Statistical Physics

Oxford Physics

Second year physics course

A. A. Schekochihin
(with thanks to S. J. Blundell)

Problem Sets 3–4: Kinetic Theory

Michaelmas Term 2014

Introductory Note

Lectures

I will not be following any single book, so I advise you to attend lectures and take notes (a very useful skill to learn). My hand-written notes (prepared in 2012) are available via the course webpage, `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/A1/`, *but they are just that—lecture notes—and so come with* no guarantee of legibility or book-level transparency of structure. *This year, by popular demand, I intend to produce a LaTeX'ed version of these Notes. The current version of this write-up will always be linked from the course webpage; it will accrete sections as we proceed. I will be very grateful for your feedback: comments, error corrections, views etc. I do of course hope that you might find these Notes helpful, but whether you do or don't, you must not regard them as the sole source to learn from.*

Oxford has 99 libraries and you are missing out if you have not yet become an avid explorer of the world of books. Learning a subject and making sense of it from a variety of sources is an essential part of higher education—and indeed it is part of the thrill of one's intellectual formation to find oneself free to decide whom to believe and what does and doesn't make sense. I will give you reading suggestions, both specific ones based on the Reading List, and others, designed for you to explore the subject laterally or in more depth—but don't stop there, you do not want to be intellectual clones of me, so make your own decisions what to read!

Of the books on the Reading List, I particularly like Blundell & Blundell, Pauli, Schrödinger, and Landau & Lifshitz. The first two are on the undergraduate level, the third does not deal with Kinetic Theory and will become relevant in HT, and "Landaushitz"—Vol. 10 does everything on a very high level of analytical sophistication, so reading it will be a challenge and you should not despair if you find it hard. If you prefer a much more ponderous and meticulously precise mathematical treatment in the old Cambridge style, Chapman & Cowling can be your bible of Kinetic Theory. This said, I'll do it all largely my way.

The course will be quite mathematical, possibly more so than you have so far experienced. But Physics has been a mathematical subject since Newton and we would be moving backwards if we did it A-level style. Learning to describe and predict Nature mathematically is one of the most impressive achievements of our civilisation. So become civilised!

Please ask questions during the lectures or by email (to `a.schekochihin1@physics.ox.ac.uk`*). I will appreciate real-time feedback.*

Problem Sets

Problem Set 3 covers the material of Lectures 1-3. Start working on it at the end of Week 6.

Problem Set 4 will cover the rest of Kinetic Theory and is intended as vacation work.

Questions that may prove difficult (more so than anything you are likely to face in an exam) or that deal with lateral issues are marked with (*). Skip them if you must, although I do hope you will relish the challenge rather than seek the minimum-energy state.

Some Useful Constants

Boltzmann's constant	k_B	$1.3807 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Proton rest mass	m_p	$1.6726 \times 10^{-27}\,\mathrm{kg}$
Avogadro's number	N_A	$6.022 \times 10^{23}\,\mathrm{mol^{-1}}$
Standard molar volume		$22.414 \times 10^{-3}\,\mathrm{m^3\,mol^{-1}}$
Molar gas constant	R	$8.315\ \mathrm{J\,mol^{-1}\,K^{-1}}$
1 pascal (Pa)		$1\,\mathrm{N\,m^{-2}}$
1 standard atmosphere		$1.0132 \times 10^{5}\,\mathrm{Pa}\,(\mathrm{N\,m^{-2}})$
1 bar (= 1000 mbar)		$10^{5}\,\mathrm{N\,m^{-2}}$
Stefan–Boltzmann constant	σ	$5.67 \times 10^{-8}\ \mathrm{Wm^{-2}K^{-4}}$

PROBLEM SET 3: Particle Distributions

Calculating Averages

3.1 a) If θ is a continuous random variable which is uniformly distributed between 0 and π, write down an expression for $p(\theta)$. Hence find the value of the following averages:

(i) $\langle\theta\rangle$

(ii) $\langle\theta - \frac{\pi}{2}\rangle$

(iii) $\langle\theta^2\rangle$

(iv) $\langle\theta^n\rangle$ (for the case $n \geq 0$)

(v) $\langle\cos\theta\rangle$

(vi) $\langle\sin\theta\rangle$

(vii) $\langle|\cos\theta|\rangle$

(viii) $\langle\cos^2\theta\rangle$

(ix) $\langle\sin^2\theta\rangle$

(x) $\langle\cos^2\theta + \sin^2\theta\rangle$

Check that your answers are what you expect.

b) If particle velocities are distributed isotropically, how are their angles distributed? Is the angle between the velocity vector and a fixed axis (chosen by you) distributed uniformly? Why? Answer these questions for the case of a 2- and 3-dimensional world.

