boldsymbol{B}$ is the solution of (17.38) (since $\boldsymbol{\nabla}\times\boldsymbol{B} = \alpha\boldsymbol{B} = \alpha\boldsymbol{\nabla}\times\boldsymbol{A}$, we have $\boldsymbol{B} = \alpha\boldsymbol{A} + \boldsymbol{\nabla}\chi$ and the χ term vanishes under volume integration).

Thus, the prescription for finding force-free equilibria is

—solve (17.38), get $\boldsymbol{B} = \boldsymbol{B}(\alpha)$, parametrically dependent on α,
—calculate $H(\alpha)$ according to (17.39),
—set $H(\alpha) = H_0$, where H_0 is the initial value of helicity, hence calculate $\alpha = \alpha(H_0)$ and complete the solution by using this α in $\boldsymbol{B} = \boldsymbol{B}(\alpha)$.

Note that it is possible for this procedure to return multiple solutions. In that case, the solution with the smallest energy must be the right one (if a system relaxed to a local minimum, one can always imagine it being knocked out of it by some perturbation and falling to a lower energy).

Exercise 17.1. Force-free fields in 2D.[138] Show that for incompressible MHD confined to the 2D plane (x, y), the quantity $\int \mathrm{d}^2\boldsymbol{r}\, A_z^2$ is conserved, except for resistive dissipation and under suitable assumptions about what happens at the boundaries of the domain (this 2D invariant is sometimes called "anastrophy"). Work out the 2D version of J. B. Taylor relaxation and show that the resulting equilibrium field is a linear force-free field.

17.4. *Relaxed Force-Free State of a Cylindrical Pinch*

Let us illustrate how the procedure derived in §17.3 works by considering again the case of cylindrical and axial symmetry [see (17.5)]. The z component of (17.38) gives us the following equation for $B_z(r)$:

$$B_z'' + \frac{1}{r}B_z' + \alpha^2 B_z = 0. \qquad (17.40)$$

[138]The 2020 exam question was based on this.

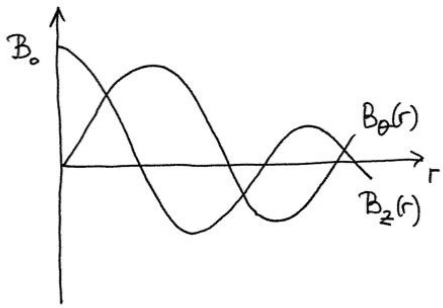


FIGURE 84. Relaxed cylindrical pinch.

This is a Bessel equation, whose solution, subject to $B_z(0) = B_0$ and $B_z(\infty) = 0$, is

$$\boxed{B_z(r) = B_0 J_0(\alpha r)}\,. \tag{17.41}$$

We can now calculate the azimuthal field as follows

$$\alpha B_\theta = (\boldsymbol{\nabla} \times \boldsymbol{B})_\theta = -B_z' \quad \Rightarrow \quad \boxed{B_\theta(r) = B_0 J_1(\alpha r)}\,. \tag{17.42}$$

This gives us an interesting twisted field (Fig. 84), able to maintain itself in equilibrium without help from pressure gradients.

Finally, we calculate its helicity according to (17.39): assuming that the length of the cylinder is L, its radius R and so its volume $V = \pi R^2 L$, we have

$$\begin{aligned} H &= \frac{1}{\alpha}\int \mathrm{d}\boldsymbol{r}\, B^2 = \frac{2\pi L B_0^2}{\alpha}\int_0^R \mathrm{d}r r\left[J_0^2(\alpha r) + J_1^2(\alpha r)\right] \\ &= \frac{B_0^2 V}{\alpha^2}\left[J_0^2(\alpha R) + 2J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R}\, J_1(\alpha R) J_2(\alpha R)\right]. \end{aligned} \tag{17.43}$$

If we solve this for $\alpha = \alpha(H)$, our solution is complete.

Exercise 17.2. Work out what happens in the general case of $\partial/\partial\theta \neq 0$ and $\partial/\partial z \neq 0$ and whether the simple symmetric solution obtained above is the correct relaxed, minimum-energy state (not always, it turns out). This is not a trivial exercise. The solution is in Taylor & Newton (2015, §9), where you will also find much more on the subject of J. B. Taylor relaxation, relaxed states and much besides—all from the original source.

There are other useful variational principles—other in the sense that the constraints that are imposed are different from helicity conservation. The need for them arises when one considers magnetic equilibria in domains that do not completely enclose the field lines, i.e., when $\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} \neq 0$ at the boundary. One example of such a variational principle, also yielding a force-free field (although not necessarily a linear one), is given in Q-1(e). A specific example of such a field arises in Q-9(f).

17.5. *Decay of MHD Turbulence*

Let me show you how one can use the idea of relaxation at constant helicity to work out, non-rigorously but rather convincingly, I think, the decay law (with time) of some arbitrary initial magnetic configuration. As I argued at the end of §17.1.2, such a configuration would normally go unstable, drive flows, and eventually rearrange itself into a force-free state—without, as we now know, changing its net helicity. Let us imagine that this is all happening in empty space (no interesting boundaries) and that initial magnetic energy is $\langle B_0^2\rangle$ (average over volume), consisting of random, tangled magnetic fields correlated on scale $\sim \ell_0$. How will the magnetic energy $\langle B^2 \rangle$ and the field correlation scale ℓ change as functions of time? This is known as the problem of the decay of MHD turbulence.

Let us do some "twiddle algebra". As per the relaxation philosophy articulated above, let us assume that the magnetic energy will decay at a rate that is independent of resistivity, to wit, comparable to the inverse dynamical time $\sim u/\ell$, where u is the typical size, and ℓ the typical scale, of the flows in the system:

$$\frac{\mathrm{d}B^2}{\mathrm{d}t} \sim -\frac{uB^2}{\ell} \sim -\frac{B^3}{\ell}, \tag{17.44}$$

where B is the typical size of the fields. The last step is reasonable if we assume that the flows are "Alfvénic", $u \sim B$ (with the understanding that B is measured in velocity units, as in §16.2.5), i.e., driven by (or in concert with) the magnetic forces. Conservation of helicity allows us to relate ℓ to B: since $A \sim B\ell$,

$$H = V\langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle \sim VB^2\ell \sim \text{const} \quad \Rightarrow \quad \ell \propto B^{-2}. \tag{17.45}$$

Using this in (17.44), we get

$$\frac{\mathrm{d}B^2}{\mathrm{d}t} \propto -B^5 \quad \Rightarrow \quad \langle B^2 \rangle \propto t^{-2/3}, \quad \ell \propto t^{2/3}. \tag{17.46}$$

Thus, we expect the energy to decay as aa power-law decay with time, whose exponent is fixed by the assumptions of conserved helicity and Alfvénicity of flows. As their energy decays, the magnetic structures will become larger and larger, also in a power-law fashion. A Taylor state will be reached only when they finally bump into some physical boundaries.

This may all look rather uncontroversial and not very difficult to grasp, but it is, in fact, the tip of a sizeable iceberg of past and current research. I have swept a number of nuances under the rug here, e.g., whether the decay and merger of magnetic structures really happens on an ideal-MHD time scale (not obviously and not always!), and also have not addressed such questions as what might happen if $H = 0$ but some local links and twists of the magnetic field still impose local topological constraints on its decay. If this topic interests you, read my review (Schekochihin 2022, §12) and/or the paper by Hosking & Schekochihin (2021), and follow the paper trail from there.

Exercise 17.3. Decay of 2D MHD turbulence. Show that in 2D, the decay laws are

$$\langle B^2 \rangle \propto t^{-1}, \qquad \ell \propto t^{1/2}. \tag{17.47}$$

17.6. *Parker's Problem and Topological MHD*

Coming soon... On topology in MHD, a very mathematically minded student might enjoy the book by Arnold & Khesin (1999).

---

18. MHD Stability and Instabilities

We now wish to take a more general view of the MHD stability problem: given some static[139] equilibrium (some ρ_0, p_0, $\boldsymbol{B}_0$ and $\boldsymbol{u}_0 = 0$), will this equilibrium be stable to small perturbations of it, i.e., will these perturbations grow or decay?

There are two ways to answer this question:

1) Carry out the normal-mode analysis, i.e., linearise the MHD equations around the given equilibrium, just as we did when we studied MHD waves in §16.1, and see if any of the frequencies (solutions of the dispersion relation) turn out to be complex, with positive imaginary parts (growth rates). This approach has the advantage of being direct

[139] A treatment of the more general case of a dynamic equilibrium, $\boldsymbol{u}_0 \neq 0$, can be found in the excellent textbook by Davidson (2016).

and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do and one might be willing to settle for less: just being able to prove that some configuration is stable or that certain types of perturbations might grow. Hence the the second approach:

2) Check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system *increasing*. If so, then the equilibrium is stable—this is called *the energy principle* and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy *decreasing*, that equilibrium is unstable. The advantage of this second approach is that we do not need to solve the (linearised) MHD equations in order to pronounce on stability, just to examine the properties of the perturbed energy functional.

It should be already quite clear how to do the normal-mode analysis, at least conceptually, so I shall focus on the second approach.

18.1. *Energy Principle*

Recall what the total energy in MHD is (§14.10)

$$\mathscr{E} = \int \mathrm{d}\boldsymbol{r} \left(\frac{\rho u^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right) \equiv \int \mathrm{d}\boldsymbol{r}\, \frac{\rho u^2}{2} + W. \tag{18.1}$$

As we saw in §16.1, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements $\boldsymbol{\xi}$—we will work this out shortly for a general equilibrium, but for now, let us accept that this will be true.[140] As $\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$ by definition of $\boldsymbol{\xi}$, we have

$$\mathscr{E} = \int \mathrm{d}\boldsymbol{r}\, \frac{1}{2}\rho_0 \left| \frac{\partial \boldsymbol{\xi}}{\partial t} \right|^2 + W_0 + \delta W_1[\boldsymbol{\xi}] + \delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] + \dots, \tag{18.2}$$

where we have kept terms up to second order in ξ and so W_0 is the equilibrium part of W (i.e., its value for $\boldsymbol{\xi} = 0$), $\delta W_1[\boldsymbol{\xi}]$ is linear in $\boldsymbol{\xi}$, $\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}]$ is bilinear (quadratic), etc. Energy must be conserved to all orders, so

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = \int \mathrm{d}\boldsymbol{r}\, \underbrace{\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}}_{\equiv \boldsymbol{F}[\boldsymbol{\xi}]} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} + \delta W_1 \left[\frac{\partial \boldsymbol{\xi}}{\partial t} \right] + \delta W_2 \left[\frac{\partial \boldsymbol{\xi}}{\partial t}, \boldsymbol{\xi} \right] + \delta W_2 \left[\boldsymbol{\xi}, \frac{\partial \boldsymbol{\xi}}{\partial t} \right] + \dots = 0. \tag{18.3}$$

This must be true at all times, including at $t = 0$, when $\boldsymbol{\xi}$ and $\partial\boldsymbol{\xi}/\partial t$ can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$,

$$\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\eta} \cdot \boldsymbol{F}[\boldsymbol{\xi}] + \delta W_1[\boldsymbol{\eta}] + \delta W_2[\boldsymbol{\eta}, \boldsymbol{\xi}] + \delta W_2[\boldsymbol{\xi}, \boldsymbol{\eta}] + \dots = 0. \tag{18.4}$$

In the first order, this tells us that

$$\delta W_1[\boldsymbol{\eta}] = 0, \tag{18.5}$$

which is good to know because it means that δW_1 disappears from (18.2) (there are no first-order energy perturbations). In the second order, we get

$$\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\eta} \cdot \boldsymbol{F}[\boldsymbol{\xi}] = -\delta W_2[\boldsymbol{\eta}, \boldsymbol{\xi}] - \delta W_2[\boldsymbol{\xi}, \boldsymbol{\eta}]. \tag{18.6}$$

[140] In fact, also the fully nonlinear dynamics can be completely expressed in terms of displacements if the MHD equations are written in Lagrangian coordinates (see §14.13).

Let $\boldsymbol{\eta} = \boldsymbol{\xi}$. Then (18.6) implies

$$\boxed{\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] = -\frac{1}{2}\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\xi}\cdot\boldsymbol{F}[\boldsymbol{\xi}]}\,. \tag{18.7}$$

This is the part of the perturbed energy in (18.2) that can be both positive and negative. The *Energy Principle* is

$$\boxed{\delta W_2[\boldsymbol{\xi}, \boldsymbol{\xi}] > 0 \text{ for any } \boldsymbol{\xi} \quad \Leftrightarrow \quad \text{equilibrium is stable}} \tag{18.8}$$

(Bernstein *et al.* 1958). Before we are in a position to prove this, we must do some preparatory work.

18.1.1. *Properties of the Force Operator*

Since the right-hand side of (18.6) is symmetric with respect to swapping $\boldsymbol{\xi} \leftrightarrow \boldsymbol{\eta}$, so must be the left-hand side:

$$\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\eta}\cdot\boldsymbol{F}[\boldsymbol{\xi}] = \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\xi}\cdot\boldsymbol{F}[\boldsymbol{\eta}]. \tag{18.9}$$

Therefore, *operator* $\boldsymbol{F}[\boldsymbol{\xi}]$ *is self-adjoint.* Since, by definition,

$$\boldsymbol{F}[\boldsymbol{\xi}] = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}, \tag{18.10}$$

the eigenmodes of this operator satisfy

$$\boldsymbol{\xi}(t, \boldsymbol{r}) = \boldsymbol{\xi}_n(\boldsymbol{r}) e^{-i\omega_n t} \quad \Rightarrow \quad \boldsymbol{F}[\boldsymbol{\xi}_n] = -\rho_0 \omega_n^2 \boldsymbol{\xi}_n. \tag{18.11}$$

As always for self-adjoint operators, we can prove a number of useful statements.

1) The eigenvalues $\{\omega_n^2\}$ *are real.*

Proof. If (18.11) holds, so must

$$\boldsymbol{F}[\boldsymbol{\xi}_n^*] = -\rho_0 (\omega_n^2)^* \boldsymbol{\xi}_n^*, \tag{18.12}$$

provided $\boldsymbol{F}$ has no complex coefficients (we shall confirm this explicitly in §18.2.1). Taking the full scalar products (including integarting over space) of (18.11) with $\boldsymbol{\xi}_n^*$ and of (18.12) with $\boldsymbol{\xi}_n$ and subtracting one from the other, we get

$$-\left[\omega_n^2 - (\omega_n^2)^*\right] \underbrace{\int \mathrm{d}\boldsymbol{r}\, \rho_0 |\boldsymbol{\xi}_n|^2}_{>0} = \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\xi}_n^* \cdot \boldsymbol{F}[\boldsymbol{\xi}_n] - \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{\xi}_n \cdot \boldsymbol{F}[\boldsymbol{\xi}_n^*] = 0$$

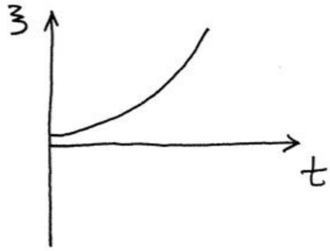
$$\Rightarrow \quad \boxed{\omega_n^2 = (\omega_n^2)^*}\,, \quad \text{q.e.d.} \tag{18.13}$$

This result implies that, if any MHD equilibrium is *unstable*, at least one of the eigenvalues must be $\omega_n^2 < 0$ and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 85a), rather than growing oscillations (also known as "overstabilities"; see Fig. 85b).

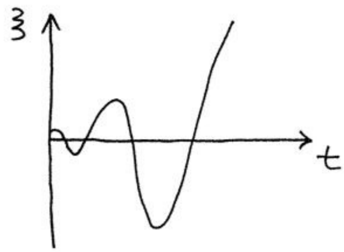
2) The eigenmodes $\{\boldsymbol{\xi}_n\}$ *are orthogonal.*

Proof. Taking the full scalar products of (18.11) with $\boldsymbol{\xi}_m$ (assuming $m \neq n$ and non-degeneracy of $\omega_{m,n}^2$), and of the analogous equation

$$\boldsymbol{F}[\boldsymbol{\xi}_m] = -\rho_0 \omega_m^2 \boldsymbol{\xi}_m \tag{18.14}$$



(a) Instability.



(b) "Overstability" (does not happen in MHD).

FIGURE 85. MHD instabilities.

with $\boldsymbol{\xi}_n$ and subtracting them, we get[141]

$$-\underbrace{(\omega_n^2-\omega_m^2)}_{\neq 0}\int \mathrm{d}\boldsymbol{r}\,\rho_0\boldsymbol{\xi}_n\cdot\boldsymbol{\xi}_m=\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{\xi}_m\cdot\boldsymbol{F}[\boldsymbol{\xi}_n]-\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{\xi}_n\cdot\boldsymbol{F}[\boldsymbol{\xi}_m]=0$$

$$\Rightarrow\quad \boxed{\int \mathrm{d}\boldsymbol{r}\,\rho_0\boldsymbol{\xi}_n\cdot\boldsymbol{\xi}_m=\delta_{nm}\int \mathrm{d}\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}_n|^2},\quad \text{q.e.d.}\qquad (18.15)$$

18.1.2. *Proof of the Energy Principle*

Let us assume completeness of the set of eigenmodes $\{\boldsymbol{\xi}_n\}$ (not, in fact, an indispensable assumption, but we shall not worry about this nuance here; see Kulsrud 2005, §7.2). Then any displacement at any given time t can be decomposed as

$$\boldsymbol{\xi}(t,\boldsymbol{r})=\sum_n a_n(t)\boldsymbol{\xi}_n(\boldsymbol{r}).\qquad (18.16)$$

The energy perturbation (18.7) is

$$\begin{aligned}\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]&=-\frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{\xi}\cdot\boldsymbol{F}[\boldsymbol{\xi}]=-\frac{1}{2}\sum_{nm}a_na_m\int \mathrm{d}\boldsymbol{r}\,\boldsymbol{\xi}_n\cdot\boldsymbol{F}[\boldsymbol{\xi}_m]\\&=\frac{1}{2}\sum_{nm}a_na_m\omega_m^2\underbrace{\int \mathrm{d}\boldsymbol{r}\,\rho_0\boldsymbol{\xi}_n\cdot\boldsymbol{\xi}_m}_{\text{use (18.15)}}=\frac{1}{2}\sum_n a_n^2\omega_n^2\int \mathrm{d}\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}_n|^2.\end{aligned}\qquad (18.17)$$

By the same token,

$$K[\boldsymbol{\xi},\boldsymbol{\xi}]\equiv\frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}|^2=\frac{1}{2}\sum_n a_n^2\int \mathrm{d}\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}_n|^2.\qquad (18.18)$$

Then, if we arrange $\omega_1^2\leqslant\omega_2^2\leqslant\dots$, the smallest eigenvalue is

$$\omega_1^2=\min_{\boldsymbol{\xi}}\frac{\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]}{K[\boldsymbol{\xi},\boldsymbol{\xi}]}.\qquad (18.19)$$

Therefore,

• condition (18.8) is *sufficient* for stability because, if $\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]>0$ for all possible $\boldsymbol{\xi}$, then the smallest eigenvalue $\omega_1^2>0$, and so all eigenvalues are positive, $\omega_n^2\geqslant\omega_1^2>0$;

• condition (18.8) is *necessary* for stability because, if the equilibrium is stable, then all eigenvalues are positive, $\omega_n^2>0$, whence $\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]>0$ in view of (18.17), q.e.d.

[141] Note that in view of (18.13), we can take $\{\boldsymbol{\xi}_n\}$ to be real.

18.2. *Explicit Calculation of δW_2*

Now that we know that we need the sign of δW_2 to ascertain stability (or otherwise), it is worth working out δW_2 as an explicit function of $\boldsymbol{\xi}$. It is a second-order quantity, but (18.7) tells us that all we need to calculate is $\boldsymbol{F}[\boldsymbol{\xi}]$ to first order in $\boldsymbol{\xi}$, i.e., we just need to linearise the MHD equations around an arbitrary static equilibrium. The procedure is the same as in §16.1, but without assuming ρ_0, p_0 and $\boldsymbol{B}_0$ to be spatially homogeneous.

18.2.1. *Linearised MHD Equations*

Thus, generalising somewhat the procedure adopted in (16.3–16.5), we have

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0 \quad\Rightarrow\quad \frac{\partial\delta\rho}{\partial t} = -\boldsymbol{\nabla}\cdot\left(\rho_0\frac{\partial\boldsymbol{\xi}}{\partial t}\right)$$

$$\Rightarrow\quad \boxed{\delta\rho = -\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi})}\,, \tag{18.20}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)p = -\gamma p\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad\Rightarrow\quad \frac{\partial\delta p}{\partial t} = -\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\boldsymbol{\nabla}p_0 - \gamma p_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}$$

$$\Rightarrow\quad \boxed{\delta p = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0 - \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}}\,, \tag{18.21}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) \quad\Rightarrow\quad \frac{\partial\delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\times\boldsymbol{B}_0\right)$$

$$\Rightarrow\quad \boxed{\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)}\,. \tag{18.22}$$

Note that again $\delta\rho$, δp and $\delta\boldsymbol{B}$ are all expressed as linear operators on $\boldsymbol{\xi}$—and so $\delta W = \delta\int d\boldsymbol{r}\,\left[B^2/8\pi + p/(\gamma-1)\right]$ must also be some operator involving $\boldsymbol{\xi}$ and its gradients but not $\partial\boldsymbol{\xi}/\partial t$ (as we assumed in §18.1).

Finally, we deal with the momentum equation (to which we add gravity as this will give some interesting instabilities):

$$\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) = -\boldsymbol{\nabla}p + \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} + \rho\boldsymbol{g}. \tag{18.23}$$

This gives us

$$\begin{aligned}\boldsymbol{F}[\boldsymbol{\xi}] = \rho_0\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} &= -\boldsymbol{\nabla}\delta p + \frac{(\boldsymbol{\nabla}\times\boldsymbol{B}_0)\times\delta\boldsymbol{B}}{4\pi} + \frac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\times\boldsymbol{B}_0}{4\pi} + \delta\rho\,\boldsymbol{g}\\ &= \boldsymbol{\nabla}\left(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0 + \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right) - \boldsymbol{g}\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}) + \frac{\boldsymbol{j}_0\times\delta\boldsymbol{B}}{c} + \frac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\times\boldsymbol{B}_0}{4\pi},\end{aligned} \tag{18.24}$$

where $\boldsymbol{j}_0 = c(\boldsymbol{\nabla}\times\boldsymbol{B}_0)/4\pi$, we have used (18.20) and (18.21) for $\delta\rho$ and δp, respectively, and $\delta\boldsymbol{B}$ is given by (18.22).

18.2.2. *Energy Perturbation*

Now we can use (18.24) in (18.7) to calculate explicitly

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\left[\underbrace{-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0 + \gamma p_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right)}_{\substack{=(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)\,\boldsymbol{\nabla}\cdot\boldsymbol{\xi}+\gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2\\ \text{after integration by parts}}} + (\boldsymbol{g}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi})\right.$$

$$\left.\underbrace{-\frac{(\boldsymbol{j}_0\times\delta\boldsymbol{B})\cdot\boldsymbol{\xi}}{c}}_{=\frac{\boldsymbol{j}_0\cdot(\boldsymbol{\xi}\times\delta\boldsymbol{B})}{c}} \quad \underbrace{-\frac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\times\boldsymbol{B}_0}{4\pi}\cdot\boldsymbol{\xi}}_{\substack{=\frac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\cdot(\boldsymbol{\xi}\times\boldsymbol{B}_0)}{4\pi}\\ \text{by parts}\\ =\frac{(\delta\boldsymbol{B}\times\boldsymbol{\nabla})\cdot(\boldsymbol{\xi}\times\boldsymbol{B}_0)}{4\pi}\\ =\frac{\delta\boldsymbol{B}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)]}{4\pi}\\ =\frac{|\delta\boldsymbol{B}|^2}{4\pi},\ \text{using (18.22)}}}\right]. \tag{18.25}$$

Thus, we have arrived at a standard textbook (e.g., Kulsrud 2005) expression for the energy perturbation (this expression is non-unique because one can do various integrations by parts):

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\left[(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)\,\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + (\boldsymbol{g}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi})\right.$$
$$\left. + \frac{\boldsymbol{j}_0\cdot(\boldsymbol{\xi}\times\delta\boldsymbol{B})}{c} + \frac{|\delta\boldsymbol{B}|^2}{4\pi}\right], \tag{18.26}$$

where $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)$. Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilising. The terms that are not sign-definite and so potentially destabilising involve equilibrium gradients of pressure, density and magnetic field (currents). It is perhaps not a surprise to learn that Nature, with its fundamental yearning for thermal equilibrium, might dislike gradients—while it is of course not a rule that all such inhomogeneities render the system unstable, we will see that they often do, usually when gradients exceed certain critical thresholds.

All we need to do now is calculate δW_2 according to (18.26) for any equilibrium that interests us and see if it can be negative for any class of perturbations (or show that it is positive for all perturbations).

18.3. *Interchange Instabilities*

As the first and simplest example of how one does stability calculations using the Energy Principle, we will (perhaps disappointingly) consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic *stratified atmosphere:*

$$\rho_0 = \rho_0(z) \quad \text{and} \quad p_0 = p_0(z) \quad \text{satisfying} \quad \boxed{\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g} \tag{18.27}$$

(gravity acts downward, against the z direction, $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$).

18.3.1. *Formal Derivation of the Schwarzschild Criterion*

With $\boldsymbol{B}_0 = 0$ and the hydrostatic equilibrium (18.27), (18.26) becomes

$$\begin{aligned}\delta W_2 &= \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\left[\xi_z p_0' \boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 - g\xi_z(\rho_0'\xi_z + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\right]\\ &= \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\left[2p_0'\xi_z\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 - \rho_0' g\xi_z^2\right], \end{aligned}\tag{18.28}$$

where we have used $\rho_0 g = -p_0'$. We see that δW_2 depends on ξ_z and $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$. Let us treat them as independent variables and minimise δW_2 with respect to them (i.e., *seek the most unstable possible situation*):

$$\frac{\partial}{\partial(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})}\left[\begin{array}{c}\text{integrand}\\ \text{of (18.28)}\end{array}\right] = 2p_0'\xi_z + 2\gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) = 0 \quad\Rightarrow\quad \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = -\frac{p_0'}{\gamma p_0}\,\xi_z. \tag{18.29}$$

Substituting this back into (18.28), we get

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\left(-\frac{p_0'^2}{\gamma p_0} - \rho_0' g\right)\xi_z^2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\frac{\rho_0 g}{\gamma}\underbrace{\left(\frac{p_0'}{p_0} - \gamma\frac{\rho_0'}{\rho_0}\right)}_{=\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{p_0}{\rho_0^\gamma}}\xi_z^2. \tag{18.30}$$

By the Energy Principle, the system is stable iff

$$\delta W_2 > 0 \quad\Leftrightarrow\quad \boxed{\frac{\mathrm{d}\ln s_0}{\mathrm{d}z} > 0}\,, \tag{18.31}$$

where $s_0 = p_0/\rho_0^\gamma$ is the entropy function. The inequality (18.31) is the *Schwarzschild criterion* for convective stability.[142] If this criterion is broken, there will be an instability, called the *interchange instability*.

This calculation illustrates both the power and the weakness of the method:

—on the one hand, we have obtained a stability criterion quite quickly and without having to solve the underlying equations,

—on the other hand, while we have established the condition for instability, we have as yet absolutely no idea what is going on physically.

18.3.2. *Physical Picture*

We can remedy the latter problem by examining what type of displacements give rise to $\delta W_2 < 0$ when the Schwarzschild criterion is broken. Recalling (18.20) and (18.21) and specialising to the displacements given by (18.29) (as they are the ones that minimise δW_2), we get

$$\frac{\delta p}{p_0} = -\frac{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0}{p_0} - \gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = -\frac{p_0'}{p_0}\,\xi_z - \gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0, \tag{18.32}$$

$$\frac{\delta\rho}{\rho_0} = -\frac{1}{\rho_0}\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}) = -\frac{\rho_0'}{\rho_0}\,\xi_z - \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \frac{1}{\gamma}\left(-\gamma\frac{\rho_0'}{\rho_0} + \frac{p_0'}{p_0}\right) = \frac{1}{\gamma}\underbrace{\frac{\mathrm{d}\ln s_0}{\mathrm{d}z}}_{\substack{<0\\ \text{(unstable)}}}\xi_z. \tag{18.33}$$

[142] We studied perturbations of a stably stratified atmosphere in §16.2.8 and Q-6, where we saw that these perturbations indeed did not grow provided the entropy scale length $1/L_s = \mathrm{d}\ln s_0/\mathrm{d}z$ was positive.

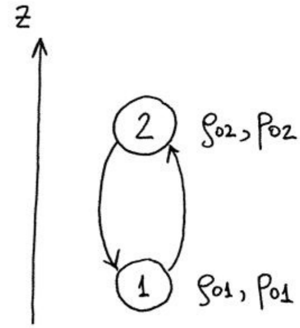


FIGURE 86. Interchange instability.

Thus, the offending perturbations maintain themselves in pressure balance (i.e., they are *not* sound waves) and locally increase or decrease density for blobs of fluid that fall ($\xi_z < 0$) or rise ($\xi_z > 0$), respectively.

This gives us some handle on the situation: if we imagine a blob of fluid slowly rising (slowly, so $\delta p = 0$) from the denser nether regions of the atmosphere to the less dense upper ones, then we can ask whether staying in pressure balance with its surroundings will require the blob to expand ($\delta\rho < 0$) or contract ($\delta\rho > 0$). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly) and the system will be unstable. The direction of the entropy gradient determines which of these two scenarios is realised.

18.3.3. *Intuitive Rederivation of the Schwarzschild Criterion*

We can use this physical intuition to derive the Schwarzschild criterion directly. Consider two blobs, at two different vertical locations, lower (1) and upper (2), where the equilibrium densities and pressures are ρ_{01}, p_{01} and ρ_{02}, p_{02}. Now interchange these two blobs (Fig. 86). Inside the blobs, the new densities and pressures are ρ_1, p_1 and ρ_2, p_2.

Requiring the blobs to stay in pressure balance with their local surroundings gives

$$p_1 = p_{02}, \quad p_2 = p_{01}. \tag{18.34}$$

Requiring the blobs to rise or fall adiabatically, i.e., to satisfy $p/\rho^\gamma = \text{const}$, and then using pressure balance (18.34) gives

$$\frac{p_{01}}{\rho_{01}^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_{02}}{\rho_1^\gamma} \quad \Rightarrow \quad \frac{\rho_1}{\rho_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma}. \tag{18.35}$$

Requiring that the buoyancy of the rising blob overcome gravity, i.e., that the weight of the displaced fluid be larger than the weight of the blob,

$$\rho_{02} g > \rho_1 g, \tag{18.36}$$

gives the condition for instability:

$$\rho_1 < \rho_{02} \quad \Leftrightarrow \quad \frac{\rho_1}{\rho_{02}} = \frac{\rho_{01}}{\rho_{02}}\left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma} < 1 \quad \Leftrightarrow \quad \boxed{\frac{p_{02}}{\rho_{02}^\gamma} < \frac{p_{01}}{\rho_{01}^\gamma}}. \tag{18.37}$$

This is exactly the same as the Schwarzschild condition (18.31) for the interchange instability (and this is why the instability is called that).

Note that, while this is of course a much simpler and more intuitive agrument than the application of the Energy Principle, it only gives us a particular example of the kind of perturbation that would be *unstable* under particular conditions, not any general criterion of what equilibria might be guaranteed to be stable.

In Q-10, we will explore how the above considerations can be generalised to an equilibrium that also features a non-zero magnetic field.

18.4. *Instabilities of a Pinch*

As our second (also classic) example, we consider the stability of a z-pinch equilibrium (§17.1.1, Fig. 80):

$$\boldsymbol{B}_0 = B_0(r)\hat{\boldsymbol{\theta}}, \quad \boldsymbol{j}_0 = j_0(r)\hat{\boldsymbol{z}} = \frac{c}{4\pi r}\,(rB_0)'\hat{\boldsymbol{z}}, \quad p_0'(r) = -\frac{1}{c}\,j_0 B_0 = -\frac{B_0(rB_0)'}{4\pi r}. \tag{18.38}$$

Since we are going to have to work in cylindrical coordinates, we must first write all the terms in (18.26) in these coordinates and with the equilibrium (18.38):

$$\begin{aligned}
(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) &= \xi_r p_0'\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\xi_r + \cancel{\frac{1}{r}\frac{\partial\xi_\theta}{\partial\theta}} + \frac{\partial\xi_z}{\partial z}\right)\\
&= p_0'\frac{\xi_r^2}{r} + p_0'\xi_r\left(\frac{\partial\xi_r}{\partial r} + \frac{\partial\xi_z}{\partial z}\right),
\end{aligned} \tag{18.39}$$

$$\gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 = \gamma p_0\left(\frac{1}{r}\frac{\partial}{\partial r}\,r\xi_r + \frac{1}{r}\frac{\partial\xi_\theta}{\partial\theta} + \frac{\partial\xi_z}{\partial z}\right)^2, \tag{18.40}$$

$$\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0) = \hat{\boldsymbol{r}}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\,\xi_r B_0\right) + \hat{\boldsymbol{\theta}}\left(-\frac{\partial}{\partial z}\,\xi_z B_0 - \frac{\partial}{\partial r}\,\xi_r B_0\right) + \hat{\boldsymbol{z}}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\,\xi_z B_0\right), \tag{18.41}$$

$$\begin{aligned}
\frac{\boldsymbol{j}_0\cdot(\boldsymbol{\xi}\times\delta\boldsymbol{B})}{c} &= \underbrace{\frac{j_0}{c}}_{=-\frac{p_0'}{B_0}}(\xi_r\delta B_\theta - \xi_\theta\delta B_r) = p_0'\left[\xi_r\left(\frac{\partial\xi_z}{\partial z} + \frac{\partial\xi_r}{\partial r} + \xi_r\frac{B_0'}{B_0}\right) + \cancel{\xi_\theta\frac{1}{r}\frac{\partial\xi_r}{\partial\theta}}\right],\\
&= p_0'\xi_r\left(\frac{\partial\xi_z}{\partial z} + \frac{\partial\xi_r}{\partial r}\right) + \frac{p_0'B_0'}{B_0}\,\xi_r^2,
\end{aligned} \tag{18.42}$$

$$\frac{|\delta\boldsymbol{B}|^2}{4\pi} = \frac{B_0^2}{4\pi r^2}\left[\left(\frac{\partial\xi_r}{\partial\theta}\right)^2 + \left(\frac{\partial\xi_z}{\partial\theta}\right)^2\right] + \underbrace{\frac{B_0^2}{4\pi}\left(\frac{\partial\xi_z}{\partial z} + \frac{\partial\xi_r}{\partial r} + \xi_r\frac{B_0'}{B_0}\right)^2}_{=\frac{B_0^2}{4\pi}\left(\frac{\partial\xi_z}{\partial z} + \frac{\partial\xi_r}{\partial r}\right)^2 + \frac{2B_0B_0'}{4\pi}\,\xi_r\left(\frac{\partial\xi_z}{\partial z} + \frac{\partial\xi_r}{\partial r}\right) + \frac{B_0'^2}{4\pi}\,\xi_r^2}. \tag{18.43}$$

The terms that are crossed out have been dropped because they combine into a full derivative with respect to θ and so, upon substitution into (18.26), vanish under integration. Assembling all this together, we have

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r} \left\{ \underbrace{\left(p_0' + \frac{p_0' r B_0'}{B_0} + \frac{r B_0'^2}{4\pi}\right)}_{=\,2p_0' + \frac{B_0^2}{4\pi r}} \frac{\xi_r^2}{r} + 2\underbrace{\left(p_0' + \frac{B_0 B_0'}{4\pi}\right)}_{=\,-\frac{B_0^2}{4\pi r}} \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r}\right) \right.$$
$$\left. + \gamma p_0 (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2}\left[\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right] + \frac{B_0^2}{4\pi}\left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r}\right)^2 \right\}$$
$$= \frac{1}{2}\int \mathrm{d}\boldsymbol{r} \left\{ 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi}\left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r}\right)^2 \right.$$
$$\left. + \gamma p_0 (\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2}\left[\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right] \right\}, \qquad (18.44)$$

where, in simplifying the first two terms in the integrand, we used the equilibrium equation (18.38):

$$p_0' = -\frac{B_0^2}{4\pi r} - \frac{B_0 B_0'}{4\pi} \quad \Rightarrow \quad \frac{r B_0'^2}{4\pi} = -\frac{p_0' r B_0'}{B_0} - \frac{B_0 B_0'}{4\pi} = -\frac{p_0' r B_0'}{B_0} + p_0' + \frac{B_0^2}{4\pi r}. \qquad (18.45)$$

Finally, after a little further tiding up,

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r} \left\{ 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi}\left(r\frac{\partial}{\partial r}\frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z}\right)^2 + \gamma p_0 \left(\frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z}\right)^2 \right.$$
$$\left. + \frac{B_0^2}{4\pi r^2}\left[\left(\frac{\partial \xi_r}{\partial \theta}\right)^2 + \left(\frac{\partial \xi_z}{\partial \theta}\right)^2\right] \right\}. \qquad (18.46)$$

18.4.1. *Sausage Instability*

Let us first consider axisymmetric perturbations: $\partial/\partial\theta = 0$. Then δW_2 depends on two variables only:

$$\xi_r \quad \text{and} \quad \eta \equiv \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}. \qquad (18.47)$$

Indeed, unpacking all the r derivatives in (18.46), we get

$$\boxed{\delta W_2 = \frac{1}{2}\int \mathrm{d}\boldsymbol{r} \left[2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi}\left(\eta - \frac{\xi_r}{r}\right)^2 + \gamma p_0 \left(\eta + \frac{\xi_r}{r}\right)^2 \right]}. \qquad (18.48)$$

We shall treat ξ_r and η as independent variables and minimise δW_2 with respect to η:

$$\frac{\partial}{\partial \eta}\left[\begin{array}{c}\text{integrand}\\ \text{of (18.48)}\end{array}\right] = 2\frac{B_0^2}{4\pi}\left(\eta - \frac{\xi_r}{r}\right) + 2\gamma p_0 \left(\eta + \frac{\xi_r}{r}\right) = 0 \quad \Rightarrow \quad \eta = \frac{1-\gamma\beta/2}{1+\gamma\beta/2}\frac{\xi_r}{r}, \qquad (18.49)$$

where, as usual, $\beta = 8\pi p_0/B_0^2$. Putting this back into (18.48), we get

$$\delta W_2 = \int \mathrm{d}\boldsymbol{r}\, p_0 \left[\frac{r p_0'}{p_0} + \frac{1}{\beta}\left(\frac{\gamma\beta}{1+\gamma\beta/2}\right)^2 + \frac{\gamma}{2}\left(\frac{2}{1+\gamma\beta/2}\right)^2\right]\frac{\xi_r^2}{r^2}$$
$$= \int \mathrm{d}\boldsymbol{r}\, p_0 \left(r\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} + \frac{2\gamma}{1+\gamma\beta/2}\right)\frac{\xi_r^2}{r^2}. \qquad (18.50)$$

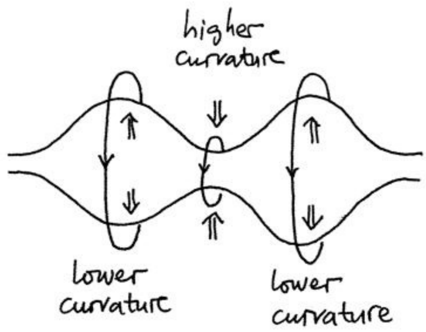


FIGURE 87. Sausage instability.

There will be an instability ($\delta W_2 < 0$) if (but not only if, because we are considering the restricted set of axisymmetric displacements)

$$\boxed{-r\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} > \frac{2\gamma}{1+\gamma\beta/2}}\,, \tag{18.51}$$

i.e., when the pressure gradient is too steep, the equilibrium is unstable.

What sort of instability is this? Recall that the perturbations that we have identified as making $\delta W_2 < 0$ are axisymmetric, have some radial and axial displacements and are *compressible:* from (18.49),

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \eta + \frac{\xi_r}{r} = \frac{2}{1+\gamma\beta/2}\frac{\xi_r}{r}. \tag{18.52}$$

They are illustrated in Fig. 87. The mechanism of this aptly named *sausage instability* is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further.

Exercise 18.1. Convince yourself that the displacements that have been identified cause magnetic perturbations that are consistent with the cartoon in Fig. 87.

18.4.2. *Kink Instability*

Now consider non-axisymmetric perturbations ($\partial/\partial\theta \neq 0$) to see what other instabilities might be there. First of all, since we now have θ variation, δW_2 depends on ξ_θ. However, in (18.46), ξ_θ only appears in the third term, where it is part of $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$, which enters quadratically and with a positive coefficient γp_0. We can treat $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ as an independent variable, alongside ξ_r and ξ_z, and minimise δW_2 with respect to it. Obviously, the energy perturbation is minimal when

$$\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0, \tag{18.53}$$

i.e., the most dangerous non-axisymmetric perturbations are incompressible (unlike for the case of the axisymmetric sausage mode in §18.4.1: there we could not—and did not—have such incompressible perturbations because we did not have ξ_θ at our disposal, to be chosen in such a way as to enforce incompressibility).

To carry out further minimisation of δW_2, it is convenient to Fourier transform our displacements in the θ and z directions—both are directions of symmetry (i.e., the equilibrium profiles do not vary in these directions), so this can be done with impunity:

$$\boldsymbol{\xi} = \sum_{m,k} \boldsymbol{\xi}_{mk}(r)\, e^{i(m\theta+kz)}. \tag{18.54}$$

Then (18.46) (with $\nabla\cdot\boldsymbol{\xi}=0$) becomes, by Parseval's theorem (the operator $\boldsymbol{F}[\boldsymbol{\xi}]$ being self-adjoint; see §18.1.1),

$$\boxed{\delta W_2 = \frac{1}{2}\sum_{m,k} 2\pi L_z \int_0^\infty \mathrm{d}r\, r\left\{2p_0'\frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi}\left[\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r} + ik\xi_z\right|^2 + \frac{m^2}{r^2}\left(|\xi_r|^2+|\xi_z|^2\right)\right]\right\}}. \tag{18.55}$$

As ξ_z and ξ_z^* only appear algebraically in (18.55) (no r derivatives), it is easy to minimise δW_2 with respect to them: setting the derivative of the integrand with respect to either ξ_z or ξ_z^* to zero, we get

$$-ik\left(r\frac{\partial}{\partial r}\frac{\xi_r}{r} + ik\xi_z\right) + \frac{m^2}{r^2}\,\xi_z = 0 \quad\Rightarrow\quad \xi_z = \frac{ikr^3}{m^2+k^2r^2}\frac{\partial}{\partial r}\frac{\xi_r}{r}. \tag{18.56}$$

Putting this back into (18.55) and assembling terms, we get

$$\begin{aligned}\delta W_2 = \sum_{m,k}\pi L_z\int_0^\infty \mathrm{d}r\, r\Bigg\{&2p_0\left(\frac{rp_0'}{p_0}+\frac{m^2}{\beta}\right)\frac{|\xi_r|^2}{r^2}\\ &+\frac{B_0^2}{4\pi}\underbrace{\left[\left(1-\frac{k^2r^2}{m^2+k^2r^2}\right)^2+\frac{m^2k^2r^2}{(m^2+k^2r^2)^2}\right]}_{=\frac{m^2}{m^2+k^2r^2}}\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}\right|^2\Bigg\}.\end{aligned} \tag{18.57}$$

The second term here is always stabilising. The most unstable modes will be ones with $k\to\infty$, for which the stabilising term is as small as possible. The remaining term will allow $\delta W_2 < 0$ and, therefore, an instability, if

$$\boxed{-r\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} > \frac{m^2}{\beta}}. \tag{18.58}$$

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest m, viz., $m=1$.

Note that another way of writing the instability condition (18.58) is

$$-rp_0' = \frac{B_0^2}{4\pi} + \frac{rB_0B_0'}{4\pi} > m^2\frac{B_0^2}{8\pi} \quad\Rightarrow\quad \boxed{r\frac{\mathrm{d}\ln B_0}{\mathrm{d}r} > \frac{m^2}{2} - 1}, \tag{18.59}$$

where we have used the equilibrium equation (18.38).

What does this instability look like? The unstable perturbations are incompressible:

$$\nabla\cdot\boldsymbol{\xi} = 0 \quad\Rightarrow\quad \frac{1}{r}\frac{\partial}{\partial r}r\xi_r + \frac{im}{r}\,\xi_\theta + ik\xi_z = 0. \tag{18.60}$$

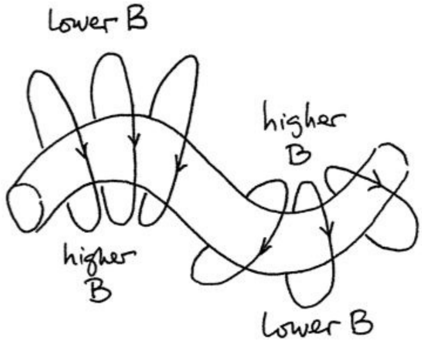


FIGURE 88. Kink instability.

Setting $m = 1$ and using (18.56), we find

$$i\xi_\theta = -\frac{\partial}{\partial r} r\xi_r + \underbrace{\frac{k^2 r^4}{m^2 + k^2 r^2}}_{\substack{\approx r^2 \\ \text{as } k \to \infty}} \frac{\partial}{\partial r}\frac{\xi_r}{r} \approx -2\xi_r \quad \text{and} \quad \xi_z \ll \xi_r. \tag{18.61}$$

The basic cartoon (Fig. 88) is as follows: the flux surfaces are bent, with a twist (to remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher B to those with lower B, thermal pressure in the latter (convex) areas becomes uncompensated, the field lines open up further, etc. This is called the *kink instability.*

Similar methodology can be used to show that, unlike the z pinch, the θ pinch (§17.1.1, Fig. 81) is always stable: see Q-11.

Exercise 18.2. Find the value of β above which the 3D modes go unstable at lower pressure gradients than the 2D ones do.

18.5. *Further Reading on MHD Instabilities*

There are very many of these, easily a whole course's worth. They are an interesting topic. A founding text is the old, classic, super-meticulous monograph by Chandrasekhar (2003). In the context of toroidal (fusion) plasmas, you want to learn the so-called *ballooning theory*, a tour de force of theoretical plasma physics, which, like the relaxation theory, is associated with J. B. Taylor's name—so his lectures, Taylor & Newton (2015), are a good starting point (the original paper on the subject is Connor *et al.* 1979). In the unlikely event that you have an appetite for more energy-principle calculations in the style of §18.4, the book by Freidberg (2014) will teach you more than you ever wanted to know. In astrophysics, MHD instabilities have been a hot topic since the early 1990s, not least due the realisation by Balbus & Hawley (1991) that the magnetorotational instability (MRI) is responsible for triggering turbulence and, therefore, maintaining momentum transport in accretion flows—the Oxford lecture notes by Balbus (2015) are an excellent place to start learning about this subject (this is also an opportunity to learn how to handle equilibria that are *not static*,e.g., most interestingly, featuring rotating and shear flows). Another excellent set of lecture notes on astrophysical fluid dynamics is Ogilvie (2016), this one originating from Cambridge Part III.

As with everything in physics, the frontier in this subject is nonlinear phenomena. One very attractive theoretical topic has been the theory of explosive instabilities and erupting flux tubes by Steve Cowley and his co-workers: the founding (quite pedagogically written) paper was

Cowley & Artun (1997), a key more recent one is Cowley *et al.* (2015); follow the paper trail from there for various refinements and applications (from space to tokamaks).

---

19. Magnetic Reconnection

Most of our discussion revolved around properties of ideal MHD equations. It is, in fact, quite essential to study resistive effects, even when resistivity is very small, because many ideal solutions have a natural tendency to develop ever smaller spatial gradients, which can only be regularised by resistivity (we touched on this, e.g., in §17.2.2). The key linear result here is the *tearing mode*, a resistive instability associated with the propensity of magnetic-field lines to reconnect—change their topology in such a way as to release some of their energy. This will be discussed in §19.1.

But here again the frontier is nonlinear: the theory of *magnetic reconnection*. Tearing modes, in their nonlinear stage, tend to lead to formation of *current sheets* (which is, in fact, a general tendency of X-point solutions in MHD: see Q-5), and how reconnection happens after that has been a subject of frenetic research since mid-20th century. Magnetic reconnection is believed to be a key player in a host of plasma phenomena, from solar flares to the so-called "sawtooth crash" in tokamaks, to MHD turbulence. Kulsrud (2005, §14) has a good introduction to the history and the basics of the subject, coming from a live witness and key contributor. There has been much going on in it in the last two decades, many of the advances occurring on the collisionless reconnection front requiring kinetic theory (some key names to search for in the extensive recent literature are W. Daughton, J. Drake, J. Egedal), but even within MHD, the discovery of the *plasmoid instability* (amounting to the realisation that current sheets are tearing unstable; see Loureiro *et al.* 2007) has led to a new theory of resistive MHD reconnection (Uzdensky *et al.* 2010), a development that I (obviously) find important. I will go through all that in §§19.2 and 19.3.

Even more recently, magnetic reconnection became intimately intertwined with the theory of MHD turbulence (§16.4)—you will find an account of this in my (hopefully pedagogical) review, Schekochihin (2022). Its Appendix D also contains a "reconnection primer" covering tearing modes, current sheets and related topics in the most straightforward non-rigorous way that I could manage.

19.1. *Tearing Mode*

Coming soon... For now, see `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/PartIIIMHD/LecturesL05/sec16_TearingMode.pdf`, Schekochihin (2022, Appendix D.1), Boldyrev & Loureiro (2018), Parra (2019*a*), or Kunz (2020). Other good places to read about tearing are Taylor & Newton (2015), the original paper by Furth *et al.* (1963), or standard textbooks (e.g., Sturrock 1994, §17).

19.2. *Sweet–Parker Reconnection*

See Schekochihin (2022, Appendix D.4.1).

Exercise 19.1. Plasmoid instability. See Schekochihin (2022, Appendix D.4.2) and Loureiro *et al.* (2007).

19.3. *Fast MHD Reconnection*

See Schekochihin (2022, Appendix D.6) and Uzdensky *et al.* (2010).

19.4. *Non-MHD Reconnection*

Go to §24.4.

Magnetohydrodynamics Problem Set

1. Clebsch Coordinates. As $\nabla\cdot\boldsymbol{B}=0$, it is always possible to find two scalar functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$ such that

$$\boldsymbol{B}=\nabla\alpha\times\nabla\beta. \tag{19.1}$$

(a) Argue that any magnetic field line can be described by the equations

$$\alpha=\text{const},\quad \beta=\text{const}. \tag{19.2}$$

This means that (α,β,ℓ), where ℓ is the distance (arc length) along the field line, are a good set of curvilinear coordinates, known as the *Clebsch coordinates*.

(b) Show that the magnetic flux through any area S in the (x,y) plane is

$$\Phi=\int_{\tilde{S}}\mathrm{d}\alpha\,\mathrm{d}\beta, \tag{19.3}$$

where $\tilde{S}$ is the area S in new coordinates after transforming $(x,y)\to(\alpha(x,y,0),\beta(x,y,0))$.

(c) Show that if (19.1) holds at time $t=0$ and α and β are evolved in time according to

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}=0,\quad \frac{\mathrm{d}\beta}{\mathrm{d}t}=0, \tag{19.4}$$

where $\mathrm{d}/\mathrm{d}t$ is the convective derivative, then (19.1) correctly describes the magnetic field at all $t>0$.

(d) Argue from the above that magnetic flux is frozen into the flow and magnetic field lines move with the flow.

(e*) Show that the field that minimises the magnetic energy within some domain subject to the constraint that the values of α and β are fixed at the boundary of this domain (i.e., that the "footpoints" of the field lines are fixed) is a force-free field.[143]

A prototypical example of the kind of fields that arise from the variational principle in (e) is the "arcade" fields describing magnetic loops sticking out of the Sun's surface, with footpoints anchored at the surface. One such field will be considered in Q-9(f) and more can be found in Sturrock (1994, §13).

---

2. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with number density $n_{\rm gal}=1\ \text{cm}^{-3}$ undergoing a gravitational collapse to a spherical star with number density $n_{\rm star}=10^{26}\ \text{cm}^{-3}$. The magnetic field in the galactic plasma is $B_{\rm gal}\sim 3\times10^{-6}$ G. Assuming that flux is frozen, estimate the magnetic field in a star. Find out if this is a good estimate. If not, how, in your view, could we account for the discrepancy?

3. Flux Concentration. Consider a simple 2D model of incompressible convective motion (Fig. 89):

$$\boldsymbol{u}=U\left(-\sin\frac{\pi x}{L}\,\cos\frac{\pi z}{L},\ 0,\ \cos\frac{\pi x}{L}\,\sin\frac{\pi z}{L}\right). \tag{19.5}$$

(a) In the neighbourhood of the stagnation point $(0,0,0)$, linearise the flow, assume vertical magnetic field, $\boldsymbol{B}=(0,0,B(t,x))$ and derive an evolution equation for $B(t,x)$,

[143] This is based on the 2017 exam question.

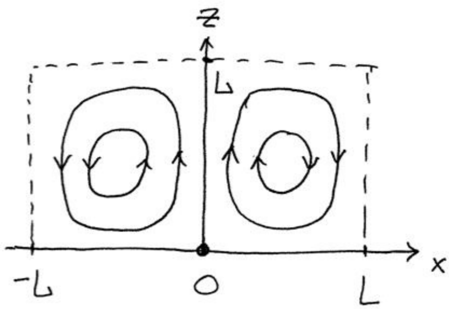


FIGURE 89. Convective cells from Q-3.

including both advection by the flow and Ohmic diffusion. Suppose the field is initially uniform, $B(t = 0, x) = B_0 = \text{const}$. It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\text{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.

(b) Find the steady-state solution of your equation. Assume $B(x) = B(-x)$ and use flux conservation to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?

(c*) Obtain the time-dependent solution of your equation for B and confirm that it indeed converges to your steady-state solution. Find the time scale on which this happens.

Hint. The following changes of variables may prove useful: $\xi = \sqrt{\pi\text{Rm}}\, x/L$, $\tau = \pi U t/L$, $X = \xi e^{\tau}$, $s = (e^{2\tau} - 1)/2$.

(d) Can you think of a quick heuristic argument based on the induction equation that would tell you that all these answers were to be expected?

4. Zeldovich (1956) Antidynamo Theorem. Consider an arbitrary 2D velocity field: $\boldsymbol{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (i.e., in a system that can be enclosed within some volume outside which there are no fields or flows), this velocity field cannot be a dynamo, i.e., any initial magnetic field will always eventually decay.

Hint. Consider separately the evolution equations for B_z and for the magnetic field in the (x, y)-plane. Show that B_z decays by working out the time evolution of the volume integral of B_z^2. Then write B_x, B_y in terms of one scalar function (which must be possible because $\partial B_x/\partial x + \partial B_y/\partial y = 0$) and show that it decays as well.

Q-5 is optional.

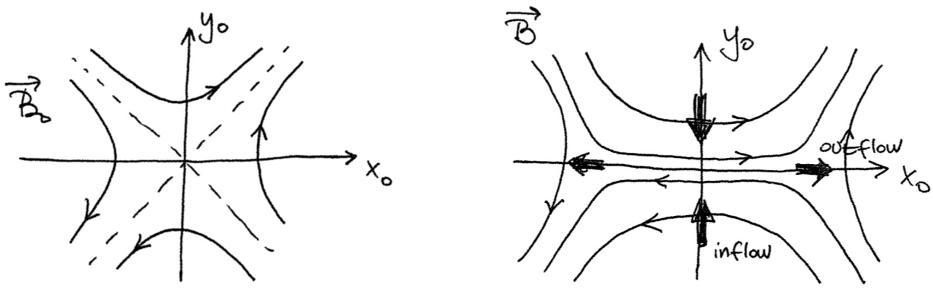
---

5. *X*-Point Collapse. Consider the following initial magnetic-field configuration:

$$\boldsymbol{B}_0(\boldsymbol{r}_0) = B_0\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_0 \varPsi(x_0, y_0), \tag{19.6}$$

where $\boldsymbol{r}_0 = (x_0, y_0, z_0)$, $B_0 = \text{const}$, and

$$\varPsi(x_0, y_0) = \frac{x_0^2 - y_0^2}{2}. \tag{19.7}$$



(a) X-point at $t = 0$. (b) Collapsing sheet at $t > 0$.

FIGURE 90. X-point collapse.

This is called an X*-point* (Fig. 90a).

(a) Use the Lagrangin MHD equation (14.87), where $\boldsymbol{r} = (x, y, z)$, and seek a solution in the form

$$x = \xi(t)x_0, \qquad y = \eta(t)y_0, \qquad z = z_0. \tag{19.8}$$

Show that ξ and η satisfy the following equations

$$\ddot{\xi} = \eta\left(\frac{1}{\eta^2} - \frac{1}{\xi^2}\right), \qquad \ddot{\eta} = \xi\left(\frac{1}{\xi^2} - \frac{1}{\eta^2}\right). \tag{19.9}$$

(b) Consider the possibility that, as time goes on, $\eta(t)$ becomes ever smaller, $\eta \to 0$, while $\xi(t)$ tends to a constant, $\xi \to \xi_c$. Show that the solution that has this property is

$$\xi(t) \approx \xi_c + \frac{9}{4}\left(\frac{2}{9\xi_c}\right)^{1/3}(t_c - t)^{4/3}, \qquad \eta(t) \approx \left(\frac{9\xi_c}{2}\right)^{1/3}(t_c - t)^{2/3} \tag{19.10}$$

as $t \to t_c$, where t_c is some finite time. This is called the *Syrovatskiĭ (1971) solution.*

(c) Calculate the magnetic field as a function of time and convince yourself that the Syrovatskii solution describes the initial X-point configuration collapsing explosively to a sheet along the x axis. What happens after t reaches t_c?

(d) Do a similar calculation, but for incompressible Lagrangian MHD, i.e., assuming $J = 1 = \text{const}$ (which is now the equation that determines the total pressure; cf. §16.2.5). Show that the solution in this case is

$$\xi(t) = \Lambda(t), \quad \eta(t) = \frac{1}{\Lambda(t)}, \tag{19.11}$$

where $\Lambda(t)$ is an arbitrary function of time. Take $\Lambda(t) = e^{\lambda t}$ and show that this solution corresponds to an exponentially collapsing X-point. This is called the *Chapman & Kendall (1963) solution.* Can this evolution continue forever?

(e) Show that the fluid flow associated with a collapsing solution consists of an inflow (into the "sheet") and an outflow (from the "sheet"). Going back to the general solution (19.11), assume that the outflow velocity u_x at a given fixed Lagrangian position $x_{\rm out}$ is equal to some known constant $u_{\rm out}$ (i.e., as the "sheet" collapses and gets longer, the outflow from its ends is always the same). Find $\xi(t)$ and $\eta(t)$ in this case. This solution is due to Uzdensky & Loureiro (2016) (read their paper to find out what the use of it is).

6. MHD Waves in a Stratified Atmosphere. The generalisation of iMHD to the case of a stratified atmosphere is explained in §16.2.8. Convince yourself that you understand how the SMHD equations and the SMHD ordering arise and then study them as follows.

(a) Work out all SMHD waves (both their frequencies and the corresponding eigenvectors). It is convenient to choose the coordinate system in such a way that $\boldsymbol{k} = (k_x, 0, k_z)$, where z is the vertical direction (the direction of gravity). The mean magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{b}_0$ is assumed to be straight and uniform, at a general angle to z. We continue referring to the projection of the wave number onto the magnetic-field direction as $k_\parallel = \boldsymbol{k}\cdot\boldsymbol{b}_0 = k_x b_{0x} + k_z b_{0z}$. Note that in the case of $B_0 = 0$, you are dealing with stratified hydrodynamics, not MHD—the waves that you obtain in this case are the well known *gravity waves*, or *"g-modes"*.

(b) Explain the physical nature of the perturbations (what makes the fluid oscillate) in the special cases (i) $k_z = 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{z}}$, (ii) $k_z = 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{x}}$, (iii) $k_x = 0$, (iv) $k_z \neq 0$, $k_x \neq 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{z}}$.

(c) Under what conditions are the perturbations you have found unstable? What is the physical mechanism for the instability? What role does the magnetic field play (stabilising or destabilising) and why? Cross-check your answers with §18.3 and Q-10.

(d) Find the conserved energy (a quadratic quantity whose integral over space stays constant) for the full nonlinear SMHD equations (16.97–16.100). Give a physical interpretation of the quantity that you have obtained—why should it be conserved?

Do either Q-7 or Q-8.

---

7. Electron MHD. In certain physical regimes (roughly realised, for example, in the solar-wind and other kinds of astrophysical turbulence at scales smaller than the ion Larmor radius; see Schekochihin *et al.* 2009 or Boldyrev *et al.* 2013), plasma turbulence can be described by an approximation in which the magnetic field is frozen into the electron flow $\boldsymbol{u}_e$, while ions are considered motionless, $\boldsymbol{u}_i = 0$. In this approximation, Ohm's law becomes[144]

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c}. \tag{19.12}$$

Here $\boldsymbol{u}_e$ can be expressed directly in terms of $\boldsymbol{B}$ because the current density in a plasma consisting of motionless hydrogen ions ($n_i = n_e$) and moving electrons is

$$\boldsymbol{j} = en_e(\boldsymbol{u}_i - \boldsymbol{u}_e) = -en_e\boldsymbol{u}_e, \tag{19.13}$$

but, on the other hand, $\boldsymbol{j}$ is known via Ampère's law. Here n_e is the electron number density and e the electron charge.

(a) Using this and Faraday's law, show that the evolution equation for the magnetic field in this approximation is

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = -d_i\boldsymbol{\nabla}\times[(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]}, \tag{19.14}$$

where the magnetic field has been rescaled to Alfvénic velocity units, $\boldsymbol{B}/\sqrt{4\pi m_i n_i} \to \boldsymbol{B}$, and $d_i = c/\omega_{\rm pi}$ is the ion inertial scale ("ion skin depth"), $\omega_{\rm pi} = \sqrt{4\pi e^2 n_i/m_i}$.

---

[144]Strictly speaking, the generalised Ohm's law in this approximation also contains an electron-pressure gradient (see, e.g., Goedbloed & Poedts 2004), but that vanishes upon substitution of $\boldsymbol{E}$ into Faraday's law.

Equation (19.14) is the equation of *Electron MHD (EMHD)*, completely self-consistent for $\boldsymbol{B}$.

(b) Show that magnetic energy is conserved by (19.14). Is magnetic helicity conserved? Does J. B. Taylor relaxation work and what kind of field will be featured in the relaxed state? Is it obvious that this field is a good steady-state solution of (19.14)?

(c) Consider infinitesimal perturbations of a straight-field equilibrium, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}$, and show that they are helical waves with the dispersion relation

$$\boxed{\omega = \pm k_\parallel v_{\mathrm{A}} k d_i}\,. \tag{19.15}$$

These are called *Kinetic Alfvén Waves (KAW)*.

(d) Now consider finite perturbations and argue that the appropriate ordering in which linear and nonlinear physics can coexist while perturbations remain small is

$$|\delta\boldsymbol{b}| \sim \frac{\delta B}{B} \sim \frac{k_\parallel}{k} \ll 1. \tag{19.16}$$

Under this ordering, show that the magnetic field can be represented as

$$\frac{\delta\boldsymbol{B}}{B_0} = \frac{1}{v_{\mathrm{A}}}\,\hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\Psi + \hat{\boldsymbol{z}}\,\frac{\delta B}{B} \tag{19.17}$$

and the evolution equations for Ψ and $\delta B/B_0$ are

$$\boxed{\frac{\partial\Psi}{\partial t} = v_{\mathrm{A}}^2 d_i\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta B}{B_0}, \quad \frac{\partial}{\partial t}\frac{\delta B}{B_0} = -d_i\boldsymbol{b}\cdot\boldsymbol{\nabla}\nabla_\perp^2\Psi}\,, \tag{19.18}$$

where $\boldsymbol{b}\cdot\boldsymbol{\nabla}$ is given by (16.115). These are the equations of *Reduced Electron MHD*.

(e) Check that the conservation of magnetic energy and the KAW dispersion relation (19.15) are recovered from (19.18). Is there any other conservation law?

8. Hydrodynamics of Rotating Fluid.[145] Most of this question is not on MHD, but deals with equations describing a somewhat analogous system: also embedded into an external field and supporting anisotropic wave-like perturbations. It is an incompressible fluid rotating at angular velocity $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is the unit vector in the direction of the z axis. The velocity field $\boldsymbol{u}$ in such a fluid satisfies the following equation

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}p + 2\boldsymbol{u}\times\boldsymbol{\Omega}, \tag{19.19}$$

where pressure p is found from the incompressibility condition $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, the last term on the right-hand side is the Coriolis force, the centrifugal force has been absorbed into p, and viscosity has been ignored.

(a) Consider infinitesimal perturbations of a static ($\boldsymbol{u}_0 = 0$), homogeneous equilibrium of (19.19). Show that the system supports waves with the dispersion relation

$$\boxed{\omega = \pm 2\Omega\frac{k_\parallel}{k}}\,. \tag{19.20}$$

These are called *inertial waves*. Here $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ (without loss of generality); the subscripts refer to directions perpendicular and parallel to the axis of rotation.

[145] This is based on the 2018 exam question.

(b) In the case $k_\parallel \ll k_\perp$, determine the direction of propagation of the inertial waves. Determine also the relationship between the components of the velocity vector $\boldsymbol{u}$ associated with the wave. Comment on the polarisation of the wave.

(c) When rotation is strong, i.e., when $\Omega \gg ku$, perturbations in a rotating system are anisotropic with $\epsilon = k_\parallel / k_\perp \ll 1$. Order the linear and nonlinear time scales to be similar to each other and work out the ordering of all relevant quantities, namely, $\boldsymbol{u}_\perp$ (horizontal velocity), $u_\parallel$ (vertical velocity), δp (perturbed pressure), ω, Ω, $k_\parallel$, $k_\perp$ with respect to each other and to ϵ. Using this ordering, show that the motions of a rotating fluid satisfy the following reduced equations

$$\boxed{\frac{\partial}{\partial t}\nabla_\perp^2 \Phi + \left\{\Phi, \nabla_\perp^2 \Phi\right\} = 2\Omega \frac{\partial u_\parallel}{\partial z}, \quad \frac{\partial u_\parallel}{\partial t} + \left\{\Phi, u_\parallel\right\} = -2\Omega \frac{\partial \Phi}{\partial z}}\,, \tag{19.21}$$

where the "Poisson bracket" is defined by (16.114) and Φ is the stream function of the perpendicular velocity, i.e., to the lowest order in ϵ, $\boldsymbol{u}_\perp^{(0)} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Phi$. Note that, in order to obtain the above equations, you will need to work out $\boldsymbol{\nabla}_\perp \cdot \boldsymbol{u}_\perp$ to both the lowest and next order in ϵ, i.e., both $\boldsymbol{\nabla}_\perp \cdot \boldsymbol{u}_\perp^{(0)}$ and $\boldsymbol{\nabla}_\perp \cdot \boldsymbol{u}_\perp^{(1)}$.

(d) Show that any purely horizontal flows in a strongly rotating fluid must be exactly two-dimensional (i.e., constant along the axis of rotation).

(e) For a strongly rotating, incompressible, highly electrically conducting fluid embedded in a strong uniform magnetic field $\boldsymbol{B}_0$ parallel to the axis of rotation, discuss qualitatively under what conditions you would expect anisotropic ($k_\parallel \ll k_\perp$) Alfvénic and slow-wave-like (pseudo-Alfvénic) perturbations to be decoupled from each other?

There are certain interesting similarities between MHD turbulence and turbulence in rotating fluid systems described by (19.21) and, indeed, also turbulence in stratified environments that we dealt with in §16.2.8 and Q-6. If you would like to know more, see Nazarenko & Schekochihin (2011) and follow the paper trail from there.

---

9. Grad–Shafranov Equation. Consider static MHD equilibria (17.1) in cylindrical coordinates (r, θ, z) and assume axisymmetry, $\partial/\partial\theta = 0$.

(a) Using the solenoidality of the magnetic field, show that any axisymmetric such field can be expressed in the form

$$\boldsymbol{B} = I\boldsymbol{\nabla}\theta + \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta, \tag{19.22}$$

where I and ψ are functions of r and z and $\boldsymbol{\nabla}\theta = \hat{\boldsymbol{\theta}}/r$ ($\hat{\boldsymbol{\theta}}$ is the unit basis vector in the θ direction). Show that magnetic surfaces are surfaces of $\psi = \text{const}$.

(b) Using the force balance, show that $\boldsymbol{\nabla} I \times \boldsymbol{\nabla}\psi = 0$ and $\boldsymbol{\nabla} p \times \boldsymbol{\nabla}\psi = 0$ and hence argue that

$$I = I(\psi) \quad \text{and} \quad p = p(\psi) \tag{19.23}$$

are functions of ψ only (i.e., they are constant on magnetic surfaces).

(c) Again from the force balance, show that $\psi(r, z)$ satisfies the *Grad–Shafranov equation*

$$\boxed{-\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) = 4\pi r^2 \frac{\mathrm{d}p}{\mathrm{d}\psi} + I\,\frac{\mathrm{d}I}{\mathrm{d}\psi}}\,. \tag{19.24}$$

This defines the shape of an axisymmetric equilibrium, given the profiles $p(\psi)$ and $I(\psi)$.

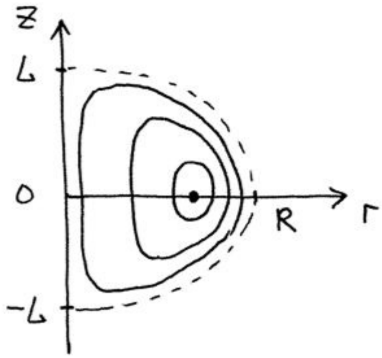


FIGURE 91. A simple equilibrium from Q-9, superficially resembling the poloidal cross-section of a tokamak.

(d) Show that in cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z = 0$), (19.24) reduces to (17.8).

(e*) Assume $I(\psi) = \text{const}$ (so the azimuthal field $B_\theta = I/r$ is similar to the magnetic field from a central current) and $p(\psi) = a\psi$, where a is some constant. Find a solution of (19.24) that gives rise to magnetic surfaces that resemble nested tori, but with "D-shaped" cross section (Fig. 91; this looks a bit like the modern tokamaks). If you stipulate that p must vanish at $r = 0$ and at $r = R$ along the $z = 0$ axis and also at $z = \pm L$ along the $r = 0$ axis and that the maximum pressure at $r < R$ is p_0, show that the corresponding magnetic surfaces are described by

$$\psi = 2\sqrt{\frac{2\pi p_0}{1 + R^2/4L^2}}\, r^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{L^2}\right). \tag{19.25}$$

Where is the (azimuthal) magnetic axis of these surfaces? What is the value of a?

(f) Seek solutions to (19.24) that are linear force-free fields. Show that in this case, (19.24) reduces to the Bessel equation (a substitution $\psi = rf(r,z)$ will prove useful). Set $B_z(0,0) = B_0$. Find solutions of two kinds: (i) ones in a semi-infinite domain $z \geqslant 0$, with the field vanishing exponentially at $z \to \infty$; (ii) ones periodic in z. If you also impose the boundary condition $B_r = 0$ at $r = R$, how can this be achieved? Can either of these solutions be the result of J. B. Taylor relaxation of an MHD system? If so, how would one decide whether it is more or less likely to be the correct relaxed state than the solution derived in §17.4?

You will find the solution of the type (i) in Sturrock (1994, §13) (who also shows how to construct many other force-free fields, useful in various physical and astrophysical contexts). Think of this solution in the context of Q-1(e). The solution of type (ii) is a particular case of the general ($\partial/\partial\theta \neq 0$) equilibrium solution derived and discussed in Taylor & Newton (2015, §9). However, the axisymmteric solution is not very useful because, as they show, depending on the values of helicity and of R, the true relaxed state is either the cylidrically and axially symmetric solution derived in §17.4 or one which also has variation in the θ direction.

Do either Q-10 or Q-11.

---

10. Magnetised Interchange Instability. Consider the same set up as in §18.3, but

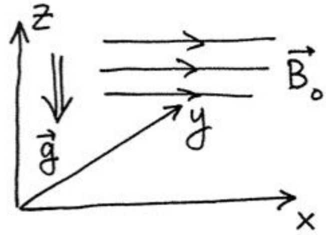


FIGURE 92. Magnetised atmosphere.

now the stratified atmosphere is threaded by straight horizontal magnetic field (Fig. 92):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \boldsymbol{B}_0 = B_0(z)\hat{\boldsymbol{x}}, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(p_0 + \frac{B_0^2}{8\pi}\right) = -\rho_0 g. \tag{19.26}$$

We shall be concerned with the stability of this equilibrium.

(a) For simplicity, assume $\partial\boldsymbol{\xi}/\partial x = 0$. This rules out any perturbations of the magnetic-field direction, $\delta\boldsymbol{b} = 0$, so there will be no field-line bending, no restoring curvature forces. For this restricted set of perturbations, work out δW_2 and observe that, like in the unmagnetised case considered in §18.3, it depends only on $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ and ξ_z. Minimise δW_2 with respect to $\boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ and show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta}\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{B_0}{\rho_0} < 0 \tag{19.27}$$

is a sufficient condition for instability (the *magnetised interchange instability*). Would you be justified in expecting stability if the condition (19.27) were not satisfied?

(b) Explain how this instability operates and rederive (19.27) by considering interchanging blobs (or, rather, flux tubes), in the spirit of §18.3.3.

If field-line bending is allowed ($\partial\xi/\partial x \neq 0$), another instability emerges, the *Parker (1966) instability*. Do investigate. Another thing to investigate is what happens when magnetised atmospheres are metastable: it turns out that they are nonlinearly unstable and have a tendency to rearrange themselves in rather whimsical ways—a very beautiful theory of this is Hosking *et al.* (2025), where, curiously, some ideas from the statistical mechanics of collisionless plasmas (§10) come methodologically handy.

---

11. Stability of the θ Pinch. Consider the following cylindrically and axially symmetric equilibrium:

$$\boldsymbol{B}_0 = B_0(r)\hat{\boldsymbol{z}}, \quad \boldsymbol{j}_0 = j_0(r)\hat{\boldsymbol{\theta}} = -\frac{c}{4\pi}B_0'(r)\hat{\boldsymbol{\theta}}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(p_0 + \frac{B_0^2}{8\pi}\right) = 0 \tag{19.28}$$

(a θ pinch; see §17.1.1, Fig. 81). Consider general displacements of the form

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{mk}(r)e^{im\theta+ikz}. \tag{19.29}$$

Show that the θ pinch is always stable. Specifically, you should be able to show that

$$\delta W_2 = \pi L_z\int_0^\infty \mathrm{d}r\, r\left\{\gamma p_0|\boldsymbol{\nabla}\cdot\boldsymbol{\xi}|^2 + \frac{B_0^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial\xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right|^2\right]\right\} > 0, \tag{19.30}$$

where L_z is the length of the cylinder.

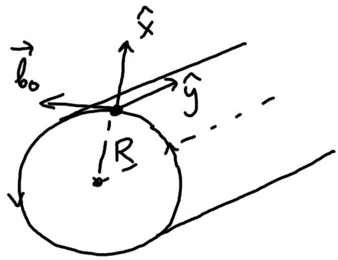


FIGURE 93. Local coordinates in a z pinch.

12. Local Analysis of z-Pinch Instabilities. For the purposes of this question, it will be convenient to consider the MHD momentum and induction equations in the form

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u}=-\boldsymbol{\nabla}_\perp\left(p+\frac{B^2}{8\pi}\right)-\boldsymbol{b}\nabla_\parallel p+\frac{B^2}{4\pi}\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}, \tag{19.31}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}\right)\ln B=-\boldsymbol{\nabla}_\perp\cdot\boldsymbol{u}_\perp, \tag{19.32}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}\right)\boldsymbol{b}=\left(\nabla_\parallel\boldsymbol{u}_\perp\right)\cdot(\mathsf{I}-\boldsymbol{bb}), \tag{19.33}$$

where $\boldsymbol{B}=B\boldsymbol{b}$, $\boldsymbol{\nabla}_\perp=(\mathsf{I}-\boldsymbol{bb})\cdot\boldsymbol{\nabla}$, $\nabla_\parallel=\boldsymbol{b}\cdot\boldsymbol{\nabla}$, $\boldsymbol{u}_\perp=\boldsymbol{u}\cdot(\mathsf{I}-\boldsymbol{bb})$, and we shall assume a z-pinch equilbrium. We shall work in local coordinates (x,y,z) chosen so that $\hat{\boldsymbol{x}}$ is in the radial direction, $\hat{\boldsymbol{z}}=\boldsymbol{b}_0$, and $\hat{\boldsymbol{y}}=\boldsymbol{b}_0\times\hat{\boldsymbol{x}}$ (Fig. 93). In this notation, in a z pinch, the relationships between the local values of the equilibrium quantities and their gradients are

$$\frac{1}{R}-\frac{1}{L_B}=\frac{\beta}{2L_p},\quad \boldsymbol{b}_0\cdot\boldsymbol{\nabla}\boldsymbol{b}_0=-\frac{\hat{\boldsymbol{x}}}{R},\quad \boldsymbol{\nabla}\cdot\boldsymbol{b}_0=0,\quad \boldsymbol{b}_0\cdot\boldsymbol{\nabla}\hat{\boldsymbol{x}}=\frac{\boldsymbol{b}_0}{R}, \tag{19.34}$$

where $\beta=8\pi p_0/B_0^2$ (as usual), R is the radius of curvature of the magnetic field ($=r$, the radius of the cylindrical flux surface), and $L_B^{-1}=-\mathrm{d}\ln B_0/\mathrm{d}x$ and $L_p^{-1}=-\mathrm{d}\ln p_0/\mathrm{d}x$ are the local gradient scale lengths of the magnetic and thermal pressure, respectively.

(a) Show that (19.31–19.33) follow from the conventional MHD equations and that (19.34) is indeed the z-pinch equilibrium of these equations.

(b) Linearise (19.31–19.33) for $\boldsymbol{u}=\partial\boldsymbol{\xi}/\partial t$, $B=B_0+\delta B$, $\boldsymbol{b}=\boldsymbol{b}_0+\delta\boldsymbol{b}$ together with the MHD equations for $\rho=\rho_0+\delta\rho$ and $p=p_0+\delta p$ (keeping in mind that $\perp$ and $\parallel$ in those equations are with respect to the exact field direction, so it contains both $\boldsymbol{b}_0$ and $\delta\boldsymbol{b}$). This is an exercise equivalent to what was done in §18.2.1, but performed on a somewhat different form of MHD equations and with the purpose of analysing linear modes directly (as in §16.1) rather than via the energy principle (as in §18.4). Eye-balling your result, you should be able to discern how various MHD modes (Alfvén waves, slow waves, sound waves) are going to manifest themselves in this equilibrium.

(c) Show that the displacements parallel to $\boldsymbol{b}_0$ satisfy

$$\frac{\partial^2\xi_\parallel}{\partial t^2}=c_{\mathrm{s}}^2\nabla_\parallel\left(\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp+\nabla_\parallel\xi_\parallel+\frac{\xi_x}{R}\right), \tag{19.35}$$

where $\perp$ and $\parallel$ are now with respect to $\boldsymbol{b}_0$.

(d) Fourier transform in time ($\partial_t\to-i\omega$) and in the two homogeneous spatial dimensions ($\hat{\boldsymbol{y}}\cdot\boldsymbol{\nabla}\to ik_y$, $\boldsymbol{b}_0\cdot\boldsymbol{\nabla}\to im/r\equiv ik_\parallel$). Considering the equation for ξ_y and assuming perturbations satisfying the ordering

$$\frac{\partial}{\partial t} \sim k_{\parallel} v_{\mathrm{A}} \ll k_y c_{\mathrm{s}}, \qquad \frac{\delta p}{p_0} \sim \frac{\delta B}{B_0} \sim \partial_y \xi_y, \tag{19.36}$$

show that, to lowest order, these perturbations will be pressure-balanced. Using this fact to calculate $\boldsymbol{\nabla}_{\perp} \cdot \boldsymbol{\xi}_{\perp}$, show that

$$\left(\omega^2 - \omega_{\mathrm{S}}^2\right) \xi_{\parallel} = -i\omega_R \omega_{\mathrm{S}} \xi_x, \tag{19.37}$$

where $\omega_{\mathrm{S}}^2 = k_{\parallel}^2 c_{\mathrm{s}}^2/(1+\gamma\beta/2)$ and $\omega_R^2 = 4c_{\mathrm{s}}^2/R^2(1+\gamma\beta/2)$ (note that $\gamma\beta/2 = c_{\mathrm{s}}^2/v_{\mathrm{A}}^2$).

(e) Assuming further that, for the perturbations of interest, $\partial_x \ll k_y$, consider the equation for ξ_x and show that

$$\left(\omega^2 - \omega_{\mathrm{A}}^2 + \omega_*^2 - \omega_R^2\right) \xi_x = i\omega_R \omega_{\mathrm{S}} \xi_{\parallel}, \tag{19.38}$$

where $\omega_{\mathrm{A}}^2 = k_{\parallel}^2 v_{\mathrm{A}}^2$ and $\omega_*^2 = 2c_{\mathrm{s}}^2/\gamma R L_p$. Hence derive the dispersion relation.

(f) Derive the conditions for the 2D ($k_{\parallel} = 0$) and 3D ($k_{\parallel} \neq 0$) perturbations to be destabilised and show that these reproduce precisely the sausage (18.51) and kink (18.58) instability conditions.

(g) Study the solutions of your dispersion relation and describe what happens where in the parameter space $(k_{\parallel} R, \beta, R/L_p)$: what kind of perturbations are unstable and with what growth rates. You can do this analytically and/or numerically. You may find it useful to consider various asymptotic limits (e.g., $\beta \ll 1$, $\beta \gg 1$, low frequency, high frequency, etc.). Keep in mind that, in a z pinch, if $k_{\parallel} \neq 0$, then $k_{\parallel} R \geqslant 1$.

IUCUNDI (iterum) ACTI LABORES.

PART IV

Kinetic Theory of Plasmas in a Strong Magnetic Field

20. Kinetic MHD

In Part I of these Lectures, where kinetic theory was introduced, my starting point was the Vlasov–Landau equation (1.30), coupled to Maxwell's equations (1.23–1.26) for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of charge densities and currents. I then quickly set $\boldsymbol{B} = 0$ and spent all of my time investigating a collisionless plasma in the electrostatic approximation, described by the Vlasov–Poisson system (1.50–1.51). On a (seemingly) completely separate track, in Part III, I looked at the fluid dynamics of a conducting medium threaded by magnetic fields—while I often called this medium "plasma", I refused to engage with its microscopic structure and instead derived evolution equations for its mass density, velocity and pressure from the basic requirements imposed by conservation laws (§§14.1, 14.2, and 14.10).

Kinetic descriptions are often harder to wrap one's mind around than fluid ones and so we like imagining plasmas as fluids, characterised by density, velocity, and also pressure, temperature or some generalisations thereof. Even in the very kinetic lectures on such things as longitudinal plasma waves (§3), to inject some physical intuition, I occasionally appealed to fluid-dynamical descriptions: e.g., the "Langmuir hydrodynamics" (Exercise 3.1, then §8.1.1) and the "hydrodynamics of sound waves" (Exercise 3.6, then §8.1.2). Such descriptions reduce the dimensionality of the phase space from 6 to 3 and thus make things "simpler", expressing everything in terms of "intuitive" quantities—but why are these quantities intuitive?

Well, they are, of course, densities of conserved integrals (total number of particles, momentum, energy), but our predilection for limiting ourselves to them unless we absolutely must do otherwise (like in the case of Landau damping, §5.4, or phase-space turbulence, §§12–13) is rooted in the fact that gases that we are used to (e.g., Earth's atmosphere) are very collisional. In application to the Vlasov–Landau equation (1.30), this would mean that the collision term in the right-hand side of the equation were dominant, i.e., the typical collision rate for each species α were large compared to the typical frequencies associated with particle motions, waves, etc.:

$$\nu_\alpha \gg \omega, k v_{\mathrm{th}\alpha}. \tag{20.1}$$

Then, to lowest order in the large-ν_α expansion (e.g., Helander & Sigmar 2005; Parra 2019*a*; Kunz 2021),

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0 \quad \Rightarrow \quad f_\alpha(t,\boldsymbol{r},\boldsymbol{v}) = \frac{n_\alpha(t,\boldsymbol{r})}{\left[\pi v_{\mathrm{th}\alpha}^2(t,\boldsymbol{r})\right]^{3/2}} \exp\left[-\frac{|\boldsymbol{v}-\boldsymbol{u}_\alpha(t,\boldsymbol{r})|^2}{v_{\mathrm{th}\alpha}^2(t,\boldsymbol{r})}\right], \tag{20.2}$$

i.e., the distribution function of species α is a local Maxwellian with, for each species, some particle-number density n_α, flow velocity $\boldsymbol{u}_\alpha$, and temperature $T_\alpha = m_\alpha v_{\mathrm{th}\alpha}^2/2$, all functions of time (slow compared to ν_α) and space.[146] The dirty secret of the "ideal" MHD description is that, to get closed equations, I did, in fact, need collisions to be dominant—you can try to spot now where that happened in §14 (**exercise**); I will point this out in what follows.

In this part of the Lectures, I shall be interested in a physical regime where collisions

[146] It is convenient to let different species have different flow velocities and temperatures because these often equalise at a lesser rate than each species' distribution converges to a Maxwellian: see texts cited above (20.2), or any other textbook on plasma physics.

are not so dominant, viz.,

$$\nu_\alpha \sim \omega, kv_{\mathrm{th}\alpha}, \qquad \text{or even} \qquad \nu_\alpha \ll \omega, kv_{\mathrm{th}\alpha}, \tag{20.3}$$

but I will make a concerted effort to preserve as much of the fluid approach as possible. Indeed, we shall see that, in the presence of a strong magnetic field ("strong" in a sense shortly to be quantified), one can preserve quite a lot of it—thus making an explicit and aesthetically pleasing, as well as physically intuitive, link between kinetics and MHD.

20.1. *Peculiar Kinetics*

I would like to start with a preliminary technical step that will facilitate formulating plasma kinetics explicitly as a generalisation of the fluid theory. Let us change variables in the Vlasov–Landau equation (1.30) as follows:

$$(t, \boldsymbol{r}, \boldsymbol{v}) \to (t, \boldsymbol{r}, \boldsymbol{w}), \qquad \boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_\alpha(t, \boldsymbol{r}), \tag{20.4}$$

where $\boldsymbol{w}$ is called the "peculiar velocity" and, by definition,

$$\boldsymbol{u}_\alpha(t, \boldsymbol{r}) = \frac{1}{n_\alpha(t, \boldsymbol{r})} \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{20.5}$$

$$n_\alpha(t, \boldsymbol{r}) = \int \mathrm{d}\boldsymbol{v}\, f_\alpha(t, \boldsymbol{r}, \boldsymbol{v}), \tag{20.6}$$

whatever f_α might be (not, generally, a Maxwellian). Thus, the mean flow of species α ("fluid", or "ordered", motion) is explicitly separated from the particle's "disordered" motion. Under (20.4),

$$\left(\frac{\partial}{\partial t}\right)_{\boldsymbol{v}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{w}} + \left(\frac{\partial \boldsymbol{w}}{\partial t}\right)_{\boldsymbol{v}} \cdot \frac{\partial}{\partial \boldsymbol{w}} = \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{w}} - \frac{\partial \boldsymbol{u}_\alpha}{\partial t} \cdot \frac{\partial}{\partial \boldsymbol{w}}, \tag{20.7}$$

$$(\boldsymbol{\nabla})_{\boldsymbol{v}} = (\boldsymbol{\nabla})_{\boldsymbol{w}} + (\boldsymbol{\nabla} \boldsymbol{w})_{\boldsymbol{v}} \cdot \frac{\partial}{\partial \boldsymbol{w}} = (\boldsymbol{\nabla})_{\boldsymbol{w}} - (\boldsymbol{\nabla} \boldsymbol{u}_\alpha)_{\boldsymbol{v}} \cdot \frac{\partial}{\partial \boldsymbol{w}}, \tag{20.8}$$

$$\frac{\partial}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{w}} \tag{20.9}$$

(the subscripts tell you what is held constant while taking derivatives). The kinetic equation in these new variables is, therefore,

$$\frac{\partial f_\alpha}{\partial t} + (\boldsymbol{u}_\alpha + \boldsymbol{w}) \cdot \boldsymbol{\nabla} f_\alpha + \left\{ \frac{q_\alpha}{m_\alpha} \left[\boldsymbol{E} + \frac{(\boldsymbol{u}_\alpha + \boldsymbol{w}) \times \boldsymbol{B}}{c} \right] - \frac{\partial \boldsymbol{u}_\alpha}{\partial t} - (\boldsymbol{u}_\alpha + \boldsymbol{w}) \cdot \boldsymbol{\nabla} \boldsymbol{u}_\alpha \right\} \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{w}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}. \tag{20.10}$$

Let us introduce compact notation for a convective derivative with respect to the mean flow of species α,

$$\frac{\mathrm{d}}{\mathrm{d}t_\alpha} \equiv \left(\frac{\partial}{\partial t}\right)_{\boldsymbol{w}} + \boldsymbol{u}_\alpha \cdot (\boldsymbol{\nabla})_{\boldsymbol{w}}, \tag{20.11}$$

and group terms in (20.10) nicely:

$$\frac{\mathrm{d} f_\alpha}{\mathrm{d}t_\alpha} + \boldsymbol{w} \cdot \boldsymbol{\nabla} f_\alpha + \Bigg[\frac{q_\alpha}{m_\alpha} \frac{\boldsymbol{w} \times \boldsymbol{B}}{c} - \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u}_\alpha + \underbrace{\frac{q_\alpha}{m_\alpha} \left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c} \right) - \frac{\mathrm{d} \boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha}}_{\substack{\equiv\, \boldsymbol{a}_\alpha \\ \text{acceleration} \\ \text{independent of } \boldsymbol{w}}} \Bigg] \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{w}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}. \tag{20.12}$$

The terms in the square bracket involving $\boldsymbol{u}_\alpha$ are the non-inertial forces that the particle experiences in a frame moving with the mean flow of the species α.

20.2. *Momentum Equations*

The definition (20.5) of the mean flow velocity now becomes a constraint on the new distribution function‘$f_\alpha(t,\boldsymbol{r},\boldsymbol{w})$:

$$\int \mathrm{d}\boldsymbol{w}\,\boldsymbol{w} f_\alpha = 0. \tag{20.13}$$

Taking the first $\boldsymbol{w}$ moment of (20.12), $\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}$(20.12), and using (20.13) gives us an evolution equation for $\boldsymbol{u}_\alpha$—this is exactly equivalent to taking the first $\boldsymbol{v}$ moment of (1.30):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_\alpha}\underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w} f_\alpha}_{=0} + \boldsymbol{\nabla}\cdot\underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\boldsymbol{w} f_\alpha}_{\substack{\equiv\,\mathsf{P}_\alpha\\ \text{pressure}\\ \text{tensor}}}\\ + \underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\left(\frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{w}\times\boldsymbol{B}}{c} - \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\alpha + \boldsymbol{a}_\alpha\right)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{w}}}_{\substack{\text{integrate by parts, use (20.13),}\\ \text{get } -m_\alpha n_\alpha \boldsymbol{a}_\alpha}}\\ = \underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha \boldsymbol{w}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}}_{\substack{\equiv\,\boldsymbol{R}_\alpha\\ \text{friction force}}}.\end{aligned} \tag{20.14}$$

Thus, we have a force balance:

$$\boldsymbol{\nabla}\cdot\mathsf{P}_\alpha - m_\alpha n_\alpha \boldsymbol{a}_\alpha = \boldsymbol{R}_\alpha. \tag{20.15}$$

Substituting for $\boldsymbol{a}_\alpha$ from (20.12), we get the momentum equation for species α:

$$\boxed{m_\alpha n_\alpha \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = -\boldsymbol{\nabla}\cdot\mathsf{P}_\alpha + q_\alpha n_\alpha\left(\boldsymbol{E} + \frac{\boldsymbol{u}_\alpha\times\boldsymbol{B}}{c}\right) + \boldsymbol{R}_\alpha}\,. \tag{20.16}$$

To summarise, our new kinetic equation is (20.12), and it depends on the fields $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{u}_\alpha$, which are calculated from f_α and each other via Maxwell's equations (1.23–1.26) and the momentum equation (20.16). We are on the brink of being able to work with MHD variables. Recall that what we cared about in §14 was the flow of mass:

$$\rho = \sum_\alpha m_\alpha n_\alpha, \qquad \boldsymbol{u} = \frac{1}{\rho}\sum_\alpha m_\alpha n_\alpha \boldsymbol{u}_\alpha. \tag{20.17}$$

Let us see how the mass density and the mass-flow velocity evolve with time.

20.3. *Continuity Equation*

Taking the $\int \mathrm{d}\boldsymbol{w}$ (zeroth) moment of (20.12), we get

$$\frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha} + \int \mathrm{d}\boldsymbol{w}\left(\frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{w}\times\boldsymbol{B}}{c} - \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\alpha\right)\cdot\frac{\partial f_\alpha}{\partial \boldsymbol{w}} = 0, \tag{20.18}$$

where all other terms vanish by (20.13) and by the conservation of particles by collisions. After integration by parts, so does the middle term in (20.18), and we are left with

$$\frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha} + n_\alpha \boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha = 0. \tag{20.19}$$

Equivalently, and more conventionally,

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{\nabla}\cdot(n_\alpha \boldsymbol{u}_\alpha) = 0, \tag{20.20}$$

the continuity equation for species α, whence, after summation over species, $\sum_\alpha m_\alpha$(20.20), we get the mass-continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0}\,, \tag{20.21}$$

the same as in MHD (14.2).

20.4. *Mass Flow*

Now let us sum the momentum equation (20.16) over species:

$$\sum_\alpha m_\alpha n_\alpha \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = -\boldsymbol{\nabla}\cdot\mathsf{P} + \sigma\boldsymbol{E} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c}, \tag{20.22}$$

where $\mathsf{P} = \sum_\alpha \mathsf{P}_\alpha$ is the total pressure tensor,

$$\sigma \equiv \sum_\alpha q_\alpha n_\alpha = \frac{\boldsymbol{\nabla}\cdot\boldsymbol{E}}{4\pi} \tag{20.23}$$

by Gauss's law (14.9), and

$$\boldsymbol{j} \equiv \sum_\alpha q_\alpha n_\alpha \boldsymbol{u}_\alpha = \frac{c}{4\pi}\left(\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{20.24}$$

by Ampère's law (14.12). The terms involving $\boldsymbol{E}$ will be negligible, but we shall need a good estimate of the size of the electric field to prove this (in §20.5).

There is some tedious work to be done on the left-hand side of (20.22):

$$\sum_\alpha m_\alpha n_\alpha \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha} = \underbrace{\sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t_\alpha} m_\alpha n_\alpha \boldsymbol{u}_\alpha}_{\text{use (20.17)}} - \underbrace{\sum_\alpha m_\alpha \boldsymbol{u}_\alpha \frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha}}_{\text{use (20.19)}}$$

$$= \frac{\partial}{\partial t}\rho\boldsymbol{u} + \underbrace{\sum_\alpha \left[\boldsymbol{u}_\alpha\cdot\boldsymbol{\nabla} m_\alpha n_\alpha \boldsymbol{u}_\alpha + m_\alpha n_\alpha \boldsymbol{u}_\alpha \boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha\right]}_{\substack{=\boldsymbol{\nabla}\cdot\sum_\alpha m_\alpha n_\alpha \boldsymbol{u}_\alpha\boldsymbol{u}_\alpha \\ \text{and let } \boldsymbol{u}_\alpha = \boldsymbol{u}+\boldsymbol{u}'_\alpha\text{, where } \boldsymbol{u}'_\alpha \text{ is the} \\ \text{drift velocity of species } \alpha \text{ with respect to} \\ \text{the mass flow}}}$$

$$= \rho\frac{\partial\boldsymbol{u}}{\partial t} + \underbrace{\boldsymbol{u}\frac{\partial\rho}{\partial t}}_{\substack{\text{use}\\ (20.21)}} + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}\boldsymbol{u} + \underbrace{\sum_\alpha m_\alpha n_\alpha\left(\cancel{\boldsymbol{u}'_\alpha\boldsymbol{u}} + \cancel{\boldsymbol{u}\boldsymbol{u}'_\alpha} + \boldsymbol{u}'_\alpha\boldsymbol{u}'_\alpha\right)}_{\substack{\text{vanishings due to zero net drift:}\\ \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}'_\alpha = 0}}\right]$$

$$= \underbrace{\rho\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)}_{\equiv\,\mathrm{d}\boldsymbol{u}/\mathrm{d}t} + \boldsymbol{\nabla}\cdot\sum_\alpha m_\alpha n_\alpha \boldsymbol{u}'_\alpha\boldsymbol{u}'_\alpha. \tag{20.25}$$

Assembling this with (20.22), we arrive at the following evolution equation for the mean mass flow:

$$\boxed{\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\left(\mathsf{P} + \sum_\alpha m_\alpha n_\alpha \boldsymbol{u}'_\alpha\boldsymbol{u}'_\alpha\right) + \sigma\boldsymbol{E} + \frac{\boldsymbol{j}\times\boldsymbol{B}}{c}}\,. \tag{20.26}$$

20.5. *Ideal Ohm's Law*

To get the promised estimate for the size of $\boldsymbol{E}$, let us return to (20.16) and rearrange it so:

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_\alpha\times\boldsymbol{B}}{c} + \cancel{\frac{\boldsymbol{\nabla}\cdot\mathsf{P}_\alpha}{q_\alpha n_\alpha}} - \cancel{\frac{\boldsymbol{R}_\alpha}{q_\alpha n_\alpha}} + \cancel{\frac{m_\alpha}{q_\alpha}\frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha}}. \tag{20.27}$$

All the crossed-out terms are small, as follows. The last term compared to the first is

$$\frac{\left|\dfrac{m_\alpha}{q_\alpha}\dfrac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha}\right|}{\left|\dfrac{\boldsymbol{u}_\alpha\times\boldsymbol{B}}{c}\right|} \sim \frac{c m_\alpha\omega\cancel{u_\alpha}}{q_\alpha\cancel{u_\alpha}B} \sim \frac{\omega}{\Omega_\alpha} \ll 1, \tag{20.28}$$

where $\Omega_\alpha = q_\alpha B/cm_\alpha$ is the Larmor frequency of the particles of species α. We assume that the dynamics are always slow compared to the particles' Larmor motion:

$$k u_\alpha \sim \omega \ll \Omega_\alpha. \tag{20.29}$$

This is the sense in which the magnetic field shall be required to be "strong". Note that this has nothing to do with either the value of plasma beta (16.24) or with whether the magnetic field is dynamically strong enough to affect the mean (fluid) motion [cf. (14.45)].