3.2 a) Consider an isotropic distribution of particle velocities: $f(\mathbf{v}) = f(v)$, where $v = |\mathbf{v}|$ is the particle speed. In 3D, what is the distribution of the speeds, $\tilde{f}(v)$?

Please note that the notation I use is different from Blundell & Blundell: $f(\mathbf{v})d^3\mathbf{v}$ is velocity distribution in 3D, normalised to 1; when it is isotropic, $f(\mathbf{v}) = f(v)$ (same letter used, f, although if I had been more mathematically fastidious, I would have used a different letter); the speed distribution is $\tilde{f}(v)dv$. In contrast, Blundell & Blundell use $f(v)$ for the speed distribution and $g(\mathbf{v})$ for velocity distribution.

b) Calculate the following averages of velocity components in terms of averages of speed ($\langle v\rangle$, $\langle v^2\rangle$, etc.)

(i) $\langle v_i\rangle$, where $i = x, y, z$

(ii) $\langle|v_i|\rangle$, where $i = x, y, z$

(iii) $\langle v_i^2\rangle$, where $i = x, y, z$

(iv) $\langle v_i v_j\rangle$, where $i, j = x, y, z$ (any index can designate any of the components)

(v) $\langle v_i v_j v_k\rangle$, where $i, j = x, y, z$

You can do them all by direct integration with respect to angles, but think carefully whether this is necessary in all cases. You may be able to obtain the answers in a quicker way by symmetry considerations (being lazy often spurs creative thinking).

Hint for (iv). Here is a smart way of doing this. $\langle v_i v_j\rangle$ is a symmetric rank-2 tensor (i.e., a tensor, or matrix, with two indices). Since the velocity distribution is isotropic, this tensor must be rotationally invariant (i.e., not change under rotations of the coordinate frame). The only symmetric rank-2 tensor that has this property is a constant times Kronecker delta δ_{ij}. So it must be that $\langle v_i v_j \rangle = C\delta_{ij}$, where C can only depend on the distribution of *speeds* v (not vectors $\mathbf{v}$). Can you figure out what C is? Is it the same in 2D and in 3D? This is a much simpler derivation than doing velocity integrals directly, but it is worth checking the result by direct integration to convince yourself that the symmetry magic works.

c*) Calculate $\langle v_i v_j v_k v_l \rangle$, where $i, j, k, l = x, y, z$ (any index can designate any of the components) — in terms of averages of powers of v.

Hint. Doing this by direct integration is *a lot* of work. Generalise the symmetry argument given above: see what symmetric rotationally invariant rank-4 tensors (i.e., tensors with 4 indices) you can cook up: it turns out that they have to be products of Kronecker deltas, e.g., $\delta_{ij}\delta_{kl}$; what other combinations are there? Then $\langle v_i v_j v_k v_l \rangle$ must be a linear combination of these tensors, with coefficients that depend on some moments (averages) of v. By examining the symmetry properties of $\langle v_i v_j v_k v_l \rangle$, work out what these coefficients are (if you have done question b(iv) above, you'll know what to do). How does the answer depend on the dimensionality of the world (2D, 3D)?