Moving on to the friction term in (20.27) and using the fact that the friction (drag) force on species α by other species is (e.g., Helander & Sigmar 2005; Parra 2019*a*; Kunz

2021)

$$\boldsymbol{R}_\alpha = -m_\alpha n_\alpha \sum_{\alpha'} \nu_{\alpha\alpha'} \left(\boldsymbol{u}_\alpha - \boldsymbol{u}_{\alpha'}\right), \tag{20.30}$$

where $\nu_{\alpha\alpha'}$ is the interspecies collision rate, we can estimate

$$\frac{\left|\dfrac{\boldsymbol{R}_\alpha}{q_\alpha n_\alpha}\right|}{\left|\dfrac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c}\right|} \sim \frac{c m_\alpha \cancel{n_\alpha} \nu_{\alpha\alpha'} \cancel{u_\alpha}}{q_\alpha \cancel{n_\alpha} \cancel{u_\alpha} B} \sim \frac{\nu_{\alpha\alpha'}}{\Omega_\alpha} \ll 1. \tag{20.31}$$

Thus, collisions too are to be assumed slower than the Larmor motion. In kinetic theory, such a plasma is sometimes called *magnetised* or *weakly collisional*; if, at the same time, $\nu_{\alpha\alpha'} \gg \omega, kv_{\mathrm{th}\alpha}$, it is called *dilute* (by Balbus 2004; I shall come back to this limit in §22.3).

Finally, let us deal with the pressure term in (20.27):

$$\frac{\left|\dfrac{\boldsymbol{\nabla}\cdot\mathbf{P}_\alpha}{q_\alpha n_\alpha}\right|}{\left|\dfrac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c}\right|} \sim \frac{ckm_\alpha \cancel{n_\alpha} v_{\mathrm{th}\alpha}^2}{q_\alpha \cancel{n_\alpha} u_\alpha B} \sim \underbrace{\frac{kv_{\mathrm{th}\alpha}^2}{u_\alpha \Omega_\alpha} \sim \frac{kv_{\mathrm{th}i}^2}{u_i \Omega_i}}_{\substack{\text{assuming}\\ T_i \sim T_e}} \sim \frac{k\rho_i}{\mathrm{Ma}} \ll 1, \tag{20.32}$$

where $\mathrm{Ma} \sim u_i/v_{\mathrm{th}i}$ is the Mach number (cf. §16.2) and $\rho_i = v_{\mathrm{th}i}/\Omega_i$ is the ion Larmor radius (not to be confused with the ion mass density, which will always be called $m_i n_i$). To enable the neglect of the pressure term, we may assume

$$k\rho_i \ll \mathrm{Ma} \sim 1. \tag{20.33}$$

In fact, if $\mathrm{Ma} \sim 1$, the restriction that $k\rho_i \ll 1$ already follows from (20.29): $kv_{\mathrm{th}i} \sim ku_i \ll \Omega_i$.

What, however, if we want $\mathrm{Ma} \ll 1$, as we will do, in §21.5? Typically, in such an approximation, the pressure tensor will split into a slower- and faster-varying parts, $\mathbf{P}_\alpha = \mathbf{P}_{0\alpha} + \delta\mathbf{P}_\alpha$, where $\delta\mathbf{P}_\alpha \sim \mathrm{Ma}\mathbf{P}_{0\alpha}$. Then $\boldsymbol{\nabla}\cdot\mathbf{P}_\alpha \sim \boldsymbol{\nabla}\cdot\delta\mathbf{P}_\alpha \sim k\mathrm{Ma}\mathbf{P}_{0\alpha}$, so Ma in (20.32) cancels and $k\rho_i \ll 1$ is all that is needed. In this case, $k\rho_i \ll 1$ is an additional constraint, not equivalent to (20.29). Keeping the latter but allowing $k\rho_i \sim 1$ leads to a more sophisticated approximation, called *gyrokinetics* (§24.1). I will return to the possibility of subsonic flows (specifically, $\mathrm{Ma} \sim k\rho_i \ll 1$) in §§20.11 and 21.5.

Thus, to reiterate, we have assumed the ordering

$$\boxed{\nu_{\alpha\alpha'} \sim \omega \sim ku_\alpha \sim kv_{\mathrm{th}i} \ll \Omega_\alpha}\,, \tag{20.34}$$

and deduced, to lowest order in this approximation, that

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_\alpha \times \boldsymbol{B}}{c}, \tag{20.35}$$

an equation sometimes referred to as the "ideal Ohm's law". It gives us two things.

First, it implies immediately that the part of the mean flow velocity locally perpendicular to the magnetic field is

$$\boldsymbol{u}_{\perp\alpha} = c\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} = \boldsymbol{u}_\perp, \tag{20.36}$$

idependent of species, i.e., perpendicularly to $\boldsymbol{B}$, all species move in concert at the $\boldsymbol{E}\times\boldsymbol{B}$

drift velocity. This is the sense in which the magnetic field "fluidises" even a collisionless plasma.

Secondly, we can use (20.35) to estimate the size of $\boldsymbol{E}$ in (20.24) and (20.26):

$$\frac{\left|\dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t}\right|}{|\boldsymbol{\nabla}\times\boldsymbol{B}|} \sim \frac{\omega u_\alpha \cancel{B}}{c^2 k \cancel{B}} \sim \frac{u_\alpha^2}{c^2} \ll 1, \tag{20.37}$$

$$\frac{|\sigma\boldsymbol{E}|}{\left|\dfrac{\boldsymbol{j}\times\boldsymbol{B}}{c}\right|} \sim \frac{\cancel{k}E^2}{\cancel{k}B^2} \sim \frac{u_\alpha^2}{c^2} \ll 1. \tag{20.38}$$

These are, of course, exactly the same estimates as I made in §14.3 based on the MHD Ohm's law (14.13). In the absence of resistivity (i.e., in neglect of collisions), that is the same as (20.35), if use (20.36) to rewrite it as

$$\boxed{\boldsymbol{E} + \frac{\boldsymbol{u}\times\boldsymbol{B}}{c} = 0}\,. \tag{20.39}$$

To get $\eta\boldsymbol{j}$ in the right-hand side, write (20.27) for electrons, $\alpha = e$, and keep the friction force $\boldsymbol{R}_e$. A systematic derivation can be found, e.g., in the texts cited above (20.30).

20.6. *Almost MHD*

An immediate consequence of (20.36) is that all species drifts with respect to the mean mass flow can only be parallel to $\boldsymbol{B}$:

$$\boldsymbol{u}'_\alpha = u'_{\|\alpha}\boldsymbol{b}. \tag{20.40}$$

With this additional information, and with the neglect of the electric field allowed by (20.37) and (20.38), the momentum equation (20.26) becomes

$$\boxed{\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}\cdot\left(\mathsf{P} + \sum_\alpha m_\alpha n_\alpha u'^2_{\|\alpha}\boldsymbol{b}\boldsymbol{b} + \frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right)}\,, \tag{20.41}$$

where the Lorentz force has been expressed as the divergence of the Maxwell stress according to (14.20). Another immediate consequence of (20.39) is a closed equation for $\boldsymbol{B}$ in terms of $\boldsymbol{u}$: from Faraday's law (14.11),

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\boldsymbol{E} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u}}\,, \tag{20.42}$$

the familiar (ideal) MHD induction equation (§14.5).

Together with the mass-continuity equation (20.21), (20.41) and (20.42) look exactly like the first three ideal MHD equations (14.57–14.59), but with the scalar pressure replaced by the pressure tensor plus the additional stress associated with the species drifts. In a strongly collisional environment, the lowest-order approximation of (20.16) in the large-ν_α expansion is $\boldsymbol{R}_\alpha = 0$, so, by (20.30), there are no species drifts. In the same expansion, the pressure tensor is isotropic, $\mathsf{P} = p\mathsf{I}$, and thus MHD is recovered. This, however, requires the collision rate to be larger than all other frequencies, in particular, the Larmor frequency, breaking the assumption (20.34). Let us see what can be deduced about P without breaking this assumption.

20.7. *Gyrotropic Plasma*

Let us go back to our kinetic equation (20.12) and make the following observation about one of its terms:

$$\frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{w}\times\boldsymbol{B}}{c}\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}} = -\Omega_\alpha\left(\frac{\partial f_\alpha}{\partial\vartheta}\right)_{w_\perp,w_\parallel}, \tag{20.43}$$

where $\Omega_\alpha = q_\alpha B/cm_\alpha$ is again the Larmor frequency and I have made a change of variables in the velocity space:

$$\boldsymbol{w} = w_\perp\cos\vartheta\,\hat{\boldsymbol{x}} + w_\perp\sin\vartheta\,\hat{\boldsymbol{y}} + w_\parallel\,\hat{\boldsymbol{z}}, \tag{20.44}$$

with the triplet of orts $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{z}})$ chosen in such a way that $\hat{\boldsymbol{z}}$ is aligned with the local direction of the magnetic field, $\hat{\boldsymbol{z}} = \boldsymbol{b}$. Then

$$\left(\frac{\partial\boldsymbol{w}}{\partial\vartheta}\right)_{w_\perp,w_\parallel} = -w_\perp\sin\vartheta\,\hat{\boldsymbol{x}} + w_\perp\cos\vartheta\,\hat{\boldsymbol{y}} = -\boldsymbol{w}\times\boldsymbol{b}, \tag{20.45}$$

whence

$$(\boldsymbol{w}\times\boldsymbol{b})\cdot\frac{\partial}{\partial\boldsymbol{w}} = -\left(\frac{\partial}{\partial\vartheta}\right)_{w_\perp,w_\parallel}, \tag{20.46}$$

and (20.43) follows. We can, therefore, rewrite (20.12) so:

$$\Omega_\alpha\left(\frac{\partial f_\alpha}{\partial\vartheta}\right)_{w_\perp,w_\parallel} = \frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha} + \boldsymbol{w}\cdot\boldsymbol{\nabla}f_\alpha + (\boldsymbol{a}_\alpha - \boldsymbol{w}\cdot\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}} - \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}. \tag{20.47}$$

It is not hard to see that, if one assumes the ordering (20.34), all terms on the right-hand side of (20.47) are small compared to its left-hand side—the Larmor motion dominates. Therefore, to lowest order in the large-Ω_α expansion,

$$\Omega_\alpha\left(\frac{\partial f_\alpha}{\partial\vartheta}\right)_{w_\perp,w_\parallel} = 0 \quad\Rightarrow\quad \boxed{f_\alpha = f_\alpha(t,\boldsymbol{r},w_\perp,w_\parallel)}\,, \tag{20.48}$$

i.e., the distribution function does not depend on the gyroangle. Such a plasma is called *gyrotropic*. The evolution equation for f_α can be found by averaging (20.47) over ϑ and using (20.48): $\langle\text{rhs of (20.47)}\rangle_\vartheta = 0$.

I shall do this calculation in §20.9, but first, in pursuit of instant gratification, let me explore the immediate consequence of gyrotropy for the structure of the pressure tensor:

$$\mathsf{P}_\alpha = \int\mathrm{d}\boldsymbol{w}\,m_\alpha\boldsymbol{w}\boldsymbol{w}f_\alpha(t,\boldsymbol{r},w_\perp,w_\parallel) = m_\alpha\int\mathrm{d}w_\perp w_\perp\int\mathrm{d}w_\parallel f_\alpha(t,\boldsymbol{r},w_\perp,w_\parallel)\int\mathrm{d}\vartheta\,\boldsymbol{w}\boldsymbol{w}. \tag{20.49}$$

The gyroangle integral is easy:

$$\int\mathrm{d}\vartheta\,\boldsymbol{w}\boldsymbol{w} = 2\pi\langle\boldsymbol{w}\boldsymbol{w}\rangle_\vartheta = 2\pi\left[\frac{w_\perp^2}{2}(\mathsf{I}-\boldsymbol{b}\boldsymbol{b}) + w_\parallel^2\boldsymbol{b}\boldsymbol{b}\right]. \tag{20.50}$$

Therefore, the pressure tensor is diagonal,

$$\boxed{\mathsf{P}_\alpha = p_{\perp\alpha}(\mathsf{I}-\boldsymbol{b}\boldsymbol{b}) + p_{\parallel\alpha}\boldsymbol{b}\boldsymbol{b}}\,, \tag{20.51}$$

with the perpendicular and parallel pressures defined as

$$p_{\perp\alpha} = \int\mathrm{d}\boldsymbol{w}\,\frac{m_\alpha w_\perp^2}{2}f_\alpha, \qquad p_{\parallel\alpha} = \int\mathrm{d}\boldsymbol{w}\,m_\alpha w_\parallel^2 f_\alpha. \tag{20.52}$$

This gives us the final form of the momentum equation (20.41):

$$\boxed{\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla}\left(p_\perp + \frac{B^2}{8\pi}\right) + \boldsymbol{\nabla}\cdot\left[\boldsymbol{bb}\left(p_\perp - p'_\parallel + \frac{B^2}{4\pi}\right)\right]}\,, \tag{20.53}$$

where the combined pressures are

$$p_\perp = \sum_\alpha p_{\perp\alpha}, \qquad p_\parallel = \sum_\alpha p_{\parallel\alpha}, \qquad p'_\parallel = p_\parallel + \sum_\alpha m_\alpha n_\alpha u'^2_{\parallel\alpha}. \tag{20.54}$$

Thus, the difference between MHD and "kinetic MHD" boils down to $p_\perp - p'_\parallel \neq 0$ and to the need to calculate $p_\perp$ and $p'_\parallel$ kinetically. Manifestly, this pressure difference matters only if

$$|p_\perp - p'_\parallel| \gtrsim \frac{B^2}{4\pi} \quad\Leftrightarrow\quad \frac{|p_\perp - p'_\parallel|}{p_\parallel} \gtrsim \frac{B^2}{4\pi p_\parallel} \equiv \frac{2}{\beta_\parallel}, \tag{20.55}$$

i.e., *not* in low-beta plasmas (of which more later—their physics is different; see §23). Plasmas where both (20.55) and (20.34) are satisfied are usually in space (although, with ever more powerful lasers, it might be possible to get them in the lab: see Meinecke *et al.* 2022).

Exercise 20.1. Pressure tensor with gyroviscosity. Here you have an opportunity to hone your tensor-algebra and gyroaveraging skills, which is boring but useful—and, as a reward, you will get an expression for the pressure tensor that includes some non-gyrotropic corrections (which will later prove to play an interesting physical role: see Exercises 20.3 and 20.4).

(a) Convince yourself that the following identity holds exactly:

$$\boldsymbol{ww} = \frac{w_\perp^2}{2}(\mathsf{I} - \boldsymbol{bb}) + w_\parallel^2\boldsymbol{bb} + \frac{\partial\mathsf{T}}{\partial\vartheta}, \tag{20.56}$$

where, in, respectively, vector and index notation,

$$\mathsf{T} = \left(w_\parallel\boldsymbol{b} + \frac{\boldsymbol{w}_\perp}{4}\right)(\boldsymbol{w}_\perp\times\boldsymbol{b}) + (\boldsymbol{w}_\perp\times\boldsymbol{b})\left(w_\parallel\boldsymbol{b} + \frac{\boldsymbol{w}_\perp}{4}\right), \tag{20.57}$$

$$T_{ij} = \frac{1}{4}M_{ijkl}w_kw_l, \quad \text{where} \quad M_{ijkl} = (\delta_{ik} + 3b_ib_k)\epsilon_{jln}b_n + \epsilon_{iln}b_n(\delta_{jk} + 3b_jb_k). \tag{20.58}$$

Show therefore that the pressure tensor is, exactly,

$$P_{ij} = p_\perp\delta_{ij} - (p_\perp - p_\parallel)b_jb_j \underbrace{-\frac{1}{4}M_{ijkl}\int\mathrm{d}\boldsymbol{w}\,w_kw_l\frac{\partial f}{\partial\vartheta}}_{\equiv\, G_{ij}}, \tag{20.59}$$

suppressing the species index α here and in everything that follows. Use the kinetic equation (20.47) to show that, again exactly,

$$\int\mathrm{d}\boldsymbol{w}\,w_kw_l\frac{\partial f}{\partial\vartheta} = \frac{1}{\mathit{\Omega}}\left[\frac{\mathrm{d}P_{kl}}{\mathrm{d}t} + \nabla_m q_{mkl} + (\delta_{mn}P_{kl} + \delta_{kn}P_{ml} + \delta_{ln}P_{ml})\nabla_m u_n - C_{kl}\right], \tag{20.60}$$

where $q_{mkl} = m\int\mathrm{d}\boldsymbol{w}\,w_mw_kw_lf$ is the heat-flux tensor and $C_{kl} = m\int\mathrm{d}\boldsymbol{w}\,w_kw_l(\partial f/\partial t)_{\mathrm{c}}$.

(b) To obtain the next-order correction to the gyrotropic pressure tensor (20.51), one simply needs to calculate the last term in (20.59) via (20.60) assuming f to be gyrotropic (which it is to lowest order) in the right-hand side of the latter equation. Then gyroaveraging can be carried out in all velocity integrals similarly to how it was done in (20.49). Do this and show that

$$\boxed{\mathsf{G} = \frac{1}{\mathit{\Omega}}\left[\frac{\boldsymbol{b}\times\mathsf{S}\cdot(\mathsf{I} + 3\boldsymbol{bb}) - (\mathsf{I} + 3\boldsymbol{bb})\cdot\mathsf{S}\times\boldsymbol{b}}{4} + \boldsymbol{b}\,(\boldsymbol{\sigma}\times\boldsymbol{b}) + (\boldsymbol{\sigma}\times\boldsymbol{b})\,\boldsymbol{b}\right]}\,, \tag{20.61}$$

with the auxiliary notation

$$\mathsf{S} = [p_\perp \nabla \boldsymbol{u} + \nabla(q_\perp \boldsymbol{b})] + [p_\perp \nabla \boldsymbol{u} + \nabla(q_\perp \boldsymbol{b})]^T, \tag{20.62}$$

$$\boldsymbol{\sigma} = (p_\perp - p_\parallel)\left(\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \boldsymbol{b}\cdot\nabla\boldsymbol{u}\right) + (3q_\perp - q_\parallel)\,\boldsymbol{b}\cdot\nabla\boldsymbol{b}, \tag{20.63}$$

where $q_\perp = m\int \mathrm{d}\boldsymbol{w}\, w_\parallel (w_\perp^2/2) f$ and $q_\parallel = m\int \mathrm{d}\boldsymbol{w}\, w_\parallel^3 f$ are the parallel fluxes of the perpendicular and parallel heat, respectively. If you find this calculation difficult, some intermediate steps are shown in Appendix A of Schekochihin *et al.* (2010).

The tensor (20.61) is called the *gyroviscosity tensor*. With $p_\perp = p_\parallel$ and no heat fluxes, it first appeared in the paper by Braginskii (1965), who assumed sonic flows, $u \sim v_{\mathrm{th}i}$, a regime in which the heat fluxes worked out to be smaller. They got upgraded to compete with subsonic flows by Mikhailovskii & Tsypin (1971).

Exercise 20.2. Next-order corrections to Ohm's law. (a) Use the electron momentum equation to determine the next-order (in $k\rho_i$) corrections to the evolution of the magnetic field. Namely, in (20.27) with $\alpha = e$, retain the electron-pressure term, using the lowest-order (gyrotropic) approximation for it (what do you have to assume to do that while being allowed to drop the other terms?). Assume further that there are only two species in a plasma, electrons and a single species of ions, whose flow dominates the mass flow $\boldsymbol{u}$. Hence work out $\boldsymbol{u}_e$ in terms of $\boldsymbol{u}$ and the current density $\boldsymbol{j}$, which you can express in terms of $\boldsymbol{B}$ via Ampère's law. Show therefore that the magnetic field evolves according to

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\left\{\boldsymbol{u}\times\boldsymbol{B} + \frac{c}{en_e}\nabla\cdot\left[\left(p_{\perp e} + \frac{B^2}{8\pi}\right)\mathsf{I} - \boldsymbol{b}\boldsymbol{b}\left(p_{\perp e} - p_{\parallel e} + \frac{B^2}{8\pi}\right)\right]\right\}}. \tag{20.64}$$

The extra magnetic terms in this equation are known as the "Hall term" (and give rise to the Electron MHD approximation considered in Q-7 of the MHD Problem Set). In the absence of pressure anisotropy, pressure and density gradients, do they upset the conservation of magnetic flux?

(b) Show that (20.64) can be written in the form

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}_{\mathrm{eff}}\times\boldsymbol{B}) - \frac{c\nabla n_e\times\nabla p_{\parallel e}}{en_e^2} + \left(c\nabla\frac{p_{\perp e} - p_{\parallel e}}{en_e}\right)\times\frac{\nabla B}{B}, \tag{20.65}$$

where $\boldsymbol{u}_{\mathrm{eff}}$ is some effective velocity, which you must work out. Argue that the magnetic flux is frozen provided the last two terms on the right-hand side can be neglected. The first of these is known as the Biermann (1950) battery, a popular (in astrophysical and laser-plasma literature) mechanism for making magnetic fields from scratch, with no seed.

20.8. *Firehose Instability*

I will discuss the ways in which a pressure anistropy might arise later on, after we learn how to calculate $p_\perp$ and $p'_\parallel$, but first I want to take a break from deriving equations and extract some fun physics from what we have got. If you have done Exercise 16.9, you have a pretty good idea of what is about happen.

Let us take our new momentum equation (20.53), together with the induction equation (20.42), linearise them, and see what can be learned. The linearisation will be around a putative equilibrium with

$$\rho = \rho_0 = \text{const}, \quad p_\perp = p_{\perp 0} = \text{const}, \quad p'_\parallel = p'_{\parallel 0} = \text{const}, \quad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} = \text{const}, \tag{20.66}$$

and no flows [cf. (16.1)].

The linearised induction equation (20.42) is, as in MHD [cf. (16.5)],

$$-\omega\delta\boldsymbol{B} = B_0\left(k_\parallel\boldsymbol{u} - \hat{\boldsymbol{z}}\boldsymbol{k}\cdot\boldsymbol{u}\right), \tag{20.67}$$

turning, for $\delta\boldsymbol{B}_\perp/B_0 = \delta\boldsymbol{b}$ and $\delta B_\parallel = \delta B$, into

$$-\omega\delta\boldsymbol{b} = k_\parallel \boldsymbol{u}_\perp, \tag{20.68}$$

$$-\omega\frac{\delta B}{B_0} = -\boldsymbol{k}_\perp\cdot\boldsymbol{u}_\perp. \tag{20.69}$$

The linearised momentum equation (20.53) is, noting that $\boldsymbol{k}\cdot\delta\boldsymbol{b} = -k_\parallel\delta B/B_0$,

$$\begin{aligned}
-\omega\rho_0\boldsymbol{u} &= -\boldsymbol{k}\left(\delta p_\perp + \frac{B_0\delta B}{4\pi}\right)\\
&\quad + \boldsymbol{k}\cdot\left[(\hat{\boldsymbol{z}}\,\delta\boldsymbol{b} + \delta\boldsymbol{b}\,\hat{\boldsymbol{z}})\left(p_{\perp 0} - p'_{\parallel 0} + \frac{B_0^2}{4\pi}\right) + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\left(\delta p_\perp - \delta p'_\parallel + \frac{B_0\delta B}{2\pi}\right)\right]\\
&= -\boldsymbol{k}_\perp\left(\delta p_\perp + \frac{B_0\delta B}{4\pi}\right)\\
&\quad - \hat{\boldsymbol{z}}k_\parallel \underbrace{\left[\cancel{\delta p_\perp} + \cancel{\frac{B_0\delta B}{4\pi}} + \frac{\delta B}{B_0}\left(p_{\perp 0} - p'_{\parallel 0} + \cancel{\frac{B_0^2}{4\pi}}\right) - \cancel{\delta p_\perp} + \delta p'_\parallel - \cancel{\frac{B_0\delta B}{2\pi}}\right]}_{=\,\delta p'_\parallel + (p_{\perp 0} - p'_{\parallel 0})\frac{\delta B}{B_0}}\\
&\quad + \delta\boldsymbol{b}\,k_\parallel\left(p_{\perp 0} - p'_{\parallel 0} + \frac{B_0^2}{4\pi}\right).
\end{aligned} \tag{20.70}$$

Obviously, we cannot complete the job because we need the kinetic equation to calculate $\delta p_\perp$ and $\delta p'_\parallel$, but we *can* extract a piece that is entirely independent of any kinetic perturbations: let us take the perpendicular part of (20.70),

$$-\,\omega\rho_0\boldsymbol{u}_\perp = -\boldsymbol{k}_\perp\left(\delta p_\perp + \frac{B_0\delta B}{4\pi}\right) + \delta\boldsymbol{b}\,k_\parallel\left(p_{\perp 0} - p'_{\parallel 0} + \frac{B_0^2}{4\pi}\right), \tag{20.71}$$

and cross it with $\boldsymbol{k}_\perp$, noting that, from (20.68), $\delta\boldsymbol{b} = -k_\parallel\boldsymbol{u}_\perp/\omega$:

$$-\,\omega\rho_0\,\boldsymbol{k}_\perp\times\boldsymbol{u}_\perp = k_\parallel\left(p_{\perp 0} - p'_{\parallel 0} + \frac{B_0^2}{4\pi}\right)\left(-\frac{k_\parallel}{\omega}\,\boldsymbol{k}_\perp\times\boldsymbol{u}_\perp\right). \tag{20.72}$$

You guessed it right, these are Alfvén waves (cf. §16.1.1), with the dispersion relation

$$\boxed{\omega^2 = k_\parallel^2\left(\frac{p_{\perp 0} - p'_{\parallel 0}}{\rho_0} + v_A^2\right)}\,, \tag{20.73}$$

where, as in (16.10), $v_A = B_0/\sqrt{4\pi\rho_0}$. Like in MHD, these waves do not care about (do not involve) compressive perturbations—and, therefore, they do not care about kinetics, just about the equilibrium stress. Obvioulsy, to get the rest of the linear modes, we shall need kinetics.

But look at the terrifying thing that is manifest in (20.73): if

$$\frac{p_{\perp 0} - p'_{\parallel 0}}{\rho_0} + v_A^2 < 0 \quad\Leftrightarrow\quad \boxed{\frac{p'_{\parallel 0} - p_{\perp 0}}{p_{\parallel 0}} > \frac{B_0^2}{4\pi p_{\parallel 0}} \equiv \frac{2}{\beta_\parallel}}\,, \tag{20.74}$$

the Alfvén wave ceases to be a wave and becomes an instability—this is called the *firehose instability* (discovered in the days of yore by the giants of old: Rosenbluth 1956; Chandrasekhar *et al.* 1958; Parker 1958; Vedenov & Sagdeev 1958). Its physics is clear enough. Pressure anisotropy provides an additional stress, which weakens the restoring

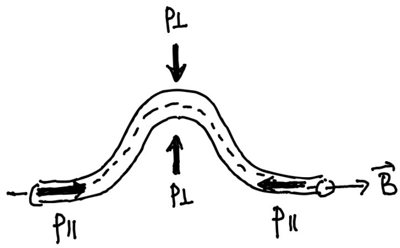


FIGURE 94. Firehose instability: magnetic flux tube buckling in a pressure-anisotropic environment ($p_\parallel > p_\perp$).

magnetic-tension force and, when large enough, turns the restoring force negative, so field lines lose elasticity and develop a tendency to buckle (Fig. 94).

A truly bad implication of (20.73) is that the instability's growth rate

$$\gamma = k_\parallel \sqrt{\frac{p'_{\parallel 0} - p_\perp}{\rho_0} - v_{\rm A}^2} \propto k_\parallel, \tag{20.75}$$

so the fastest-growing modes are at $k_\parallel \to \infty$, i.e., outside our approximation $k\rho_i \ll 1$. Thus, the equations that we are deriving are ill-posed in physical regimes where they can develop large enough (negative) pressure anisotropy to satisfy (20.74). Keep this in mind. We shall see later on (§22.1) that positive pressure anisotropy, $p_{\perp 0} > p'_{\parallel 0}$, is also bad.

Exercise 20.3. Peak growth rate of the firehose instability. Using the results of Exercises 20.1 and 20.2 (no need to have derived them to do this exercise), and assuming, as in §20.8, a homogeneous equilibrium with a non-zero negative ($p_\perp < p_\parallel$) pressure anisotropy and no heat fluxes ($q_\perp = q_\parallel = 0$), and also a single ion species, show that the dispersion relation for the growing firehose perturbations with $k_\perp = 0$ is

$$\boxed{\frac{\omega}{\Omega_i} = \pm\frac{(k_\parallel \rho_i)^2}{4}(1-\delta) + \frac{i|k_\parallel\rho_i|}{\sqrt{2}}|\Lambda|^{1/2}\sqrt{1-\frac{k_\parallel^2}{k_0^2}}, \quad k_0\rho_i = \frac{2\sqrt{2}}{|1-\delta|}|\Lambda|^{1/2}}, \tag{20.76}$$

where $\rho_i = v_{{\rm th}i}/\Omega_i$, $v_{{\rm th}i} = \sqrt{2p_{\parallel i}/m_i n_i}$,

$$\Lambda = \frac{p_{\perp i} - p_{\parallel i} + p_{\perp e} - p_{\parallel e}}{p_{\parallel i}} + \frac{2}{\beta_{\parallel i}} < 0, \quad \beta_{\parallel i} = \frac{8\pi p_{\parallel i}}{B^2}, \quad \delta = \frac{p_{\perp i} - p_{\parallel i} - (p_{\perp e} - p_{\parallel e})}{p_{\parallel i}} - \frac{2}{\beta_{\parallel i}}, \tag{20.77}$$

and I have suppressed 0's in the subscripts of the equilibrium quantities. Find the peak growth rate and the wave number at which it is reached.

Note that in the above calculation, you should drop the electron gyroviscosity tensor G_e because $\Omega_e \gg \Omega_i$. Besides this, what do you need to assume in order for your calculation, based on including the gyroviscosity tensor perturbatively, to be valid?

Exercise 20.4. Gyrothermal instability. Generalise the result of Exercise 20.3 to an equilibrium with non-zero ion heat fluxes and study again the stability of $k_\perp = 0$ perturbations. You should find that an instability exists provided

$$\boxed{\left(\frac{2q_{\perp i} - q_{\parallel i}}{p_{\parallel i} v_{{\rm th}i}}\right)^2 > \frac{(1-\delta)^2}{2}\Lambda}. \tag{20.78}$$

Study what happens very far from, and very close to, marginal stability. Note that even when there is no pressure anisotropy, there is an instability, driven purely by the ion heat fluxes—it is called the *gyrothermal instability (GTI)* (Schekochihin *et al.* 2010; Rosin *et al.* 2011).

You must be wondering what happens in nature: these instabilities grow, then what? How do they saturate? There is some research literature on this subject, which is sufficiently fresh that, instead of distilling it into a textbook-style exposition, I would rather invite you to read it. Here is a short, super-biased selection, whence you are free, however, to follow the paper trail into both the past and the future:

• Kunz *et al.* (2014) and Melville *et al.* (2016, and references therein)—how the firehose saturates. Basically, firehose fluctuations scatter particles and thus make the plasma effectively more collisional, pinning its pressure anisotropy to the instability threshold (20.74). The instability also has another way of saturating, not involving particle scattering: see Rosin *et al.* (2011), where a very long and very systematic kinetic calculation is done for a very simplified case—an exercise of mostly pedagogical use. A short precursor of that work, based, unlike Rosin's paper, entirely on kinetic MHD, without higher-order terms, was Schekochihin *et al.* (2008)—if your enthusiasm for wading through algebra is limited, that might prove to be a more manageable read.
• Squire *et al.* (2017, and references therein)—what happens if you try to propagate a finite-amplitude Alfvén wave through a high-beta plasma, but it gets interrupted by the loss of tension due to pressure anisotropy that it itself develops.
• Bott *et al.* (2021, 2025)—MHD turbulence in a firehose-infested plasma, e.g., the solar wind, which the particle scattering by firehoses renders effectively collisional and so quite MHD-like (in certain respects).
• Zhdankin *et al.* (2023)—firehose instability in a relativistic, radiating plasma (pushed to negative pressure anisotropy by synchrotron radiation depleting perpendicular energy of particles), which is, again, made effectively collisional as a consequence.

The interaction of large-scale MHD dynamics at high beta with a sea of firehose fluctuations usually triggered by it at small scales is an active research topic—do jump in!

20.9. *Gyroaveraged Kinetic Equation*

Going back to deriving equations, let me do what I promised to do after (20.48): gyroaverage the kinetic equation (20.47) and thus derive the evolution equation for f_α valid to lowest order in the KMHD expansion (20.34).

I shall process the right-hand side of (20.47) term by term. The first term:

$$\left\langle\left(\frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha}\right)_{\boldsymbol{w}}\right\rangle_\vartheta = \left(\frac{\mathrm{d}f_\alpha}{\mathrm{d}t_\alpha}\right)_{w_\perp,w_\parallel} + \cancel{\left\langle\left(\frac{\mathrm{d}w_\perp}{\mathrm{d}t_\alpha}\right)_{\boldsymbol{w}}\right\rangle_\vartheta}\frac{\partial f_\alpha}{\partial w_\perp} + \cancel{\left\langle\left(\frac{\mathrm{d}w_\parallel}{\mathrm{d}t_\alpha}\right)_{\boldsymbol{w}}\right\rangle_\vartheta}\frac{\partial f_\alpha}{\partial w_\parallel}, \quad (20.79)$$

where the last two terms vanish by the following property. Since $w_\perp = \sqrt{w^2 - w_\parallel^2}$ and $w_\parallel = \boldsymbol{w}\cdot\boldsymbol{b}$, any time or space derivatives of these taken at constant $\boldsymbol{w}$ and then gyroveraged are

$$\left\langle\left(\partial w_\parallel\right)_{\boldsymbol{w}}\right\rangle_\vartheta = \langle(\partial\boldsymbol{b})\cdot\boldsymbol{w}\rangle_\vartheta = w_\parallel(\partial\boldsymbol{b})\cdot\boldsymbol{b} = 0, \quad (20.80)$$

$$\langle(\partial w_\perp)_{\boldsymbol{w}}\rangle_\vartheta = -\frac{w_\parallel}{w_\perp}\left\langle\left(\partial w_\parallel\right)_{\boldsymbol{w}}\right\rangle_\vartheta = 0. \quad (20.81)$$

Now use the same kind of calculation to gyroaverage the second term of (20.47):

$$\begin{aligned}\left\langle\boldsymbol{w}\cdot(\boldsymbol{\nabla}f_\alpha)_{\boldsymbol{w}}\right\rangle_\vartheta &= w_\parallel\left(\nabla_\parallel f_\alpha\right)_{w_\perp,w_\parallel} + \underbrace{\left\langle\boldsymbol{w}\cdot(\boldsymbol{\nabla}w_\perp)_{\boldsymbol{w}}\right\rangle_\vartheta\frac{\partial f_\alpha}{\partial w_\perp} + \left\langle\boldsymbol{w}\cdot\left(\boldsymbol{\nabla}w_\parallel\right)_{\boldsymbol{w}}\right\rangle_\vartheta\frac{\partial f_\alpha}{\partial w_\parallel}}_{=\langle\boldsymbol{w}\cdot(\boldsymbol{\nabla}\boldsymbol{b})\cdot\boldsymbol{w}\rangle_\vartheta\left(-\frac{w_\parallel}{w_\perp}\frac{\partial f_\alpha}{\partial w_\perp}+\frac{\partial f_\alpha}{\partial w_\parallel}\right)} \\ &= w_\parallel\nabla_\parallel f_\alpha + \frac{\nabla_\parallel B}{B}\frac{w_\perp}{2}\left(w_\parallel\frac{\partial f_\alpha}{\partial w_\perp} - w_\perp\frac{\partial f_\alpha}{\partial w_\parallel}\right),\end{aligned} \quad (20.82)$$

where the gyroveraged expression was calculated using (20.50):

$$\langle \boldsymbol{w}\boldsymbol{w}\rangle_\vartheta : \boldsymbol{\nabla}\boldsymbol{b} = \frac{w_\perp^2}{2}\boldsymbol{\nabla}\cdot\boldsymbol{b} + \left(w_\parallel^2 - \frac{w_\perp^2}{2}\right)\underbrace{\boldsymbol{b}\boldsymbol{b}:\boldsymbol{\nabla}\boldsymbol{b}}_{=0} = -\frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}. \tag{20.83}$$

The gyroverage of the third term in (20.47) is

$$\begin{aligned}\left\langle \boldsymbol{a}_\alpha\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}}\right\rangle_\vartheta &= \boldsymbol{a}_\alpha\cdot\left(\underbrace{\left\langle\frac{\partial w_\perp}{\partial\boldsymbol{w}}\right\rangle_\vartheta}_{=\left\langle\frac{\boldsymbol{w}-w_\parallel\boldsymbol{b}}{w_\perp}\right\rangle_\vartheta=0}\frac{\partial f_\alpha}{\partial w_\perp} + \underbrace{\left\langle\frac{\partial w_\parallel}{\partial\boldsymbol{w}}\right\rangle_\vartheta}_{=\boldsymbol{b}}\frac{\partial f_\alpha}{\partial w_\parallel}\right) = \boldsymbol{a}_\alpha\cdot\boldsymbol{b}\frac{\partial f_\alpha}{\partial w_\parallel}\\ &= \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{\mathrm{d}\boldsymbol{u}_\alpha}{\mathrm{d}t_\alpha}\cdot\boldsymbol{b}\right)\frac{\partial f_\alpha}{\partial w_\parallel}.\end{aligned} \tag{20.84}$$

And finally the fourth term, where we again need (20.50):

$$\begin{aligned}-\left\langle\boldsymbol{w}\cdot(\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{w}}\right\rangle_\vartheta &= -\left\langle\boldsymbol{w}\cdot(\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\cdot\left(\frac{\boldsymbol{w}-w_\parallel\boldsymbol{b}}{w_\perp}\frac{\partial f_\alpha}{\partial w_\perp} + \boldsymbol{b}\frac{\partial f_\alpha}{\partial w_\parallel}\right)\right\rangle_\vartheta\\ &= -\langle\boldsymbol{w}\boldsymbol{w}\rangle_\vartheta : (\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\frac{1}{w_\perp}\frac{\partial f_\alpha}{\partial w_\perp} + \boldsymbol{b}\boldsymbol{b}:(\boldsymbol{\nabla}\boldsymbol{u}_\alpha)\,w_\parallel\left(\frac{w_\parallel}{w_\perp}\frac{\partial f_\alpha}{\partial w_\perp} - \frac{\partial f_\alpha}{\partial w_\parallel}\right)\\ &= \underbrace{(\boldsymbol{b}\boldsymbol{b}:\boldsymbol{\nabla}\boldsymbol{u}_\alpha - \boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha)}_{=\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t_\alpha}}\frac{w_\perp}{2}\frac{\partial f_\alpha}{\partial w_\perp} - w_\parallel\left(\nabla_\parallel\boldsymbol{u}_\alpha\right)\cdot\boldsymbol{b}\frac{\partial f_\alpha}{\partial w_\parallel}.\end{aligned} \tag{20.85}$$

Assembling (20.79), (20.82), (20.84), and (20.85), we get the desired gyroaveraged kinetic equation:

$$\boxed{\frac{Df_\alpha}{Dt_\alpha} + \frac{1}{B}\frac{DB}{Dt_\alpha}\frac{w_\perp}{2}\frac{\partial f_\alpha}{\partial w_\perp} + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{u}_\alpha}{Dt_\alpha}\cdot\boldsymbol{b} - \frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}\right)\frac{\partial f_\alpha}{\partial w_\parallel} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}}, \tag{20.86}$$

where the derivative along the (gyroaveraged) particle trajectory has been introduced:

$$\frac{D}{Dt_\alpha} = \frac{\mathrm{d}}{\mathrm{d}t_\alpha} + w_\parallel\nabla_\parallel = \frac{\partial}{\partial t} + \left(\boldsymbol{u}_\alpha + w_\parallel\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}. \tag{20.87}$$

It will be a refrain of these Lectures that things often become simpler, or more analytically convenient, or more physically revealing, when the variables describing the phase space of gyrating and drifting particles are changed. It is, in my view, the right attitude that there is no intrinsically "right" set of variables, just opportunistic choices. In the next two subsections, I will explore some of the rearrangements of the kinetic equation (20.86) associated with some such choices.

20.9.1. *Hiding Species Drifts*

The only place where the species drifts (20.40) appear is

$$\frac{D}{Dt_\alpha} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla} + \underbrace{\left(u'_{\parallel\alpha} + w_\parallel\right)}_{\equiv w'_\parallel}\nabla_\parallel = \frac{\mathrm{d}}{\mathrm{d}t} + w'_\parallel\nabla_\parallel \equiv \frac{D}{Dt}, \tag{20.88}$$

where I have shoved the drifts back into the parallel velocity variable, so

$$v_\parallel = u_{\parallel\alpha} + w_\parallel = u_\parallel + w'_\parallel. \tag{20.89}$$

Making this transformation of variables in (20.86) turns all the appearances of $\boldsymbol{u}_\alpha$ into $\boldsymbol{u}$:

$$\boxed{\frac{Df_\alpha}{Dt} + \frac{1}{B}\frac{DB}{Dt}\frac{w_\perp}{2}\frac{\partial f_\alpha}{\partial w_\perp} + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{u}}{Dt}\cdot\boldsymbol{b} - \frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}\right)\frac{\partial f_\alpha}{\partial w_\parallel'} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}}\,. \tag{20.90}$$

In this formulation, the drifts, if we want them, are moments of f_α:[147]

$$u_{\parallel\alpha}' = \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{w}\, w_\parallel' f_\alpha. \tag{20.91}$$

In fact, we do not want them all that much because the only place where we really need them for our MHD-like equations is as part of the expression for $p_{\parallel\alpha}'$ [see (20.54)], but that too is readily computed as a moment of f_α in our new parallel velocity variable:

$$p_{\parallel\alpha}' = \underbrace{\int \mathrm{d}\boldsymbol{w}\, m_\alpha w_\parallel^2 f_\alpha}_{=p_{\parallel\alpha}} + m_\alpha n_\alpha u_{\parallel\alpha}'^2 = \int \mathrm{d}\boldsymbol{w}\, m_\alpha w_\parallel'^2 f_\alpha. \tag{20.92}$$

All this is really just about which part of the particle motion is interpreted as "internal"—the peculiar velocity with respect to the mean flow $\boldsymbol{u}_\alpha$ of each species or with respect to the mean mass flow $\boldsymbol{u}$.

Sometimes (e.g., in the original derivation of KMHD by Kulsrud 1964) the parallel velocity variable $v_\parallel$ is left alone entirely, with no separation into mean and peculiar motion. Then, straightforwardly,

$$\boxed{\frac{Df_\alpha}{Dt} + \frac{1}{B}\frac{DB}{Dt}\frac{w_\perp}{2}\frac{\partial f_\alpha}{\partial w_\perp} + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{u}_\perp}{Dt}\cdot\boldsymbol{b} - \frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}\right)\frac{\partial f_\alpha}{\partial v_\parallel} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}}\,. \tag{20.93}$$

While this is slightly awkward because it makes $u_\parallel$ determined from the mass-flow evolution equation (20.53) redundant (as $u_\parallel$ can now also be determined directly from f_α), this form of kinetic equation presents certain technical advantages for some of the calculations that follow (in §§21 and 22.1).

Exercise 20.5. Derive Kulsrud's KMHD kinetic equation (20.93).

20.9.2. *Magnetic Moment*

Another transformation of variables that I would like to discuss is far more interesting (and consequential) physically. Let

$$(t, \boldsymbol{r}, w_\perp, w_\parallel') \to (t, \boldsymbol{r}, \mu, w_\parallel'), \qquad \mu = \frac{w_\perp^2}{2B}, \tag{20.94}$$

where μ is the "first adiabatic invariant", a.k.a., the "magnetic moment", of the particle motion. Physically, it is the angular momentum of a gyrating particle: if r is the (velocity-dependent) radius of gyration and $w_\perp$ its speed, then

$$m w_\perp r = \frac{m w_\perp^2}{\Omega} = \frac{m^2 c w_\perp^2}{qB} \propto \mu. \tag{20.95}$$

This is conserved by the gyromotion, as we shall prove directly in §21.3, but are about to see now from the form that the gyroaveraged kinetic equation takes in the variables (20.94).

---

[147] If we do not make this change of variables, the drifts have to be determined from the projection of (20.16) onto $\boldsymbol{b}$.

Under the transformation (20.94),

$$\left(\frac{Df_\alpha}{Dt}\right)_{w_\perp} = \left(\frac{Df_\alpha}{Dt}\right)_\mu + \left(\frac{D\mu}{Dt}\right)_{w_\perp}\frac{\partial f_\alpha}{\partial \mu} = \frac{Df_\alpha}{Dt} - \underbrace{\frac{w_\perp^2}{2B^2}\frac{DB}{Dt}}_{=\frac{\mu}{B}\frac{DB}{Dt}}\frac{\partial f_\alpha}{\partial \mu}, \tag{20.96}$$

$$\frac{\partial f_\alpha}{\partial w_\perp} = \frac{\partial \mu}{\partial w_\perp}\frac{\partial f_\alpha}{\partial \mu} = \frac{w_\perp}{B}\frac{\partial f_\alpha}{\partial \mu}. \tag{20.97}$$

If this is substituted into (20.90), the first two terms in the kinetic equation reveal themselves as always having wanted to have a cancellation and turn into the time derivative along the particle trajectory at constant μ. That is a manifestation of the conservation of μ by the particle motion—μ is then a kinetic variable with respect to which there are no derivatives in the kinetic equation, it enters as a parameter. The resulting form of the gyroaveraged kinetic equation is the simplest of all:

$$\boxed{\frac{Df_\alpha}{Dt} + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{u}}{Dt}\cdot\boldsymbol{b} - \mu\nabla_\parallel B\right)\frac{\partial f_\alpha}{\partial w_\parallel'} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}}\,. \tag{20.98}$$

This tells us simply that gyrating particles stream, conserve μ, and are subject to three parallel forces: the parallel electric field, the parallel part of the non-inertial force associated with the frame moving at the flow velocity, and the "mirror force" $-\mu\nabla_\parallel B$, which is an effective parallel force felt by the little circular current that is the particle's Larmor orbit.

20.10. *Kinetic MHD Equations Completed: Calculating $E_\parallel$*

So we are almost done with the derivation of a closed system of equations. From the gyroveraged kinetic equation (20.98), we can calculate

$$\rho = \sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{w}\, f_\alpha, \quad p_\perp = \sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{w}\, \mu B f_\alpha, \quad p_\parallel' = \sum_\alpha m_\alpha \int \mathrm{d}\boldsymbol{w}\, w_\parallel'^2 f_\alpha, \tag{20.99}$$

where, if we use the variables (20.94),

$$\int \mathrm{d}\boldsymbol{w} = 2\pi \int \mathrm{d}w_\perp w_\perp \int w_\parallel' = 2\pi B \iint \mathrm{d}\mu \mathrm{d}w_\parallel'. \tag{20.100}$$

This makes the continuity equation (20.21) redundant, and ρ, $p_\perp$ and $p_\parallel'$ go directly into the momentum equation (20.53), which also needs $\boldsymbol{B}$—it gets it from the induction equation (20.42). Then the resulting $\boldsymbol{u}$, alongside B and $\boldsymbol{b}$, get reinserted into the kinetic equation (20.98). The system is closed except we still need $E_\parallel$. Let us figure out how to get it.

Note first that the ideal Ohm's law (20.39) only gave us a non-zero $\boldsymbol{E}_\perp$, which, evidently, was much larger than $E_\parallel$. Well, the only equation from the original Vlasov–Maxwell set that we have not yet properly used is Gauss's law (20.23). The part of it containing $\boldsymbol{\nabla}\cdot\boldsymbol{E}$ is negligble: in the same spirit as we did the estimates in §20.5, we have

$$\frac{|\boldsymbol{\nabla}\cdot\boldsymbol{E}|}{4\pi e n_e} \sim \frac{kuB}{4\pi e n_e c} \sim \frac{ku\Omega_e}{\omega_{\rm pe}^2} \sim k\lambda_{\rm De}\frac{u}{v_{\rm th e}}\frac{\Omega_e}{\omega_{\rm pe}} \sim k\lambda_{\rm De}\mathrm{Ma}_i\sqrt{\frac{m_e}{m_i}}\frac{\Omega_e}{\omega_{\rm pe}} \ll 1. \tag{20.101}$$

This is very safely small in a number of small parameters. Physically, quasineutrality is violated only at $\omega \sim \omega_{\rm pe}$, but we will always stay well clear of that: $\omega \ll \Omega_e \ll \omega_{\rm pe}$.

Thus, we are left with

$$\boxed{\sum_\alpha q_\alpha n_\alpha = \sum_\alpha q_\alpha \int \mathrm{d}\boldsymbol{w}\, f_\alpha = 0}\,, \tag{20.102}$$

the quasineutrality constraint—from which $E_\parallel$ is determined implicitly, by the requirement that whatever f_α comes out of (20.98) must unfailingly satisfy (20.102). This completes the "hybrid" fluid-kinetic system known as *Kinetic MHD (KMHD)*.

Exercise 20.6. Energy budget in KMHD.[148] The total energy of a plasma is (cf. §14.10)

$$\mathscr{E} = \int \mathrm{d}\boldsymbol{r} \left[\sum_\alpha \left(\frac{m_\alpha n_\alpha u_\alpha^2}{2} + \frac{3}{2} p_\alpha \right) + \frac{B^2}{8\pi} \right]. \tag{20.103}$$

(a) Starting from the KMHD equations (§20.10), prove that the total energy is conserved, viz.,

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = 0, \tag{20.104}$$

assuming no energy flows through any boundaries of the system. Use any set of kinetic variables that you consider convenient for this purpose. If you find that handling the general case with species drifts is challenging (it is certainly somewhat tedious), assume that there are no species drifts. Cross-referencing your derivation with its MHD counterpart in §14.10 will help you see (some of) what is new in the KMHD description.

(b) Define the temperature of species α to be $T_\alpha = p_\alpha/n_\alpha$. Show that its evolution equation can be written in the following form:

$$\frac{3}{2} n_\alpha \frac{\mathrm{d}T_\alpha}{\mathrm{d}t_\alpha} = p_{\parallel\alpha} \frac{1}{n_\alpha} \frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha} - \boldsymbol{\nabla} \cdot (q_\alpha \boldsymbol{b}) + (p_{\perp\alpha} - p_{\parallel\alpha}) \frac{1}{B} \frac{\mathrm{d}B}{\mathrm{d}t_\alpha}, \tag{20.105}$$

where q_α is the heat flux and you may ignore any heating or cooling arising from collisions with particles of other species. What is the physical meaning of each of the three terms on the right-hand side of (20.105)?

Exercise 20.7. Reduced KMHD. The next step after deriving MHD equations in §14 was to consider what happens if a strong straight magnetic field is imposed externally. In §16.3, this led to Reduced MHD via the assumptions of small perturbations and strong anisotropy along the imposed field ($k_\parallel \ll k_\perp$). Work out how to derive a similar approximation for KMHD, leading to Reduced KMHD (or Kinetic RMHD). You may assume $f = f_0 + \delta f$, where the equilibrium distribution is homogeneous in space and, in general, anisotropic: $f_0 = f_0(w_\perp, w_\parallel)$. This is a good independent-study project, but if you want this study to be closely guided, refer to Kunz *et al.* (2015). A somewhat less demanding version of this calculation that you might want to study first is one where you assume f_0 to be a Maxwellian (so no equilibrium pressure anisotropy)—you will find the relevant derivations in Schekochihin *et al.* (2007, 2009) (in the second of these papers, KRMHD is arrived at by a different route, via gyrokinetics; see §24.1).

20.11. *High-Flow vs. Low-Flow Drift Kinetics*

Let me rewrite the gyroaveraged kinetic equation (20.98) in Kulsrud's original formulation (20.93), i.e., using $v_\parallel$ instead of $w'_\parallel$, but still retaining μ as the perpendicular velocity variable. By the same straightforward step as with (20.93), this simply involves shifting the parallel flow $u_\parallel$ back into the kinetic velocity variable and replacing $\boldsymbol{u}$ with $\boldsymbol{u}_\perp$ everywhere: our kinetic particles now live in a perpendicularly drifting "fluid" frame. As we saw in (20.36), this drift is just the particles' $\boldsymbol{E} \times \boldsymbol{B}$ drift, sometimes denoted

$$\boldsymbol{v}_E = c \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \tag{20.106}$$

---

[148] This is based on (a part of) the 2022 exam question.

The gyroaveraged kinetic equation in this language,

$$\boxed{\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v}_E\cdot\boldsymbol{\nabla} f_\alpha + v_\parallel \nabla_\parallel f_\alpha + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{v}_E}{Dt}\cdot\boldsymbol{b} - \mu\nabla_\parallel B\right)\frac{\partial f_\alpha}{\partial v_\parallel} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}}, \tag{20.107}$$

is often called the *drift-kinetic equation (DKE)*, stressing that it describes a drifting kinetic population. Obviously, the $\boldsymbol{E}\times\boldsymbol{B}$ drift is still determined as the perpendicular part of the mass-flow velocity, $\boldsymbol{v}_E = \boldsymbol{u}_\perp$, given by the momentum equation (20.53) [cf. Kulsrud's formulation (20.93)].

Recall that, in deriving this equation, we assumed, *inter alia*, that the mass flows are sonic in size, i.e., $u \sim v_E \sim v_{\mathrm{th}i}$, or $\mathrm{Ma}\sim 1$ [see (20.33)]. In drift kinetics, this is called the *high-flow ordering*, and it is this ordering that allows the $\boldsymbol{E}\times\boldsymbol{B}$ drift and the parallel particle streaming to enter on equal grounds in (20.107). It is, however, entirely possible, and indeed common, for plasma dynamics to be subsonic—indeed I spent quite a bit of time and energy discussing just such possibilities in MHD, in §§16.2–16.3. In drift kinetics, this is the *low-flow ordering*, whereby, specifically, one assumes

$$\mathrm{Ma} \sim k\rho_i \ll 1. \tag{20.108}$$

As I discussed very cursorily after (20.33), this does not break the arguments of §20.5 leading to the ideal Ohm's law (20.39). However, it certainly reduces the DKE (20.107) to a worrying degree: since now $v_E \ll v_{\mathrm{th}i}$, we are left with

$$\frac{\partial f_\alpha}{\partial t} + v_\parallel \nabla_\parallel f_\alpha + \left(\frac{q_\alpha}{m_\alpha}E_\parallel - \mu\nabla_\parallel B\right)\frac{\partial f_\alpha}{\partial v_\parallel} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}. \tag{20.109}$$

The reason this is worrying is that we now have a kinetic equation that contains no perpendicular motion of the particles—it is just 1D kinetics along the magnetic field! But we know that zero must never be taken for an answer in asymptotic theory, and that, therefore, this disappearance of an important element of physics means that we must go to next order to recapture it.[149] In §21, I will explain how to do it, and then do it.

21. Drift Kinetics from Particle Motion

We are ready to start studying the full implications of KMHD (at least in the high-flow ordering), but this time, I will go for delayed, rather than instant, gratification, and spend some time rewinding back to basics and discussing how kinetic equations are derived directly from particle motion. This will serve three purposes:

—solidify the physical interpretation of KMHD in terms of particle motion and μ conservation;
—offer a general scheme for systematic improvement of approximations;
—set us up for the derivation of drift kinetics in the low-flow ordering, the need for which was signalled in §20.11, and which is an important analytical framework for, especially, fusion physics.

This section largely follows the lecture notes of Parra (2019*b*), who based his exposition on Hazeltine (1973).

[149] Alternatively, we may have to infer that the terms that we count as large in our ordering will in fact turn out to be smaller, and, therefore, must be assumed to be so *a priori*: e.g., recall, that in RMHD, $k_\parallel/k_\perp \sim \mathrm{Ma} \ll 1$ [see (16.104)]. This will be hinted at in §21.6 and then done properly when gyrokinetics is introduced in §24.1.

21.1. *Relationship Between Particle Motion and Kinetic Equations*

Let me briefly go back to basics. Suppose we have a collection of particles in a phase space described by the coordinates $\boldsymbol{Q}$. Two examples that will be relevant in what follows are $\boldsymbol{Q} = (\boldsymbol{r},\boldsymbol{v})$ and $\boldsymbol{Q} = (\boldsymbol{r},\mu,v_\parallel,\vartheta)$. Suppose the motion of an individual particle is described by

$$\dot{\boldsymbol{Q}} = \boldsymbol{V}(\boldsymbol{Q},t), \tag{21.1}$$

where $\boldsymbol{V}$ is the generalised phase velocity, e.g., for $\boldsymbol{Q} = (\boldsymbol{r},\boldsymbol{v})$,

$$\boldsymbol{V}(\boldsymbol{r},\boldsymbol{v},t) = \left(\boldsymbol{v},\, \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}(\boldsymbol{r},t) + \frac{\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r},t)}{c}\right]\right). \tag{21.2}$$

Then, by exactly the same argument as, e.g., for the derivation of the continuity equation in §14.1, but applied to phase space, the conservation of particles implies

$$\boxed{\frac{\partial P}{\partial t} + \frac{\partial}{\partial \boldsymbol{Q}}\cdot\left(\boldsymbol{V}P\right) = 0}\,, \tag{21.3}$$

where $P(\boldsymbol{Q},t)$ is the phase-space density of particles. For $\boldsymbol{Q} = (\boldsymbol{r},\boldsymbol{v})$ and $P = f$, via (21.2), this gets us the Vlasov equation (1.30).[150]

Now let us assume that $\boldsymbol{Q} \neq (\boldsymbol{r},\boldsymbol{v})$, so there exists a transformation of variables

$$(\boldsymbol{r},\boldsymbol{v}) \to \boldsymbol{Q}(\boldsymbol{r},\boldsymbol{v},t). \tag{21.4}$$

The phase-space density in the new variables still satisfies (21.3) but $P \neq f$. Indeed the derivation of (21.3) required P to be normalised in such a way that it integrated to the (conserved!) total number of particles N. Therefore,

$$\int \mathrm{d}\boldsymbol{Q}\, P = N = \iint \mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\, f = \int \mathrm{d}\boldsymbol{Q}\, \underbrace{\left|\det\frac{\partial(\boldsymbol{r},\boldsymbol{v})}{\partial\boldsymbol{Q}}\right|}_{\equiv J} f, \tag{21.5}$$

where $J(\boldsymbol{Q},t)$ is the Jacobian of the transformation. Thus, if we stick with the definition of f in the "original" variables $(\boldsymbol{r},\boldsymbol{v})$, then, in any other variables,

$$P(\boldsymbol{Q},t) = J(\boldsymbol{Q},t) f\big(\boldsymbol{r}(\boldsymbol{Q},t),\boldsymbol{v}(\boldsymbol{Q},t),t\big). \tag{21.6}$$

Colloquially, I shall often refer to f in the new variables as $f(\boldsymbol{Q},t)$. For this f, (21.3) implies

$$\boxed{\frac{\partial Jf}{\partial t} + \frac{\partial}{\partial \boldsymbol{Q}}\cdot\left(\dot{\boldsymbol{Q}}Jf\right) = 0}\,. \tag{21.7}$$

By way of example, let us consider $\boldsymbol{Q} = (\boldsymbol{r},\mu,v_\parallel,\vartheta)$, the variables in which we ended up writing the KMHD kinetic equation (20.98), except with $w'_\parallel$ replaced by $v_\parallel$ as in (20.93) (this will turn out to be the most germane set for calculations involving direct consideration of particle motion). The transformation (21.4) is defined by

$$\mu = \frac{|(\boldsymbol{v}-\boldsymbol{u})\cdot(\mathsf{I}-\boldsymbol{bb})|^2}{2B}, \quad v_\parallel = \boldsymbol{v}\cdot\boldsymbol{b}, \quad \boldsymbol{v} = \boldsymbol{u}\cdot(\mathsf{I}-\boldsymbol{bb}) + \sqrt{2B\mu}\,(\hat{\boldsymbol{x}}\cos\vartheta + \hat{\boldsymbol{y}}\sin\vartheta) + v_\parallel\boldsymbol{b}. \tag{21.8}$$

The Jacobian is

$$J = \left|\det\frac{\partial(\boldsymbol{r},\boldsymbol{v})}{\partial\boldsymbol{Q}}\right| = \left|\det\frac{\partial\boldsymbol{v}}{\partial(\mu,v_\parallel,\vartheta)}\right| = \left|\det\begin{pmatrix} \sqrt{\dfrac{B}{2\mu}}\cos\vartheta & \sqrt{\dfrac{B}{2\mu}}\sin\vartheta & 0 \\ 0 & 0 & 1 \\ -\sqrt{2B\mu}\sin\vartheta & \sqrt{2B\mu}\cos\vartheta & 0\end{pmatrix}\right| = B. \tag{21.9}$$

---

[150] Without the collision term, obviously. Or, to be precise, if we use the exact $\boldsymbol{E}$ and $\boldsymbol{B}$, it gets us the Klimontovich equation (1.35), then we separate the macroscopic fields, shove the microscopic ones into the collision operator, then neglect the latter and get the Vlasov equation (§1.7).

The kinetic equation (21.7) is, therefore,

$$\frac{\partial Bf}{\partial t} + \boldsymbol{\nabla}\cdot(\dot{\boldsymbol{r}}Bf) + \frac{\partial}{\partial\mu}(\dot{\mu}Bf) + \frac{\partial}{\partial v_\parallel}(\dot{v}_\parallel Bf) + \frac{\partial}{\partial\vartheta}\left(\dot{\vartheta}Bf\right) = 0. \tag{21.10}$$

It turns out this can be rewritten simply as

$$\boxed{\frac{\partial f}{\partial t} + \dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla}f + \dot{\mu}\frac{\partial f}{\partial\mu} + \dot{v}_\parallel\frac{\partial f}{\partial v_\parallel} + \dot{\vartheta}\frac{\partial f}{\partial\vartheta} = 0}\,. \tag{21.11}$$

This might seem like a non-obvious step, requiring proof for the particular variable transformation that I have chosen. In fact, (21.7) always implies

$$\boxed{\frac{\partial f}{\partial t} + \dot{\boldsymbol{Q}}\cdot\frac{\partial f}{\partial\boldsymbol{Q}} = 0} \tag{21.12}$$

because, by the conservation of phase-space volume,

$$\frac{\partial J}{\partial t} + \frac{\partial}{\partial\boldsymbol{Q}}\cdot(\dot{\boldsymbol{Q}}J) = 0. \tag{21.13}$$

Proof. This is perhaps obvious [take (21.7) with $f = 1$], but here it is explicitly. First observe that phase volume is conserved in the $(\boldsymbol{r},\boldsymbol{v})$ space: for an infinitesimal volume $\delta\Gamma$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\delta\Gamma}\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} &= \frac{\mathrm{d}}{\mathrm{d}t}\delta x\delta y\delta z\delta v_x\delta v_y\delta v_z = \left(\iint_{\delta\Gamma}\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\right)\left(\frac{\delta\dot{x}}{\delta x} + \frac{\delta\dot{y}}{\delta y} + \frac{\delta\dot{z}}{\delta z} + \frac{\delta\dot{v}_x}{\delta v_x} + \frac{\delta\dot{v}_y}{\delta v_y} + \frac{\delta\dot{v}_z}{\delta v_z}\right)\\ &= \left(\iint_{\delta\Gamma}\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v}\right)\left(\frac{\partial}{\partial\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \frac{\partial}{\partial\boldsymbol{v}}\cdot\dot{\boldsymbol{v}}\right) = 0\end{aligned} \tag{21.14}$$

(obviously, because it is a Hamiltonian system; cf. §1.6). But this then implies

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\delta\Gamma}\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{v} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\delta\Gamma}\mathrm{d}\boldsymbol{Q}\,J(\boldsymbol{Q},t) = \frac{\mathrm{d}}{\mathrm{d}t}J(\boldsymbol{Q},t)\prod_{i=1}^{6}\delta Q_i\\ = \left[\int_{\delta\Gamma}\mathrm{d}\boldsymbol{Q}\,J(\boldsymbol{Q},t)\right]\left(\sum_{i=1}^{6}\frac{\delta\dot{Q}_i}{\delta Q_i} + \frac{1}{J}\frac{\mathrm{d}J}{\mathrm{d}t}\right) &= \left[\int_{\delta\Gamma}\mathrm{d}\boldsymbol{Q}\,J(\boldsymbol{Q},t)\right]\left[\frac{\partial}{\partial\boldsymbol{Q}}\cdot\dot{\boldsymbol{Q}} + \frac{1}{J}\left(\frac{\partial J}{\partial t} + \dot{\boldsymbol{Q}}\cdot\frac{\partial J}{\partial\boldsymbol{Q}}\right)\right].\end{aligned} \tag{21.15}$$

The expression in the second square bracket must, therefore, be equal to 0. But it is the same as (21.13), q.e.d.

Thus, any time we want a kinetic equation in some new set of variables, instead of carrying out variable transformations in the Vlasov equation, we can just work out $\dot{\boldsymbol{Q}}$ for a single particle and substitute that into (21.12). This turns out to be a useful method.

Exercise 21.1. Calculate the Jacobian of the transformation $(\boldsymbol{r},\boldsymbol{v}) \to (\boldsymbol{r},\mu,\varepsilon,\vartheta)$, where $\varepsilon = mw^2/2$, the kinetic energy of the particle's peculiar motion.

21.2. *General Perturbation Theory for Drift-Kinetic Equations*

The reason it is useful is that it allows us to construct a systematic perturbation theory for drift-kinetic equations describing particle motion in a strong magnetic field. The perturbation theory is in the small parameter $k\rho_\alpha$, which it is conventional in plasma literature to call ρ_*. We are already familiar with the lowest-order DKE (20.98), but sometimes one wants or needs to go to higher orders, typically just one, $O(\rho_*)$. Here is the least painful method for doing this.

Let me name $\boldsymbol{Q} = (\boldsymbol{r},\mu,v_\parallel)$, excluding ϑ. This is my new, reduced phase space, and I am about to describe how to construct an evolution equation for the distribution function averaged over ϑ. I must use (21.10) rather than (21.11) because I want to be able to average out the

ϑ derivative:

$$\frac{\partial Bf}{\partial t} + \frac{\partial}{\partial Q_i}\dot{Q}_i Bf + \frac{\partial}{\partial \vartheta}\dot{\vartheta}Bf = 0 \tag{21.16}$$

and, after gyroaveraging,

$$\frac{\partial B\langle f\rangle}{\partial t} + \frac{\partial}{\partial Q_i}B\langle\dot{Q}_i f\rangle = 0. \tag{21.17}$$

Let us now separate the gyroveraged parts of all quantities from the ϑ-dependent ones:

$$f = \langle f\rangle + \delta f, \qquad \dot{Q}_i = \langle\dot{Q}_i\rangle + \delta\dot{Q}_i. \tag{21.18}$$

This gives us, from (21.17),

$$B\left(\frac{\partial\langle f\rangle}{\partial t} + \langle\dot{Q}_i\rangle\frac{\partial\langle f\rangle}{\partial Q_i}\right) + \langle f\rangle\underbrace{\left(\frac{\partial B}{\partial t} + \frac{\partial}{\partial Q_i}\langle\dot{Q}_i\rangle B\right)}_{\substack{=0 \\ \text{gyroveraged (21.13)}}} + \frac{\partial}{\partial Q_i}B\langle\delta\dot{Q}_i\delta f\rangle = 0. \tag{21.19}$$

Thus, the evolution equation for the gyroveraged distribution function is

$$\boxed{\frac{\partial\langle f\rangle}{\partial t} + \langle\dot{Q}_i\rangle\frac{\partial\langle f\rangle}{\partial Q_i} + \frac{1}{B}\frac{\partial}{\partial Q_i}B\langle\delta\dot{Q}_i\delta f\rangle = 0}\,. \tag{21.20}$$

This is all exact. Now let us do the perturbation theory in ρ_*. We shall work out the equations for the particle motion later on (§21.3), but it is already obvious (§20.7) that

$$\dot{\vartheta} = -\Omega\left[1 + O(\rho_*)\right]. \tag{21.21}$$

By the same argument as in §20.7, to lowest order in ρ_*, (21.16) then just tells us that f is independent of ϑ, i.e., $f = \langle f\rangle$. Therefore, δf is $O(\rho_*)$ compared to $\langle f\rangle$, and so the third term in (21.20) is $O(\rho_*)$ compared to the first two: to lowest order,

$$\frac{\partial\langle f\rangle}{\partial t} + \langle\dot{Q}_i\rangle\frac{\partial\langle f\rangle}{\partial Q_i} = 0. \tag{21.22}$$

Thus, the gyroveraged distribution function simply evolves with the gyroveraged particle trajectories.

If we want the next-order correction (and you will see in §21.5 that sometimes we do), we need δf. Now it is convenient to use (21.11) and substitute (21.21) in it:

$$\frac{\partial f}{\partial t} + \dot{Q}_i\frac{\partial f}{\partial Q_i} + (-\Omega + \dots)\frac{\partial f}{\partial\vartheta} = 0. \tag{21.23}$$

Using (21.18) and the fact that (21.22) holds to lowest order, we get

$$\Omega\frac{\partial\delta f}{\partial\vartheta} = \delta\dot{Q}_i\frac{\partial\langle f\rangle}{\partial Q_i}. \tag{21.24}$$

Since $\delta\dot{Q}_i$ all have zero gyroverages, they must be representable as derivatives with respect to ϑ: say,

$$\delta\dot{Q}_i = -\Omega\frac{\partial\delta Q_i}{\partial\vartheta}, \tag{21.25}$$

where δQ_i is *defined* by this equation.[151] Therefore, integrating (21.24) gives us

$$\delta f = -\delta Q_i\frac{\partial\langle f\rangle}{\partial Q_i}. \tag{21.26}$$

[151]You can also think of (21.25) as following simply from $\delta\dot{Q}_i = \dot{\vartheta}(\partial\delta Q_i/\partial\vartheta)$ to lowest order, via (21.21). Formally, this would require proving that gyroveraging and time differentiation of Q_i commute to lowest order, but we need not bother with this.

Inserting this into (21.20), we get the first-order approximation to the evolution equation for $\langle f\rangle$:

$$\frac{\partial\langle f\rangle}{\partial t}+\langle\dot{Q}_i\rangle\frac{\partial\langle f\rangle}{\partial Q_i}=\frac{1}{B}\frac{\partial}{\partial Q_i}B\langle\delta\dot{Q}_i\delta Q_j\rangle\frac{\partial\langle f\rangle}{\partial Q_j}$$
$$=\cancel{\langle\delta\dot{Q}_i\delta Q_j\rangle\frac{\partial^2\langle f\rangle}{\partial Q_i\partial Q_j}}+\left(\frac{1}{B}\frac{\partial}{\partial Q_i}B\langle\delta\dot{Q}_i\delta Q_j\rangle\right)\frac{\partial\langle f\rangle}{\partial Q_j}. \tag{21.27}$$

The first of these terms vanishes because, using (21.25), we find that the correlator

$$\langle\delta\dot{Q}_i\delta Q_j\rangle=-\Omega\left\langle\frac{\partial\delta Q_i}{\partial\vartheta}\delta Q_j\right\rangle=\Omega\left\langle\delta Q_i\frac{\partial\delta Q_j}{\partial\vartheta}\right\rangle=-\langle\delta Q_i\delta\dot{Q}_j\rangle \tag{21.28}$$

is antisymmetric with respect to swapping $i\leftrightarrow j$. The second term on the right-hand side of (21.27) can now be combined with the second term on the left-hand side to give us the following DKE, valid to two leading orders in ρ_*:

$$\boxed{\frac{\partial\langle f\rangle}{\partial t}+\dot{Q}_i^{\rm eff}\frac{\partial\langle f\rangle}{\partial Q_i}=0}\,,\qquad\boxed{\dot{Q}_i^{\rm eff}=\langle\dot{Q}_i\rangle+\frac{1}{B}\frac{\partial}{\partial Q_j}B\langle\delta Q_j\delta\dot{Q}_i\rangle}\,, \tag{21.29}$$

where I have rearranged the equation to look like an advection of the gyroveraged distribution by an effective gyroaveraged phase-space flow.

It should be obvious how one would, if one cared to, derive further, higher-order DKEs. This is rarely, if ever, necessary.

So this is how far one can get in general. To give (21.29) a specific form, one must work out the various gyroaverages of the particle motion that enter into $\dot{Q}_i^{\rm eff}$. I shall do this in what follows, pointing out interesting physics as I go along.

21.3. *Particle Motion in a Strong Magnetic Field*

21.3.1. *Constant Fields*

Let us start with a first-year undergraduate problem: motion of a charged particle in crossed, constant $\boldsymbol{E}$ and $\boldsymbol{B}$ fields:

$$\dot{\boldsymbol{v}}=\frac{q}{m}\left(\boldsymbol{E}+\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right). \tag{21.30}$$

This splits into parallel motion,

$$\dot{v}_\parallel=\frac{q}{m}E_\parallel\quad\Rightarrow\quad v_\parallel(t)=v_{\parallel 0}+\frac{q}{m}E_\parallel t, \tag{21.31}$$

and perpendicular motion,

$$\dot{\boldsymbol{v}}_\perp=\frac{q}{m}\left(\boldsymbol{E}_\perp+\frac{\boldsymbol{v}_\perp\times\boldsymbol{B}}{c}\right). \tag{21.32}$$

The latter equation consists of a homogeneous and inhomogeneous bits, and has a particular integral

$$\boldsymbol{v}_\perp=c\frac{\boldsymbol{E}_\perp\times\boldsymbol{B}}{B^2}\equiv\boldsymbol{v}_E, \tag{21.33}$$

the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity—in a frame moving with this velocity, the particle sees no perpendicular electric field. The full solution of (21.32) is, therefore,

$$\boldsymbol{v}_\perp=\boldsymbol{w}_\perp+\boldsymbol{v}_E,\quad\text{where}\quad\dot{\boldsymbol{w}}_\perp=\Omega\,\boldsymbol{w}_\perp\times\boldsymbol{b},\quad\Omega=\frac{qB}{mc}. \tag{21.34}$$

Note that

$$\frac{{\rm d}}{{\rm d}t}w_\perp^2=0\quad\Rightarrow\quad w_\perp={\rm const}. \tag{21.35}$$

If we let $\boldsymbol{w}_\perp=w_\perp(\hat{\boldsymbol{x}}\cos\vartheta+\hat{\boldsymbol{y}}\sin\vartheta)$, where everything is constant except ϑ, then (21.34) gives us

$$\dot{\boldsymbol{w}}_\perp=w_\perp(-\hat{\boldsymbol{x}}\sin\vartheta+\hat{\boldsymbol{y}}\cos\vartheta)\,\dot{\vartheta}=\Omega w_\perp(\hat{\boldsymbol{x}}\sin\vartheta-\hat{\boldsymbol{y}}\cos\vartheta)\quad\Rightarrow\quad\dot{\vartheta}=-\Omega\quad\Rightarrow\quad\vartheta=\vartheta_0-\Omega t, \tag{21.36}$$

the Larmor motion (or gyromotion) of the particle. Assembling all this together, the particle's velocity is

$$\boxed{\boldsymbol{v}(t)=v_\parallel(t)\boldsymbol{b}+\boldsymbol{v}_E+w_\perp\left[\hat{\boldsymbol{x}}\cos\vartheta(t)+\hat{\boldsymbol{y}}\sin\vartheta(t)\right]}\,. \tag{21.37}$$

The particle's position is the integral of the above:

$$\boxed{\boldsymbol{r}(t)=\boldsymbol{r}_0+\left(v_{\parallel 0}t+\frac{qE_\parallel t^2}{2m}\right)\boldsymbol{b}+\boldsymbol{v}_E t-\underbrace{\frac{w_\perp}{\Omega}\left[\hat{\boldsymbol{x}}\sin\vartheta(t)-\hat{\boldsymbol{y}}\cos\vartheta(t)\right]}_{=\frac{\boldsymbol{w}_\perp(t)\times\boldsymbol{B}}{\Omega}}}\,. \tag{21.38}$$

21.3.2. *Slowly Varying Fields*

The idea now is that if $\boldsymbol{E}$ and $\boldsymbol{B}$ are functions of t and $\boldsymbol{r}$, particle motion will look similar to (21.38) as long as the characteristic time and length scales of the fields' variation are long compared to the Larmor frequency Ω and radius $\rho=w_\perp/\Omega$:

$$\rho_*\equiv k\rho\ll 1,\qquad \frac{\omega}{\Omega}\ll 1. \tag{21.39}$$

Note that if $\omega\sim kv_{\rm th}$, then $\rho_*\sim\omega/\Omega$, so the two small parameters are the same [cf. (20.34)].

The other key thing to order is the size of the electric field: from (21.31),

$$\omega v_{\rm th}\sim\frac{q}{m}E_\parallel\quad\Rightarrow\quad E_\parallel\sim\frac{m\omega v_{\rm th}}{q}, \tag{21.40}$$

and from (21.33),

$$E_\perp\sim\frac{v_E B}{c}\sim\frac{qB}{mc}\frac{mv_E}{q}\sim\frac{\Omega}{\omega}\frac{v_E}{v_{\rm th}}\frac{m\omega v_{\rm th}}{q}\sim\frac{\rm Ma}{\rho_*}E_\parallel. \tag{21.41}$$

Thus, $E_\perp\gg E_\parallel$ as long as ${\rm Ma}\gg\rho_*$—this is the high-flow ordering (§§20.11 and 21.4); in contrast, if ${\rm Ma}\sim\rho_*$, then $E_\perp\sim E_\parallel$, the low-flow ordering (§§20.11 and 21.5).

With these assumptions about the separation of scales, and with the experience of working with constant fields (§21.3.1), the motion of the particle in slowly varying fields is begging to be formally expressed as (21.37), but where now everything depends on t both directly and via the $\boldsymbol{r}(t)$ dependence of the fields, that dependence being slow ($\sim\omega$) except for ϑ. Recalling §20.9.2, we know that we will want to express $w_\perp=2B\mu$. Formally, this is a time-dependent transformation of variables

$$(\boldsymbol{r},\boldsymbol{v})\to(\boldsymbol{r},\mu,v_\parallel,\vartheta), \tag{21.42}$$

and we need to calculate $\dot{\boldsymbol{r}}$, $\dot\mu$, and $\dot v_\parallel$ if we want to write the kinetic equation (21.12) in the new variables.

Most straightforwardly,

$$\dot{\boldsymbol{r}}=v_\parallel\boldsymbol{b}+\boldsymbol{v}_E+\boldsymbol{w}_\perp, \tag{21.43}$$

and, anticipating the need to gyroaverage (§21.2),

$$\boxed{\langle\dot{\boldsymbol{r}}\rangle=v_\parallel\boldsymbol{b}+\boldsymbol{v}_E}\,. \tag{21.44}$$

Slightly more work will be needed to calculate $\dot v_\parallel$ and $\dot\mu$. In preparation for that, note that, with no approximations yet, we have, for any field $\psi(\boldsymbol{r},t)$,

$$\dot\psi=\left(\frac{\partial\psi}{\partial t}\right)_{\boldsymbol{r},\boldsymbol{v}}+\dot{\boldsymbol{r}}\cdot\left(\frac{\partial\psi}{\partial\boldsymbol{r}}\right)_{\boldsymbol{v},t}=\underbrace{\frac{\partial\psi}{\partial t}+(v_\parallel\boldsymbol{b}+\boldsymbol{v}_E+\boldsymbol{w}_\perp)\cdot\boldsymbol{\nabla}\psi}_{\equiv\frac{D\psi}{Dt}}. \tag{21.45}$$

Therefore, since $v_\parallel=\boldsymbol{v}\cdot\boldsymbol{b}$, we have

$$\dot v_\parallel=\underbrace{\dot{\boldsymbol{v}}\cdot\boldsymbol{b}}_{=\frac{q}{m}E_\parallel}+\left(\frac{D\boldsymbol{b}}{Dt}+\boldsymbol{w}_\perp\cdot\boldsymbol{\nabla}\boldsymbol{b}\right)\cdot\left(\cancel{v_\parallel\boldsymbol{b}}+\boldsymbol{v}_E+\boldsymbol{w}_\perp\right). \tag{21.46}$$

Gyroveraging this expression involves calculating

$$\langle \boldsymbol{w}_\perp \boldsymbol{w}_\perp \rangle : \boldsymbol{\nabla} \boldsymbol{b} = \frac{w_\perp^2}{2} (\mathsf{I} - \boldsymbol{b}\boldsymbol{b}) : \boldsymbol{\nabla} \boldsymbol{b} = \frac{w_\perp^2}{2} \boldsymbol{\nabla} \cdot \boldsymbol{b} = -\frac{w_\perp^2}{2B} \nabla_\parallel B = -\mu \nabla_\parallel B. \tag{21.47}$$

Therefore,

$$\boxed{\langle \dot{v}_\parallel \rangle = \frac{q}{m} E_\parallel - \frac{D\boldsymbol{v}_E}{Dt} \cdot \boldsymbol{b} - \mu \nabla_\parallel B}\,. \tag{21.48}$$

These are, as before [cf. §20.9.2 and (20.107)], the parallel electric force, the parallel projection of the non-inertial force due to the acceleration of the $\boldsymbol{E} \times \boldsymbol{B}$ frame, and the mirror force.

Turning to the evolution of the magnetic moment, we have

$$\dot{\mu} = \frac{w_\perp \dot{w}_\perp}{B} - \underbrace{\frac{w_\perp^2}{2B^2}}_{=\frac{\mu}{B}} \left(\frac{DB}{Dt} + \boldsymbol{w}_\perp \cdot \boldsymbol{\nabla} B \right). \tag{21.49}$$

Since $\boldsymbol{w}_\perp = (\boldsymbol{v} - \boldsymbol{v}_E) \cdot (\mathsf{I} - \boldsymbol{b}\boldsymbol{b})$ and $w_\perp = \sqrt{\boldsymbol{w}_\perp \cdot \boldsymbol{w}_\perp}$, we have

$$\begin{aligned}
\dot{w}_\perp &= \frac{\boldsymbol{w}_\perp \cdot \dot{\boldsymbol{w}}_\perp}{w_\perp} = \frac{\boldsymbol{w}_\perp}{w_\perp} \cdot \left(\dot{\boldsymbol{v}} - \dot{\boldsymbol{v}}_E - v_\parallel \dot{\boldsymbol{b}} \right) \\
&= \frac{\boldsymbol{w}_\perp}{w_\perp} \cdot \left[\cancel{\frac{q}{m} E_\parallel} \boldsymbol{b} + \Omega \cancel{\boldsymbol{w}_\perp} \times \boldsymbol{b} - \frac{D\boldsymbol{v}_E}{Dt} - \boldsymbol{w}_\perp \cdot \boldsymbol{\nabla} \boldsymbol{v}_E - v_\parallel \left(\frac{D\boldsymbol{b}}{Dt} + \boldsymbol{w}_\perp \cdot \boldsymbol{\nabla} \boldsymbol{b} \right) \right] \\
&= -\frac{1}{w_\perp} \left[\boldsymbol{w}_\perp \cdot \left(\frac{D\boldsymbol{v}_E}{Dt} + v_\parallel \frac{D\boldsymbol{b}}{Dt} \right) + \boldsymbol{w}_\perp \boldsymbol{w}_\perp : \left(\boldsymbol{\nabla} \boldsymbol{v}_E + v_\parallel \boldsymbol{\nabla} \boldsymbol{b} \right) \right], \qquad (21.50) \\
\langle \dot{w}_\perp \rangle &= -\frac{w_\perp}{2} \left(\boldsymbol{\nabla} \cdot \boldsymbol{v}_E - \boldsymbol{b}\boldsymbol{b} : \boldsymbol{\nabla} \boldsymbol{v}_E + v_\parallel \boldsymbol{\nabla} \cdot \boldsymbol{b} \right). \qquad (21.51)
\end{aligned}$$

Now observe that, since, identically, $\boldsymbol{E} = -\boldsymbol{v}_E \times \boldsymbol{B}/c + E_\parallel \boldsymbol{b}$, Faraday's law implies

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c \boldsymbol{\nabla} \times \boldsymbol{E} = -\boldsymbol{v}_E \cdot \boldsymbol{\nabla} \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{v}_E - \boldsymbol{B} \boldsymbol{\nabla} \cdot \boldsymbol{v}_E - c \boldsymbol{\nabla} \times \left(E_\parallel \boldsymbol{b} \right). \tag{21.52}$$

Dotting this with $\boldsymbol{B}/B^2$ gives us

$$\frac{1}{B} \left(\frac{\partial B}{\partial t} + \boldsymbol{v}_E \cdot \boldsymbol{\nabla} B \right) = \boldsymbol{b}\boldsymbol{b} : \boldsymbol{\nabla} \boldsymbol{v}_E - \boldsymbol{\nabla} \cdot \boldsymbol{v}_E - \frac{c}{B} \underbrace{\boldsymbol{b} \cdot \left(\boldsymbol{\nabla} \times E_\parallel \boldsymbol{b} \right)}_{= E_\parallel \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})}. \tag{21.53}$$

Using this expression in (21.51) yields

$$\begin{aligned}
\langle \dot{w}_\perp \rangle &= \frac{w_\perp}{2} \left[\frac{1}{B} \left(\frac{\partial B}{\partial t} + \boldsymbol{v}_E \cdot \boldsymbol{\nabla} B \right) + \frac{c}{B} E_\parallel \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) + v_\parallel \frac{\nabla_\parallel B}{B} \right] \\
&= \frac{\mu}{w_\perp} \left[\frac{DB}{Dt} + c E_\parallel \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b}) \right]. \qquad (21.54)
\end{aligned}$$

If we now use this to get the gyroaverage of (21.49), we find

$$\boxed{\langle \dot{\mu} \rangle = \mu \frac{c E_\parallel}{B} \boldsymbol{b} \cdot (\boldsymbol{\nabla} \times \boldsymbol{b})}\,. \tag{21.55}$$

Using the $E_\parallel$ estimate (21.40), (21.48) and (21.55) imply

$$\frac{\langle \dot{\mu} \rangle}{\mu} \sim k v_{\rm th} \frac{\omega}{\Omega} \ll \frac{\langle \dot{v}_\parallel \rangle}{v_\parallel} \sim \omega \sim k v_{\rm th}. \tag{21.56}$$

Thus, μ changes much more slowly than anything else—it is an adiabatic invariant of the particle motion (in fact, μ is the lowest-order approximation to an invariant that is conserved to all orders, with exponential precision: see Kruskal 1958, 1962).

Finally, the calculation of $\dot{\vartheta}$ is the same as (21.36):

$$\dot{\boldsymbol{w}}_\perp = \Omega \boldsymbol{w}_\perp \times \boldsymbol{b} + \dots \quad \Rightarrow \quad \dot{\vartheta} = -\Omega + \dots \tag{21.57}$$

The higher-order terms in $\dot{\boldsymbol{w}}_\perp$, which we already saw in the square bracket of the second line of (21.50), are there but will never be needed.

21.4. *High-Flow Drift Kinetics: KMHD Regained*

With these results, we can return to the derivation of drift-kinetic equations and immediately obtain the lowest-order result (21.22):

$$\frac{\partial\langle f\rangle}{\partial t} + \langle\dot{\boldsymbol{r}}\rangle\cdot\boldsymbol{\nabla}\langle f\rangle + \langle\dot{v}_\parallel\rangle\frac{\partial\langle f\rangle}{\partial v_\parallel} + \cancel{\langle\dot{\mu}\rangle\frac{\partial\langle f\rangle}{\partial\mu}} = 0. \tag{21.58}$$

Using (21.44), (21.48), and (21.55), we recover the high-flow DKE (20.107), a.k.a. the KMHD kinetic equation:

$$\boxed{\frac{\partial\langle f\rangle}{\partial t} + \left(\boldsymbol{v}_E + v_\parallel\boldsymbol{b}\right)\cdot\boldsymbol{\nabla}\langle f\rangle + \left(\frac{q}{m}E_\parallel - \frac{D\boldsymbol{v}_E}{Dt}\cdot\boldsymbol{b} - \mu\nabla_\parallel B\right)\frac{\partial\langle f\rangle}{\partial v_\parallel} = 0}\,. \tag{21.59}$$

Check. Now to something new.

Exercise 21.2. Use the same method to derive kinetic equations for $\langle f\rangle$ in the following sets of variables:
(a) $(\boldsymbol{r}, w_\perp, w_\parallel)$ [answer: (20.86)];
(b) $(\boldsymbol{r}, \mu, \varepsilon)$, where $\varepsilon = m_\alpha w^2/2$ is the kinetic energy of the peculiar motion of the particle of species α [confirm that the ways in which your answer is different from (22.42) make sense].

21.5. *Low-Flow Drift Kinetics: Going to Higher Order*

To go to next order in ρ_*, let us put into operation the general scheme (21.29). Besides $\langle\dot{\boldsymbol{r}}\rangle$, $\langle\dot{v}_\parallel\rangle$, and $\langle\dot{\mu}\rangle$ given by (21.44), (21.48) and (21.55), respectively, we will need the ϑ-dependent parts of $\dot{\boldsymbol{r}}$, $\dot{v}_\parallel$, and $\dot{\mu}$, which we can get by substracting the above expression from (21.43), (21.46), and (21.49):

$$\delta\dot{\boldsymbol{r}} = \boldsymbol{w}_\perp, \tag{21.60}$$

$$\delta\dot{v}_\parallel = \boldsymbol{w}_\perp\cdot\left[\frac{D\boldsymbol{b}}{Dt} + \cancel{(\boldsymbol{\nabla}\boldsymbol{b})\cdot\boldsymbol{v}_E}\right] + \left[\boldsymbol{w}_\perp\boldsymbol{w}_\perp - \frac{w_\perp^2}{2}(\mathbf{I} - \boldsymbol{b}\boldsymbol{b})\right] : \boldsymbol{\nabla}\boldsymbol{b}, \tag{21.61}$$

$$\delta\dot{\mu} = -\frac{1}{B}\left\{\boldsymbol{w}_\perp\cdot\left[\cancel{\frac{D\boldsymbol{v}_E}{Dt}} + v_\parallel\frac{D\boldsymbol{b}}{Dt} + \mu\boldsymbol{\nabla}B\right] + \left[\boldsymbol{w}_\perp\boldsymbol{w}_\perp - \frac{w_\perp^2}{2}(\mathbf{I} - \boldsymbol{b}\boldsymbol{b})\right] : \left(\cancel{\boldsymbol{\nabla}\boldsymbol{v}_E} + v_\parallel\boldsymbol{\nabla}\boldsymbol{b}\right)\right\}. \tag{21.62}$$

We can drop all terms involving $\boldsymbol{v}_E$ because $v_E \ll v_{\rm th}$ in the low-flow ordering and we only require the lowest-order expressions for δQ_i, as they will go into the quadratic corrections in $\dot{Q}_i^{\rm eff}$. By the same token, assuming that the rate of evolution of the magnetic field is $\omega \sim kv_E$,

$$\frac{D\boldsymbol{b}}{Dt} \approx v_\parallel\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} = v_\parallel\boldsymbol{\kappa}, \tag{21.63}$$

where $\boldsymbol{\kappa}$ is the curvature of the magnetic field line.

In order to get $\delta\boldsymbol{r}$, $\delta v_\parallel$, and $\delta\mu$ defined by (21.25), we must integrate (21.60–21.62) with respect to ϑ. This is easily done by noticing that

$$\boldsymbol{w}_\perp = w_\perp\left(\hat{\boldsymbol{x}}\cos\vartheta + \hat{\boldsymbol{y}}\sin\vartheta\right) = w_\perp\frac{\partial}{\partial\vartheta}\left(\hat{\boldsymbol{x}}\sin\vartheta - \hat{\boldsymbol{y}}\cos\vartheta\right) = \frac{\partial}{\partial\vartheta}\boldsymbol{w}_\perp\times\boldsymbol{b} \tag{21.64}$$

and, less obvious fact (which you derived in Exercise 20.1) that

$$\boldsymbol{w}_\perp\boldsymbol{w}_\perp - \frac{w_\perp^2}{2}(\mathbf{I} - \boldsymbol{b}\boldsymbol{b}) = \frac{\partial}{\partial\vartheta}\frac{1}{4}\left[\boldsymbol{w}_\perp(\boldsymbol{w}_\perp\times\boldsymbol{b}) + (w_\perp\times\boldsymbol{b})\boldsymbol{w}_\perp\right]. \tag{21.65}$$

Using these integration rules and retaining only the lowest-order terms, we get

$$\delta \boldsymbol{r} \approx -\frac{\boldsymbol{w}_\perp \times \boldsymbol{b}}{\Omega}, \tag{21.66}$$

$$\delta v_\parallel \approx -\frac{1}{\Omega}\left\{v_\parallel(\boldsymbol{w}_\perp \times \boldsymbol{b})\cdot\boldsymbol{\kappa} + \frac{1}{4}\left[\boldsymbol{w}_\perp(\boldsymbol{w}_\perp\times\boldsymbol{b}) + (w_\perp\times\boldsymbol{b})\boldsymbol{w}_\perp\right] : \boldsymbol{\nabla}\boldsymbol{b}\right\}, \tag{21.67}$$

$$\delta\mu \approx \frac{1}{B\Omega}\left\{(\boldsymbol{w}_\perp\times\boldsymbol{b})\cdot\left(v_\parallel^2\boldsymbol{\kappa} + \mu\boldsymbol{\nabla}B\right) + \frac{v_\parallel}{4}\left[\boldsymbol{w}_\perp(\boldsymbol{w}_\perp\times\boldsymbol{b}) + (w_\perp\times\boldsymbol{b})\boldsymbol{w}_\perp\right] : \boldsymbol{\nabla}\boldsymbol{b}\right\}. \tag{21.68}$$

With these in hand, we now must calculate

$$\dot{\boldsymbol{r}}^{\rm eff} = \boldsymbol{v}_E + v_\parallel\boldsymbol{b} + \frac{1}{B}\boldsymbol{\nabla}\cdot\left(B\langle\delta\boldsymbol{r}\delta\dot{\boldsymbol{r}}\rangle\right) + \frac{\partial}{\partial v_\parallel}\langle\delta v_\parallel\delta\dot{\boldsymbol{r}}\rangle + \frac{\partial}{\partial\mu}\langle\delta\mu\delta\dot{\boldsymbol{r}}\rangle, \tag{21.69}$$

$$\dot{v}_\parallel^{\rm eff} = \underbrace{\frac{q}{m}E_\parallel - \frac{D\boldsymbol{v}_E}{Dt}\cdot\boldsymbol{b}}_{\approx \frac{q}{m}\left(\boldsymbol{b} + \frac{v_\parallel}{\Omega}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\boldsymbol{E}} - \mu\nabla_\parallel B - \frac{1}{B}\boldsymbol{\nabla}\cdot\left(B\langle\delta v_\parallel\delta\dot{\boldsymbol{r}}\rangle\right) + \frac{\partial}{\partial\mu}\langle\delta\mu\delta\dot{v}_\parallel\rangle, \tag{21.70}$$

$$\dot{\mu}^{\rm eff} = \underbrace{\mu\frac{cE_\parallel}{B}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})}_{=\frac{q}{m}E_\parallel\frac{v_{\rm B}}{B}} - \frac{1}{B}\boldsymbol{\nabla}\cdot\left(B\langle\delta\mu\delta\dot{\boldsymbol{r}}\rangle\right) - \frac{\partial}{\partial v_\parallel}\langle\delta\mu\delta\dot{v}_\parallel\rangle. \tag{21.71}$$

I have made a number of simplifications. In the expressions for $\langle\dot{v}_\parallel\rangle$ and $\langle\dot{\mu}\rangle$, we must now keep two lowest order in ρ_*, but not more than that. Therefore, in $\dot{v}_\parallel^{\rm eff}$, we can approximate

$$\frac{D\boldsymbol{v}_E}{Dt}\cdot\boldsymbol{b} \approx v_\parallel(\nabla_\parallel\boldsymbol{v}_E)\cdot\boldsymbol{b} = -v_\parallel(\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b})\cdot\frac{c\boldsymbol{E}\times\boldsymbol{b}}{B} = -\frac{q}{m}\frac{v_\parallel}{\Omega}(\boldsymbol{b}\times\boldsymbol{\kappa})\cdot\boldsymbol{E}. \tag{21.72}$$

In $\dot{\mu}^{\rm eff}$, it was convenient to introduce a new quantity

$$\boxed{v_{\rm B} = \frac{\mu B}{\Omega}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})}\,, \tag{21.73}$$

called the *Baños drift*, whose physical significance will emerge shortly. As far as the quadratic corrections are concerned, because of the symmetry (21.28), we only need to calculate four different correlators. These calculations are greatly reduced by the observation that any gyroaverages involving an odd number of occurrances of $\boldsymbol{w}_\perp$ vanish, as does anything that has the form $\left\langle \text{scalar}\frac{\partial}{\partial\vartheta}\text{scalar}\right\rangle$. This includes in particular the terms where scalar $= [\boldsymbol{w}_\perp(\boldsymbol{w}_\perp\times\boldsymbol{b}) + (w_\perp\times\boldsymbol{b})\boldsymbol{w}_\perp] : \boldsymbol{\nabla}\boldsymbol{b}$, thus obviating the unpleasant second terms in (21.61) and (21.68).