3.3 The probability distribution of molecular speeds in a gas in thermal equilibrium is a Maxwellian: a molecule of mass m will have a velocity in a 3-dimensional interval $[v_x, v_x + dv_x] \times [v_y, v_y + dv_y] \times [v_z, v_z + dv_z]$ (denoted $d^3\mathbf{v}$) with probability

$$f(\mathbf{v})d^3\mathbf{v} \propto e^{-v^2/v_{\rm th}^2} d^3\mathbf{v},$$

where $v_{\rm th} = \sqrt{2k_BT/m}$ is the "thermal speed," T temperature, k_B Boltzmann's constant, and I have used the proportionality sign ($\propto$) because the normalisation constant has been omitted (work it out by integrating $f(\mathbf{v})$ over all velocities).

a) Given the Maxwellian distribution, what is the distribution of speeds, $\tilde{f}(v)$? Calculate the mean speed $\langle v \rangle$ and the mean inverse speed $\langle 1/v \rangle$. Show that $\langle v \rangle \langle 1/v \rangle = 4/\pi$.

b) Calculate $\langle v^2 \rangle$, $\langle v^3 \rangle$, $\langle v^4 \rangle$, $\langle v^5 \rangle$.

c*) Work out a general formula for $\langle v^n \rangle$. What is larger, $\langle v^{27} \rangle^{1/27}$ or $\langle v^{56} \rangle^{1/56}$? Do you understand why that is, qualitatively?

Hint. Consider separately odd and even n. Use $\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$. [These things are worked out in Blundell & Blundell, but do try to figure them out yourself!]

d*) What is the distribution of speeds $\tilde{f}(v)$ in an n-dimensional world (for general n)?

Pressure

3.4 Remind yourself how one calculates pressure from a particle distribution function. Let us consider an *anisotropic* system, where there exists one (and only one) special direction in space (call it z), which affects the distribution of particle velocities (an example of such a situation is a gas of charged particles in a straight magnetic field).

a) How many variables does the distribution function now depend on? (Recall that in the isotropic case, it depended only on one, v.) Write down the most general form of the distribution function under these symmetries — what is the appropriate transformation of variables from (v_x, v_y, v_z)?

b) What is the expression for pressure $p_{\|}$ (in terms of averages of those new velocity variables) that the gas will exert on a wall perpendicular to the z axis? (It is called $p_{\|}$ because it is due to particles whose velocities have non-zero projections onto the special direction z.) What is $p_{\perp}$, pressure on a wall parallel to z?

c) Now consider a wall with a normal $\hat{\mathbf{n}}$ at an angle θ to z. What is the pressure on this wall in terms of $p_{\|}$ and $p_{\perp}$?

Effusion

3.5 a) Show that the number of molecules hitting unit area of a surface per unit time with speeds between v and $v + dv$ and angles between θ and $\theta + d\theta$ to the normal is

$$d\tilde{\Phi}(v,\theta) = \frac{1}{2} n v \tilde{f}(v) dv \, \sin\theta \cos\theta \, d\theta,$$

where $\tilde{f}(v)$ is the distribution of particle speeds.

b) Show that the average value of $\cos\theta$ for these molecules is $\frac{2}{3}$.

c) Using the results of Q3.3, show that for a gas obeying the Maxwellian distribution, the average energy of all the molecules is $(3/2)k_{\mathrm{B}}T$, but the average energy of those hitting the surface is $2k_{\mathrm{B}}T$.

3.6 a) A Maxwellian gas effuses through a small hole to form a beam. After a certain distance from the hole, the beam hits a screen. Let v_1 be the most probable speed of atoms that, during a fixed interval of time, hit the screen. Let v_2 be the most probable speed of atoms situated, at any instant, between the small hole and the screen. Find expressions for v_1 and v_2. Why are these two speeds different?

b) You have calculated the most probable speed (v_1) for molecules of mass m which have effused out of an enclosure at temperature T. Now calculate their mean speed $\langle v \rangle$. Which is the larger and why?

3.7 A vessel contains a monatomic gas at temperature T. Use Maxwell's distribution of speeds to calculate the mean kinetic energy of the molecules.

Molecules of the gas stream through a small hole into a vacuum. A box is opened for a short time and catches some of the molecules. Assuming the box is thermally insulated, calculate the final temperature of the gas trapped in the box.

3.8 *This question requires you to think geometrically.*

a) A gas effuses into a vacuum through a small hole of area A. The particles are then collimated by passing through a very small circular hole of radius a, in a screen a distance d from the first hole. Show that the rate at which particles emerge from the circular hole is $\frac{1}{4} n A \langle v \rangle (a^2/d^2)$, where n is the particle density and $\langle v \rangle$ is the average speed. (Assume no collisions take place after the gas effuses and that $d \gg a$.)

b) Show that if a gas were allowed to leak into an evacuate sphere and the particles condensed where they first hit the surface they would form a uniform coating.