21.5.1. *Magnetic Drifts*

Let us calculate the quadratic corrections to the gyroaveraged particle motion (21.69). The first of them requires the correlator

$$\langle\delta r_i\delta\dot{r}_j\rangle = -\left\langle\frac{\boldsymbol{w}_\perp\times\boldsymbol{b}}{\Omega}\boldsymbol{w}_\perp\right\rangle_{ij} = -\frac{\epsilon_{imn}\langle w_m w_j\rangle b_n}{\Omega} = -\frac{w_\perp^2}{2\Omega}\epsilon_{imn}\left(\delta_{mj} - b_m b_j\right)b_n = -\frac{\mu B}{\Omega}\epsilon_{ijn}b_n. \tag{21.74}$$

Therefore, the first quadratic corection in (21.69) is

$$\frac{1}{B}\boldsymbol{\nabla}\cdot\left(B\langle\delta\boldsymbol{r}\delta\dot{\boldsymbol{r}}\rangle\right) = \frac{1}{B}\boldsymbol{\nabla}\times\left(B\,\frac{\mu B}{\Omega}\,\boldsymbol{b}\right) = \frac{\mu}{\Omega}\left(B\cancel{\boldsymbol{\nabla}\times\boldsymbol{b}} - \cancel{\boldsymbol{b}\times\boldsymbol{\nabla}B}\right). \tag{21.75}$$

Note that $\boldsymbol{\nabla}$ is taken at constant μ, so the factor $\mu B/\Omega = \mu mc/q$ goes right through the gradient. Both terms will cancel in a moment. The correlator in the second quadratic correction in (21.69) is

$$\langle\delta v_\parallel \delta\dot{\boldsymbol{r}}\rangle = -v_\parallel \boldsymbol{\kappa}\cdot\left\langle\frac{\boldsymbol{w}_\perp\times\boldsymbol{b}}{\Omega}\boldsymbol{w}_\perp\right\rangle = -v_\parallel\frac{\mu B}{\Omega}\boldsymbol{b}\times\boldsymbol{\kappa} \tag{21.76}$$

$$\Rightarrow\quad \frac{\partial}{\partial v_\parallel}\langle\delta v_\parallel \delta\dot{\boldsymbol{r}}\rangle = v_{\mathrm{B}}\boldsymbol{b} - \frac{\mu B}{\Omega}\cancel{\boldsymbol{\nabla}\times\boldsymbol{b}}, \tag{21.77}$$

the last step being a consequence of the vector identity

$$\boldsymbol{b}\times\boldsymbol{\kappa} = -\boldsymbol{b}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b}) + \boldsymbol{\nabla}\times\boldsymbol{b}, \tag{21.78}$$

deployed to arrange for the calcellation of the $\boldsymbol{\nabla}\times\boldsymbol{b}$ term with its counterpart in (21.75). Finally, the correlator in the third quadratic correction in (21.69) is

$$\langle\delta\mu\delta\dot{\boldsymbol{r}}\rangle = \frac{v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B}{B}\cdot\left\langle\frac{\boldsymbol{w}_\perp\times\boldsymbol{b}}{\Omega}\boldsymbol{w}_\perp\right\rangle = \frac{\mu}{\Omega}\boldsymbol{b}\times\left(v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B\right) \tag{21.79}$$

$$\Rightarrow\quad \frac{\partial}{\partial\mu}\langle\delta\mu\delta\dot{\boldsymbol{r}}\rangle = \frac{v_\parallel^2}{\Omega}\boldsymbol{b}\times\boldsymbol{\kappa} + \cancel{2}\,\frac{\mu}{\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}B. \tag{21.80}$$

Note the cancellation of the factor of 2 with the second term in (21.75). We are ready to assemble (21.69):

$$\boxed{\dot{\boldsymbol{r}}^{\mathrm{eff}} = \boldsymbol{v}_E + \left(v_\parallel + v_{\mathrm{B}}\right)\boldsymbol{b} + \boldsymbol{v}_{\mathrm{d}}}\,,\qquad \boxed{\boldsymbol{v}_{\mathrm{d}} = \frac{1}{\Omega}\boldsymbol{b}\times\left(v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B\right)}\,. \tag{21.81}$$

Thus, going to next order in ρ_* elicited two new drifts: a small additional mean motion along the field lines, called the Baños drift (21.73), and two *magnetic drifts* acros the field: the *curvature drift* and the $\boldsymbol{\nabla}B$ *drift*, associated with two effective forces experienced by a gyrating particle, perhaps already familiar to you from undergraduate textbooks.

21.5.2. *Magnetic Forces*

We already have one of the ingredients needed to work out the quadratic corrections to the parallel acceleration (21.70): using (21.76), we get

$$-\frac{1}{B}\boldsymbol{\nabla}\cdot\left(B\langle\delta v_\parallel\delta\dot{\boldsymbol{r}}\rangle\right) = \frac{v_\parallel\mu}{\Omega}\boldsymbol{\nabla}\cdot(B\boldsymbol{b}\times\boldsymbol{\kappa}) = \frac{v_\parallel\mu}{\Omega}\left[\cancel{(\boldsymbol{b}\times\boldsymbol{\kappa})\cdot\boldsymbol{\nabla}B} + B\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa})\right], \tag{21.82}$$

where again, like in (21.75), $\mu B/\Omega$ was unaffected by $\boldsymbol{\nabla}$. The correlator required for the second quadratic correction in (21.70) is

$$\langle\delta\mu\delta\dot{v}_\parallel\rangle = \frac{v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B}{B}\cdot\left\langle\frac{\boldsymbol{w}_\perp\times\boldsymbol{b}}{\Omega}\boldsymbol{w}_\perp\right\rangle\cdot\boldsymbol{\kappa}v_\parallel = -\frac{v_\parallel\mu}{\Omega}\left[\left(v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B\right)\times\boldsymbol{b}\right]\cdot\boldsymbol{\kappa}$$

$$= -\frac{v_\parallel\mu^2}{\Omega}(\boldsymbol{b}\times\boldsymbol{\kappa})\cdot\boldsymbol{\nabla}B \tag{21.83}$$

$$\Rightarrow\quad \frac{\partial}{\partial\mu}\langle\delta\mu\delta\dot{v}_\parallel\rangle = -\cancel{2}\,\frac{v_\parallel\mu}{\Omega}(\boldsymbol{b}\times\boldsymbol{\kappa})\cdot\boldsymbol{\nabla}B. \tag{21.84}$$

Note that the factor of 2 cancels with one of the terms in (21.82). Assembling (21.70), we get

$$\boxed{\dot{v}_\parallel^{\mathrm{eff}} = \underbrace{\left(\boldsymbol{b}+\frac{v_\parallel}{\Omega}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\frac{q}{m}\boldsymbol{E}-\mu\boldsymbol{\nabla}B\right)}_{\equiv a_\parallel} + \frac{v_\parallel\mu B}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa})}\,. \tag{21.85}$$

Terms have combined nicely, but let us finish the calculation before discussing physics.

To get the quadratic corrections to the rate of change of the magnetic moment (21.71), we use (21.83) and (21.79):

$$-\frac{\partial}{\partial v_\parallel}\langle\delta\mu\delta\dot{v}_\parallel\rangle = \cancel{\frac{\mu^2}{\Omega}(\boldsymbol{b}\times\boldsymbol{\kappa})}\cdot\boldsymbol{\nabla}B, \tag{21.86}$$

$$\begin{aligned}-\frac{1}{B}\boldsymbol{\nabla}\cdot\big(B\langle\delta\mu\delta\dot{\boldsymbol{r}}\rangle\big) &= -\frac{\mu}{\Omega}\boldsymbol{\nabla}\cdot\left[\boldsymbol{b}\times\left(v_\parallel^2\boldsymbol{\kappa}+\mu\boldsymbol{\nabla}B\right)\right]\\ &= -\frac{v_\parallel^2\mu}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa}) - \underbrace{\frac{\mu^2}{\Omega}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})\boldsymbol{b}\cdot\boldsymbol{\nabla}B}_{=\frac{v_{\rm B}}{B}\mu\nabla_\parallel B} - \cancel{\frac{\mu^2}{\Omega}(\boldsymbol{b}\times\boldsymbol{\kappa})}\cdot\boldsymbol{\nabla}B.\end{aligned} \tag{21.87}$$

Assembling (21.71), we get

$$\boxed{\dot{\mu}^{\rm eff} = \frac{v_{\rm B}}{B}\underbrace{\left(\frac{q}{m}E_\parallel - \mu\nabla_\parallel B\right)}_{\approx\, a_\parallel} - \frac{v_\parallel^2\mu}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa})}\,. \tag{21.88}$$

Note that the parallel acceleration $a_\parallel$ that enters this expression is really the same one as in (21.85), except the $\boldsymbol{b}\times\boldsymbol{\kappa}$ bit need not be there to this order in ρ_*.

21.5.3. *Low-Flow Drift-Kinetic Equation*

With (21.81), (21.85) and (21.88) in hand, we can write the DKE valid to two orders in ρ_*:

$$\begin{aligned}\frac{D\langle f\rangle}{Dt} &= \frac{\partial\langle f\rangle}{\partial t} + (\boldsymbol{v}_E+\boldsymbol{v}_{\rm d})\cdot\boldsymbol{\nabla}_\perp\langle f\rangle + \left(v_\parallel+v_{\rm B}\right)\nabla_\parallel\langle f\rangle\\ &= -\left[a_\parallel + \frac{v_\parallel\mu B}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa})\right]\frac{\partial\langle f\rangle}{\partial v_\parallel} - \left[\frac{v_{\rm B}}{B}a_\parallel - \frac{v_\parallel^2\mu}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa})\right]\frac{\partial\langle f\rangle}{\partial\mu}.\end{aligned} \tag{21.89}$$

The $\boldsymbol{b}\times\boldsymbol{\kappa}$ terms are a bit of an eyesore. Let us make them pretty: using (21.78), we get

$$\frac{v_\parallel\mu}{\Omega}\boldsymbol{\nabla}\cdot(\boldsymbol{b}\times\boldsymbol{\kappa}) = -\frac{v_\parallel\mu}{\Omega}\boldsymbol{\nabla}\cdot\left(\boldsymbol{b}\,\frac{v_{\rm B}\Omega}{\mu B}\right) = -\frac{v_\parallel}{B}\boldsymbol{\nabla}\cdot(v_{\rm B}\boldsymbol{b}) = v_\parallel\left(\frac{v_{\rm B}}{B^2}\nabla_\parallel B - \frac{\nabla_\parallel v_{\rm B}}{B}\right) = -\nabla_\parallel\frac{v_\parallel v_{\rm B}}{B}, \tag{21.90}$$

recalling yet again that all gradients are taken at constant μ and $v_\parallel$. When physically untransparent terms like this turn up in pairs, this is often an indication that they really want to cancel—they are only there because the equation is written in "wrong" variables, and the cancellation can be consummated by an appropriate transformation to the "right" ones. It is not hard to guess what this transformation is: to wit,

$$\bar{v}_\parallel = v_\parallel + v_{\rm B}, \qquad \bar{\mu} = \mu - \frac{v_\parallel v_{\rm B}}{B}. \tag{21.91}$$

In these adjusted variables, and using (21.90), (21.89) becomes

$$\begin{aligned}\frac{D\langle f\rangle}{Dt} &= \frac{\partial\langle f\rangle}{\partial t} + (\boldsymbol{v}_E+\boldsymbol{v}_{\rm d})\cdot\boldsymbol{\nabla}_\perp\langle f\rangle + \bar{v}_\parallel\nabla_\parallel\langle f\rangle + \underbrace{\frac{Dv_{\rm B}}{Dt}}_{\approx\,\cancel{v_\parallel\nabla_\parallel v_{\rm B}}}\frac{\partial\langle f\rangle}{\partial\bar{v}_\parallel} - \underbrace{v_\parallel\left(\frac{D}{Dt}\frac{v_{\rm B}}{B}\right)}_{\approx\,\cancel{v_\parallel^2\nabla_\parallel\frac{v_{\rm B}}{B}}}\frac{\partial\langle f\rangle}{\partial\bar{\mu}}\\ &= \underbrace{\Bigg(-a_\parallel + v_\parallel\underbrace{B\nabla_\parallel\frac{v_{\rm B}}{B}}_{=\,\cancel{v_\parallel\nabla_\parallel v_{\rm B}} - \frac{v_\parallel v_{\rm B}}{B}\nabla_\parallel B}\Bigg)}_{=\,-a_\parallel,\ \text{but with}\ \mu\to\bar{\mu}}\left(\frac{\partial\langle f\rangle}{\partial\bar{v}_\parallel} - \cancel{\frac{v_{\rm B}}{B}\frac{\partial\langle f\rangle}{\partial\bar{\mu}}}\right) - \left(\cancel{\frac{v_{\rm B}}{B}a_\parallel} + \cancel{v_\parallel^2\nabla_\parallel\frac{v_{\rm B}}{B}}\right)\frac{\partial\langle f\rangle}{\partial\bar{\mu}},\end{aligned} \tag{21.92}$$

where only terms in the lowest two orders in ρ_* and Ma (which means the lowest order in all places where $v_{\rm B}$ is involved) have been retained. Taking account of all the cancellations, we end up with a very nicely compact equation:

$$\boxed{\frac{\partial\langle f\rangle}{\partial t}+(\boldsymbol{v}_E+\boldsymbol{v}_{\rm d})\cdot\boldsymbol{\nabla}_\perp\langle f\rangle+\bar{v}_\parallel\nabla_\parallel\langle f\rangle+\left(\boldsymbol{b}+\frac{\bar{v}_\parallel}{\Omega}\boldsymbol{b}\times\boldsymbol{\kappa}\right)\cdot\left(\frac{q}{m}\boldsymbol{E}-\bar{\mu}\boldsymbol{\nabla}B\right)\frac{\partial\langle f\rangle}{\partial\bar{v}_\parallel}=0}\,, \tag{21.93}$$

where $\boldsymbol{v}_{\rm d}$ is given by (21.81). The absence of a derivative with respect to $\bar{\mu}$ means that $\bar{\mu}$ is the adiabatic invariant conserved to $O(\rho_*^2)$—the existence of such a thing was promised after (21.56), with a reference to Kruskal (1958, 1962) (who proved conservation to all orders; just the second order had been done shortly before him by Hellwig 1955).

The most important new feature of (21.93) compared with (21.59) is the magnetic drifts $\boldsymbol{v}_{\rm d}$, which enter on par with $\boldsymbol{v}_E$ in the low-flow ordering. All other corrections are directly competing with larger terms, so will not be important, even though it would have been immoral to drop them in the low-flow ordering.

21.6. *Anisotropic Drift Kinetics*

There is another kind of immorality that is inherent in the low-flow DK equation (21.93): keeping subdominant terms really is something that ought not to be done. Either they are physically important and then they must be able to compete legitimately with everything else in the same, lowest, order, or they are not there to lowest order and are, therefore, unimportant. It is clear that perpendicular drifts *are* physically important, so the problem is the ordering. How one can fix this problem is, in fact, quite obvious: the lowest-order terms in (21.93) ought to be downgraded to be of the same size as (some of) the next-order effects that have been distilled in our expansion. Physically, it is easy to see the way forward if we recall how, in the derivation of RMHD, the general tendency of the gradients parallel to the magnetic field to be much smaller than the gradients perpendicular to it was weaponised to "reduce" the MHD equations, making them simpler (§16.3). Thus, if we keep the low-flow ordering of $\boldsymbol{v}_E\sim\rho_* v_{\rm th}$ but also assume that $\nabla_\parallel\sim\rho_*\boldsymbol{\nabla}_\perp$,[152] that brings the parallel streaming term in (21.93) down to the level of the perpendicular advection by the electric and magnetic drifts. Assuming, accordingly, that $\partial/\partial t$ and $E_\parallel$ are small enough to match this slower dynamics, we can rewrite (21.93) as follows:

$$\boxed{\frac{\partial\langle f\rangle}{\partial t}+(\boldsymbol{v}_E+\boldsymbol{v}_{\rm d})\cdot\boldsymbol{\nabla}_\perp\langle f\rangle+v_\parallel\nabla_\parallel\langle f\rangle+\left(\frac{q}{m}E_\parallel-\mu\nabla_\parallel B\right)\frac{\partial\langle f\rangle}{\partial v_\parallel}=0}\,. \tag{21.94}$$

Much of the work that we have done above thus gets tossed away: the only surviving additional low-flow term is the advection by the magnetic drifts $\boldsymbol{v}_{\rm d}$. For all the useful applications, however, this is all that is needed.

---

22. Kinetic MHD Continued

22.1. *Mirror Instability and Barnes Damping*

Now that we have derived the KMHD (drift-kinetic) equation, let us use it. Recall that in a simple, homogeneous, static equilibrium considered in §20.8, magnetic perturbations $\delta\boldsymbol{b}$ and δB satisfied (20.68) and (20.69), respectively, tying them to $\boldsymbol{u}_\perp$, which, in turn, obeyed (20.71). In §20.8, I extracted the $\delta\boldsymbol{b}$ part of this equation, which contained the Alfvén waves and the firehose instability. These modes had $\delta\boldsymbol{b}=-k_\parallel\boldsymbol{u}_\perp/\omega$ perturbations and everything else was $=0$. Now I want to extract modes with $\delta B\neq 0$. Since, according to (20.69), $\delta B/B_0=\boldsymbol{k}_\perp\cdot\boldsymbol{u}_\perp/\omega$, one can do this simply by dotting (20.71) with $\boldsymbol{k}_\perp$:

$$-\omega\rho_0\boldsymbol{k}_\perp\cdot\boldsymbol{u}_\perp=-k_\perp^2\left(\delta p_\perp+\frac{B_0\delta B}{4\pi}\right)+\boldsymbol{k}_\perp\cdot\delta\boldsymbol{b}\,k_\parallel\left(p_{\perp0}-p'_{\parallel0}+\frac{B_0^2}{4\pi}\right). \tag{22.1}$$

[152] There is nuance: one assumes this for the perturbations but not for the equilibrium fields. Thus, the scheme, as described here, is not quite rigorous—it is made so in §24.1.

Since $\boldsymbol{k}_\perp \cdot \delta\boldsymbol{b} = -k_\parallel \delta B/B_0$, this can be rewritten as follows:

$$\boxed{\omega^2 \frac{\delta B}{B_0} = k_\perp^2 \left(\frac{\delta p_\perp}{\rho_0} + v_{\rm A}^2 \frac{\delta B}{B_0} \right) + k_\parallel^2 \left(\frac{p_{\perp 0} - p'_{\parallel 0}}{\rho_0} + v_{\rm A}^2 \right) \frac{\delta B}{B_0}}\,. \tag{22.2}$$

Physically, this is a balance between, on the left-hand side, ion inertia and, on the right-hand side, the combined perpendicular pressure and the pressure-anisotropy-modified tension force.

Once we calculate $\delta p_\perp$, which we can do from the linearised kinetic equation, we will be done. Note that I have not done anything about the parallel part of the full linearised momentum equation (20.70), which is

$$-\omega \rho_0 u_\parallel = k_\parallel \left[\delta p'_\parallel + \left(p_{\perp 0} - p'_{\parallel 0} \right) \frac{\delta B}{B_0} \right], \tag{22.3}$$

but this can be obviated if one chooses the parallel kinetic variable to be $v_\parallel$ rather than $w'_\parallel$. Then (22.3) is entirely contained in the linearised kinetic equation.

22.1.1. *Formal Derivation of Dispersion Relation*

Thus, the kinetic equation that we shall use will be (20.107). In our equilibrium, $\boldsymbol{v}_E = 0$, $E_\parallel = 0$, and $\boldsymbol{B} = B_0 \hat{\boldsymbol{z}}$, so

$$\frac{\partial f_0}{\partial t} + v_\parallel \frac{\partial f_0}{\partial z} = 0, \tag{22.4}$$

and we are free to choose any f_0 as long as it is not a function of t or z. Let us then have

$$f = f_0(\mu, v_\parallel) + \delta f(t, \boldsymbol{r}, \mu, v_\parallel). \tag{22.5}$$

We will only need the perpendicular pressure [defined in (20.99)]:

$$\begin{aligned} p_\perp &= \int \mathrm{d}\boldsymbol{w}\, \frac{m w_\perp^2}{2} f = 2\pi m B^2 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \mu f = 2\pi m B_0^2 \left(1 + 2\frac{\delta B}{B_0} \right) \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \mu (f_0 + \delta f) \\ &= p_{\perp 0} + \underbrace{2 p_{\perp 0} \frac{\delta B}{B_0} + 2\pi m B_0^2 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \mu\, \delta f}_{= \delta p_\perp}\,. \end{aligned} \tag{22.6}$$

There is a subtlety here, which it is important to understand in the context of some calculations to follow. The equilibrium distribution $f_0(\mu, v_\parallel)$ is independent of $\boldsymbol{r}$ if one holds μ and $v_\parallel$ constant. But that is not the same as holding $w_\perp$ and $v_\parallel$ constant, except when $B = B_0$. Therefore, denoting $\mu_0 = w_\perp^2/2B_0$,

$$\begin{aligned} \int \mathrm{d}\boldsymbol{w}\, \frac{m w_\perp^2}{2} f_0(\mu, v_\parallel) &= \int \mathrm{d}\boldsymbol{w}\, \frac{m w_\perp^2}{2} f_0\left(\mu_0 \left(1 + \frac{\delta B}{B_0} \right)^{-1}, v_\parallel \right) \\ &\approx \int \mathrm{d}\boldsymbol{w}\, \frac{m w_\perp^2}{2} f_0(\mu_0, v_\parallel) - \frac{\delta B}{B_0} \int \mathrm{d}\boldsymbol{w}\, \frac{m w_\perp^2}{2} \left(\mu \frac{\partial f_0}{\partial \mu} \right)_{v_\parallel} (\mu_0, v_\parallel) \\ &= p_{\perp 0} - \frac{\delta B}{B_0} 2\pi m \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, J B \mu^2 \frac{\partial f_0}{\partial \mu} = p_{\perp 0} + 2\frac{\delta B}{B_0} p_{\perp 0}, \end{aligned} \tag{22.7}$$

where the second line is obtained by Taylor expansion in small $\delta B/B_0$, the last step is achieved via integration by parts, $J = B$ is the Jacobian of the transformation $(w_\perp, w_\parallel) \to (\mu, v_\parallel)$, and the distinction between B and B_0 in the second term does not matter to the linear order. The outcome is the same as in (22.6), but there it was achieved by a slightly different route: changing the integration variables $(w_\perp, w_\parallel) \to (\mu, v_\parallel)$ first, then expanding JB in small δB.

The two methods are equivalent, but the one involving expanding f_0 can be easier to follow for certain choices of variables, as, e.g., in Exercise 22.1.

Exercise 22.1. Explore what would happen to (22.7) if, instead of $(\boldsymbol{r}, \mu, v_\parallel)$, you used the variables $(\boldsymbol{r}, \mu, \varepsilon)$ given by (22.39). Be mindful of the fact that, for these new variables, $J = B/m|v_\parallel|$, where $|v_\parallel| = \sqrt{2(\varepsilon/m - \mu B)}$, and that

$$\left(\frac{\partial f_0}{\partial \mu}\right)_\varepsilon = \left(\frac{\partial f_0}{\partial \mu}\right)_{v_\parallel} - \frac{B}{v_\parallel}\left(\frac{\partial f_0}{\partial v_\parallel}\right)_\mu. \tag{22.8}$$

Continue the calculation of the mirror dispersion relation in these variables and make sure that you arrive at the same result (22.23). This exercise is pure algebra, but unless you can do this algebra with ease, you have not quite got the hang of (the technical side of) this subject. This is also a way to work through the following calculation in an active, rather than passive, mode.

To calculate δf, we linearise (20.107), which gives[153]

$$\delta f = -\frac{i}{\omega - k_\parallel v_\parallel}\left(\frac{q}{m}E_\parallel - ik_\parallel \mu \delta B\right)\frac{\partial f_0}{\partial v_\parallel}. \tag{22.9}$$

Recall that $E_\parallel$ is supposed to be determined by the quasineutrality (20.102). By the same token as (22.6), or (22.7),

$$n = 2\pi B \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, f = n_0 + \underbrace{n_0\frac{\delta B}{B_0} + 2\pi B_0 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \delta f}_{=\,\delta n}. \tag{22.10}$$

Therefore, (22.9) must respect

$$\sum_\alpha q_\alpha 2\pi B_0 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \delta f_\alpha = 0. \tag{22.11}$$

As it should, (22.11) gives us the parallel electric field:

$$E_\parallel = ik_\parallel \frac{\delta B}{B_0} \frac{\displaystyle\sum_\alpha q_\alpha 2\pi B_0^2 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \mu \frac{1}{\omega - k_\parallel v_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}}{\displaystyle\sum_\alpha \frac{q_\alpha^2}{m_\alpha} 2\pi B_0 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \frac{1}{\omega - k_\parallel v_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}} \equiv ik_\parallel \frac{\delta B}{B_0}\chi(\omega, k_\parallel). \tag{22.12}$$

Putting this back into (22.9), we get

$$\delta f = \frac{k_\parallel}{\omega - k_\parallel v_\parallel}\frac{\delta B}{B_0}\left[\frac{q}{m}\chi(\omega, k_\parallel) - \mu B_0\right]\frac{\partial f_0}{\partial v_\parallel}. \tag{22.13}$$

Finally, from (22.6),

$$\delta p_\perp = \left[2p_{\perp 0} + k_\parallel 2\pi m B_0^2 \iint \mathrm{d}v_\parallel \mathrm{d}\mu\, \mu \frac{(q/m)\chi(\omega, k_\parallel) - \mu B_0}{\omega - k_\parallel v_\parallel}\frac{\partial f_0}{\partial v_\parallel}\right]\frac{\delta B}{B_0}. \tag{22.14}$$

This is now ripe for substitution back into (22.2). The resulting dispersion relation can be derived and solved for the general f_0 (Exercise 22.2), but I prefer to simplify by showing you the popular case of a "bi-Maxwellian" equilibrium with no species drifts [cf. (6.45)]:

$$f_0(\mu, v_\parallel) = \frac{n_0}{\pi^{3/2} v_{\mathrm{th}\perp}^2 v_{\mathrm{th}\parallel}} \exp\left(-\frac{2\mu B_0}{v_{\mathrm{th}\perp}^2} - \frac{v_\parallel^2}{v_{\mathrm{th}\parallel}^2}\right). \tag{22.15}$$

[153] This is obviously a shorthand for a Laplace-transform solution along the same lines as in §3.

For such an equilibrium,

$$2\pi B_0 \int \mathrm{d}\mu\, f_0 = \frac{n_0}{\sqrt{\pi} v_{\mathrm{th}\parallel}}\, e^{-v_\parallel^2/v_{\mathrm{th}\parallel}^2} \equiv F_\mathrm{M}(v_\parallel), \tag{22.16}$$

$$2\pi B_0^2 \int \mathrm{d}\mu\, \mu f_0 = \frac{v_{\mathrm{th}\perp}^2}{2} F_\mathrm{M}(v_\parallel), \tag{22.17}$$

$$2\pi B_0^3 \int \mathrm{d}\mu\, \mu^2 f_0 = \frac{v_{\mathrm{th}\perp}^4}{2} F_\mathrm{M}(v_\parallel). \tag{22.18}$$

Therefore, carrying out the μ integral in (22.14),

$$\begin{aligned}\delta p_\perp &= \left\{2p_{\perp 0} - \frac{v_{\mathrm{th}\perp}^2}{v_{\mathrm{th}\parallel}^2}\left[q\chi(\omega,k_\parallel) - m v_{\mathrm{th}\perp}^2\right] \underbrace{\int \mathrm{d}v_\parallel \frac{k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel} F_\mathrm{M}(v_\parallel)}_{= -n_0\left[1+\zeta Z(\zeta)\right]}\right\}\frac{\delta B}{B_0} \\ &= \left\{2p_{\perp 0} + \frac{p_{\perp 0}}{p_{\parallel 0}}\left[n_0 q\chi(\omega,k_\parallel) - 2p_{\perp 0}\right]\left[1+\zeta Z(\zeta)\right]\right\}\frac{\delta B}{B_0},\end{aligned} \tag{22.19}$$

where $\zeta = \omega/|k_\parallel| v_{\mathrm{th}\parallel}$ and $Z(\zeta)$ is the plasma dispersion function (3.89); note that $p_{\perp,\parallel 0} = m n_0 v_{\mathrm{th}\perp,\parallel}^2/2$.

By a similar set of calculations,

$$\chi(\omega,k_\parallel) = \frac{\displaystyle\sum_\alpha n_\alpha q_\alpha \frac{p_{\perp\alpha}}{p_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]}{\displaystyle\sum_\alpha \frac{n_\alpha^2 q_\alpha^2}{p_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]}, \tag{22.20}$$

where I am now dropping 0's in the subscripts of the equilibrium quantities. This χ goes into (22.19), which, before it can be used in (22.2), has to be summed over species, because it is the total pressure perturbation that matters for the force balance. Normalising it to $\rho_0 v_\mathrm{A}^2 = B_0^2/4\pi$ and introducing $\beta_{\perp,\parallel\alpha} = 8\pi p_{\perp,\parallel\alpha}/B_0^2$, we get

$$\frac{\delta p_\perp}{\rho_0 v_\mathrm{A}^2} = \frac{\delta B}{B_0}\left\{\underbrace{\sum_\alpha \beta_{\perp\alpha} - \sum_\alpha \beta_{\perp\alpha}\frac{p_{\perp\alpha}}{p_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]}_{= -\sum_\alpha \beta_{\perp\alpha}\left[\underbrace{\left(\frac{p_{\perp\alpha}}{p_{\parallel\alpha}} - 1\right)}_{\equiv \Delta_\alpha} + \frac{p_{\perp\alpha}}{p_{\parallel\alpha}}\zeta_\alpha Z(\zeta_\alpha)\right]} + X(\omega,k_\parallel)\right\}, \tag{22.21}$$

where the last term comes from the contribution in (22.19) involving χ (which originates from $E_\parallel$):

$$X(\omega,k_\parallel) = \frac{\left(\displaystyle\sum_\alpha n_\alpha q_\alpha \frac{p_{\perp\alpha}}{p_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]\right)^2}{\displaystyle 2\sum_\alpha \frac{n_\alpha^2 q_\alpha^2}{\beta_{\parallel\alpha}}\left[1+\zeta_\alpha Z(\zeta_\alpha)\right]}. \tag{22.22}$$

The dispersion relation now follows from (22.2):

$$\boxed{\omega^2 = k_\perp^2 v_\mathrm{A}^2 \Bigg\{ \underbrace{1 - \sum_\alpha \beta_{\perp\alpha}\left[\varDelta_\alpha + \frac{p_{\perp\alpha}}{p_{\|\alpha}}\,\zeta_\alpha Z(\zeta_\alpha)\right]}_{\equiv -\varLambda} + X \Bigg\} + k_\|^2 v_\mathrm{A}^2 \underbrace{\left(1 + \frac{1}{2}\sum_\alpha \beta_{\|\alpha}\varDelta_\alpha\right)}_{\equiv Y}}\,. \tag{22.23}$$

This is obviously a complicated transcendental equation, which can be studied *ad nauseam*. I am not going to do this, but, in §22.1.2, will rather consider a simple but revealing limit.

Exercise 22.2. Mirror dispersion relation for a general equilibrium. This exercise is very boring. Instead of a bi-Maxwellian (22.15), consider a general homogeneous equilibrium $f_{0\alpha}(\mu, v_\|)$. Define

$$-2\pi B_0 \int \mathrm{d}\mu\, \frac{1}{v_\|}\frac{\partial f_{0\alpha}}{\partial v_\|} \equiv \frac{m_\alpha n_\alpha}{p_{\|\alpha}} F_{0\alpha}(v_\|), \tag{22.24}$$

$$-2\pi B_0^2 \int \mathrm{d}\mu\, \frac{\mu}{v_\|}\frac{\partial f_{0\alpha}}{\partial v_\|} \equiv \frac{p_{\perp\alpha}}{p_{\|\alpha}} F_{1\alpha}(v_\|), \tag{22.25}$$

$$-2\pi B_0^3 \int \mathrm{d}\mu\, \frac{\mu^2}{v_\|}\frac{\partial f_{0\alpha}}{\partial v_\|} \equiv \frac{2p_{\perp\alpha}^2}{m_\alpha n_\alpha p_{\|\alpha}} F_{2\alpha}(v_\|), \tag{22.26}$$

$$\int \mathrm{d}v_\|\, F_{l\alpha}(v_\|) \equiv n_\alpha C_{l\alpha}, \quad l = 0, 1, 2, \tag{22.27}$$

$$\int \mathrm{d}v_\| \frac{F_{l\alpha}(v_\|)}{v_\| - \omega/|k_\||} \equiv \frac{n_\alpha}{v_{\mathrm{th}\|\alpha}} Z_{l\alpha}(\zeta_\alpha), \quad \zeta_\alpha \equiv \frac{\omega}{|k_\||v_{\mathrm{th}\|\alpha}}, \quad v_{\mathrm{th}\|\alpha} \equiv \sqrt{\frac{2p_{\|\alpha}}{m_\alpha n_\alpha}}. \tag{22.28}$$

For a bi-Maxwellian $f_{0\alpha}$, all $C_{l\alpha} = 1$ and $Z_{l\alpha}(\zeta_\alpha) = Z(\zeta_\alpha)$. With the above definitions, prove that the general mirror dispersion relation is the same as (22.23) but with the expression in the figure brackets replaced by

$$\Big\{\dots\Big\} \to 1 - \sum_\alpha \beta_{\perp\alpha}\left[\left(\frac{p_{\perp\alpha}}{p_{\|\alpha}} C_{2\alpha} - 1\right) + \frac{p_{\perp\alpha}}{p_{\|\alpha}}\,\zeta_\alpha Z_{2\alpha}(\zeta_\alpha)\right] + X, \tag{22.29}$$

$$X = \frac{\left(\displaystyle\sum_\alpha n_\alpha q_\alpha \frac{p_{\perp\alpha}}{p_{\|\alpha}}\left[C_{1\alpha} + \zeta_\alpha Z_{1\alpha}(\zeta_\alpha)\right]\right)^2}{2\displaystyle\sum_\alpha \frac{n_\alpha^2 q_\alpha^2}{\beta_{\|\alpha}}\left[C_{0\alpha} + \zeta_\alpha Z_{0\alpha}(\zeta_\alpha)\right]}. \tag{22.30}$$

If, unlikely though it seems, you have enjoyed this Exercise (and/or Exercises 20.3 and 20.4), you will be thrilled to know that firehose and mirror instabilities are a tiny tip of a large iceberg of microinstabilities driven by pressure anisotropies and heat fluxes in high-beta plasmas. An encyclopaedia of these is Bott *et al.* (2024), going well beyond the kinetic-MHD approximation and offering as complete and unified a treatment as currently seems possible, as well as a comprehensive reference list covering the long history of the subject.

22.1.2. *High-Beta Limit*

The magic ordering is

$$\frac{1}{\beta} \sim \varDelta \sim \zeta \ll 1. \tag{22.31}$$

You will see that this is the right thing to assume once we have worked through what it implies.

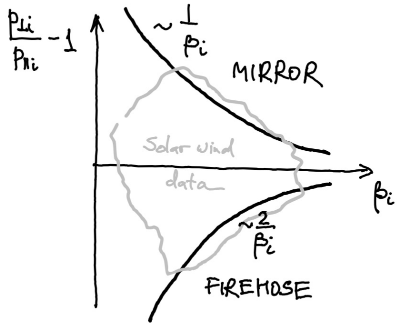


FIGURE 95. Stability boundaries for the firehose (20.74) and mirror (22.36) instabilities. In gray, a mock-up of the solar-wind data (the "Brazil plot") is shown, as per Hellinger *et al.* (2006), Bale *et al.* (2009), and Chen *et al.* (2016).

Under this assumption, the $E_\parallel$ contribution in (22.23) is small: indeed, the denominator of X in (22.22) is $O(1/\beta)$, whereas the numerator is $O(1/\beta^2)$, because, both pressure anisotropy and ζ_α being small, it vanishes to lowest order by quasineutrality. Furthermore, since $\zeta_\alpha \ll 1$, $Z(\zeta_\alpha) \approx i\sqrt{\pi}$ [see (3.97)]. We can also safely drop ζ_e compared to ζ_i everywhere, since $v_{\mathrm{th}e} \gg v_{\mathrm{th}i}$. Finally,

$$\frac{\omega^2}{k_\parallel^2 v_\mathrm{A}^2} \sim \zeta_i^2 \beta_i \sim \frac{1}{\beta_i} \ll 1, \tag{22.32}$$

so the left-hand side of (22.23) (ion inertia) can be neglected. This leaves us with

$$-\mathit{\Lambda} - i\beta_i\zeta_i\sqrt{\pi} + \frac{k_\parallel^2}{k_\perp^2} Y = 0. \tag{22.33}$$

Clearly, $-i\zeta_i = -i\omega/|k_\parallel|v_{\mathrm{th}i} = \gamma/|k_\parallel|v_{\mathrm{th}i}$ is real, and we find

$$\boxed{\gamma = \frac{|k_\parallel| v_{\mathrm{th}i}}{\sqrt{\pi}\beta_i}\left(\mathit{\Lambda} - \frac{k_\parallel^2}{k_\perp^2} Y\right)}\,. \tag{22.34}$$

This is an instability, called *mirror instability*,[154] provided

$$\boxed{\mathit{\Lambda} = \sum_\alpha \beta_{\perp\alpha}\left(\frac{p_{\perp\alpha}}{p_{\parallel\alpha}} - 1\right) - 1 > 0}\,, \tag{22.35}$$

i.e., provided the pressure anistropy is positive and large enough compared to $1/\beta$. For simplicity, let us assume momentarily that electrons have no anisotropy, in which case the instability condition is

$$\mathit{\Delta}_i = \frac{p_{\perp i} - p_{\parallel i}}{p_{\parallel i}} > \frac{1}{\beta_i}. \tag{22.36}$$

Thus, as I promised at the end of §20.8, positive pressure anisotropies in KMHD are just as destabilising as the negative ones, and so the stable region in the parameter space $(\mathit{\Delta}_i, \beta_i)$ is an ever-narrower band as $\beta_i \to \infty$ (while at $\beta_i \to 0$, none of this

[154] The realisation that positive pressure anisotropies are destabilising goes back to the same people whom I quoted as the forefathers of the firehose instability in §20.8, but the first definitive kinetic calculation appears to be due to Hasegawa (1969).

matters). This actually checks out rather well observationally in the solar wind, which seems to restrict itself to the stable regime quite effectively, with increased level of fluctuations (presumably firehose- and mirror-induced) at the margins, acting to keep pressure anisotropy under control in some way that is increasingly well, but perhaps not yet completely, understood. Fig. 95 is a cartoon; the iconic observational plot from the solar wind, colloquially known as the "Brazil plot" on account of the shape of the region occupied by the data, is in the paper by Bale *et al.* (2009) (although in fairness, the same plot first appeared in Hellinger *et al.* 2006, albeit without the information about the fluctuation level; Chen *et al.* 2016 have the most up-to-date version of this plot, taking account of multiple species and their drifts).

Like the firehose, the mirror instability breaks through the constraints of the KMHD approximation and renders our equations ill-posed. Indeed, accounting for the stabilising effect of large $k_\parallel$ in (22.34), it is not hard to see that the maximum growth rate is achieved at

$$k_\parallel = k_\perp\sqrt{\frac{\Lambda}{3Y}} \quad \Rightarrow \quad \gamma_{\max} = \frac{2\Lambda^{3/2}}{3\sqrt{3\pi Y}\beta_i}k_\perp v_{\mathrm{th}i}. \tag{22.37}$$

This explodes at $k_\perp \to \infty$, so clearly the maximum growth rate is reached outside the $k_\perp\rho_i \ll 1$ limit. Note that near the mirror threshold ($\Lambda \ll 1$), the most unstable structures are highly anisotropic, $k_\parallel \ll k_\perp$.

A detailed calculation that includes the finite-Larmor-radius effects and thus captures the fastest-growing mirror modes can be found in the tutorial by Hellinger (2007). You will also find a somewhat more systematic (than mine) treatment of the linear theory of the mirror instability done under the KMHD approximation in the lecture notes by Parra (2019*b*), while I shall now move on to discuss the physics of this instability.

22.1.3. *Physics of Mirror Instability and Barnes Damping*

Looking back at (22.2), it is evident that the "unstable" part of the dispersion relation (22.23) comes entirely from the perpendicular pressure balance

$$\delta p_\perp + \frac{B_0\delta B}{4\pi} \approx \left[-\sum_\alpha \beta_{\perp\alpha}\left(\frac{p_{\perp\alpha}}{p_{\parallel\alpha}} - 1\right) + \sqrt{\pi}\beta_i\frac{\gamma}{|k_\parallel|v_{\mathrm{th}i}} + 1\right]\frac{B_0\delta B}{4\pi} \approx 0, \tag{22.38}$$

where the first term in the square bracket is the perpendicular-pressure perturbation associated with non-resonant particles, the second one with the resonant ones (the $\sqrt{\pi}$ factor is a tell-tale signature of the Landau resonance!), and the last one is the magnetic-pressure perturbation. We see that the effect of positive background pressure anisotropy is to provide a positive perturbed perpendicular pressure in the magnetic troughs ($\delta B < 0$), counteracting the magnetic pressure (and vice versa in the regions of $\delta B > 0$). In the unstable regime, the total non-resonant plus magnetic pressure turns negative, so the system, dynamically, wants to create greater magnetic rarefactions and compressions. In MHD, this would create a destablised magnetosonic wave (cf. §16.1.6), but in a collisionless plasma, it is time-aperiodic because the dominant effect of time evolution comes in not via inertia, as in MHD, but via the perpendicular pressure perturbation arising from the resonant particles and acting to restore pressure balance.

In order to see the origin, and the physics, of the resonant piece of $\delta p_\perp$, it is quite useful to write the kinetic equation using yet another set of variables:

$$(\boldsymbol{r}, \mu, v_\parallel) \to (\boldsymbol{r}, \mu, \varepsilon), \qquad \varepsilon = m\mu B + \frac{mv_\parallel^2}{2}, \tag{22.39}$$

where ε is the particle's kinetic energy (excluding the $\boldsymbol{E}\times\boldsymbol{B}$ drift). In these variables,

$$\left(\frac{Df}{Dt}\right)_{\mu,v_\parallel} = \left(\frac{Df}{Dt}\right)_{\mu,\varepsilon} + \underbrace{\left(\frac{D\varepsilon}{Dt}\right)_{\mu,v_\parallel}}_{=\,m\mu\frac{DB}{Dt}} \frac{\partial f}{\partial\varepsilon} = \frac{Df}{Dt} + m\mu\left(\frac{\mathrm{d}B}{\mathrm{d}t} + \cancel{v_\parallel\nabla_\parallel B}\right)\frac{\partial f}{\partial\varepsilon}, \tag{22.40}$$

$$\left(\cdots - \mu\nabla_\parallel B\right)\frac{\partial f}{\partial v_\parallel} = m\left(\cdots - \cancel{\mu\nabla_\parallel B}\right)v_\parallel\frac{\partial f}{\partial\varepsilon}. \tag{22.41}$$

Upon substitution of this into the kinetic equation (20.107), the $\mu v_\parallel\nabla_\parallel B$ term in (22.40) cancels with the mirror force in (22.41), and we are left with

$$\boxed{\frac{Df}{Dt} + m\left[v_\parallel\left(\frac{q}{m}E_\parallel - \frac{D\boldsymbol{v}_E}{Dt}\cdot\boldsymbol{b}\right) + \mu\frac{\mathrm{d}B}{\mathrm{d}t}\right]\frac{\partial f}{\partial\varepsilon} = 0}\,. \tag{22.42}$$

It is not hard to see that the resonant part of the pressure perturbation comes from the last term in (22.42) (you already know this if you have done Exercise 22.1). This effect is often called the *betatron acceleration*, or *magnetic pumping*, referring to what happens when magnetic troughs ($\delta B < 0$) are damped ($\gamma < 0$), giving energy to particles; the mirror force $-\mu\nabla_\parallel B$ in this case plays the same role as the electric field does in the standard Landau resonance. If we focus on the lived experience of the δB perturbations rather than of the particles, their damping is called *Barnes (1966) damping*, a.k.a., *transit-time damping*. This is a textbook result: in the absence of pressure anisotropy, (22.38) gives immediately

$$\boxed{\gamma \approx -\frac{|k_\parallel|v_{\mathrm{th}i}}{\sqrt{\pi}\beta_i}}\,. \tag{22.43}$$

Physically, the perturbed magnetic pressure naturally opposes formation of δB perturbations and, to maintain pressure balance, the resonant particles's pressure must compensate by having $\gamma < 0$—consequently, perturbations are damped. In contrast, if the additional non-resonant perpendicular pressure perturbation due to positive background pressure anisotropy effectively reverses the effect of the magnetic pressure, then the resonant pressure must have $\gamma > 0$ to compensate for that—energy is moved from particles to δB, mirror instability is operative.

As I did in the case of the firehose, let me point you to some research literature (again selected with extreme bias) on the subject of how mirrors grow and saturate. Two now-classic papers, Southwood & Kivelson (1993) and Kivelson & Southwood (1996), are worth reading for the qualitative physical picture that they give of both linear and nonlinear (respectively) physics involved. Rincon *et al.* (2015) offer a systematic (but very compressed) exposition of how to derive a (weakly) nonlinear theory of the mirror instability.

Exercise 22.3. Work through their paper and reproduce their results.

A much more qualitative, hand-waving picture is available from Melville *et al.* (2016) (building on the simulations of Kunz *et al.* 2014). Unlike the firehose, the mirror instability in the (early) nonlinear stage leads not to particle scattering but to particle trapping in troughs ($\delta B < 0$), effectively imposing a check on the destabilising action of the pressure anisotropy.

Exercise 22.4. Free energy in Kinetic MHD.[155] (a) Assume that the distribution of par-

[155] This is based on the 2023 exam question.

ticles consists of a spatially homogeneous mean and a small (but not necessarily infinitesimally small) perturbation:

$$f_\alpha = f_{0\alpha}(\mu,\varepsilon) + \delta f_\alpha(t,\boldsymbol{r},\mu,\varepsilon), \qquad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}(t,\boldsymbol{r}), \tag{22.44}$$

where $\mu = w_\perp^2/2B$ and $\varepsilon = m_\alpha w^2/2$ (cf. Exercise 21.2b). Show that δf_α satisfies

$$\frac{D\delta f_\alpha}{Dt_\alpha} = -m_\alpha\left[w_\parallel\left(\frac{q_\alpha}{m_\alpha}E_\parallel - \frac{D\boldsymbol{u}_\alpha}{Dt_\alpha}\cdot\boldsymbol{b}\right) + \mu\frac{\mathrm{d}B}{\mathrm{d}t_\alpha}\right]\frac{\partial f_{0\alpha}}{\partial\varepsilon}. \tag{22.45}$$

Introduce a quantity that generalises $T\delta S$ from Maxwellian kinetics [see (5.16)] to arbitrary f_0, and has previously turned up as Kruskal–Oberman available energy (9.30):

$$\mathscr{A} = \int\mathrm{d}\boldsymbol{r}\sum_\alpha\int\mathrm{d}\boldsymbol{w}\,\frac{\delta f_\alpha^2}{2(-\partial f_{0\alpha}/\partial\varepsilon)}. \tag{22.46}$$

Show that it evolves according to

$$\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}t} = -\int\mathrm{d}\boldsymbol{r}\sum_\alpha \mathsf{P}[\delta f_\alpha] : \boldsymbol{\nabla}\boldsymbol{u}_\alpha, \qquad \mathsf{P}[\delta f_\alpha] = p_\perp[\delta f_\alpha](\mathsf{I} - \boldsymbol{bb}) + p_\parallel[\delta f_\alpha]\boldsymbol{bb}, \tag{22.47}$$

where the $\mathsf{P}[\delta f_\alpha]$ is the part of the pressure tensor associated with δf_α:

$$p_\perp[\delta f_\alpha] = \int\mathrm{d}\boldsymbol{w}\,\frac{m_\alpha w_\perp^2}{2}\,\delta f_\alpha, \qquad p_\parallel[\delta f_\alpha] = \int\mathrm{d}\boldsymbol{w}\,m_\alpha w_\parallel^2\delta f_\alpha. \tag{22.48}$$

Note that, in the $(w_\perp, w_\parallel)$ and (μ,ε) variables, respectively,

$$\int\mathrm{d}\boldsymbol{w} = 2\pi\iint\mathrm{d}w_\parallel\mathrm{d}w_\perp w_\perp = 2\pi\iint\mathrm{d}\varepsilon\mathrm{d}\mu\,J, \tag{22.49}$$

with the Jacobian $J = B/m_\alpha|w_\parallel|$, where $|w_\parallel| = \sqrt{2(\varepsilon/m - \mu B)}$.

(b) Prove that the total kinetic energy $\mathscr{K}$ of the particles' peculiar motion satisfies exactly

$$\frac{\mathrm{d}\mathscr{K}}{\mathrm{d}t} = -\int\mathrm{d}\boldsymbol{r}\sum_\alpha\mathsf{P}[f_\alpha] : \boldsymbol{\nabla}\boldsymbol{u}_\alpha, \tag{22.50}$$

where $\mathsf{P}[f_\alpha]$ is the KMHD pressure tensor, whose components are defined analogously to (22.48) but with the exact f_α instead of δf_α.[156] Hence, using the fact that the total energy is conserved (Exercise 20.6), show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{A} + \int\mathrm{d}\boldsymbol{r}\left(\sum_\alpha\frac{m_\alpha n_\alpha u_\alpha^2}{2} + \frac{B^2}{8\pi}\right)\right] = \int\mathrm{d}\boldsymbol{r}\sum_\alpha\mathsf{P}[f_{0\alpha}] : \boldsymbol{\nabla}\boldsymbol{u}_\alpha. \tag{22.51}$$

Recall that since $f_{0\alpha}$ is a function of μ and ε and the exact B is involved in the definition of μ, $\mathsf{P}[f_{0\alpha}]$ contains both the mean pressure and part of the perturbed pressure (the non-resonant part; see Exercise 22.1).

(c) Hence show that the generalised free energy

$$\begin{aligned}\mathscr{F} = \mathscr{A} + \int\mathrm{d}\boldsymbol{r}\,\Bigg\{\sum_\alpha\frac{m_\alpha n_\alpha u_\alpha^2}{2} &+ \left[1 - \sum_\alpha\frac{\beta_{\parallel\alpha}}{2}\left(1 - \frac{p_{\perp\alpha}}{p_{\parallel\alpha}}\right)\right]\frac{\delta B_\perp^2}{8\pi}\\ &+ \left[1 - \sum_\alpha\beta_{\perp\alpha}\left(\frac{p_{\perp\alpha}}{p_{\parallel\alpha}}C_{2\alpha} - 1\right)\right]\frac{\delta B_\parallel^2}{8\pi}\Bigg\},\end{aligned} \tag{22.52}$$

is conserved, to lowest non-trivial order in perturbations:

$$\boxed{\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = 0}\,. \tag{22.53}$$

[156] This is particularly straightforward if you start with the drift-kinetic equation in $(\boldsymbol{r},\mu,\varepsilon)$ variables written in the form (21.7) (gyroaveraged, of course).

Here $p_{\perp\alpha}$, $p_{\parallel\alpha}$, $\beta_{\perp\alpha}$, $\beta_{\parallel\alpha}$ are all mean quantities, which can be taken to be constant in time and space, and the constant $C_{2\alpha}$ is defined by

$$\frac{2p_{\perp\alpha}^2}{p_{\parallel\alpha}}C_{2\alpha} = -2\pi m_\alpha^2 B_0^2 \iint \mathrm{d}\varepsilon\mathrm{d}\mu\, J\mu^2 \frac{\partial f_{0\alpha}}{\partial\varepsilon} \tag{22.54}$$

[cf. (22.27); $C_{2\alpha} = 1$ for a bi-Maxwellian mean distribution]. The following facts will come handy in the proof:

$$\left(\frac{\partial f_0}{\partial\mu}\right)_\varepsilon = B\left[\frac{1}{w_\perp}\left(\frac{\partial f_0}{\partial w_\perp}\right)_{w_\parallel} - \frac{1}{w_\parallel}\left(\frac{\partial f_0}{\partial w_\parallel}\right)_{w_\perp}\right], \tag{22.55}$$

which is one step from (22.8),

$$\frac{\mathrm{d}\ln B}{\mathrm{d}t_\alpha} = \boldsymbol{bb} : \boldsymbol{\nabla}\boldsymbol{u}_\alpha - \boldsymbol{\nabla}\cdot\boldsymbol{u}_\alpha, \tag{22.56}$$

which follows from Faraday's law and (20.35), and was once used in the derivation (20.86), and

$$\ln B = \ln\sqrt{B_0^2 + 2\boldsymbol{B}_0\cdot\delta\boldsymbol{B} + |\delta\boldsymbol{B}|^2} \approx \ln B_0 + \frac{\delta B_\parallel}{B_0} + \frac{\delta B_\perp^2 - \delta B_\parallel^2}{2B_0^2}, \tag{22.57}$$

which is the Taylor expansion of the logarithm to second order. You may also assume that all-space integrals of full derivatives vanish and $\int \mathrm{d}\boldsymbol{r}\,\delta\boldsymbol{B} = 0$.

(d) Discuss what the conservation of the quadratic quantity (22.52) implies for stability of KMHD with respect to small perturbations: can you formulate a sufficient condition for stability? How is it compatible with the existence of the firehose and mirror instabilities?

As far as I know, the idea of a generalised free energy in the form (22.52) originates from Kunz *et al.* (2015), who worked it out for KRMHD (see Exercise 20.7), and later for gyrokinetics (Kunz *et al.* 2018).

22.2. *Origin of Pressure Anisotropy*

Up to this point, I have taken the formal approach of picking a simple, (locally) homogeneous equilibrium and considering small perturbations around it. The result is that, if

$$\frac{|p_\perp - p_\parallel|}{p_\parallel} \gtrsim \frac{1}{\beta}, \tag{22.58}$$

any such equilibrium would be violently unstable to fast, microscale modes, which, in fact, take us outside the KMHD description. Their effect, once they saturate, is presumably to limit all dynamics to the stable part of the parameter space—how that happens is a live research question, but the emerging answer is, crudely, that the plasma will, thanks to those instabilities, develop an effective collisionality that pins pressure anisotropies locally to the marginal levels.[157] Let us now inquire how ubiquitous, or otherwise, the need for such non-KMHD adjustments of KMHD dynamics is likely to be.

At $\beta \ll 1$, there is clearly no problem—I shall return to this limit in due course (in §23 and onwards)—but at $\beta \gg 1$ (or even $\beta \gtrsim 1$), it is possible to show by a very elementary argument that pressure anisotropies will be endemic, and so must be the associated instabilities.

As we have seen, in the KMHD limit, each particle conserves its magnetic moment:

$$\mu = \text{const.} \tag{22.59}$$

---

[157] There is an additional interesting dynamical tendency for the plasma to be reluctant to have motions that cross into the unstable regions, but that does not appear to be enough on its own: the relevant paper to read about this is Squire *et al.* (2023) (and follow the paper trail to $t \to \pm\infty$).

But the mean of this over all velocities, for particles of species α, is

$$\langle\mu\rangle_{\boldsymbol{w}} = \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{w}\,\mu f_\alpha = \frac{p_{\perp\alpha}}{m_\alpha n_\alpha B} = \text{const.} \tag{22.60}$$

For the purposes of a qualitative discussion, let me pretend momentarily (and, in general, wrongly) that $n_\alpha = \text{const}$. Then (22.60) means that (locally in a fluid element) every time plasma dynamics change B, the pressure $p_{\perp\alpha}$ must change proportionally:

$$\frac{1}{p_{\perp\alpha}}\frac{\mathrm{d}p_{\perp\alpha}}{\mathrm{d}t_\alpha} \sim \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t_\alpha} - \nu_\alpha\frac{p_{\perp\alpha}-p_{\parallel\alpha}}{p_{\perp\alpha}} \tag{22.61}$$

(the last term allows for the breaking of this principle by collisions, "true" or effective, that want to isotropise pressure). Thus, physically, we expect any dynamics that increase B to lead to $p_\perp > p_\parallel$, triggering the mirror instability, and any that decrease B to lead to $p_\perp < p_\parallel$, triggering the firehose (I will make this conclusion more solid and quantitative shortly). Since the magnetic field described by the MHD induction equation, as it still is in KMHD (§20.6), is frozen into fluid motions, it is quite difficult to move anything without changing B (§§14.7–14.9).

One simple way to see why the microinstabilities are not only inevitable but, in fact, indispensable for high-β plasma dynamics to make physical sense, is to think of the kinematic dynamo (§15). Take a magnetic field that is sufficiently weak to exert no Lorentz force on the motions ($B^2/8\pi \ll \rho u^2/2$) but sufficiently strong to magnetise particles ($\Omega_\alpha \gg ku, \nu_\alpha$). A generic chaotic 3D flow will amplify such a field multi-fold—in MHD. But in KMHD, every time B is doubled, so should be $p_\perp$, which is proportional to the total thermal energy of the plasma. Clearly, this is impossible, so the μ conservation must be broken—and it is, by the effect of saturated microinstabilities on particles.

You will find a formal proof, and numerical confirmation, of the impossibility of dynamo action in KMHD in the paper by Helander *et al.* (2016). Nevertheless, dynamo does work in a collisionless, magnetised plasma—thanks to the microinstabilities (St-Onge & Kunz 2018). Another interesting effect of microinstabilities is associated with compressive motions. The presence of n_α in (22.60) implies that any dynamics that change the density of the plasma will also produce pressure anisotropies and thus trigger instabilities. The simplest such a dynamical phenomenon is a sound wave: Kunz *et al.* (2020) launched one (numerically) into a magnetised, collisionless, high-β plasma and discovered that, unlike in a textbook electrostatic plasma (§3.11), it was able to propagate, like in MHD (§16.1.2), because it created its own local effective collisionality by triggering firehose and mirror fluctuations that scattered particles.

22.2.1. *CGL Equations*

Let us now work some of this out more formally. For that, we shall derive evolution equations for $p_{\perp\alpha}$ and $p_{\parallel\alpha}$ from the KMHD kinetic equation. Its most convenient form for this purpose is (20.86), in $(w_\perp, w_\parallel)$ variables: dropping the species indices as usual, to avoid clutter (but still meaning $\boldsymbol{u}_\alpha$ by $\boldsymbol{u}$, etc.), it is

$$\frac{Df}{Dt} + \frac{1}{B}\frac{DB}{Dt}\frac{w_\perp}{2}\frac{\partial f}{\partial w_\perp} + \left(\frac{q}{m}E_\parallel - \frac{D\boldsymbol{u}}{Dt}\cdot\boldsymbol{b} - \frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}\right)\frac{\partial f}{\partial w_\parallel} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}. \tag{22.62}$$

Multiply this by $mw_\perp^2/2$ and integrate over velocities, $\int \mathrm{d}\boldsymbol{w} = 2\pi\iint \mathrm{d}w_\parallel \mathrm{d}w_\perp w_\perp$:

$$\begin{aligned}
&\frac{\mathrm{d}p_\perp}{\mathrm{d}t} + \boldsymbol{b}\cdot\boldsymbol{\nabla}\underbrace{\int \mathrm{d}\boldsymbol{w}\, w_\parallel \frac{mw_\perp^2}{2} f}_{\substack{\equiv q_\perp \\ \parallel \text{ flux} \\ \text{of } \perp \text{ heat}}} + \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t}\underbrace{2\pi\iint \mathrm{d}w_\parallel \mathrm{d}w_\perp \frac{mw_\perp^4}{4}\frac{\partial f}{\partial w_\perp}}_{\substack{= -2p_\perp \\ \text{by parts}}} \\
&\quad + \underbrace{\frac{\nabla_\parallel B}{B} 2\pi\iint \mathrm{d}w_\parallel \mathrm{d}w_\perp w_\parallel \frac{mw_\perp^4}{4}\frac{\partial f}{\partial w_\perp}}_{\substack{= 2q_\perp \boldsymbol{\nabla}\cdot\boldsymbol{b} \\ \text{by parts}}} \\
&\quad + \underbrace{\int \mathrm{d}\boldsymbol{w}\, \frac{mw_\perp^2}{2}\left(\cancel{\frac{q}{m}E_\parallel} - \cancel{\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\boldsymbol{b}} - w_\parallel \boldsymbol{bb}:\boldsymbol{\nabla}\boldsymbol{u} - \cancel{\frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}}\right)\frac{\partial f}{\partial w_\parallel}}_{\substack{= p_\perp \boldsymbol{bb}:\boldsymbol{\nabla}\boldsymbol{u} = p_\perp\left(\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t}\right) \\ \text{by parts, then using induction and continuity equations}}} = -\nu\left(p_\perp - p_\parallel\right).
\end{aligned} \tag{22.63}$$

The collision term can be derived formally, but it is also intuitive that the effect of collisions should be to push the two pressures closer to each other. Assembling terms, one gets

$$\boxed{p_\perp \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{p_\perp}{nB} = -\boldsymbol{\nabla}\cdot(q_\perp\boldsymbol{b}) - q_\perp\boldsymbol{\nabla}\cdot\boldsymbol{b} - \nu\left(p_\perp - p_\parallel\right)}\,. \tag{22.64}$$

This is the statement (22.60) of conservation of the mean magnetic moment in a fluid element (associated with an individual species α), broken only by collisions and by heat fluxes, i.e., in the case of the latter, by the leakage of (perpendicular) energy along the field lines from/to the fluid element.

Now multiply (22.62) by $mw_\parallel^2$ and integrate, using the same tricks:

$$\begin{aligned}
&\frac{\mathrm{d}p_\parallel}{\mathrm{d}t} + \boldsymbol{b}\cdot\boldsymbol{\nabla}\underbrace{\int \mathrm{d}\boldsymbol{w}\, mw_\parallel^3 f}_{\substack{\equiv q_\parallel \\ \parallel \text{ flux} \\ \text{of } \parallel \text{ heat}}} + \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t}\underbrace{2\pi\iint \mathrm{d}w_\parallel \mathrm{d}w_\perp mw_\parallel^2 \frac{w_\perp^2}{2}\frac{\partial f}{\partial w_\perp}}_{\substack{= -p_\parallel \\ \text{by parts}}} \\
&\quad + \underbrace{\frac{\nabla_\parallel B}{B} 2\pi\iint \mathrm{d}w_\parallel \mathrm{d}w_\perp mw_\parallel^3 \frac{w_\perp^2}{2}\frac{\partial f}{\partial w_\perp}}_{\substack{= q_\parallel \boldsymbol{\nabla}\cdot\boldsymbol{b} \\ \text{by parts}}} \\
&\quad + \underbrace{\int \mathrm{d}\boldsymbol{w}\, mw_\parallel^2\left(\cancel{\frac{q}{m}E_\parallel} - \cancel{\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\boldsymbol{b}} - w_\parallel \boldsymbol{bb}:\boldsymbol{\nabla}\boldsymbol{u} - \frac{w_\perp^2}{2}\frac{\nabla_\parallel B}{B}\right)\frac{\partial f}{\partial w_\parallel}}_{\substack{= 3p_\parallel \boldsymbol{bb}:\boldsymbol{\nabla}\boldsymbol{u} + 2q_\perp\frac{\nabla_\parallel B}{B} = 3p_\parallel\left(\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t}\right) - 2q_\perp\boldsymbol{\nabla}\cdot\boldsymbol{b} \\ \text{by parts, then using induction and continuity equations,} \\ \text{and } \int \mathrm{d}\boldsymbol{w}\, w_\parallel f = 0}} = -2\nu\left(p_\parallel - p_\perp\right).
\end{aligned} \tag{22.65}$$

The form of the collision term ensures that collisions conserve total energy $p_\perp + p_\parallel/2$.

Assembling terms, one gets

$$\boxed{p_\parallel \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{p_\parallel B^2}{n^3} = -\boldsymbol{\nabla}\cdot\left(q_\parallel \boldsymbol{b}\right) + 2q_\perp \boldsymbol{\nabla}\cdot\boldsymbol{b} - 2\nu\left(p_\parallel - p_\perp\right)}\,. \tag{22.66}$$

Equations (22.64) and (22.66) (often with 0's on their right-hand sides), which I promised you some time ago in (14.66) and Exercise 16.8, are called *Chew–Goldberger–Law (CGL) equations* (Chew *et al.* 1956), or, sometimes, the *double-adiabatic approximation*. "Double" because, while (22.64) confirms the conservation of the first adiabatic invariant μ modulo heat fluxes and collisions, (22.66) suggests that there is another adiabatic invariant in the system, also modulo heat fluxes and collisions, whose conservation controls the evolution of the parallel pressure (and reinforces the tendency towards positive/negative pressure anisotropy when B is increased/decreased).

Exercise 22.5. CGL RMHD and CGL waves. If you did not do Exercise 16.8 when studying MHD, do it now, and learn why slow waves are fast in CGL MHD.

22.2.2. *Longitudinal Invariant*

Let us work out what this second invariant might be. By analogy with (22.60), if it is some quantity associated with each individual particle, then

$$\frac{p_{\parallel\alpha} B^2}{m_\alpha n_\alpha^3} = \frac{1}{n_\alpha}\int \mathrm{d}\boldsymbol{w} \left(\frac{w_\parallel B}{n_\alpha}\right)^2 f_\alpha = \mathrm{const} \tag{22.67}$$

should represent the mean of this quantity over all velocities. The mean square, rather, it turns out, of a quantity called the *longitudinal invariant*:

$$J = w_\parallel \ell_\alpha, \tag{22.68}$$

where ℓ_α is the length along the field line of a fluid element moving with the species α. Recall that an infinitesimal separation vector $\delta\boldsymbol{r}$ between two points frozen into the fluid flow $\boldsymbol{u}$ satisfies the same equation as $\boldsymbol{B}/n$, provided the magnetic field and density are evolved by the same flow (§14.7):[158]

$$\frac{\mathrm{d}\delta\boldsymbol{r}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r}) \approx \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{B}}{n} = \frac{\boldsymbol{B}}{n}\cdot\boldsymbol{\nabla}\boldsymbol{u}. \tag{22.69}$$

Therefore, $\ell_\alpha \propto B/n_\alpha$, so $J \propto w_\parallel B/n_\alpha$ and the conservation of $\langle J^2\rangle_{\boldsymbol{w}}$ in a fluid element gives us (22.67).

This type of adiabaticity is related to particles getting trapped (or just slowed down) in magnetic mirrors as they stream along the field lines (Fig. 96). Then they have a *second adiabatic invariant*

$$J = \oint w_\parallel \mathrm{d}\ell = \mathrm{const}, \tag{22.70}$$

where the integral is taken between the bounce points and conservation is assured provided the bounce frequency is large: $\omega_\mathrm{b} \sim k_\parallel w_\parallel \gg ku$ (bouncing is faster than fluid motion).

Parenthetically, let me explain why bouncing happens. Ignoring fluid motions, assumed slow

[158] Note that $\boldsymbol{B}$ is frozen into $\boldsymbol{u}_\perp$, so also satisfies the induction equation with either $\boldsymbol{u}$ or $\boldsymbol{u}_\alpha$: the induction equation does not know about the parallel component of the flow, even if that is less obvious when the double vector product is unravelled into three terms like in (20.42) (which can, therefore, be written with $\boldsymbol{u}_\alpha$ instead of $\boldsymbol{u}$).

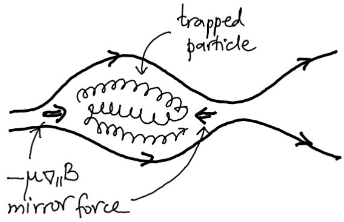


FIGURE 96. Gyrating particle trapped in a magnetic mirror.

($v_E \ll v_{\rm th}$), and the slow change of $\boldsymbol{B}$ due to them, the equations of the gyroveraged particle motion, to lowest order in ρ_*, are (§21.3.2)

$$\dot{\boldsymbol{r}} = v_\parallel \boldsymbol{b}, \quad \dot{\mu} = 0, \quad \dot{v}_\parallel = \frac{q}{m}E_\parallel - \mu\nabla_\parallel B. \tag{22.71}$$

Besides the conserved μ, there is a conserved energy:

$$H = \frac{mv_\parallel^2}{2} + q\varphi(\boldsymbol{r}) + m\mu B(\boldsymbol{r}) \quad \Rightarrow \quad \dot{H} = mv_\parallel \dot{v}_\parallel + q\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla}\varphi + m\mu\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla}B = 0. \tag{22.72}$$

The motion along the field is then described by

$$\dot{\ell} = v_\parallel = \pm\sqrt{\frac{2}{m}\left[H - q\varphi(\ell) - \mu B(\ell)\right]}. \tag{22.73}$$

The particle will bounce wherever the expression under the square root vanishes. Like always in classical mechanics, these oscillations have an associated adiabatic invariant, given by (22.70) (Gardner 1959; Kruskal 1962).

22.2.3. *Pressure Anisotropy from CGL Equations*

Finally, let us work out the evolution of pressure anisotropy from the CGL equations (22.64) and (22.66):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(p_\perp - p_\parallel\right) = \left(p_\perp + 2p_\parallel\right)\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} + \left(p_\perp - 3p_\parallel\right)\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t}\\ - \boldsymbol{\nabla}\cdot\left[(q_\perp - q_\parallel)\boldsymbol{b}\right] - 3q_\perp\boldsymbol{\nabla}\cdot\boldsymbol{b} - 3\nu\left(p_\perp - p_\parallel\right).\end{aligned} \tag{22.74}$$

We see that, as anticipated at the beginning of this section, increasing/decreasing B causes positive/negative pressure anisotropy. Increasing/decreasing density produces the opposite effect, and heat fluxes can also cause pressure anistropies (by carrying in or out along the field lines preferentially particles with larger perpendicular or parallel energies). Collisions always, of course, strive to isotropise pressure.

22.2.4. *Braginskii Viscosity*

Therefore, in the collisional limit, $\nu \gg \omega \sim ku$ (but still $\nu \ll \Omega$), one expects the pressure anisotropy to be small and dominated by the balance between the collisional isotropisation and the three pressure-anisotropy production mechanisms: assuming $p_\perp - p_\parallel \ll p_\perp \approx p_\parallel$, this balance gives us

$$\frac{p_\perp - p_\parallel}{p_\parallel} \approx \frac{1}{\nu}\left\{\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} - \frac{2}{3}\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}t} - \frac{\boldsymbol{\nabla}\cdot\left[(q_\perp - q_\parallel)\boldsymbol{b}\right] - 3q_\perp\boldsymbol{\nabla}\cdot\boldsymbol{b}}{3p_\parallel}\right\}. \tag{22.75}$$

Ignoring the heat fluxes and using the induction and continuity equations to rewrite the above expression in terms of flow velocities, one gets

$$p_\perp - p_\parallel \approx \frac{p_\parallel}{\nu}\left(\boldsymbol{bb}:\boldsymbol{\nabla u} - \frac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{u}\right). \tag{22.76}$$

If you substitute this into the momentum equation (20.53), where the pressure anisotropy appears as a parallel stress, you will recognise its effect as that of a parallel collisional viscosity—known as the *Braginskii (1965) viscosity*. Indeed, if $\boldsymbol{B}$ were held straight and constant,

$$\rho\frac{\mathrm{d}u_\parallel}{\mathrm{d}t} = \dots + \frac{p_\parallel}{\nu}\nabla_\parallel^2 u_\parallel. \tag{22.77}$$

Except of course now you know that, if $\boldsymbol{B}$ is not held straight and constant, such a viscosity can be violently (if microscopically) destabilising.

Exercise 22.6. Viscous heating. Assuming the rate ν_α of the collisional relaxation of pressure anisotropy of species α to be larger than the rate of dynamical evolution of all quantities and assuming also an incompressible mass flow, no species drifts, and no heat fluxes, use (20.105) to show that the local rate of viscous heating of the plasma is nowhere negative and is non-zero in all fluid elements where the magnetic field's strength is changing.

Braginskii viscosity may be destabilising at microscales, but at the scale of the fluid motions that give rise to it, it is manifestly a viscosity, so its effect will be to dissipate those motions. That is, dynamically speaking, the effect of the pressure-anisotropy stress on the motions that give rise to it is to oppose those motions, making it more difficult, in a KMHD plasma, to change B or n. This dynamical tendency of such a plasma to avoid changing B in particular is called *magneto-immutability* (a term coined by Squire *et al.* 2019, who also noted a useful analogy with the way in which both pressure and the "bulk viscosity" in conventional fluids tend to oppose compressible motions). This effect turns out to be more general than the collisional limit considered here, extending also to collisionless KMHD plasmas (Squire *et al.* 2023). Making sense of how magneto-immutable plasma dynamics work brings us to the current research frontier (Majeski *et al.* 2024), which I invite you to visit on your own, armed with what you have learned here.

Exercise 22.7. Relativistic KMHD and CGL. Work out KMHD and CGL equations in the relativistic framework (general or at least special). This is actually a research-level exercise, as this does not appear to have been done in a definitive way, and I am not aware of an extant pedagogic exposition of the current state of the art. The paper by Gedalin & Oiberman (1995) seems to be a sensible and at least somewhat comprehensible start. See if you can do better!

22.3. *Braginskii MHD*

22.3.1. *Heat Fluxes*

22.3.2. *Magnetothermal Instability*

Exercise 22.8. HBI. See Quataert (2008).

Exercise 22.9. Magnetoviscous instability. See Balbus (2004).

22.4. *KMHD Energy Principle*

The refrain of the KMHD theory is that everything is unstable, on large scales, on small scales, on even smaller scales, etc. It is an almighty mess caused by plasma's general unhappiness with, more or less, anything other than a uniform Maxwellian equilibrium—not really a surprise for the aficionados of thermodynamics! Everything, therefore, is well-nigh-always turbulent, and what one really needs is some sort of effective mean-field theory. We do not have that, so we are still at the stage where we must examine closely the 1001 ways in which unstable configurations can be unstable, with a hope of integrating all this knowledge one day into some general principles of kinetic stability.

This section is coming in due course…

---

23. Electrostatic Drift Kinetics

So far I have focused on physical regimes in which the dynamics of the magnetic field were of natural interest—this gave a special significance to the $\beta \gg 1$ limit, pressure anisotropies and the associated instabilities. I am now going to explore the extreme opposite regime—one in which $\boldsymbol{B}$ does not change at all. Intuitively, this is an appropriate approximation when $\beta \ll 1$, but I will postpone a careful consideration of exactly when this *electrostatic limit* is valid (till §23.4), and instead first explore what kind of wild life exists in this part of the plasma jungle.

So let us assume $\boldsymbol{B} = B_0\hat{\boldsymbol{z}} = \text{const}$ and $\delta\boldsymbol{B} = 0$ exactly. This implies

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = 0 \quad \Rightarrow \quad \boxed{\boldsymbol{E} = -\boldsymbol{\nabla}\varphi}\,, \tag{23.1}$$

an electrostatic electric field, and so the DK equation (20.107) [or, indeed, its low-flow version (21.93), where most terms disappear because $\boldsymbol{B}$ does not vary] is

$$\frac{\partial f}{\partial t} + \boldsymbol{v}_E \cdot \boldsymbol{\nabla}_\perp f + v_\parallel \nabla_\parallel f + \frac{q}{m} E_\parallel \frac{\partial f}{\partial v_\parallel} = 0. \tag{23.2}$$

Since $E_\parallel = -\nabla_\parallel \varphi$, $\boldsymbol{v}_E = (c/B_0)\hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \varphi$, and, therefore,

$$\boldsymbol{v}_E \cdot \boldsymbol{\nabla}_\perp f = \frac{c}{B_0}\hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_\perp \varphi \times \boldsymbol{\nabla}_\perp f) = \frac{c}{B_0}\left(\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial \varphi}{\partial y}\frac{\partial f}{\partial x}\right) \equiv \frac{c}{B_0}\left\{\varphi, f\right\}, \tag{23.3}$$

the electrostatic DK equation has the form

$$\boxed{\frac{\partial f}{\partial t} + \frac{c}{B_0}\left\{\varphi, f\right\} + v_\parallel \nabla_\parallel f - \frac{q}{m}\left(\nabla_\parallel \varphi\right)\frac{\partial f}{\partial v_\parallel} = 0}\,. \tag{23.4}$$

Note that $\nabla_\parallel$ here is just a synonym of $\partial/\partial z$.

Consider now a static equilibrium with $\varphi = 0$. This must satisfy

$$v_\parallel \nabla_\parallel f_0 = 0, \tag{23.5}$$

so any distribution that has no dependence on z will do. This time, I want to study an equilibrium that has some profile in space, but is Maxwellian in velocity:

$$f_0(x, w_\perp, v_\parallel) = \frac{n_0(x)}{\pi^{3/2} v_{\text{th}}^3(x)}\, e^{-(w_\perp^2 + v_\parallel^2)/v_{\text{th}}^2(x)}, \qquad v_{\text{th}}(x) = \sqrt{\frac{2T_0(x)}{m}}. \tag{23.6}$$

This set up is about as simple as it gets, but we shall see that it is nevertheless interesting. It is, of course, inspired by tokamak physics, where x is the radial variation of the profiles (hot and dense inside, cold and diffuse outside), but applies in many devices, real and imaginary (e.g., various pinches; see §17.1.1), or, indeed, generally in stratified environments. Obviously, in devices, even if $\delta\boldsymbol{B}$ might indeed be negligble, $\boldsymbol{B}_0$ is never perfectly straight and homogeneous, but let us for now assume that it varies on longer scales than $n_0(x)$ or $T_0(x)$ (in a torus, that can be the distinction between the major and minor radii); I shall come back to some important effects of the variation of $\boldsymbol{B}_0$ in §24.2.1.

As always, one starts with linear physics, so here is (23.4) perturbed around the equilibrium (23.6):[159]

$$\boxed{\frac{\partial \delta f}{\partial t} + v_\parallel \nabla_\parallel \delta f = -v_\parallel \nabla_\parallel \frac{q\varphi}{T} f_0 + \frac{\rho v_{\rm th}}{2}\frac{\partial f_0}{\partial x}\frac{\partial}{\partial y}\frac{q\varphi}{T}}, \tag{23.7}$$

where $\rho = cmv_{\rm th}/qB_0$ is the Larmor radius. This is of course, two equations, one for electrons and one for ions (assuming a single species thereof). As usual, they are connected via quasineutrality (20.102), which, as always in KMHD, determines the electric field—in the electrostatic limit, this means both $E_\parallel$ and $\boldsymbol{E}_\perp$, as they are both set by φ.

23.1. *Boltzmann Electrons*

Writing (23.7) for electrons immediately gives us something very useful. Assuming that T_e is not hugely smaller than T_i, electrons are much faster than ions, on account of their small mass:

$$\frac{v_{\rm the}}{v_{\rm thi}} = \sqrt{\frac{T_e m_i}{T_i m_e}} \gg 1. \tag{23.8}$$

If we look for solutions with $\omega \ll k_\parallel v_{\rm the}$, we can neglect all terms in (23.7) except those $\propto v_\parallel$. On the right-hand side, this requires

$$\frac{\dfrac{\rho_e v_{\rm the}}{2}\dfrac{\partial f_{0e}}{\partial x}\dfrac{\partial}{\partial y}\dfrac{e\varphi}{T_e}}{v_\parallel \nabla_\parallel \dfrac{e\varphi}{T_e} f_{0e}} \sim \frac{k_y \rho_e}{k_\parallel L_e} \ll 1, \tag{23.9}$$

where L_e is the characteristic gradient scale length of the variation of the electron density and/or temperature profiles. This leaves us with

$$v_\parallel \nabla_\parallel \delta f_e = v_\parallel \nabla_\parallel \frac{e\varphi}{T_e} f_{0e} \quad \Rightarrow \quad \delta f_e = \frac{e\varphi}{T_e} f_{0e} \quad \Rightarrow \quad \frac{\delta n_e}{n_e} = \frac{e\varphi}{T_e}. \tag{23.10}$$

This is called *Boltzmann response* for the electons [cf. (1.4)].

This is a moment to pounce: if we treat (23.10) as φ being determined by δn_e, we can instantly relate that to δn_i via quasineutrality:

$$\boxed{\frac{e\varphi}{T_e} = \frac{\delta n_e}{n_e} = \frac{\delta n_i}{n_i}}. \tag{23.11}$$

Thus, if we can solve the ion DK equation (23.7) in terms of φ, then we can calculate δn_i, which then immediately gives us φ via (23.11).

23.2. *Fluid ITG Instability*

So let us consider now the ion version of the linearised, electrostatic DKE (23.7). Of course I can (and will, in §23.3) solve it in the usual way (like I did, e.g., for the mirror instability in §22.1), but, to elicit the key physics, I shall first consider the "fluid dynamics" of this system, by taking moments of (23.7).

[159] I will routinely drop the subscript 0 on the equilibrium quantities T and n wherever this seems aesthetically justified—especially when they also carry a species subscript.

First, take the density moment, $\int \mathrm{d}\boldsymbol{w}\,(23.7)_i$:

$$\frac{\partial \delta n_i}{\partial t} + \nabla_\parallel n_i u_{\parallel i} = \frac{\rho_i v_{\mathrm{th}i}}{2}\frac{\mathrm{d}n_i}{\mathrm{d}x}\frac{\partial}{\partial y}\frac{q_i\varphi}{T_i}, \tag{23.12}$$

where $q_i = Ze$ (the ion charge). This is, obviously, the continuity equation for the total density $n_i(x) + \delta n_i$: this becomes particularly obvious if one notices that the right-hand side is just $-v_{Ex}\partial_x n_i$, the advection of the mean density by the $\boldsymbol{E}\times\boldsymbol{B}$ flow. Let

$$-\frac{\mathrm{d}\ln n_i}{\mathrm{d}x} = \frac{1}{L_n}, \qquad -\frac{\mathrm{d}\ln T_i}{\mathrm{d}x} = \frac{1}{L_T}. \tag{23.13}$$

be the definitions of the gradient scale lengths of the ion density and temperature (anticipating the need for the latter). The minus sign is a tokamak convention: the density and temperature increase towards the toroidal axis, in the negative radial direction. Then (23.12) becomes

$$\boxed{\frac{\partial}{\partial t}\frac{\delta n}{n} + \nabla_\parallel u_\parallel = -\frac{\rho v_{\mathrm{th}}}{2L_n}\frac{\partial\bar\varphi}{\partial y}}\,, \tag{23.14}$$

where, in a bid to reduce clutter, I have dropped the ion species index everywhere and denoted $\bar\varphi = Ze\varphi/T_i$. The last term can be expressed in terms of the ion density perturbation via the quasineutrality (23.11):

$$\boxed{\bar\varphi = \bar\tau^{-1}\frac{\delta n}{n}}\,, \tag{23.15}$$

where $\bar\tau = T_i/ZT_e$ (the temperature ratio is usually called τ, but I would like to include Z into it).

Secondly, take the parallel-velocity moment, $\int \mathrm{d}\boldsymbol{w}\, m_i v_\parallel (23.7)_i$:

$$\frac{\partial}{\partial t} m_i n_i u_{\parallel i} + \nabla_\parallel \delta p_{\parallel i} = -\nabla_\parallel \frac{Ze\varphi}{T_i} p_i, \tag{23.16}$$

assuming no mean parallel flow. This is the parallel momentum equation, featuring the parallel pressure force and the parallel electric force ($-\nabla_\parallel Zen_i\varphi = Zen_iE_\parallel$). Since $p = nT = mnv_{\mathrm{th}}^2/2$ and $\delta p_\parallel = T\delta n + n\delta T_\parallel$, this becomes

$$\boxed{\frac{\partial u_\parallel}{\partial t} = -\frac{v_{\mathrm{th}}^2}{2}\nabla_\parallel\left(\frac{\delta n}{n} + \frac{\delta T_\parallel}{T} + \bar\varphi\right)}\,. \tag{23.17}$$

Finally, to get an equation for $\delta T_\parallel$, take the parallel-pressure moment, $\int \mathrm{d}\boldsymbol{w}\, m_i v_\parallel^2 (23.7)_i$:

$$\frac{\partial \delta p_{\parallel i}}{\partial t} + \nabla_\parallel\left(q_{\parallel i} + 3p_i u_{\parallel i}\right) = \frac{\rho_i v_{\mathrm{th}i}}{2}\frac{\mathrm{d}p_i}{\mathrm{d}x}\frac{\partial}{\partial y}\frac{Ze\varphi}{T_i}, \tag{23.18}$$

where the expression under $\nabla_\parallel$ came from recalling the definition of the parallel heat flux in (22.65):

$$q_\parallel = \int \mathrm{d}\boldsymbol{w}\, m w_\parallel^3 f = \int \mathrm{d}\boldsymbol{w}\, m(v_\parallel - u_\parallel)^3(f_0 + \delta f) \approx \int \mathrm{d}\boldsymbol{w}\, m v_\parallel^3 \delta f - 3u_\parallel \underbrace{\int \mathrm{d}\boldsymbol{w}\, m v_\parallel^2 f_0}_{=\,p_0}. \tag{23.19}$$

Dividing (23.18) through by $p_i = n_i T_i$, using the definition of the temperature gradient

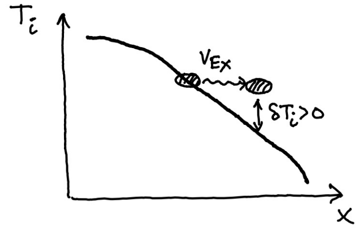


FIGURE 97. Advection of the mean temperature profile by $\boldsymbol{E}\times\boldsymbol{B}$ flow creates temperature perturbation.

scale length (23.13), and subtracting the density equation (23.14), we get

$$\boxed{\frac{\partial}{\partial t}\frac{\delta T_\parallel}{T} + \frac{\nabla_\parallel q_\parallel}{p} = -2\nabla_\parallel u_\parallel - \frac{\rho v_{\rm th}}{2L_T}\frac{\partial\bar{\varphi}}{\partial y}}\,. \tag{23.20}$$

The perturbed temperature evolves due to the heat flux, the compressional heating (the first term on the right-hand side), and the advection of the mean temperature profile by the $\boldsymbol{E}\times\boldsymbol{B}$ flow ($-v_{Ex}\partial_x T_i$, the last term; Fig. 97).

It is the last effect that turns out to be responsible for the injection of (free) energy from the mean temperature profile into the fluctuations, so let us focus on it. The heat flux can be neglected if we assume

$$\omega \gg k_\parallel v_{\rm th}, \tag{23.21}$$

and so then can be compressional heating, because $\nabla_\parallel u_\parallel \sim k_\parallel v_{\rm th}{\rm Ma}$. Let us assume (23.21) and worry later whether it is satisfied by the solution that we get. Note that this amounts to taking the *fluid limit:* $q_\parallel$ was the only quantity in our equations that still required δf to be calculated. Now that it is gone, we have a closed system! Physically, this means we are neglecting the ion Landau damping, because its rate would be $\propto e^{-(\omega/k_\parallel v_{\rm th})^2} \ll 1$, by the same approximation as I made when deriving the ion-acoustic waves in §3.8.

So, using also (23.15), (23.20) is now

$$\frac{\partial}{\partial t}\frac{\delta T_\parallel}{T} \approx -\bar{\tau}^{-1}\frac{\rho v_{\rm th}}{2L_T}\frac{\partial}{\partial y}\frac{\delta n}{n}. \tag{23.22}$$

A frequency has emerged in the right-hand side, called the *drift frequency:*

$$\omega_{*T} = \bar{\tau}^{-1}\frac{k_y\rho v_{\rm th}}{2L_T}. \tag{23.23}$$

We shall see shortly that (23.21) will be satisfied if

$$\omega_{*T} \gg \omega, \tag{23.24}$$

and, therefore,

$$\frac{\delta T_\parallel/T}{\delta n/n} \sim \frac{\omega_{*T}}{\omega} \gg 1. \tag{23.25}$$

This comes handy in dealing with the parallel-velocity equation (23.17), where only the temperature perturbation's contribution needs retaining under this approximation:

$$\frac{\partial u_\parallel}{\partial t} \approx -\frac{v_{\rm th}^2}{2}\nabla_\parallel\frac{\delta T_\parallel}{T}. \tag{23.26}$$

Finally, the density equation remains unscathed,

$$\left(\frac{\partial}{\partial t}+\bar{\tau}^{-1}\frac{\rho v_{\rm th}}{2L_n}\frac{\partial}{\partial y}\right)\frac{\delta n}{n}=-\nabla_\parallel u_\parallel, \tag{23.27}$$

provided, *a priori*, that the drift frequency associated with the density gradient is ordered as

$$\omega_{*n}\equiv\bar{\tau}^{-1}\frac{k_y\rho v_{\rm th}}{2L_n}\sim\omega, \tag{23.28}$$

requiring, for consistency with (23.24), that

$$\eta_i\equiv\frac{L_n}{L_T}\gg 1, \tag{23.29}$$

i.e., that the density gradient be much less steep than the temperature one.

Combining (23.27), (23.26), and (23.22), and Fourier-transforming the lot (in t, z, and y, but not, technically, in x), we get the *fluid ITG dispersion relation*

$$\boxed{\omega^2\left(\omega-\omega_{*n}\right)=\frac{k_\parallel^2 v_{\rm th}^2}{2}\omega_{*T}}\,. \tag{23.30}$$

The simplest case is one where $\omega_{*n}\ll\omega$, i.e., the density gradient is even weaker than anticipated in (23.28). Then the dispersion relation (23.30) becomes

$$\omega^3=\frac{k_\parallel^2 v_{\rm th}^2}{2}\omega_{*T}\quad\Rightarrow\quad\omega=\left(\frac{k_\parallel^2 v_{\rm th}^2}{2}|\omega_{*T}|\right)^{1/3}\left(-\frac{1}{2}\,{\rm sgn}\,k_y+\frac{\sqrt{3}}{2}i\right). \tag{23.31}$$

One of the three roots, which is the one written above, is unstable. This is called the *ion-temperature-gradient (ITG) instability*, which is the main culprit behind turbulent transport in tokamaks. Note that

$$\frac{\omega}{k_\parallel v_{\rm th}}\sim\left(\frac{\omega_{*T}}{k_\parallel v_{\rm th}}\right)^{1/3}\gg 1\quad\text{and}\quad\frac{\omega}{\omega_{*T}}\sim\left(\frac{k_\parallel v_{\rm th}}{\omega_{*T}}\right)^{2/3}\ll 1, \tag{23.32}$$

as per (23.21) and (23.24), respectively.

Physically, the instability feedback loop is easy to grasp. $\boldsymbol{E}\times\boldsymbol{B}$ advection of the mean temperature gradient creates $\delta T_\parallel$ via (23.20), or (23.22). This temperature perturbation has $-\nabla_\parallel\delta T_\parallel$, a parallel pressure gradient, which drives a parallel ion flow $u_\parallel$ via (23.26). A parallel compression/rarefaction associated with this flow, $\nabla_\parallel u_\parallel$, creates δn_i via (23.27). This ion density perturbation immediately results in an electron density perturbation δn_e via the quasineutrality (23.11). Electrons stream along the field lines and have to be balanced by a parallel electric field, as per the Boltzmann response (23.10). But, in an electrostatic plasma, a parallel electric field implies also a perpendicular electric field, which drives an $\boldsymbol{E}\times\boldsymbol{B}$ flow, completing the self-reinforcing cycle. *Voilà.*

Exercise 23.1. Drift waves. Consider a different limit, where $1/L_T=0$ and $\omega_{*n}\gg k_\parallel v_{\rm th}$ (or simply just $k_\parallel=0$). Work out, mathematically and physically, what happens. In constructing a physics cartoon of the drift wave, you might be helped by Parra (2019*b*).

Yet again, we have found an instability whose growth rate increases without bound as the wave number is increased, $\gamma\propto k_\parallel^{2/3}k_y^{1/3}$. We shall worry about the k_y dependence later, but the $k_\parallel$ dependence is something that we can do something about right away: the fastest-growing modes will break the fluid approximation (23.21), so we must do kinetic theory after all. Let us rewind to the ion kinetic equation (23.7) and solve it properly.

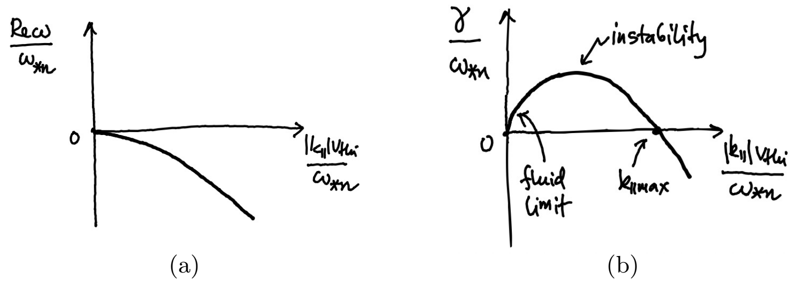


FIGURE 98. (a) Frequency and (b) growth rate of the ITG instability.

We shall assume, henceforth, that

$$\omega \sim k_\parallel v_{\rm th} \sim \omega_{*T}, \tag{23.33}$$

which is where the fastest growth (at fixed k_y) will be found.

23.3. *Kinetic ITG Instability*

The solution of (23.7) for ions ($q = Ze$) is

$$\delta f = \frac{1}{k_\parallel v_\parallel - \omega}\left\{-k_\parallel v_\parallel + \frac{k_y \rho v_{\rm th}}{2}\left[\frac{{\rm d}\ln n_0}{{\rm d}x} + \left(\frac{w_\perp^2 + v_\parallel^2}{v_{\rm th}^2} - \frac{3}{2}\right)\frac{{\rm d}\ln T_0}{{\rm d}x}\right]\right\}\bar{\varphi} f_0, \tag{23.34}$$

for the Maxwellian equilibrium (23.6). Substituting the Boltzmann response and quasineutrality (23.11), as well as the definitions of the gradient scale lengths (23.13), then integrating the above equation over velocities, gets us a disersion relation:

$$1 = -\int {\rm d}v_\parallel \frac{1}{k_\parallel v_\parallel - \omega}\left[\bar{\tau}^{-1} k_\parallel v_\parallel + \omega_{*n} + \omega_{*T}\left(\frac{v_\parallel^2}{v_{\rm th}^2} - \frac{1}{2}\right)\right]\frac{e^{-v_\parallel^2/v_{\rm th}^2}}{\sqrt{\pi} v_{\rm th}}. \tag{23.35}$$

Replacing in the above $k_\parallel v_\parallel = k_\parallel v_\parallel - \omega + \omega$ to separate a non-resonant piece of the density response, then changing the integration variable to $u = v_\parallel / v_{\rm th}$ and denoting, as usual, $\zeta = \omega/|k_\parallel| v_{\rm th}$, $\zeta_{*n} = \omega_{*n}/|k_\parallel| v_{\rm th}$, and $\zeta_{*T} = \omega_{*T}/|k_\parallel| v_{\rm th}$, we get

$$1 + \bar{\tau}^{-1} = -\int {\rm d}u \frac{1}{u - \zeta}\left[\bar{\tau}^{-1}\zeta + \zeta_{*n} + \zeta_{*T}\left(u^2 - \frac{1}{2}\right)\right]\frac{e^{-u^2}}{\sqrt{\pi}}. \tag{23.36}$$

All integrals are expressible in terms of the plasma dispersion function (3.89); in the term multiplying ζ_{*T}, this is accomplished by writing $u^2 = u^2 - \zeta^2 + \zeta^2 = (u-\zeta)(u+\zeta) + \zeta^2$ and integrating out the non-resonant bit, giving $\zeta + \zeta^2 Z(\zeta)$. The result is the *kinetic ITG dispersion relation:*

$$\boxed{1 + \bar{\tau}^{-1} + \zeta_{*T}\zeta + \left[\zeta_{*n} + \zeta_{*T}\left(\zeta^2 - \frac{1}{2}\right) + \bar{\tau}^{-1}\zeta\right] Z(\zeta) = 0}\,. \tag{23.37}$$

Please do not confuse $Z(\zeta_i)$ the plasma dispersion function with Z the dimensionless ion charge!

Exercise 23.2. Recover (23.30) from (23.37) in the fluid limit (23.21) (i.e., for $\zeta \gg 1$).

Exercise 23.3. What physics does (23.37) describe when $\omega_{*n} = \omega_{*T} = 0$?

The unstable solution of (23.37) is easy to get numerically. It generally looks like the sketch in Fig. 98. There is a very easy trick that allows one to obtain the stability boundary exactly: on this boundary, $\gamma = 0$, so ζ is real; therefore, the imaginary part of (23.37) only contains terms that are proportional to $Z(\zeta)$:

$$\zeta_{*n} + \zeta_{*T}\left(\zeta^2 - \frac{1}{2}\right) + \bar{\tau}^{-1}\zeta = 0, \tag{23.38}$$

whence also

$$1 + \bar{\tau}^{-1} + \zeta_{*T}\zeta = 0 \quad \Rightarrow \quad \zeta = -\left(1 + \bar{\tau}^{-1}\right)\zeta_{*T}^{-1}. \tag{23.39}$$

Substituting this into (23.38), we get

$$\zeta_{*T}\left(\zeta_{*n} - \frac{1}{2}\zeta_{*T}\right) + 1 + \bar{\tau}^{-1} = 0. \tag{23.40}$$

From this, we can extract the largest unstable parallel wave number:

$$\frac{|k_\parallel| v_{\rm th}}{\omega_{*n}} = \sqrt{\frac{\eta_i(\eta_i - 2)}{2(1 + \bar{\tau}^{-1})}}, \tag{23.41}$$

where $\eta_i = \omega_{*T}/\omega_{*n} = L_n/L_T$ is the ratio of the gradients, already defined in (23.29). The instability exists for $\eta_i > 2$, i.e., for temperature gradients above a critical value set by the density gradient:

$$\boxed{L_T^{-1} > L_{T{\rm c}}^{-1} = 2L_n^{-1}}\,. \tag{23.42}$$

By dimensional analysis, the peak growth rate is $\gamma_{\max} \propto \omega_{*n} F(\eta_i) \propto k_y$. Yet again, we find unstable perturbations that grow ever faster as $k_y \to \infty$, so yet again drift kinetics is inadequate and one must access finite $k_y\rho_i$ to capture the fastest-growing ITG mode.

It is actually not a coincidence that we keep finding that microinstabilities in drift kinetics explode at small scales. Drift kinetics is scale-invariant because it contains no microscales that can pin the peak growth down. The resulting UV catastrophes can be regularised either by including finite-Larmor-radius (FLR) physics (§24.1) or perpendicular dissipation terms, physical (which is, in fact, a subset of FLR physics) or artificial (as is often done in numerical simulations).

Exercise 23.4. PVG instability.[160] Instead of (23.6), assume an equilibrium distribution that also has a mean parallel flow velocity $\boldsymbol{u}_0(x) = u_0(x)\hat{\boldsymbol{z}}$:

$$f_0(x, w_\perp, v_\parallel) = \frac{n_0(x)}{\pi^{3/2} v_{\rm th}^3(x)} \exp\left\{-\frac{w_\perp^2 + \left[v_\parallel - u_0(x)\right]^2}{v_{\rm th}^2(x)}\right\}, \tag{23.43}$$

where $u_0 \sim v_{{\rm th}i}$ (or smaller). Convince yourself that electrons will remain Boltzmann, as in §23.1. In contrast, ions now have new features.

(a) Show that the perturbed ion density δn and parallel flow velocity $\delta u_\parallel$ satisfy

$$\left(\frac{\partial}{\partial t} + u_0\nabla_\parallel\right)\frac{\delta n}{n} + \nabla_\parallel \delta u_\parallel = -\frac{\rho v_{\rm th}}{2L_n}\frac{\partial\bar{\varphi}}{\partial y}, \tag{23.44}$$

$$\left(\frac{\partial}{\partial t} + u_0\nabla_\parallel\right)\delta u_\parallel = -\frac{v_{\rm th}^2}{2}\nabla_\parallel\left(\frac{\delta p_\parallel}{p} + \bar{\varphi}\right) + \frac{\rho v_{\rm th}\gamma_\parallel}{2}\frac{\partial\bar{\varphi}}{\partial y}, \tag{23.45}$$

where $\gamma_\parallel = {\rm d}u_0/{\rm d}x$ is the parallel shear rate. To start with, ignore both density and temperature gradients, $L_n^{-1} = L_T^{-1} = 0$, and conjecture that ions can be considered isobaric: $\delta p_\parallel = 0$ (we shall check this later). Determine at what wave numbers $(k_y, k_\parallel)$ there are unstable perturbations and calculate their growth rate. This is the "fluid" limit of the *parallel-velocity-gradient (PVG)*

---

[160] This is based on the 2024 exam question.

instability, which, it should be obvious from your formula, is a destabilised sound wave. Calculate the maximum growth rate given a fixed k_y.