3.9 A closed vessel is partially filled with liquid mercury; there is a hole of area $A = 10^{-7}$ m^2 above the liquid level. The vessel is placed in a region of high vacuum at $T = 273$ K and after 30 days is found to be lighter by $\Delta M = 2.4 \times 10^{-5}$ kg. Estimate the vapour pressure of mercury at 273 K. (The relative molecular mass of mercury is 200.59.)

3.10 A gas is a mixture of H_2 and HD in the proportion 7000:1. As the gas effuses through a small hole from a vessel at constant temperature into a vacuum, the composition of the remaining mixture changes. By what factor will the pressure in the vessel have fallen when the remaining mixture consists of H_2 and HD in the proportion 700:1. (H=hydrogen, D=deuterium)

3.11 (*) In the previous question, you worked out a differential equation for the time evolution of the number density of the gas in the container and then solved it (if that is not what you did, go back and think again). The container was assumed to have constant temperature. Now consider instead a thermally insulated container of volume V with a small hole of area A, containing a gas with molecular mass m. At time $t = 0$, the density is n_0 and temperature is T_0. As gas effuses out through a small hole, both density and temperature inside the container will drop. Work out their time dependence, $n(t)$ and $T(t)$ in terms of the quantities given above.

Hint. Temperature is related to the total energy of the particles in the container. Same way you calculated the flux of particles through the hole (leading to density decreasing), you can now also calculate the flux of energy, leading to temperature decreasing. As a result, you will get two differential (with respect to time) equations for two unknowns, n and T. Derive and then integrate these equations (here you will have to brush up on what learned in your 1-st year maths course).

Thermodynamic Limit

3.12 (*) Consider a large system of volume $\mathcal{V}$ containing $\mathcal{N}$ non-interacting particles. Take some fixed subvolume $V \ll \mathcal{V}$. Calculate the probability to find N particles in volume V. Now assume that both $\mathcal{N}$ and $\mathcal{V}$ tend to ∞, but in such a way that the particle number density is fixed: $\mathcal{N}/\mathcal{V} \to n = \text{const}$.

a) Show that in this limit, the probability p_N to find N particles in volume V (both N and V are fixed, $N \ll \mathcal{N}$) tends to the Poisson distribution whose average is $\langle N \rangle = nV$.

Hint. This involves proving Poisson's limit theorem. You will find inspiration or possibly even the solution in standard probability texts, e.g., Ya. G. Sinai, *Probability Theory: An Introductory Course* (Springer 1992).

b) Prove that

$$\frac{\langle (N - \langle N \rangle)^2 \rangle^{1/2}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}$$

(so fluctuations around the average are very small as $\langle N \rangle \gg 1$).

c) Show that, if $\langle N \rangle \gg 1$, p_N has its maximum at $N \approx \langle N \rangle = nV$; then show that in the vicinity of this maximum,

$$p_N \approx \frac{1}{\sqrt{2\pi nV}}\, e^{-(N-nV)^2/2nV}.$$

Hint. Use Stirling's formula for $N!$ (look it up if you don't know what that is). Taylor-expand $\ln p_N$ around $N = nV$.

The result of (a) is, of course, intuitively obvious, but it is nice to be able to prove it mathematically and even to know with what precision it holds (b) — another demonstration that the world is constructed in a sensible way.

PROBLEM SET 4: Collisions and Transport

This Problem Set is vacation work and so is longer than usual;
it contains some revision problems (at the end).

Mean Free Path

4.1 Consider a gas that is a mixture of two species of molecules: type-1 with diameter d_1, mass m_1 and mean number density n_1 and type-2 with diameter d_2, mass m_2 and mean number density n_2. If we let them collide with each other for a while (for how long? answer this after you have solved the rest of the problem), they will eventually settle into a Maxwellian equilibrium and the temperatures of the two species will be the same.

a) What will be the rms speeds of the two species?

b) Show that the combined pressure of the mixture will be $p = p_1 + p_2$ (Dalton's law).

c) What is the cross-section for the collisions between type-1 and type-2 molecules?

d) What is the mean collision rate of type-1 molecules with type-2 molecules? (here you will need to find the mean relative speed of the two types of particles, a calculation analogous to one in the lecture notes)

Hint. You will find the answers in Pauli's book, but do try to figure them out on your own.