(b) Now assume $L_n^{-1} \neq 0$, $L_T^{-1} \neq 0$, $\gamma_\parallel \neq 0$ and show that the dispersion relation for the combined ITG-PVG instability is

$$\boxed{1 + \bar{\tau}^{-1}(1 - \zeta_{*u}) + \zeta_{*T}\zeta + \left[\zeta_{*n} + \zeta_{*T}\left(\zeta^2 - \frac{1}{2}\right) + \bar{\tau}^{-1}(1 - \zeta_{*u})\,\zeta\right] Z(\zeta) = 0}\,, \tag{23.46}$$

where $\zeta = (\omega - k_\parallel u_0)/|k_\parallel| v_{\rm th}$ and $\zeta_{*u} = k_y \rho \gamma_\parallel / k_\parallel v_{\rm th}$ (that the last expression contains $k_\parallel$, rather than $|k_\parallel|$, is not a typo); the rest of the notation is the same as in §23.3.

(c) In what limit do you expect to be able to recover from (23.46) the fluid limit of the pure PVG instability obtained in (a)? Recover it and explain, *a posteriori*, why $\delta p_\parallel = 0$ was a good assumption. Argue that the maximum growth rate (at fixed k_y) calculated in the fluid limit is, in fact, only correct if ions are cold, viz., if $\bar{\tau} \ll 1$.

(d) Returning to the full dispersion relation (23.46), show that the stability boundary in the $(k_y, k_\parallel)$ space is given by

$$\frac{k_\parallel v_{\rm th}}{\omega_{*n}} = \frac{\eta_u \pm \sqrt{\eta_u^2 + \eta_i(\eta_i - 2)(1 + \bar{\tau}^{-1})/2}}{1 + \bar{\tau}^{-1}}, \tag{23.47}$$

where $\eta_u = \gamma_\parallel L_n / v_{\rm th}$. Sketch qualitatively what you expect the growth rate of the combined ITG-PVG instability at fixed k_y to look like as a function of $k_\parallel$ depending on whether η_u is positive or negative and on the values of η_i. Prove that the condition for the system to be unstable at any value of η_i is

$$|\gamma_\parallel| > \frac{v_{\rm th}}{L_n}\sqrt{\frac{1 + \bar{\tau}^{-1}}{2}}. \tag{23.48}$$

A gyrokinetic version of some of these calculations (from which the DK results required here are recoverable in the limit $k_\perp \rho_i \ll 1$) can be found in Schekochihin *et al.* (2012) (NB: without $\bar{\tau}^{-1}$ in the definitions of ω_*'s and a few other notational differences). You can also learn there how to handle ITG- and PVG-excited perturbations in the presence of a mean perpendicular shear flow, $\boldsymbol{v}_{E0} = Sx\hat{\boldsymbol{y}}$, when all growth is formally transient. Things become even more interesting when the equilibrium magnetic field is also sheared: read Newton *et al.* (2010).[161]

23.4. *Nature of the Electrostatic Limit*

I promised at the beginning of §23 to explain in a specific way how the electrostatic limit ($\delta\boldsymbol{B} = 0$) could be justified. I shall do it now.

Let us identify the point in the derivation of KMHD at which evolving $\boldsymbol{B}$ in response to a non-zero $\boldsymbol{E} \times \boldsymbol{B}$ flow became inevitable (and so electrostaticity was no longer an option). This was clearly the moment when the induction equation (20.42) emerged from the marriage of Faraday's law with the "ideal Ohm's law" (20.39). Faraday's law is hardly negotiable, but the ideal Ohm's law very much is. In §20.5, it was derived by applying the KMHD approximation to the momentum equation for (any) species α, recast as an equation for the electric field. Let us consider specifically the case $\alpha = e$ and ask what assumptions were needed to get rid of each of the three terms in (20.27) that had to be got rid of to leave us with (20.39). Neglecting both the electron-inertia term and the friction term only required Ω_e to be asymptotically large (in comparison with ω and ν_e, respectively), but the term involving the pressure tensor, even for electrons, required going to scales longer than the Larmor radius of the *ions* [see (20.32) and the discussion thereafter].

Let us now *not* assume $k\rho_i \ll 1$ and so retain the electron-pressure term:

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c} - \frac{\boldsymbol{\nabla}\cdot\mathsf{P}_e}{en_e} \quad\Rightarrow\quad E_\parallel = -\frac{(\boldsymbol{\nabla}\cdot\mathsf{P}_e)\cdot\boldsymbol{b}}{en_e} = -\frac{\nabla_\parallel p_{\parallel e} + \left(p_{\parallel e} - p_{\perp e}\right)\boldsymbol{\nabla}\cdot\boldsymbol{b}}{en_e}. \tag{23.49}$$

[161] Some similar ideas and tricks in application to sheared, stratified MHD can be learned from Howes *et al.* (2001).

The last step above was to project the equation onto the magnetic field (here an attentive reader might recall the observation made in §20.10 that the ideal Ohm's law gave $E_\| = 0$ to lowest order in $k\rho_i \ll 1$) and to use the gyrotropic approximation (20.51) for P_e, which requires assuming $k\rho_e \ll 1$ but nothing about $k\rho_i$ (cf. Exercise 20.2). Let us make two further steps, one exact, $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - (1/c)\partial\boldsymbol{A}/\partial t$, and one approximate: consider small (but finite) perturbations around an equilibrium where both $\boldsymbol{B}_0$ and f_{0e} are static and uniform (no spatial variation) and f_{0e} also has no pressure anisotropy (same as in §23.1). Then (23.49) becomes

$$\frac{1}{c}\frac{\partial A_\|}{\partial t} + \nabla_\|\varphi = \frac{\nabla_\|\delta p_{\|e}}{en_e}. \tag{23.50}$$

This is, of course, just a statement of force balance between the parallel electric field and parallel pressure gradient. Assuming also isothermal electrons (which is fine in the absence of an equilibrium electron-temperature gradient), $\delta p_{\|e} = T_e\delta n_e$, we get

$$\boxed{\frac{\partial}{\partial t}\frac{eA_\|}{cT_e} = -\nabla_\|\left(\frac{e\varphi}{T_e} - \frac{\delta n_e}{n_e}\right)}\,. \tag{23.51}$$

If the left-hand side of this equation is negligible compared to its right-hand side, we get Boltzmann electrons (23.10).[162] This makes explicit the connection between the failure of the Boltzmann closure and $\delta\boldsymbol{B} \neq 0$.

Another thing that is made explicit by (23.51) is that you cannot go all the way to the KMHD approximation, in which, once you assume $k\rho_i \ll 1$, the $\delta n_e/n_e$ term drops out, and then to the electrostatic limit ($A_\| = 0$), because then the remaining electrostatic electric field $-\nabla_\|\varphi$ would have nothing to balance it. In KMHD (and in MHD), it is balanced by the inductive bit—this was manifest already in RMHD, whose magnetic-flux equation (16.116) is the same as (23.51) with $\delta n_e/n_e$ dropped, the nonlinearity hiding inside $\nabla_\|$, and in different notation:

$$\varPsi = -\frac{A_\|}{\sqrt{4\pi m_i n_i}}, \qquad \varPhi = \frac{c\varphi}{B_0}. \tag{23.52}$$

Exercise 23.5. Prove that, under the RMHD ordering (16.104), $\delta n_e/n_e \ll e\varphi/T_e$ if $k_\perp\rho_i \ll 1$.

It is from RMHD that we can quickly get an idea of how to adjudicate the competition between the right- and left-hand sides of (23.51). Let us, for a moment, just use the RMHD stream-function equation (16.120), or (16.121), now translated into the $(\varphi, A_\|)$ language according to (23.52), whence immediately follows an estimate of the size of the magnetic response to any perturbation of φ:

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2\frac{e\varphi}{T_e} = -v_\mathrm{A}^2\nabla_\|\nabla_\perp^2\frac{eA_\|}{cT_e} \quad\Rightarrow\quad \frac{eA_\|}{cT_e} \sim \frac{\omega}{k_\| v_\mathrm{A}^2}\frac{e\varphi}{T_e} \tag{23.53}$$

[obviously, the same equation can be derived directly from the KMHD momentum equation (20.53) by application of the RMHD ordering: see Exercise 20.7]. Using the estimate (23.53) in (23.51), we see that the left-hand side can be neglected—and, therefore, the Boltzmann closure (23.10) recovered—provided $k_\| v_\mathrm{A} \gg \omega$, i.e., perturbations are electrostatic when Alfvén waves propagate very fast compared to them. Physically, when v_A is very large compared to the phase speed of the perturbations, any attempt to bend the magnetic field is Alfvénically radiated away very quickly, leaving only a very small magnetic perturbation behind (somewhat like in an incompressible fluid, where the sound speed is very large, so any pressure perturbation is radiated away by the sound waves). Mathematically, since the electrostatic perturbations that we are interested in are ITG-unstable drift waves, the condition of applicability of the electrostatic approximation becomes

$$k_\| v_\mathrm{A} \gg \omega \sim \omega_{*T} = \bar{\tau}^{-1}\frac{k_y\rho_i v_{\mathrm{th}i}}{2L_T} \quad\Leftarrow\quad \boxed{\frac{L_\|\sqrt{\beta_i}}{L_T} \ll 1}\,. \tag{23.54}$$

---

[162]It is easy to derive (23.51) from the electron DKE in the same way as (23.10) was derived in §23.1, but keeping the inductive part of $E_\|$.

The second inequality is a sufficient condition (taking $k_y\rho_i \lesssim 1$, $\bar{\tau} \sim 1$, $k_\parallel \gtrsim L_\parallel^{-1}$, and recalling that $\beta_i = v_{\mathrm{th}i}^2/v_{\mathrm{A}}^2$). This is the sense in which the electrostatic approximation is a low-beta limit. One can also call it a shallow-box limit (small $L_\parallel$) or a weakly-driven limit (small L_T^{-1}).

If one now wants to know what happens, linearly and/or nonlinearly, when the electrostatic approximation starts unravelling (β_i is too high and/or the parallel size of the device is too large and/or the gradient is too strong), it is not possible to work this out in the DK approximation because the correct ordering of frequencies must be $k_\parallel v_{\mathrm{A}} \sim \omega \sim \omega_{*T}$ and the $\delta n_e/n_e$ term in (23.51) must be kept, which requires $k_\perp\rho_i \sim 1$, clashing with what was assumed in the derivation of KMHD in §20. The way forward therefore is to work in the gyrokinetic approximation—in §24.1, I will tell you how to study it on your own.

Note that if, instead of (23.54), we assume $k_\parallel v_{\mathrm{A}} \sim \omega \gg \omega_{*T}$, this obviates the temperature gradient and takes us back to the standard KMHD regime—or the reduced version of it (KRMHD), which you had the opportunity to derive in Exercise 20.7. This corresponds to dropping the $\delta n_e/n_e$ term in (23.51). Such a physical regime, where $k_\perp\rho_i \ll 1$ and the frequencies of the perturbations are Alfvénic (and so the perturbations are fully electromagnetic) describes what happens in, e.g., the solar wind, where these perturbations are stirred up by the messy dynamics in the Sun's corona. Some reading suggestions about this topic are provided in §24.3.

23.5. *Turbulent Transport*

The reason, generally, why we care about instabilities, especially ones that are fed by gradients of the equilibrium profiles, is that these instabilities eventually saturate in a nonlinear, usually chaotic, state—turbulence—and that state, while also interesting by itself, leads to relaxation of those equilibrium profiles, known as *turbulent transport*. In §6 and in Part II of these lectures, we saw how that worked in kinetic theory (effect of δf on f_0). This time, we shall study how this works in a fluid context.

23.5.1. *Nonlinear ITG Equations*

The saturation of the ITG instability (and generally of any electrostatic DK instability) is caused by the sole surviving nonlinearity in the DK equation (23.4)—the advection of the distribution function by the $\boldsymbol{E}\times\boldsymbol{B}$ flow $\boldsymbol{v}_E$. Similarly to how it was done in MHD (§16.2.3), let us ask for the nonlinear time scale to be comparable to the linear ones, viz.,

$$k_\perp v_E \sim \omega \sim k_\parallel v_{\mathrm{th}}, \tag{23.55}$$

and thus fix the ordering of the nonlinear amplitudes. Just like with RMHD (§16.3), we end up with nonlinear equations for the perturbed fields (still small but now finitely so) that are just our linearised equations (23.7), (23.14), (23.17), and (23.20) plus the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity: the DKE

$$\boxed{\frac{\partial \delta f}{\partial t} + \boldsymbol{v}_E\cdot\boldsymbol{\nabla}_\perp\delta f + v_\parallel\nabla_\parallel\delta f = -v_\parallel\nabla_\parallel\bar{\varphi}\, f_0 + \frac{\rho v_{\mathrm{th}}}{2}\frac{\partial f_0}{\partial x}\frac{\partial\bar{\varphi}}{\partial y}}\,, \tag{23.56}$$

and its fluid moments

$$\frac{\partial}{\partial t}\bar{\tau}\bar{\varphi} = -\nabla_\parallel u_\parallel - \frac{\rho v_{\mathrm{th}}}{2L_n}\frac{\partial\bar{\varphi}}{\partial y}, \tag{23.57}$$

$$\frac{\partial u_\parallel}{\partial t} + \boldsymbol{v}_E\cdot\boldsymbol{\nabla}_\perp u_\parallel = -\frac{v_{\mathrm{th}}^2}{2}\nabla_\parallel\left[(1+\bar{\tau})\bar{\varphi} + \frac{\delta T_\parallel}{T}\right], \tag{23.58}$$

$$\frac{\partial}{\partial t}\frac{\delta T_\parallel}{T} + \boldsymbol{v}_E\cdot\boldsymbol{\nabla}_\perp\frac{\delta T_\parallel}{T} = -\frac{\nabla_\parallel q_\parallel}{p} - 2\nabla_\parallel u_\parallel - \frac{\rho v_{\mathrm{th}}}{2L_T}\frac{\partial\bar{\varphi}}{\partial y}, \tag{23.59}$$

where I have used (23.15) to write $\delta n/n = \bar{\tau}\bar{\varphi}$. Note that the $\bar{\varphi}$ equation (23.62) does not have a nonlinearity because $\boldsymbol{v}_E\cdot\boldsymbol{\nabla}_\perp = (\rho v_{\mathrm{th}}/2)\{\bar{\varphi},\dots\}$, so $\boldsymbol{v}_E\cdot\boldsymbol{\nabla}_\perp\bar{\varphi} = 0$ (this actually has some important—and wrong—consequences, which have to be dealt with in the usual spirit of never taking zero for an answer: see §24.2.2).

23.5.2. *Turbulent Diffusion*

Let us now imagine that the equilibrium distribution (23.6) is not entirely static but evolves on a time scale that is long compared to the linear and nolinear time scales present in the above equations for the turbulent perturbations. This is the same approach as I took at the very beginning of these lectures (§2.2) and followed doggedly through much of Parts I and II. Assuming that the equilibrium stays Maxwellian (i.e., that collisions, while slow compared to turbulence, are fast compared to the equilibrium evolution), this means that the temperature and density profiles evolve slowly. Their evolution can be extracted by averaging the DK equation (23.4) over y, z, and the fast time scales, leading, for temperature, e.g., to

$$\frac{\partial T_0}{\partial t} = -\frac{\partial}{\partial x}\left\langle v_{Ex}\delta T_\parallel\right\rangle + \dots, \tag{23.60}$$

where "..." represent all other terms, to be ignored for the purposes of this discussion, whose aims are purely qualitative. Thus, just like in the case of f_0 and δf, you need to know the evolution of the perturbations to work out the evolution of the equilibrium, which is encoded in the *turbulent heat flux*

$$Q_{\rm turb} = n_0\left\langle v_{Ex}\delta T_\parallel\right\rangle = -n_0 T_0 v_{\rm th}\frac{\rho}{2}\left\langle\frac{\partial\bar\varphi}{\partial y}\frac{\delta T_\parallel}{T_0}\right\rangle. \tag{23.61}$$

A sizeable chunk of the fusion literature (mostly numerical) is devoted to the calculation of this quantity as a function of the local equilibrium parameters—i.e., to determining a certain correlation function of the saturated turbulent state.

If we now recall the equations for the turbulent fields, and, in particular, the perturbed temperature (23.64), we see that it has the form

$$\frac{\partial\delta T_\parallel}{\partial t} = -v_{Ex}\frac{\partial T_0}{\partial x} + \dots \tag{23.62}$$

Integrating this with respect to time and substituting into (23.65) gives us

$$\frac{\partial T_0}{\partial t} = \frac{\partial}{\partial x}\int_{-\infty}^{t}{\rm d}t'\,\langle v_{Ex}(t)v_{Ex}(t')\rangle\,\frac{\partial T_0}{\partial x} + \dots, \tag{23.63}$$

where the difference between t' and t in the argument of $\partial T_0/\partial x$ can be ignored because T_0 changes slowly compared to the correlation time of $v_{Ex}(t)$ (an assumption that also allows one to argue away some of the terms that I have refused to write out). This is the derivation of *turbulent diffusion*, i.e., of the fact the turbulent heat flux is

$$\boxed{Q_{\rm turb} = -n_0 D_{\rm turb}\frac{\partial T_0}{\partial x} = n_0 T_0\frac{D_{\rm turb}}{L_T}}\,,\qquad \boxed{D_{\rm turb} = \int_0^\infty {\rm d}\tau\,\langle v_{Ex}(t)v_{Ex}(t-\tau)\rangle}\,. \tag{23.64}$$

Thus, to calculate the turbulent diffusivity $D_{\rm turb}$ is to calculate the time-correlation function of the turbulence, or, if one is prepared to settle for twiddle algebra and scalings (as one often must be, in studying turbulence), it is to determine the saturated amplitude and correlation time of the fluctuating $\boldsymbol{E}\times\boldsymbol{B}$ flows:

$$\boxed{D_{\rm turb} \sim v_E^2\tau_{\rm corr}}\,. \tag{23.65}$$

I shall spend the rest of §23.5 working out how this quantity scales with the equilibrium parameters, most importantly, L_T. The argument that I am about to rehearse is due to Barnes *et al.* (2011)—it was, of course, inspired by the phenomenology of MHD turbulence (§16.4).

23.5.3. *Energy Injection, Turbulent Cascade, and the Outer Scale*

Since turbulence, generally, is a multiscale phenomenon, the first question that we need to address is at what scale does one need to determine v_E and $\tau_{\rm corr}$ in order to pick up the dominant contribution to $D_{\rm turb}$. For that, we must consider how at each scale fluctuations are excited by the linear instability and broken up by the nonlinearity.

We saw in §§23.2 and 23.3 that, at any given k_y, the ITG instability's growth rate is

$$\gamma \sim \omega_{*T} \sim k_y\rho\frac{v_{\rm th}}{L_T} \propto k_y, \tag{23.66}$$

so the fastest injection of unstable perturbations is at the (ever) smaller scales. This might suggest that the DK approximation amounts to throwing the baby out with the bathwater, but we are about to see that everything is, in fact, alright (or at least enough is alright to keep us going for a while).

The energy injection by the linear instability will, at every scale, compete with nonlinear mixing—the shearing away of the perturbations by the advecting $\boldsymbol{E}\times\boldsymbol{B}$ flows, whose effect is to transfer the injected energy to smaller scales. To work out the characteristic rate at which this happens, which is, obviously, $\tau_{\rm corr}^{-1}$, we must first give a more quantitative meaning to "energy".

The $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearities that advect δf and its moments, like all advective nonlinearities, cannot change the variance of the fields that they advect but only shuffle it around scales. The right linear combination of these variances also cancels out effect of the linear terms involving $\nabla_\parallel$. The easiest way to see this is from the three moment equations (23.62–23.64), truncated for now with $q_\parallel = 0$. They then have a conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{\mathrm{d}\boldsymbol{r}}{V}\left\langle\frac{(1+\bar\tau)\bar\tau}{2}\,\bar\varphi^2 + \left(\frac{u_\parallel}{v_{\rm th}}\right)^2 + \frac{1}{4}\left(\frac{\delta T_\parallel}{T}\right)^2\right\rangle = \varepsilon - \mathscr{D}, \tag{23.67}$$

where the injection rate

$$\boxed{\varepsilon = -\frac{\rho v_{\rm th}}{4L_T}\int\frac{\mathrm{d}\boldsymbol{r}}{V}\left\langle\frac{\partial\bar\varphi}{\partial y}\frac{\delta T_\parallel}{T}\right\rangle = \frac{Q_{\rm turb}}{2nTL_T} = \frac{D_{\rm turb}}{2L_T^2} > 0} \tag{23.68}$$

is manifestly due to the background temperature gradient, and I have added the dissipation rate $\mathscr{D}$ in order to balance injection and allow for a steady state—this represents small-scale physics that has not been included in our equations, but, as we see now, has to be present either in its correct form or as a simplified model, e.g., some viscosity and thermal diffusivity (ultimately, it is always due to collisions—just like it was, e.g., in §5.2 or §12.3).

Exercise 23.6. Free energy in electrostatic DK. As you must have guessed, we are really discussing the free-energy invariant (cf. §5.2, Q-6 and Q-8 of the Kinetics Problem Set, and Exercise 22.4). Work out what it is for the DK equation (23.61) and show that the quadratic invariant appearing in (23.72) is just the sum of the squares of the first three Hermite moments of δf, whereas the full invariant is the sum of the squares of all of them (you will find this done in Schekochihin *et al.* 2016).

If we imagine *a priori*—in a moment, we shall confirm that this is true—that the energy is dominantly injected at some scale λ_0, called the *outer scale*, then, at scales below this scale, $\lambda \ll \lambda_0$, the physics will consist of the energy arriving from larger scales and then leaving, due to nonlinear shearing, towards smaller scales. It cannot linger, so its flux through each scale λ must be the same and equal to the injection rate ε. This is the usual Kolmogorov-style argument for a constant-fux energy cascade:[163]

$$\tau_{\rm corr}^{-1}\left(\frac{\delta T_\parallel}{T}\right)^2 \sim \varepsilon. \tag{23.69}$$

Here the energy is represented by the variance of the temperature fluctuations because, as we shall confirm shortly, all fields will be comparable in our model of saturated ITG turbulence:

$$\frac{\delta T_\parallel}{T} \sim \frac{u_\parallel}{v_{\rm th}} \sim \bar{\varphi} \sim \frac{\lambda v_E}{\rho v_{\rm th}} \tag{23.70}$$

(the last estimate, which will be useful shortly, follows from $v_E \sim c\nabla\varphi/B$). The time that it takes to pass the energy on across each scale λ is the correlation time at that scale, which is also, evidently, the nonlinear shearing time:

$$\tau_{\rm corr}^{-1} \sim \frac{v_E}{\lambda}. \tag{23.71}$$

Combining the last three equations gives us the scaling of the turbulent amplitudes and correlation times:

$$\boxed{v_E \sim \left(\frac{\varepsilon\rho^2 v_{\rm th}^2}{\lambda}\right)^{1/3}}, \qquad \boxed{\tau_{\rm corr}^{-1} \sim \left(\varepsilon\rho^2 v_{\rm th}^2\right)^{1/3}\lambda^{-4/3} \propto k_\perp^{4/3}}. \tag{23.72}$$

For such a turbulence, the turbulent diffusivity (23.70) associated with the motions at scale λ is

$$D_{\rm turb} \sim v_E^2\tau_{\rm corr} \sim v_E\lambda \sim (\varepsilon\rho^2 v_{\rm th}^2)^{1/3}\lambda^{2/3}, \tag{23.73}$$

so it must be dominated by the motions at the largest, outer scale, $\lambda \sim \lambda_0$. That such a scale must exist is evident from comparing the scalings of the growth rate (23.71) and the nonlinear mixing rate (23.77) with $k_\perp$ (equivalently, λ): the latter grows faster towards smaller scales than the former, so will dominate dynamics below the scale where they are comparable, which is the outer scale (Fig. 99). To reiterate: the outer scale is *not* the scale where the linear growth rate is largest but the scale below which it is smaller than the nonlinear rate (and so injection of perturbations by the linear instability at those small scales is swamped by the energy arriving from larger scales via the nonlinear cascade). So let us find the outer scale:

$$\gamma \sim \frac{\rho}{\lambda_0}\frac{v_{\rm th}}{L_T} \sim \tau_{\rm corr}^{-1} \sim \left(\varepsilon\rho^2 v_{\rm th}^2\right)^{1/3}\lambda_0^{-4/3} \quad \Rightarrow \quad \boxed{\lambda_0 \sim \frac{\varepsilon L_T^3}{\rho v_{\rm th}}}. \tag{23.74}$$

The turbulent diffusivity is, therefore, (23.78) at $\lambda \sim \lambda_0$:

$$\boxed{D_{\rm turb} \sim \varepsilon L_T^2}, \tag{23.75}$$

---

[163]This subject has, apart from brief allusions in §7.4, 8.4.6, and §12.4.2, and the redirect to my review of MHD turbulence (Schekochihin 2022) in §16.4, remained outside the purview of these lectures. The theory of turbulence, of course, deserves thorough study—and is indeed an indispensable part of modern plasma-physics education. Some old lecture notes of mine on the subject, in handwritten form, can be found here: Schekochihin (2007*a*), but, in fact, §§33-34 of Landau & Lifshitz (1987) contain almost everything you need to know. If you want to know more, books by Frisch (1995) and Davidson (2015) are modern classics.

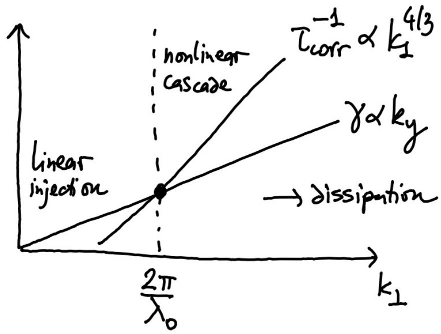


FIGURE 99. The outer scale of ITG turbulence λ_0 determined from the competition of the linear growth and nonlinear mixing rates.

a formula that is, in fact, trivial, following directly from (23.73).

The fact that we have arrived at the same place where we started suggests that, while we might have no internal contradictions in our view of the saturated turbulent state, we have not really solved anything yet—the as-yet-unfulfilled task being to determine ε as a function of L_T. To do so, we need an additional piece of physics, and, once we have it, the scaling argument that has just circled back to (23.80) will turn out to have some actual use.

23.5.4. *Parallel Coherence of Turbulent Structures and the Barnes Scaling*

The additional piece of physics is about what happens along the magnetic field. There are two ways of thinking about it, leading to equivalent results.

We saw in §§23.2 and 23.3 that, for any given k_y, the growth rate went up with $k_\parallel$, breaking the fluid approximation and achieving its greatest value at

$$k_\parallel v_{\rm th} \sim \omega_{*T}. \tag{23.76}$$

Beyond this value, at $k_\parallel v_{\rm th} \gg \omega_{*T}$, Landau damping will suppress the instability, so (23.81) is a reasonable estimate of the characteristic $k_\parallel$.

Another way of thinking about parallel dynamics is nonlinear. While a linear mode can have any coherence length in any direction, including parallel, the nonlinear perturbations cannot stay coherent unless there is some communication mechanism that allows a structure of a given scale λ to maintain coherence across some parallel scale $k_\parallel^{-1}$ over a time that it takes the perpendicular nonlinearity to decorrelate this structure at its two opposite ends along the magnetic field. The natural speed of communication in this problem is $v_{\rm th}$, so we expect, at every perpendicular scale,[164]

$$k_\parallel v_{\rm th} \sim \tau_{\rm corr}^{-1}. \tag{23.77}$$

At the outer scale, in view of (23.79), this is exactly the same condition as (23.81).

Let us now conjecture that the parallel correlation scale cannot exceed some externally imposed limit—the size of the box or the parallel scale length of the underlying

[164]At small scales, $\lambda \ll \lambda_0$, this condition tells us something about the anisotropy of turbulent structures; using (23.77), we discover $k_\parallel \sim \varepsilon^{1/3}\rho^{2/3}v_{\rm th}^{-1/3}k_\perp^{4/3}$, i.e., the turbulence gets a bit less anisotropic at smaller scales—a verifiable (and verified: see Barnes *et al.* 2011) prediction, if perhaps of little practical consequence. In MHD turbulence, the equivalent result (although implying more, rather than less, anisotropy at ever smaller scales) is called "critical balance" (§16.4; cf. §§16.3.5, 12.4.2).

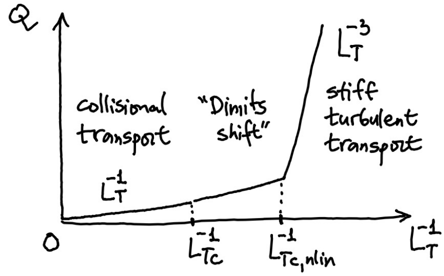


FIGURE 100. Heat flux in an ITG-unstable plasma: gentle collisional transport at temperature gradients below the linear-instability threshold [e.g., (23.42)], stiff turbulent transport (23.84) at large, unstable temperature gradients, and perhaps an intermediate region $L_{Tc}^{-1} < L_T^{-1} < L_{Tc,\mathrm{nlin}}^{-1}$ owing its existence to the Dimits shift (§24.2.3).

equilibrium,—so the outer scale is determined by the condition

$$\omega_* \sim \frac{\rho}{\lambda_0}\frac{v_{\mathrm{th}}}{L_T} \sim \frac{v_{\mathrm{th}}}{L_\parallel} \quad \Rightarrow \quad \boxed{\lambda_0 \sim \rho\frac{L_\parallel}{L_T}}. \tag{23.78}$$

Note that this is safely within the DK approximation ($k\rho \ll 1$) provided $L_\parallel \gg L_T$.

Comparing this estimate with (23.79), we can now deduce the scaling of the injection rate and, via (23.80), the transport (finally, a mildly non-trivial use of the Kolmogorov-like turbulence phenomenology):

$$\varepsilon \sim \frac{v_{\mathrm{th}}\rho^2 L_\parallel}{L_T^4} \quad \Rightarrow \quad D_{\mathrm{turb}} \sim v_{\mathrm{th}} L_\parallel \left(\frac{\rho}{L_T}\right)^2, \qquad \boxed{Q_{\mathrm{turb}} \sim nTv_{\mathrm{th}}\frac{\rho^2 L_\parallel}{L_T^3}}. \tag{23.79}$$

This is the Barnes *et al.* (2011) ITG transport scaling, which appears to be confirmed by numerical experiments[165]—this is often referred to as *stiff* transport, meaning that, for a given heat flux, the temperature gradient is more or less pinned to some marginal value beyond which (23.84) kicks in and effectively prevents the temperature gradient from rising much further (in contrast, had transport been merely collisional, rather than turbulent, we would have had $Q \propto L_T^{-1}$, a much gentler dependence than $Q_{\mathrm{turb}} \propto L_T^{-3}$). This is the bane of fusion devices, meaning that the name of the game is to determine that marginal value. Naïvely, you might think that to be the linear-stability condition (23.42), but we would not have been in this business for so long if everything had been that simple (see Fig. 100 and §24.2.3).

Exercise 23.7. Turbulent transport due to PVG instability. Inspired by the results of Exercise 23.4, work out, along the lines analogous to the above, a theory of transport of heat and (parallel) momentum by the turbulence excited predominantly by the PVG instability.

Exercise 23.8. Turbulent transport due to interchange instability in a z-pinch. Now do the same for the turbulence excited by the interchange instability in a z-pinch, as analysed

[165]The fact that the heat flux is $Q_{\mathrm{turb}} = O(\rho_*^2)$ is known as the "gyro-Bohm scaling"—a key tenet of much of the turbulent-transport theory for fusion devices. The nontrivial prediction on top of that made by Barnes *et al.* (2011) is that $Q_{\mathrm{turb}} \propto L_\parallel/L_T^3$. As far as its numerical confirmation is concerned, it certainly works in a simple slab setting (Adkins *et al.* 2023), but has been contested in toroidal geometry, where, it seems, effects associated with magnetic drifts (§21.5.1) can conspire to make $Q_{\mathrm{turb}} \propto L_T^{-1}$ instead (Nies *et al.* 2024). Things refuse to stay simple in this business.

in Q-12 , at $\beta \ll 1$. In particular, argue that

$$Q_{\mathrm{turb}} \sim nTc_{\mathrm{s}} \frac{\lambda_0^2}{R^{1/2} L_T^{3/2}}. \tag{23.80}$$

Here we have to assume $\lambda_0 \sim$ perpendicular size of the system, unconnected to its parallel size. Why is this?

---

24. A Guide for Further Study

We have only dipped our toe into the vast sea of material that opens up with the slab electrostatic DK calculations that I have gone through above and goes on to encompass many strongly magnetised plasma regimes, both electrostatic and electromagnetic, relevant to magnetically confined fusion plasmas and some astrophysical ones (e.g., the heliosphere). One day, I shall add many more pages to these Lectures to cover all this, but for now, let me give you some quick pointers to the key topics and relevant literature, much of it recent and in which I have had some involvement, that you can read to get *au fait* with the subject as I see it. If you want to be truly *au fait* with it—which means aware of how other people see it as well—you should follow the historical paper trail by looking up the literature that is cited in those papers, and the literature that is cited in that literature. Note that, since I already provided some further-reading suggestions on high-beta topics in §20.8 (firehoses), §22.1 (mirrors), and generally throughout §§20 and 22, I shall, in what follows, focus mostly on the theory that is suitable for low-beta and beta-order-unity plasmas (so, in the first instance, think of tokamaks and solar wind—but not only, as the Universe is big, made of plasma, and full of diverse physical regimes).

24.1. *Gyrokinetics*

Throughout this Part IV of the Lectures, we have kept bumping into the frustrating inevitability of KMHD/DK instabilities escaping the KMHD/DK approximation—due, as I already mentioned at the end of §23.3, to the scale invariance of the $k_\perp \rho \ll 1$ approximation. There is no point delaying any further the moment to grasp the nettle of finite-Larmor-radius physics—while for high-beta plasmas where such things as firehoses and mirrors occur, this means going all the way back to the fully magnetised Vlasov–Maxwell equations (if with some possibility for simplifying approximations: see Bott *et al.* 2024), the low-beta plasmas allow a systematic reduction of these equations to a low-frequency, small-amplitude, anisotropic approximation that is to it what RMHD (§16.3) is to MHD: *gyrokinetics (GK)*.[166]

There is a fair amount of algebra involved, so it may be a good idea to start simple: derive electrostatic GK in a slab, i.e., in the same equilibrium as I used for my treatment of the ITG instability in §23. Surprisingly, such a derivation is hard to find in print, but not impossible: a very nice pedagogical one is a lecture by Cowley (2006) at one of the (now legendary) CMPD Winter Schools at UCLA.

Exercise 24.1. ITG and PVG in GK. Derive the GK dispersion relation for the ITG and PVG instabilities in a slab (with no gradients in $\boldsymbol{B}_0$), study its solutions, making sure that you can recover the DK results (§23.3 and Exercise 23.4).

You will find the solution to this Exercise in Schekochihin *et al.* (2012).

The next step after that is to understand how to handle magnetic perturbations in the

[166]Formally, there is no problem setting up GK for $\beta \sim 1$ or $\beta \gg 1$ plasmas as well—but as soon as pressure-anisotropy-driven instabilities (or their many cousins, all catalogued in Bott's paper) are triggered, they tend to produce perturbations that break the GK approximation, i.e., GK is ill-posed for such plasmas. Indeed even at $\beta \ll 1$, it is possible to set up regimes where GK breaks down: a recently discovered, very curious, example of that is the so-called helicity-barrier phenomenon in imbalanced turbulence (Meyrand *et al.* 2021; Squire *et al.* 2022). Thus, whether GK is well-posed for any given problem is a question that should not be viewed as mere mathematical pedantry.

GK approximation—you will find the simplest exposition of electromagnetic GK (both the derivation and the treatment of waves) in the paper by Howes *et al.* (2006), where it is done for a homogeneous Maxwellian equilibrium with a straight magnetic field and no background gradients (the handwritten notes by Cowley 2004*b* are a record of his UCLA lectures on which that paper was based).

Exercise 24.2. EM GK with gradients. Work out the same equations as Howes *et al.* (2006) but including background gradients, i.e., for an equilibrium with $f_0(x)$ and, if you are strong, $\boldsymbol{B}_0(x)$. Under what conditions can you recover the electrostatic limit?

Exercise 24.3. DK from GK. Once you have the GK equation, you can be spared all the faff of having to go to second order in the ρ_* expansion in order to recover the magnetic drifts, as I did in §21.5. This is because GK makes what I mentioned in §21.6 was the winning choice: assumes anisotropy, $\nabla_\parallel \sim \rho_* \boldsymbol{\nabla}_\perp$. So derive the DK equation with magnetic drifts by taking the $k_\perp \rho \ll 1$ limit of GK—if you have not done Exercise 24.2, you can first try this derivation for the electrostatic GK. The result should resemble (21.94), except there will now be an explicit separation of f and $\boldsymbol{B}$ into equilibrium and small (but finite) perturbations. Note that there is no contradiction between working in a slab and allowing magnetic drifts, which depend on the equilibrium magnetic field having a curvature and a gradient of the field strength—in the same way as it was done for density, temperature, and parallel-velocity gradients in §23, one simply assumes that the gradients of $\boldsymbol{B}_0$ can be treated as constant across the (small) slab-like domain in which one's equations describe small-scale perturbations of a large-scale equilibrium (cf. Q-12).

You will find elements of what you need for Exercise 24.3 in Adkins *et al.* (2022) (electromagnetic) and Adkins *et al.* (2023) (electrostatic).

Finally, if you want to be enabled to mount a GK-powered attack at tokamak physics, you will find a comprehensive review of GK in a torus is Abel *et al.* (2013); there is also a tutorial by one of the fathers of GK—Catto (2019), written in a somewhat different style that might resonate with you if you have followed the DK lecture notes by Parra (2019*b*), who comes from the Catto school of plasma theory.

24.2. *Ion-Scale Instabilities and Turbulence in Magnetically Confined Plasmas*

24.2.1. *Curvature ITG Instability*

In the presence of magnetic drifts, there is another variety of ITG instability, where the link between the free-energy injection into δT by the equilibrium temperature gradient ($\partial_t \delta T \propto -L_T^{-1}\partial_y \varphi$) and the electrostatic potential is provided not by the pressure gradient ($\partial_t u_\parallel \propto -\nabla_\parallel \delta T$) and the parallel compressibility ($\partial_t \varphi \propto \partial_t \delta n \propto -\nabla_\parallel u_\parallel$), as it was in §23.2, but by the magnetic drifts, which give rise to a term in the evolution equation for φ that is proportional to $L_B^{-1}\partial_y \delta T$. This is the so-called *curvature-ITG (cITG) instability*, which is usually expected to control the turbulence in magnetically confined (fusion) plasmas close to the linear-stability threshold, whereas the slab-ITG (sITG) instability derived in §§23.2–23.3 is dominant in the fully developed turbulent regime. The cITG instability is easiest to derive in a simplified fluid setting: you will find a pedagogical treatment of it in Ivanov *et al.* (2020) (in 2D, with $k_\parallel = 0$; the case of $k_\parallel \neq 0$, containing both cITG and sITG instabilities, is done in Ivanov *et al.* 2022). Including magnetic drifts into a kinetic calculation along the lines of §23.3 is a mathematically tricky task that until quite recently remained a challenge even on the level of research literature. If you like exercises in complex analysis, you will enjoy working through the solution proposed by Ivanov & Adkins (2023).

24.2.2. *Secondary Instabilities and Zonal Flows*

As you must have noticed in my treatment of the sITG instability in §§23.2–23.3, the unstable perturbations did not need to have a non-zero k_x. The same is true about the cITG instability. The main effect of the finite-Larmor-radius (FLR) corrections that GK allows one to include (Exercise 24.1) is to quench the instabilities at $k_\perp \rho_i \gtrsim 1$, where $k_\perp^2 = k_x^2 + k_y^2$, which means that, other things being equal, the perturbations with $k_x = 0$ always grow fastest. But that raises the question of how they can ever saturate given that the nonlinearity in the DKE (23.61) is

FIGURE 101. Bill Dorland (1965-2024), a computational plasma physicist who was a key figure in the development of gyrokinetic simulations, the ITG and ETG transport paradigms, and many other things that determined the evolution of the subject since the 1990s (you will find his name behind the "*et al.*" of many papers cited in these Lectures). This picture was taken in March 2010 in Vienna by his student Anjor Kanekar.

$\{\varphi, \delta f\} = 0$ for perturbations with $k_x = 0$, so the unstable, growing modes are, in fact, exact nonlinear solutions (called *streamers*).

The answer is that these solutions are themselves unstable to secondary instabilities associated with the nonlinear term—in particular, once amplitudes become finite, there is growth of the so-called *zonal flows*, which are perturbations of $v_{Ey} \propto -\partial_x \varphi$ with $k_x \neq 0$ and $k_y = 0$, $k_\parallel = 0$. These will, upon reaching finite amplitudes themselves, shear the primary (ITG) modes and thus create a mess amenable to nonlinear saturation along the lines described in §23.5 (or sometimes along different lines: see §24.2.3). The simplest possible calculation of this secondary instability can be found in Ivanov *et al.* (2020)—and you can practice what you learn there by working it out for more complicated settings (and/or by following the paper trail and reading how other people did it). Note that (as Ivanov's paper explains carefully), in order to get this instability right, one needs a careful derivation of the nonlinear terms starting from GK—the fluid equations derived from DK, e.g., (23.62–23.64), miss crucial nonlinear terms (which are FLR terms) in the evolution equation for φ.

Secondary instabilities are, in fact, a zoo: there are many different ones. E.g., if 3D perturbations are allowed, the secondary sITG instability of the streamers produced by the primary cITG instability (i.e., an instability that extracts free energy from the gradients of the cITG-produced temperature perturbation, $\partial_y \delta T$) turns out to be consequential for the strongly turbulent saturated state (Ivanov *et al.* 2022). It can be a useful type of exercise to pick a primary instability and then work out all possible ways in which the streamers produced by it can go secondarily unstable (e.g., secondary ITG, secondary PVG, secondary Kelvin–Helmholtz—which is the instability of the $\boldsymbol{E} \times \boldsymbol{B}$ flows, etc.; two examples of now-classic papers from the fusion literature that delve into such topics are Cowley *et al.* 1991 and Rogers & Dorland 2005).

24.2.3. *Dimits Shift*

The main reason that all these details are important is that, as I hinted at the end of §23.5, the theory of turbulent transport does not quite reduce to the dichotomy between no transport below the linear instability's critical gradient $L_{T\mathrm{c}}^{-1}$ and stiff transport above it. In fact, close to the linear threshold, the zonal flows generated by the secondary instabilities (or, rather, more precisely, by the mean action of the turbulent stresses) manage to stay quasi-stable for long periods and keep the rest of the turbulence at a very low level by shearing it away. This low-transport regime (which is not described by the arguments of §23.5) endures up to a certain nonlinear threshold $L_{T\mathrm{c,nlin}}^{-1} > L_{T\mathrm{c}}^{-1}$. For $L_T^{-1} > L_{T\mathrm{c,nlin}}^{-1}$, the resilience of the zonal flows collapses and the strongly turbulent regime (§23.5) finally sets in (Fig. 100). This uplift of the effective critical temperature gradient (beyond which the transport becomes impossible to overcome) is called the *Dimits shift*, first discovered numerically (Dimits *et al.* 2000).

How and why all this happens has been a live research issue for well over 30 years. I recommend that you read Ivanov *et al.* (2020, 2022) in the first instance, but, while I believe (naturally) that what you will find there is substantially correct, it is by no means a universally accepted story (and it might not be the only story), so you ought also to look at the literature cited in those papers to become erudite in the triumphs and tribulations of the past (the most-cited review paper reflecting the *status quo ante* is Diamond *et al.* 2005; the founding text of the main rival school of thought is Rogers *et al.* 2000).

The above story line has two main limitations. First, the ion-scale dynamics that I have discussed since §23 have all been purely electrostatic. As we saw in §23.4, this has to do with the validity of the Boltzmann closure for the electrons, and a full description of the transition from electrostatic to electromagnetic regime (which happens as β_i, $L_\parallel$, or L_T^{-1} increase) requires gyrokinetic treatment. It turns out that allowing $\delta \boldsymbol{B} \neq 0$ at ion scales can have dramatic consequences for tokamak modelling: Giacomin *et al.* (2024) is the latest paper on this subject, and references therein will help you track the history of the problem; Zocco *et al.* (2015) is a rare, relatively recent, analytical paper (it is quite hard to read, but does have many useful pointers). It is, I think, fair to say that this has not been fully sorted, but can be, and should be, by an ambitious student.

Exercise 24.4. ITG in EM GK. Derive (and analyse the solutions of) the GK dispersion relation for the ITG instability in the same setting as in Exercise 24.1 (in a slab, with no spatial gradients of $\boldsymbol{B}_0$ or of the electron equilibrium distribution), assuming $k_\perp \rho_e \ll 1$, $k_\perp \rho_i \sim 1$, and electromagnetic perturbations. One of your equations will be (23.51). The general approach should be similar to the formalism set out in Schekochihin *et al.* (2019), but now you need to include the equilibrium ion-temperature gradient. The fluid limit ($k_\parallel v_{\mathrm{th}i} \ll \omega$) may be a good starting place to help with physical understanding. Pay particlar attention to the role played by the perturbation of the magnetic-field strength δB. Show that, in the limit (23.54), the electrostatic results (§§23.2 and 23.3) are restored. Work out what happens in the strange opposite limit $\omega_{*T} \gg k_\parallel v_{\mathrm{A}}$.

Secondly, I have so far been concerned only with the instabilities (and the resulting turbulence) that occur above and at the ion Larmor scale (as I mentioned above, the role of GK is mostly to cut off the dynamics at scales below ρ_i). It turns out that there is some vibrant life also between ρ_i and $\rho_e = \rho_i Z\sqrt{T_e m_e/T_i m_i}$—a scale range that, due to the smallness of the mass ratio, can be nearly two-decades wide. This vibrant life has to do with electron dynamics—the Boltzmann closure is yet again not valid at these scales (see §23.4), the electrons satisfy their own DK (or GK) equation, control the evolution of the magnetic field, can have their own instabilities, etc. The rest of this section deals with these topics.

24.3. *Electromagnetic GK Turbulence*

Howes *et al.* (2006); Schekochihin *et al.* (2009); Zocco & Schekochihin (2011); Schekochihin *et al.* (2019); Kunz *et al.* (2018)...

24.4. *Collisionless and Semicollisional Reconnection*

Zocco & Schekochihin (2011)...

24.5. *Electron-Scale Instabilities and Turbulence in Magnetically Confined Plasmas*

ETG Instability: Adkins *et al.* (2022)...
Thermo-Alfvénic Instability: Adkins *et al.* (2022)...

24.6. *Stratified Atmospheres, Fluid and Kinetic*

Xu & Kunz (2016)

Acknowledgments

I am grateful to many at Oxford who have given advice, commented, pointed out errors, asked challenging questions and generally helped me get a grip. Particular thanks to Andre Lukas and Fabian Essler, who were comrades in arms in the creation of the MMathPhys programme (which started in 2014), to Paul Dellar, James Binney, Jean-Baptiste Fouvry, Chris Hamilton, Plamen Ivanov, and Dan Kennedy, with whom I have collaborated on teaching these courses, and to Felix Parra, who originally helped design a coherent plasma-physics stream.

I am also grateful to M. Kunz, C. Pichon, F. Rincon, S. Rychkov and to the students who took the course, amongst them T. Adkins, A. Brown, R. Cooper, R. Ewart, J. Graham, C. Hamilton, M. Hardman, M. Hawes, D. Hosking, P. Ivanov, P. Luhadiya, L. Meissner, M. Nastac, V. Pratley, G. Wagner, and A. Wierzchucka, for critical comments and spotting lapses. More broadly, I am in debt to all the students who have soldiered on through these lectures and, sometimes verbally, sometimes not, projected back at me some sense of whether I was succeeding in communicating a modicum of sense.

My exposition was greatly influenced by the 1990s Princeton MHD lectures by Russell Kulsrud (published as Kulsrud 2005) and the UCLA ones by Steve Cowley (Cowley 2004*a*,*b*, 2007), as well as by the excellent books of Kadomtsev (1965), Krall & Trivelpiece (1973), Alexandrov *et al.* (1984), Zakharov *et al.* (1992), and Kingsep (1996). Section 21 drew heavily on Felix Parra's lecture notes (Parra 2019*b*). Sections 3.7, 4, 9, and 10 were written as a result of invitations by Geoffroy Lesur, Benoît Cerutti, Mickael Grech and Thomas Grismayer to lecture at the 2017, 2019, and 2023 Les Houches Schools on Plasma Physics—forcing me to face the challenge of teaching something fundamental, beautiful, but not readily available in every plasma course. I owe much of §§10.3–10.4, 11 and 12 to my collaboration with Toby Adkins, Robbie Ewart, and Michael Nastac in 2016-25.

REFERENCES

Abel, I. G., Plunk, G. G., Wang, E., Barnes, M., Cowley, S. C., Dorland, W. & Schekochihin, A. A. 2013 Multiscale gyrokinetics for rotating tokamak plasmas: fluctuations, transport and energy flows. *Rep. Prog. Phys.* **76**, 116201.

Adams, D. 1979 *The Hitch-Hiker's Guide to the Galaxy*. London: Pan Books.

Adkins, T., Ivanov, P. G. & Schekochihin, A. A. 2023 Scale invariance and critical balance in electrostatic drift-kinetic turbulence. *J. Plasma Phys.* **89**, 905890406.

Adkins, T. & Schekochihin, A. A. 2018 A solvable model of Vlasov-kinetic plasma turbulence in Fourier-Hermite phase space. *J. Plasma Phys.* **84**, 905840107.

Adkins, T., Schekochihin, A. A., Ivanov, P. G. & Roach, C. M. 2022 Electromagnetic instabilities and plasma turbulence driven by electron-temperature gradient. *J. Plasma Phys.* **88**, 905880410.

Alexandrov, A. F., Bogdankevich, L. S. & Rukhadze, A. A. 1984 *Principles of Plasma Electrodynamics*. Berlin: Springer.

Arnold, V. I. & Khesin, B. A. 1999 *Topological Methods in Hydrodynamics*. Berlin: Springer.

Balbus, S. A. 2004 Viscous shear instability in weakly magnetized, dilute plasmas. *Astrophys. J.* **616**, 857.

Balbus, S. A. 2015 *Astrophysical Fluid Dynamics*. Lecture Notes for the Oxford MMathPhys course; URL: https://www2.physics.ox.ac.uk/sites/default/files/profiles/balbus/1415-42468.pdf.

Balbus, S. A. & Hawley, J. F. 1991 A powerful local shear instability in weakly magnetized disks. I. Linear analysis. *Astrophys. J.* **376**, 214.

Bale, S. D., Kasper, J. C., Howes, G. G., Quataert, E., Salem, C. & Sundkvist, D. 2009 Magnetic fluctuation power near proton temperature anisotropy instability thresholds in the solar wind. *Phys. Rev. Lett.* **103**, 211101.

Balescu, R. 1960 Irreversible processes in ionized gases. *Phys. Fluids* **3**, 52.

Banik, U., Bhattacharjee, A. & Sengupta, W. 2024 Universal non-thermal power-law distribution functions from the self-consistent evolution of collisionless electrostatic plasmas. *Astrophys. J.* **977**, 91.

Bar-Or, B., Fouvry, J.-B. & Tremaine, S. 2021 Relaxation in a fuzzy dark matter halo. II. Self-consistent kinetic equations. *Astrophys. J.* **915**, 27.

Barnes, A. 1966 Collisionless damping of hydromagnetic waves. *Phys. Fluids* **9**, 1483.

Barnes, M., Parra, F. I. & Schekochihin, A. A. 2011 Critically balanced ion temperature gradient turbulence in fusion plasmas. *Phys. Rev. Lett.* **107**, 115003.

Batchelor, G. K. 1959 Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. General discussion and the case of small conductivity. *J. Fluid Mech.* **5**, 113.

Bedrossian, J. 2016 Nonlinear echoes and Landau damping with insufficient regularity. *E-print* arXiv:1605.06841.

Bernstein, I. B. 1958 Waves in a plasma in a magnetic field. *Phys. Rev.* **109**, 10.

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. & Kulsrud, R. M. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. London A* **244**, 17.

Bernstein, I. B., Greene, J. M. & Kruskal, M. D. 1957 Exact nonlinear plasma oscillations. *Phys. Rev.* **108**, 546.

Besse, N., Elskens, Y., Escande, D. F. & Bertrand, P. 2011 Validity of quasilinear theory: refutations and new numerical confirmation. *Plasma Phys. Control. Fusion* **53**, 025012.

Biermann, L. 1950 Über den Ursprung der Magnetfelder auf Sternen und im interstellaren Raum (miteinem Anhang von A. Schlüter). *Z. Naturforsch. A* **5**, 65.

Binney, J. J. 1999 *Functions of a Complex Variable: Theory and Applications*. Lecture Notes for the Oxford Physics course; URL: https://www-thphys.physics.ox.ac.uk/people/JamesBinney/advcalc.pdf.

Binney, J. J. 2018 *Kinetic Theory of Self-Gravitating Systems*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://www-thphys.physics.ox.ac.uk/user/JamesBinney/kt.pdf.

Bohm, D. & Gross, E. P. 1949*a* Theory of plasma oscillations. A. Origin of medium-like behavior. *Phys. Rev.* **75**, 1851.

Bohm, D. & Gross, E. P. 1949*b* Theory of plasma oscillations. B. Excitation and damping of oscillations. *Phys. Rev.* **75**, 1864.

Boldyrev, S., Horaites, K., Xia, Q. & Perez, J. C. 2013 Toward a theory of astrophysical plasma turbulence at subproton scales. *Astrophys. J.* **777**, 41.

Boldyrev, S. & Loureiro, N. F. 2018 Calculations in the theory of tearing instability. *J. Phys. Conf. Ser.* **1100**, 012003.

Bott, A. F. A., Arzamasskiy, L., Kunz, M. W., Quataert, E. & Squire, J. 2021 Adaptive critical balance and firehose instability in an expanding, turbulent, collisionless plasma. *Astrophys. J.* **922**, L35.

Bott, A. F. A., Cowley, S. C. & Schekochihin, A. A. 2024 Kinetic stability of Chapman-Enskog plasmas. *J. Plasma Phys.* **90**, 975900207.

Bott, A. F. A., Kunz, M. W., Quataert, E., Squire, J. & Arzamasskiy, L. 2025 Thermodynamics and collisionality in firehose-susceptible high-β plasmas. *E-print* arXiv:2501.13663.

Braginskii, S. I. 1965 Transport processes in a plasma. *Rev. Plasma Phys.* **1**, 205.

Breĭzman, B. N., Zakharov, V. E. & Musher, S. L. 1973 Kinetics of stimulated scattering of Langmuir waves by plasma ions. *Sov. Phys.–JETP* **37**, 658.

Buneman, O. 1958 Instability, turbulence, and conductivity in current-carrying plasma. *Phys. Rev. Lett.* **1**, 8.

Catto, P. J. 2019 Practical gyrokinetics. *J. Plasma Phys.* **85**, 925850301.

Catto, P. J. 2024 Reinterpreting radio frequency heating and current drive theory in a tokamak. *J. Plasma Phys.* **90**, 905900508.

Catto, P. J. 2025 Weak collisional steady-state linear and nonlinear plasma waves. *J. Plasma Phys.* **91**, in press.

Chandrasekhar, S. 2003 *Hydrodynamic and Hydromagnetic Stability*. Dover.

Chandrasekhar, S., Kaufman, A. N. & Watson, K. M. 1958 The stability of the pinch. *Proc. R. Soc. London A* **245**, 435.

Chapman, S. & Cowling, T. G. 1991 *The Mathematical Theory of Non-Uniform Gases*. Cambridge: Cambridge University Press, 3rd Edition.

Chapman, S. & Kendall, P. C. 1963 Liquid instability and energy transformation near a magnetic neutral line: a soluble non-linear hydromagnetic problem. *Proc. R. Soc. London A* **271**, 435.

Chavanis, P.-H. 2005 Statistical mechanics of geophysical turbulence: application to Jovian flows and Jupiter's great red spot. *Physica D* **200**, 257.

Chavanis, P.-H. 2022 Kinetic theory of collisionless relaxation for systems with long-range interactions. *Physica A* **606**, 128089.

Chavanis, P. H., Sommeria, J. & Robert, R. 1996 Statistical mechanics of two-dimensional vortices and collisionless stellar systems. *Astrophys. J.* **471**, 385.

Chen, C. H. K., Matteini, L., Schekochihin, A. A., Stevens, M. L., Salem, C. S., Maruca, B. A., Kunz, M. W. & Bale, S. D. 2016 Multi-species measurements of the firehose and mirror instability thresholds in the solar wind. *Astrophys. J.* **825**, L26.

Chertkov, M., Falkovich, G., Kolokolov, I. & Vergassola, M. 1999 Small-scale turbulent dynamo. *Phys. Rev. Lett.* **83**, 4065.

Chew, G. F., Goldberger, M. L. & Low, F. E. 1956 The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions. *Proc. R. Soc. London A* **236**, 112.

Connor, J. W., Hastie, R. J. & Taylor, J. B. 1979 High mode number stability of an axisymmetric toroidal plasma. *Proc. R. Soc. London A* **365**, 1.

Cowley, S. C. 2004*a* *Plasma Physics*. Lecture Notes for the UCLA course Physics 222a-c; URL: http://www-thphys.physics.ox.ac.uk/research/plasma/CowleyLectures/.

Cowley, S. C. 2004*b* *Turbulence in Magnetized Plasmas*. Lecture Notes for the UCLA course Physics 260; URL: http://www-thphys.physics.ox.ac.uk/research/plasma/CowleyLectures/CowleyUCLA260_2004.pdf.

Cowley, S. C. 2006 *Frozen Field Lines and Diffusion; Introduction to Gyrokinetics*. Lecture Notes for the CMPD/CMSO Winter School at UCLA; URL: http://www-thphys.physics.ox.ac.uk/research/plasma/CowleyLectures/CowleyUCLAWinterSchool_2006.pdf.

Cowley, S. C. 2007 *Plasma Turbulence*. Lecture Notes at UCLA; URL: http://www-thphys.physics.ox.ac.uk/research/plasma/CowleyLectures/CowleyUCLAPlasmaTurbulence_2007.pdf.

Cowley, S. C. & Artun, M. 1997 Explosive instabilities and detonation in magnetohydrodynamics. *Phys. Rep.* **283**, 185.

Cowley, S. C., Cowley, B., Henneberg, S. A. & Wilson, H. R. 2015 Explosive instability and erupting flux tubes in a magnetized plasma. *Proc. R. Soc. London A* **471**, 20140913.

Cowley, S. C., Kulsrud, R. M. & Sudan, R. 1991 Considerations of ion-temperature-gradient-driven turbulence. *Phys. Fluids B* **3**, 2767.

Davidson, P. A. 2015 *Turbulence: An Introduction for Scientists and Engineers* (2nd edition). Oxford: Oxford University Press.

Davidson, P. A. 2016 *Introduction to Magnetohydrodynamics* (2nd edition). Cambridge: Cambridge University Press.

Davidson, R. C. 1983 Kinetic waves and instabilities in a uniform plasma. In *Handbook of Plasma Physics, Volume 1* (ed. A. A. Galeev & R. N. Sudan), p. 519. Amsterdam: North-Holland.

Dellar, P. J. 2015 *Kinetic Theory of Gases*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://people.maths.ox.ac.uk/dellar/MMPkinetic.html.

Dellar, P. J. 2017 *Hydrodynamics of Complex Fluids*. Lecture Notes for the Oxford MMathPhys course on Advanced Fluid Dynamics; URL: http://people.maths.ox.ac.uk/dellar/MMP_Non_Newt_fluids.html.

Diamond, P. H., Itoh, S.-I. & Itoh, K. 2010 *Modern Plasma Physics. Volume 1: Physical Kinetics of Turbulent Plasmas*. Cambridge: Cambridge University Press.

Diamond, P. H., Itoh, S.-I., Itoh, K. & Hahm, T. S. 2005 Zonal flows in plasma—a review. *Plasma Phys. Control. Fusion* **47**, 35.

Dimits, A. M., Bateman, G., Beer, M. A., Cohen, B. I., Dorland, W., Hammett, G. W., Kim, C., Kinsey, J. E., Kotschenreuther, M., Kritz, A. H., Lao, L. L., Mandrekas, J., Nevins, W. M., Parker, S. E., Redd, A. J., Shumaker, D. E., Sydora, R. & Weiland, J. 2000 Comparisons and physics basis of tokamak transport models and turbulence simulations. *Phys. Plasmas* **7**, 969.

Dodin, I. Y. & Fisch, N. J. 2005 Variational formulation of Gardner's restacking algorithm. *Phys. Lett. A* **341**, 187.

Drummond, W. E. & Pines, D. 1962 Nonlinear stability of plasma oscillations. *Nucl. Fusion Suppl., Part 3* p. 1049.

DuBois, D. F., Russell, D. & Rose, H. A. 1995 Reduced description of strong Langmuir turbulence from kinetic theory. *Phys. Plasmas* **2**, 76.

Dupree, T. H. 1972 Theory of phase space density granulation in plasma. *Phys. Fluids* **15**, 334.

Elsasser, W. M. 1950 The hydromagnetic equations. *Phys. Rev.* **79**, 183.

Evans, N. W. 2020 Donald Lynden-Bell: a biographical memoir. *Biogr. Mems. Fell. R. Soc.* **69**, 333.

Ewart, R. J., Brown, A., Adkins, T. & Schekochihin, A. A. 2022 Collisionless relaxation of a Lynden-Bell plasma. *J. Plasma Phys.* **88**, 925880501.

Ewart, R. J., Nastac, M. L., Bilbao, P. J., Silva, T., Silva, L. O. & Schekochihin, A. A. 2024 Relaxation to universal non-Maxwellian equilibria in a collisionless plasma. *E-print* arXiv:2409.01742.

Ewart, R. J., Nastac, M. L. & Schekochihin, A. A. 2023 Non-thermal particle acceleration and power-law tails via relaxation to universal Lynden-Bell equilibria. *J. Plasma Phys.* **89**, 905890516.

Faddeeva, V. N. & Terent'ev, N. M. 1954 *Tables of Values of the Function* $w(z) = \exp(-z^2)(1+2i/\sqrt{\pi}\int_0^z \exp(t^2)\mathrm{d}t)$ *for Complex Argument*. Moscow: Gostekhizdat, English translation: New York: Pergamon Press, 1961.

Falkovich, G., Gawędzki, K. & Vergassola, M. 2001 Particles and fields in fluid turbulence. *Rev. Mod. Phys.* **73**, 913.

Falkovich, G. E., Kuznetsov, E. A., Newell, A. C. & Turitsyn, S. K. 2023 A scientist and a poet. *Nature Photonics* **17**, 920.

Fjørtoft, R. 1953 On the changes in the spectral distribution of kinetic energy for two-dimensional, nondivergent flow. *Tellus* **5**, 225.

Fouvry, J.-B. 2021 *Kinetic Theory of Self-Gravitating Systems*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://www2.iap.fr/users/fouvry/files_teaching/KineticTheory_Oxford_Notes.pdf.

Fouvry, J.-B. 2022 Kinetic theory of one-dimensional inhomogeneous long-range interacting N-body systems at order $1/N^2$ without collective effects. *Phys. Rev. E* **106**, 054123.

Fowler, T. K. 1963 Lyapunov's stability criteria for plasmas. *J. Math. Phys.* **4**, 559.

Fowler, T. K. 1964 Bounds on plasma instability growth rates. *Phys. Fluids* **7**, 249.

Fowler, T. K. 1968 Thermodynamics of unstable plasmas. In *Advances in Plasma Physics* (ed. A. Simon & W. B. Thompson), , vol. 1, p. 201. New York: Interscience Press.

Fowler, T. K. 2016 Speculations about plasma free energy, 50 years later. *J. Plasma Phys.* **82**, 925820501.

Freidberg, J. 2014 *Ideal MHD*. Cambridge: Cambridge University Press.

Fried, B. D. & Conte, S. D. 1961 *The Plasma Dispersion Function*. New York: Academic Press.

Frisch, U. 1995 *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge: Cambridge University Press.

Furth, H. P., Killeen, J. & Rosenbluth, M. N. 1963 Finite-resistivity instabilities of a sheet pinch. *Phys. Fluids* **6**, 459.

Furutsu, K. 1963 On the statistical theory of electromagnetic waves in a fluctuating medium (I). *J. Res. NBS* **67D**, 303.

Gardner, C. S. 1959 Adiabatic invariants of periodic classical systems. *Phys. Rev.* **115**, 791.

Gardner, C. S. 1963 Bound on the energy available from a plasma. *Phys. Fluids* **6**, 839.

Gedalin, M. & Oiberman, I. 1995 Generally covariant relativistic anisotropic magnetohydrodynamics. *Phys. Rev. E* **51**, 4901.

Giacomin, M., Kennedy, D., Casson, F. J., C J, Ajay, Dickinson, D., Patel, B. S. & Roach, C. M. 2024 On electromagnetic turbulence and transport in STEP. *Plasma Phys. Control. Fusion* **66**, 055010.

Goedbloed, H. & Poedts, S. 2004 *Principles of Magnetohydrodynamics*. Cambridge: Cambridge University Press.

Goldhirsch, I., Sulem, P.-L. & Orszag, S. A. 1987 Stability and Lyapunov stability of dynamical systems: A differential approach and a numerical method. *Physica D* **27**, 311.

Goldman, M. V. 1984 Strong turbulence of plasma waves. *Rev. Mod. Phys.* **56**, 709.

Gorev, V. V., Kingsep, A. S. & Rudakov, L. I. 1976 Strong Langmuir turbulence of a plasma. *Radiophys. Quantum Electron.* **19**, 486.

Gould, R. W., O'Neil, T. M. & Malmberg, J. H. 1967 Plasma wave echo. *Phys. Rev. Lett.* **19**, 219.

Haeff, A. 1949 The electron-wave tube—a novel method of generation and amplification of microwave energy. *Proc. IRE* **37**, 4.

Haines, M. G. 2011 A review of the dense Z-pinch. *Plasma Phys. Control. Fusion* **53**, 093001.

Hamilton, C. 2021 A simple, heuristic derivation of the Balescu-Lenard kinetic equation for stellar systems. *Mon. Not. R. Astron. Soc.* **501**, 3371.

Hamilton, C. 2022 *Kinetic Theory of Stellar Systems*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: https://hamiltonchris20.files.wordpress.com/2022/12/kinetic_theory_of_stellar_systems_lectures-8.pdf.

Hamilton, C. & Fouvry, J.-B. 2024 Kinetic theory of stellar systems: a tutorial. *Phys. Plasmas* **31**, 120901.

Hamilton, C., Modak, S. & Tremaine, S. 2024 Galactokinetics. *E-prints* arXiv:2408.03366.

Hammett, G. W., Dorland, W. & Perkins, F. W. 1992 Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetic dynamics. *Phys. Fluids B* **4**, 2052.

Hasegawa, A. 1969 Drift mirror instability of the magnetosphere. *Phys. Fluids* **12**, 2642.

Hazeltine, R. D. 1973 Recursive derivation of drift-kinetic equation. *Plasma Phys.* **15**, 77.

Helander, P. 2017 Available energy and ground states of collisionless plasmas. *J. Plasma Phys.* **83**, 715830401.

Helander, P. 2020 Available energy of magnetically confined plasmas. *J. Plasma Phys.* **86**, 905860201.

Helander, P. & Mackenbach, R. J. J. 2024 Available energy of plasmas with small fluctuations. *J. Plasma Phys.* **90**, 905900401.

Helander, P. & Sigmar, D. J. 2005 *Collisional Transport in Magnetized Plasmas*. Cambridge: Cambridge University Press.

Helander, P., Strumik, M. & Schekochihin, A. A. 2016 Constraints on dynamo action in plasmas. *J. Plasma Phys.* **82**, 905820601.

Hellinger, P. 2007 Comment on the linear mirror instability near the threshold. *Phys. Plasmas* **14**, 082105.

Hellinger, P., Trávníček, P., Kasper, J. C. & Lazarus, A. J. 2006 Solar wind proton temperature anisotropy: linear theory and WIND/SWE observations. *Geophys. Res. Lett.* **33**, L09101.

Hellwig, G. 1955 Über die Bewegung geladener Teilchen in schwach veränderlichen Magnetfeldern. *Z. Naturforsch. A* **10**, 508.

Heninger, J. M. & Morrison, P. J. 2018 An integral transform technique for kinetic systems with collisions. *Phys. Plasmas* **25**, 082118.

Holloway, J. P. & Dorning, J. J. 1991 Undamped plasma waves. *Phys. Rev. A* **44**, 3856.

Hosking, D. N. & Schekochihin, A. A. 2021 Reconnection-controlled decay of magnetohydrodynamic turbulence and the role of invariants. *Phys. Rev. X* **11**, 041005.

Hosking, D. N., Schekochihin, A. A. & Balbus, S. A. 2020 Elasticity of tangled magnetic fields. *J. Plasma Phys.* **86**, 905860511.

Hosking, D. N., Wasserman, D. & Cowley, S. C. 2025 Metastability of stratified magnetohydrostatic equilibria and their relaxation. *J. Plasma Phys.* **91**, E35.

Howes, G. G., Cowley, S. C., Dorland, W., Hammett, G. W., Quataert, E. & Schekochihin, A. A. 2006 Astrophysical gyrokinetics: basic equations and linear theory. *Astrophys. J.* **651**, 590.

Howes, G. G., Cowley, S. C. & McWilliams, J. C. 2001 Local buoyant instability of magnetized shear flows. *Astrophys. J.* **560**, 617.

Hubbard, J. 1961 The friction and diffusion coefficients of the Fokker-Planck equation in a plasma. *Proc. R. Soc. London A* **260**, 114.

Ivanov, P. G. & Adkins, T. 2023 An analytical form of the dispersion function for local linear gyrokinetics in a curved magnetic field. *J. Plasma Phys.* **89**, 905890213.

Ivanov, P. G., Schekochihin, A. A. & Dorland, W. 2022 Dimits transition in three-dimensional ion-temperature-gradient turbulence. *J. Plasma Phys.* **88**, 905880506.

Ivanov, P. G., Schekochihin, A. A., Dorland, W., Field, A. R. & Parra, F. I. 2020 Zonally dominated dynamics and Dimits threshold in curvature-driven ITG turbulence. *J. Plasma Phys.* **86**, 855860502.

Jackson, J. D. 1960 Longitudinal plasma oscillations. *J. Nuclear Energy C* **1**, 171.

Jaynes, E. T. 2003 *Probability Theory: The Logic of Science*. Cambridge: Cambridge University Press.

Kadomtsev, B. B. 1965 *Plasma Turbulence*. Academic Press.

Kadomtsev, B. B. & Pogutse, O. P. 1970 Collisionless relaxation in systems with Coulomb interactions. *Phys. Rev. Lett.* **25**, 1155.

Kadomtsev, B. B. & Pogutse, O. P. 1971 Theory of beam-plasma interaction. *Phys. Fluids* **14**, 2470.

Kadomtsev, B. B. & Pogutse, O. P. 1974 Nonlinear helical perturbations of a plasma in the tokamak. *Sov. Phys.–JETP* **38**, 575.

Kanashov, A. A. & Rubenchik, A. M. 1980 Turbulence spectra of a nonisothermal plasma. *Sov. Phys. Doklady* **25**, 631.

Kanekar, A., Schekochihin, A. A., Dorland, W. & Loureiro, N. F. 2015 Fluctuation-dissipation relations for a plasma-kinetic Langevin equation. *J. Plasma Phys.* **81**, 305810104.

Karpman, V. I. 1967 “Singular” solutions of the equations for plasma oscillations. *Sov. Phys.–JETP* **24**, 603.

Kingsep, A. S. 1996 *Introduction to Nonlinear Plasma Physics*. Moscow: MIPT Press (in Russian).

Kingsep, A. S., Rudakov, L. I. & Sudan, R. N. 1973 Spectra of strong Langmuir turbulence. *Phys. Rev. Lett.* **31**, 1482.

Kivelson, M. G. & Southwood, D. J. 1996 Mirror instability II: The mechanism of nonlinear saturation. *J. Geophys. Res.* **101**, 17365.

Klimontovich, Yu. L. 1967 *The Statistical Theory of Non-Equilibrium Processes in a Plasma*. MIT Press.

Knorr, G. 1977 Time asymptotic statistics of the Vlasov equation. *J. Plasma Phys.* **17**, 553.

Kolmes, E. J. & Fisch, N. J. 2020 Recovering Gardner restacking with purely diffusive operations. *Phys. Rev. E* **102**, 063209.

Kolmogorov, A. N. 1941 The local structure of turbulence in incompressible viscous fluid at very large Reynolds numbers. *Dokl. Acad. Nauk SSSR* **30**, 299.

Krall, N. A. & Trivelpiece, A. W. 1973 *Principles of Plasma Physics*. New York: McGraw-Hill.

Krommes, J. A. 2018 Projection-operator methods for classical transport in magnetized plasmas. Part I. Linear response, the Braginskii equations and fluctuating hydrodynamics. *J. Plasma Phys.* **84**, 925840401.

Kruskal, M. 1958 The gyration of a charged particle. *Project Matterhorn Report PM-S-33 NYO-7903,* U.S. Department of Energy, Office of Scientific and Technical Information URL: https://www.osti.gov/biblio/4332926.

Kruskal, M. 1962 Asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic. *J. Math. Phys.* **3**, 806.

Kruskal, M. D. & Oberman, C. R. 1958 On the stability of plasma in static equilibrium. *Phys. Fluids* **1**, 275.

Kulsrud, R. 1964 General stability theory in plasma physics. In *Teoria dei plasmi, Rendiconti della scuola internazionale di fisica ”Enrico Fermi”, XXV Corso* (ed. M. N. Rosenbluth), p. 54. New York: Academic Press.

Kulsrud, R. M. 2005 *Plasma Physics for Astrophysics*. Princeton: Princeton University Press.

Kunz, M. W. 2020 *Introduction to Plasma Astrophysics*. Lecture Notes for the Princeton course AST521; URL: https://www.astro.princeton.edu/~kunz/Site/AST521/AST521_lecture_notes_Kunz.pdf.

Kunz, M. W. 2021 *Irreversible Processes in Plasmas*. Lecture Notes for the Princeton course AST554; URL: https://www.astro.princeton.edu/~kunz/Site/AST554/AST554_Kunz_Lecture_Notes.pdf.

Kunz, M. W., Abel, I. G., Klein, K. G. & Schekochihin, A. A. 2018 Astrophysical gyrokinetics: turbulence in pressure-anisotropic plasmas at ion scales and beyond. *J. Plasma Phys.* **84**, 715840201.

Kunz, M. W., Schekochihin, A. A., Chen, C. H. K., Abel, I. G. & Cowley, S. C. 2015 Inertial-range kinetic turbulence in pressure-anisotropic astrophysical plasmas. *J. Plasma Phys.* **81**, 325810501.

Kunz, M. W., Schekochihin, A. A. & Stone, J. M. 2014 Firehose and mirror instabilities in a collisionless shearing plasma. *Phys. Rev. Lett.* **112**, 205003.

Kunz, M. W., Squire, J., Schekochihin, A. A. & Quataert, E. 2020 Self-sustaining sound in collisionless, high-β plasma. *J. Plasma Phys.* **86**, 905860603.

Landau, L. 1936 Transport equation in the case of Coulomb interaction. *Zh. Eksp. Teor. Fiz.* **7**, 203.

Landau, L. 1946 On the vibration of the electronic plasma. *Zh. Eksp. Teor. Fiz.* **16**, 574, English translation: *J. Phys. USSR* **10**, 25 (1946).

Landau, L. D. & Lifshitz, E. M. 1976 *Mechanics (L. D. Landau and E. M. Lifshitz's Course of Theoretical Physics, Volume 1)*. Butterworth–Heinemann.

Landau, L. D. & Lifshitz, E. M. 1987 *Fluid Mechanics (L. D. Landau and E. M. Lifshitz's Course of Theoretical Physics, Volume 6)*. Oxford: Pergamon Press.

Lau, J. Y. & Binney, J. 2021*a* Modes of a stellar system I: Ergodic systems. *Mon. Not. R. Astron. Soc.* **507**, 2241.

Lau, J. Y. & Binney, J. 2021*b* Modes of a stellar system II: Non-ergodic systems. *Mon. Not. R. Astron. Soc.* **507**, 2562.

Lenard, A. 1960 On Bogoliubov's kinetic equation for a spatially homogeneous plasma. *Ann. Phys. (New York)* **10**, 390.

Lifshitz, E. M. & Pitaevskii, L. P. 1981 *Physical Kinetics (L. D. Landau and E. M. Lifshitz's Course of Theoretical Physics, Volume 10)*. Elsevier.

Liperovskii, V. A. & Tsytovich, V. N. 1970 Spectra of Langmuir plasma turbulence. *Sov. Phys.–JETP* **30**, 682.

Loureiro, N. F., Schekochihin, A. A. & Cowley, S. C. 2007 Instability of current sheets and formation of plasmoid chains. *Phys. Plasmas* **14**, 100703.

Lynden-Bell, D. 1967 Statistical mechanics of violent relaxation in stellar systems. *Mon. Not. R. Astron. Soc.* **136**, 101.

Majeski, S., Kunz, M. W. & Squire, J. 2024 Self-organization in collisionless, high-β turbulence. *E-print* arXiv:2405.02418.

Malmberg, J. H. & Wharton, C. B. 1964 Collisionless damping of electrostatic plasma waves. *Phys. Rev. Lett.* **13**, 184.

Malmberg, J. H., Wharton, C. B., Gould, R. W. & O'Neil, T. M. 1968 Plasma wave echo experiment. *Phys. Rev. Lett.* **20**, 95.

Mazitov, R. K. 1965 Damping of plasma waves. *J. Appl. Mech. Tech. Phys.* **6**, 22.

Meinecke, J., Tzeferacos, P., Ross, J. S., Bott, A. F. A., Feister, S., Park, H.-S., Bell, A. R., Blandford, R., Berger, R. L., Bingham, R., Casner, A., Chen, L. E., Foster, J., Froula, D. H., Goyon, C., Kalantar, D., Koenig, M., Lahmann, B., Li, C., Lu, Y., Palmer, C. A. J., Petrasso, R. D., Poole, H., Remington, B., Reville, B., Reyes, A., Rigby, A., Ryu, D., Swadling, G., Zylstra, A., Miniati, F., Sarkar, S., Schekochihin, A. A., Lamb, D. Q. & Gregori, G. 2022 Strong suppression of heat conduction in a laboratory replica of galaxy-cluster turbulent plasmas. *Sci. Adv.* **8**, eabj6799.

Melville, S., Schekochihin, A. A. & Kunz, M. W. 2016 Pressure-anisotropy-driven microturbulence and magnetic-field evolution in shearing, collisionless plasma. *Mon. Not. R. Astron. Soc.* **459**, 2701.

Meyrand, R., Squire, J., Schekochihin, A. A. & Dorland, W. 2021 On the violation of the zeroth law of turbulence in space plasmas. *J. Plasma Phys.* **87**, 535870301.

Mikhailovskii, A. B. & Tsypin, V. S. 1971 Transport equations and gradient instabilities in a high pressure collisional plasma. *Plasma Phys.* **13**, 785.

Moffatt, H. K. 1978 *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge: Cambridge University Press.

Moffatt, H. K. & Saffman, P. G. 1964 Comment on "Growth of a weak magnetic field in a turbulent conducting fluid with large magnetic Prandtl number". *Phys. Fluids* **7**, 155.

Morrison, P. 2000 Hamiltonian description of Vlasov dynamics: Action-angle variables for the continuous spectrum. *Transport Theory Stat. Phys.* **29**, 397.

Morrison, P. J. 1994 The energy of perturbations for Vlasov plasmas. *Phys. Plasmas* **1**, 1447.

Musher, S. L., Rubenchik, A. M. & Zakharov, V. E. 1995 Weak Langmuir turbulence. *Phys. Rep.* **252**, 177.

Nastac, M. L., Ewart, R. J., Juno, J., Barnes, M. & Schekochihin, A. A. 2025 Universal fluctuation spectrum of Vlasov-Poisson turbulence. *E-print* arXiv:2503.17278.

Nastac, M. L., Ewart, R. J., Sengupta, W., Schekochihin, A. A., Barnes, M. & Dorland, W. D. 2024 Phase-space entropy cascade and irreversibility of stochastic heating in nearly collisionless plasma turbulence. *Phys. Rev. E* **109**, 065210.

Nazarenko, S. 2011 *Wave Turbulence*. Berlin: Springer.

Nazarenko, S. V. & Schekochihin, A. A. 2011 Critical balance in magnetohydrodynamic, rotating and stratified turbulence: towards a universal scaling conjecture. *J. Fluid Mech.* **677**, 134.

Newcomb, W. A. 1962 Lagrangian and Hamiltonian methods in magnetohydrodynamics. *Nucl. Fusion Suppl., Part 2* p. 451.

Newton, S. L., Cowley, S. C. & Loureiro, N. F. 2010 Understanding the effect of sheared flow on microinstabilities. *Plasma Phys. Control. Fusion* **52**, 125001.

Ng, C. S. & Bhattacharjee, A. 2005 Bernstein-Greene-Kruskal modes in a three-dimensional plasma. *Phys. Rev. Lett.* **95**, 245004.

Nies, R., Parra, F., Barnes, M., Mandell, N. & Dorland, W. 2024 Saturation of magnetised plasma turbulence by propagating zonal flows. *E-print* arXiv:2409.02283.

Novikov, E. A. 1965 Functionals and the random-force method in turbulence theory. *Sov. Phys.–JETP* **20**, 1290.

Nyquist, H. 1932 Regeneration theory. *Bell Syst. Tech. J.* **11**, 126.

Ogilvie, G. I. 2016 Astrophysical fluid dynamics. *J. Plasma Phys.* **82**, 205820301.

Ogilvie, G. I. & Proctor, M. R. E. 2003 On the relation between viscoelastic and magnetohydrodynamic flows and their instabilities. *J. Fluid Mech.* **476**, 389.

O'Neil, T. 1965 Collisionless damping of nonlinear plasma oscillations. *Phys. Fluids* **8**, 2255.

Ott, E. 1998 Chaotic flows and kinematic magnetic dynamos: A tutorial review. *Phys. Plasmas* **5**, 1636.

Parker, E. N. 1958 Dynamical instability in an anisotropic ionized gas of low density. *Phys. Rev.* **109**, 1874.

Parker, E. N. 1966 The dynamical state of the interstellar gas and field. *Astrophys. J.* **145**, 811.

Parra, F. I. 2019*a* *Collisional Plasma Physics*. Lecture Notes for an Oxford MMathPhys course; URL: http://www-thphys.physics.ox.ac.uk/people/FelixParra/ /CollisionalPlasmaPhysics/CollisionalPlasmaPhysics.html.

Parra, F. I. 2019*b* *Collisionless Plasma Physics*. Lecture Notes for an Oxford MMathPhys course; URL: http://www-thphys.physics.ox.ac.uk/people/FelixParra/ /CollisionlessPlasmaPhysics/CollisionlessPlasmaPhysics.html.

Peierls, R. E. 1955 *Quantum Theory of Solids*. Oxford: Clarendon Press.

Pelletier, G. 1980*a* Langmuir turbulence as a critical phenomenon. Part 1. Destruction of the statistical equilibrium of an interacting-modes ensemble. *J. Plasma Phys.* **24**, 287.

Pelletier, G. 1980*b* Langmuir turbulence as a critical phenomenon. Part 2. Application of the dynamical renormalization group method. *J. Plasma Phys.* **24**, 421.

Penrose, O. 1960 Electrostatic instabilities of a uniform non-Maxwellian plasma. *Phys. Fluids* **3**, 258.

Pfirsch, D. & Sudan, R. N. 1993 Nonlinear ideal magnetohydrodynamics instabilities. *Phys. Fluids B* **5**, 2052.

Pierce, J. R. & Hebenstreit, W. B. 1949 A new type of high-frequency amplifier. *Bell Syst. Tech. J.* **28**, 33.

Pikelner, S. B. & Tsytovich, V. N. 1969 High-frequency turbulence spectra of a plasma and acceleration of subcosmic rays. *Sov. Phys.–JETP* **28**, 507.

Quataert, E. 2008 Buoyancy instabilities in weakly magnetized low-collisionality plasmas. *Astrophys. J.* **673**, 758.

Ramos, J. J. & White, R. L. 2018 Normal-mode-based analysis of electron plasma waves with second-order Hermitian formalism. *Phys. Plasmas* **25**, 034501.

Rincon, F. 2019 Dynamo theories. *J. Plasma Phys.* **85**, 205850401.

Rincon, F., Schekochihin, A. A. & Cowley, S. C. 2015 Non-linear mirror instability. *Mon. Not. R. Astron. Soc.* **447**, L45.

Robinson, P. A. 1997 Nonlinear wave collapse and strong turbulence. *Rev. Mod. Phys.* **69**, 507.

Rogers, B. N. & Dorland, W. 2005 Noncurvature-driven modes in a transport barrier. *Phys. Plasmas* **12**, 062511.

Rogers, B. N., Dorland, W. & Kotschenreuther, M. 2000 Generation and stability of zonal flows in ion-temperature-gradient mode turbulence. *Phys. Rev. Lett.* **85**, 5336.

Rogister, A. & Oberman, C. 1968 On the kinetic theory of stable and weakly unstable plasma. Part 1. *J. Plasma Phys.* **2**, 33.

Rogister, A. & Oberman, C. 1969 On the kinetic theory of stable and weakly unstable plasma. Part 2. *J. Plasma Phys.* **3**, 119.

Rosenbluth, M. N. 1956 Stability of the pinch. *Los Alamos Laboratory Report LA-2030*.

Rosin, M. S., Schekochihin, A. A., Rincon, F. & Cowley, S. C. 2011 A non-linear theory of the parallel firehose and gyrothermal instabilities in a weakly collisional plasma. *Mon. Not. R. Astron. Soc.* **413**, 7.

Rostoker, N. 1961 Fluctuations of a plasma (I). *Nucl. Fusion* **1**, 101.

Rudakov, L. I. & Tsytovich, V. N. 1978 Strong Langmuir turbulence. *Phys. Rep.* **40**, 1.

Ruyer, C., Gremillet, L., Debayle, A. & Bonnaud, G. 2015 Nonlinear dynamics of the ion Weibel-filamentation instability: An analytical model for the evolution of the plasma and spectral properties. *Phys. Plasmas* **22**, 032102.

Sagdeev, R. Z. & Galeev, A. A. 1969 *Nonlinear Plasma Theory*. W. A. Benjamin.

Schekochihin, A. A. 2007*a* *Five Lectures on Turbulence*. Lecture Notes for the 2nd International Conference (School) on Advanced Computing and Simulation, Cambridge; URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/SummerSchool07/.

Schekochihin, A. A. 2007*b* *Fluctuation Dynamo*. Lecture Notes for École de Physique des Houches; URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/leshouches07.pdf.

Schekochihin, A. A. 2022 MHD turbulence: a biased review. *J. Plasma Phys.* **88**, 155880501.

Schekochihin, A. A. 2024 *Kinetic Theory and Statistical Physics*. Lecture Notes for the Oxford Physics course, Paper A1; URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/A1/2014/A1LectureNotes.pdf.

Schekochihin, A. A. & Cowley, S. C. 2007 Turbulence and magnetic fields in astrophysical plasmas. In *Magnetohydrodynamics: Historical Evolution and Trends* (ed. S. Molokov, R. Moreau & H. K. Moffatt), p. 85. Berlin: Springer, reprint: URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/mhdbook.pdf.

Schekochihin, A. A., Cowley, S. C. & Dorland, W. 2007 Interplanetary and interstellar plasma turbulence. *Plasma Phys. Control. Fusion* **49**, A195.

Schekochihin, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Quataert, E. & Tatsuno, T. 2009 Astrophysical gyrokinetics: kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl.* **182**, 310.

Schekochihin, A. A., Cowley, S. C., Kulsrud, R. M., Rosin, M. S. & Heinemann, T. 2008 Nonlinear growth of firehose and mirror fluctuations in astrophysical plasmas. *Phys. Rev. Lett.* **100**, 081301.

Schekochihin, A. A., Cowley, S. C., Rincon, F. & Rosin, M. S. 2010 Magnetofluid dynamics of magnetized cosmic plasma: firehose and gyrothermal instabilities. *Mon. Not. R. Astron. Soc.* **405**, 291.

Schekochihin, A. A., Cowley, S. C., Taylor, S. F., Maron, J. L. & McWilliams, J. C. 2004 Simulations of the small-scale turbulent dynamo. *Astrophys. J.* **612**, 276.

Schekochihin, A. A., Highcock, E. G. & Cowley, S. C. 2012 Subcritical fluctuations and suppression of turbulence in differentially rotating gyrokinetic plasmas. *Plasma Phys. Control. Fusion* **54**, 055011.

Schekochihin, A. A., Kawazura, Y. & Barnes, M. A. 2019 Constraints on ion versus electron heating by plasma turbulence at low beta. *J. Plasma Phys.* **85**, 905850303.

Schekochihin, A. A., Parker, J. T., Highcock, E. G., Dellar, P. J., Dorland, W. & Hammett, G. W. 2016 Phase mixing versus nonlinear advection in drift-kinetic plasma turbulence. *J. Plasma Phys.* **82**, 905820212.

Servidio, S., Chasapis, A., Matthaeus, W. H., Perrone, D., Valentini, F., Parashar, T. N., Veltri, P., Gershman, D., Russell, C. T., Giles, B., Fuselier, S. A., Phan, T. D. & Burch, J. 2017 Magnetospheric Multiscale (MMS) observation of plasma velocity-space cascade: Hermite representation and theory. *Phys. Rev. Lett.* **119**, 205101.

Severne, G. & Luwel, M. 1980 Dynamical theory of collisionless relaxation. *Astrophys. Space Sci.* **72**, 293.

Shannon, C. 1948 A Mathematical Theory of Communication. *The Bell System Technical Journal* **27**, 379.

Skoutnev, V., Hakim, A., Juno, J. & TenBarge, J. M. 2019 Temperature-dependent saturation of Weibel-type instabilities in counter-streaming plasmas. *Astrophys. J.* **872**, L28.

Southwood, D. J. & Kivelson, M. G. 1993 Mirror instability I: Physical mechanism of linear instability. *J. Geophys. Res.* **98**, 9181.

Squire, J., Kunz, M. W., Arzamasskiy, L., Johnston, Z., Quataert, E. & Schekochihin, A. A. 2023 Pressure anisotropy and viscous heating in weakly collisional plasma turbulence. *J. Plasma Phys.* **89**, 905890417.

Squire, J., Kunz, M. W., Quataert, E. & Schekochihin, A. A. 2017 Kinetic simulations of the interruption of large-amplitude shear-Alfvén waves in a high-β plasma. *Phys. Rev. Lett.* **119**, 155101.

Squire, J., Meyrand, R., Kunz, M. W., Arzamasskiy, L., Schekochihin, A. A. & Quataert, E. 2022 High-frequency heating of the solar wind triggered by low-frequency turbulence. *Nature Astron.* **6**, 715.

Squire, J., Schekochihin, A. A., Quataert, E. & Kunz, M. W. 2019 Magneto-immutable turbulence in weakly collisional plasmas. *J. Plasma Phys.* **85**, 905850114.

St-Onge, D. A. & Kunz, M. W. 2018 Fluctuation dynamo in a collisionless, weakly magnetized plasma. *Astrophys. J.* **863**, L25.

Strauss, H. R. 1976 Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks. *Phys. Fluids* **19**, 134.

Sturrock, P. A. 1966 Stochastic acceleration. *Phys. Rev.* **141**, 186.

Sturrock, P. A. 1994 *Plasma Physics. An Introduction to the Theory of Astrophysical, Geophysical and Laboratory Plasmas*. Cambridge: Cambridge University Press.

Su, C. H. & Oberman, C. 1968 Collisional damping of a plasma echo. *Phys. Rev. Lett.* **20**, 427.

Swanson, D. G. 2008 *Plasma Kinetic Theory*. CRC Press.

Syrovatskiĭ, S. I. 1971 Formation of current sheets in a plasma with a frozen-in strong magnetic field. *Sov. Phys.–JETP* **33**, 933.

Taylor, J. B. 1963 Some stable plasma equilibria in combined mirror-cusp fields. *Phys. Fluids* **6**, 1529.

Taylor, J. B. & Newton, S. L. 2015 Special topics in plasma confinement. *J. Plasma Phys.* **81**, 205810501.

Thornhill, S. G. & ter Haar, D. 1978 Langmuir turbulence and modulational instability. *Phys. Rep.* **43**, 43.

Tonks, L. & Langmuir, I. 1929 Oscillations in ionized gases. *Phys. Rev.* **33**, 195.

Tsytovich, V. N. 1970 *Nonlinear Effects in Plasma*. New York: Plenum Press.

Tsytovich, V. N. 1972 *An Introduction to the Theory of Plasma Turbulence*. Oxford: Pergamon Press.

Tsytovich, V. N. 1977 *Theory of Turbulent Plasma*. New York: Consultants Bureau.

Tsytovich, V. N. 1995 *Lectures on Non-linear Plasma Kinetics*. Berlin: Springer.

Uzdensky, D. A. & Loureiro, N. F. 2016 Magnetic reconnection onset via disruption of a forming current sheet by the tearing instability. *Phys. Rev. Lett.* **116**, 105003.

Uzdensky, D. A., Loureiro, N. F. & Schekochihin, A. A. 2010 Fast magnetic reconnection in the plasmoid-dominated regime. *Phys. Rev. Lett.* **105**, 235002.

Valentini, F., Perrone, D., Califano, F., Pegoraro, F., Veltri, P., Morrison, P. J. & O'Neil, T. M. 2012 Undamped electrostatic plasma waves. *Phys. Plasmas* **19**, 092103.

van Kampen, N. G. & Felderhof, B. U. 1967 *Theoretical Methods in Plasma Physics*. Amsterdam: North-Holland.

Vedenov, A. A. 1963 Quasilinear plasma theory (theory of a weakly turbulent plasma). *Plasma Phys. (J. Nucl. Energy C)* **5**, 169.

Vedenov, A. A., Gordeev, A. V. & Rudakov, L. I. 1967 Oscillations and instability of a weakly turbulent plasma. *Plasma Phys.* **9**, 719.

Vedenov, A. A. & Rudakov, L. I. 1965 Interaction of waves in continuous media. *Sov. Phys. Doklady* **9**, 1073.

Vedenov, A. A. & Sagdeev, R. Z. 1958 Some properties of the plasma with anisotropic distribution of the velocities of ions in the magnetic field. *Sov. Phys. Doklady* **3**, 278.

Vedenov, A. A., Velikhov, E. P. & Sagdeev, R. Z. 1961 Nonlinear oscillations of rarefied plasma. *Nucl. Fusion* **1**, 82.

Vedenov, A. A., Velikhov, E. P. & Sagdeev, R. Z. 1962 Quasilinear theory of plasma oscillations. *Nucl. Fusion Suppl., Part 2* p. 465 (in Russian).

Villani, C. 2014 Particle systems and nonlinear Landau damping. *Phys. Plasmas* **21**, 030901.

Weibel, E. S 1958 Spontaneously growing transverse waves in a plasma due to an anisotropic velocity distribution. *Phys. Rev. Lett.* **2**, 83.

Wicks, R. T., Roberts, D. A., Mallet, A., Schekochihin, A. A., Horbury, T. S. & Chen, C. H. K. 2013 Correlations at large scales and the onset of turbulence in the fast solar wind. *Astrophys. J.* **778**, 177.

Xu, R. & Kunz, M. W. 2016 Linear Vlasov theory of a magnetised, thermally stratified atmosphere. *J. Plasma Phys.* **82**, 905820507.

Zakharov, V. E. 1972 Collapse of Langmuir waves. *Sov. Phys.–JETP* **35**, 908.

Zakharov, V. E. & Karpman, V. I. 1963 On the nonlinear theory of the damping of plasma waves. *Sov. Phys.–JETP* **16**, 351.

Zakharov, V. E. & Kuznetsov, E. A. 1978 Kinetics of low- and high-frequency waves in nonlinear media. *Sov. Phys.–JETP* **48**, 458.

Zakharov, V. E., Lvov, V. S. & Falkovich, G. 1992 *Kolmogorov Spectra of Turbulence I: Wave Turbulence*. Berlin: Springer (updated online version URL: https://www.weizmann.ac.il/chemphys/lvov/research/springer-series-nonlinear-dynamics).

Zakharov, V. E., Musher, S. L. & Rubenchik, A. M. 1985 Hamiltonian approach to the description of non-linear plasma phenomena. *Phys. Rep.* **129**, 285.

Zeldovich, Ya. B. 1956 The magnetic field in the two-dimensional motion of a conducting turbulent liquid. *Zh. Eksp. Teor. Fiz.* **31**, 154, English translation: *Sov. Phys. JETP* **4**, 460 (1957).

Zeldovich, Ya. B., Ruzmaikin, A. A., Molchanov, S. A. & Sokolov, D. D. 1984 Kinematic dynamo problem in a linear velocity field. *J. Fluid Mech.* **144**, 1.

Zhdankin, V., Boldyrev, S. & Uzdensky, D. A. 2016 Scalings of intermittent structures in magnetohydrodynamic turbulence. *Phys. Plasmas* **23**, 055705.

Zhdankin, V., Kunz, M. W. & Uzdensky, D. A. 2023 Synchrotron firehose instability. *Astrophys. J.* **944**, 24.

Ziman, J. M. 1960 *Electrons and Phonons: The Theory of Transport Phenomena in Solids*. Oxford: Clarendon Press.

Zocco, A., Helander, P. & Connor, J. W. 2015 Magnetic compressibility and ion-temperature-gradient-driven microinstabilities in magnetically confined plasmas. *Plasma Phys. Control. Fusion* **57**, 085003.

Zocco, A. & Schekochihin, A. A. 2011 Reduced fluid-kinetic equations for low-frequency dynamics, magnetic reconnection, and electron heating in low-beta plasmas. *Phys. Plasmas* **18**, 102309.