4.2 Consider particles in a gas of mean number density n and collisional cross-section σ, moving with speed v (let us pretend they all have exactly the same speed).

a) What is the probability $P(t)$ for a particle to experience *no* collisions up to time t? Therefore, what is the mean time until it experiences a collision?

Hint. Work out the probability for a particle not to have a collision between t and $t+dt$. Hence work out $P(t+dt)$ in terms of $P(t)$ and the relevant parameters of the gas. You should end up with a differential equation for $P(t)$, which you can then solve. [You will find this derivation in Blundell & Blundell, but do try to figure it out yourself!]

b) What is the the probability $P(x)$ for a particle to travel a distance x before having a collision? Show that the root mean square free path is given by $\sqrt{2}\lambda_{\text{mfp}}$ where λ_{mfp} is the mean free path.

c) What is the most probable free path length?

d) What percentage of molecules travel a distance greater than (i) λ_{mfp}, (ii) $2\lambda_{\text{mfp}}$, (iii) $5\lambda_{\text{mfp}}$?

4.3 Given that the mean free path in a gas at standard temperature and pressure (S.T.P.) is about 10^3 atomic radii, *estimate* the highest allowable pressure in the chamber of an atomic beam apparatus 10^{-1} m long (if one does not want to lose an appreciable fraction of atoms through collisions).

4.4 A beam of silver atoms passing through air at a temperature of 0°C and a pressure of 1 Nm^{-2} is attenuated by a factor 2.72 in a distance of 10^{-2} m. Find the mean free path of the silver atoms and estimate the effective collision radius.

4.5 (*) Recall the example (discussed in the lecture notes) of billiard balls sensitive to the gravitational pull of a passerby. Consider now a room filled with air. Work out how long it will take for the trajectories of the molecules to be completely altered by the gravitational interaction with a stray electron appearing out of nowhere at the edge of the Universe (ignore all non-A-level physics involved).

Conductivity, Viscosity, Diffusion

4.6 a) Obtain an expression for the thermal conductivity of a classical ideal gas. Show that it depends only on temperature and the properties of individual gas molecules.

b) The thermal conductivity of argon (atomic weight 40) at S.T.P. is $1.6\times10^{-2}\ \mathrm{Wm^{-1}K^{-1}}$. Use this to calculate the mean free path in argon at S.T.P. Express the mean free path in terms of an effective atomic radius for collisions and find the value of this radius. Solid argon has a close packed cubic structure, in which, if the atoms are regarded as hard spheres, 0.74 of the volume of the structure is filled. The density of solid argon is 1.6 g cm^{-3}. Compare the effective atomic radius obtained from this information with the effective collision radius. Comment on the result.

4.7 a) Define the coefficient of viscosity. Use kinetic theory to show that the coefficient of viscosity of a gas is given, with suitable approximations, by

$$\eta = K\rho\langle v\rangle\lambda_{\mathrm{mfp}}$$

where ρ is the density of the gas, λ_{mfp} is the mean free path of the gas molecules, $\langle v\rangle$ is their mean speed, and K is a number which depends on the approximations you make.

b) In 1660 Boyle set up a pendulum inside a vessel which was attached to a pump which could remove air from the vessel. He was surprised to find that there was no observable change in the rate of damping of the swings of the pendulum when the pump was set going. Explain the observation in terms of the above formula.

Make a rough order of magnitude estimate of the lower limit to the pressure which Boyle obtained; use reasonable assumptions concerning the apparatus which Boyle might have used. [The viscosity of air at atmospheric pressure and at 293 K is 18.2 μN s m^{-2}.]

Explain why the damping is nearly independent of pressure despite the fact that fewer molecules collide with the pendulum as the pressure is reduced.

4.8 Two plane disks, each of radius 5 cm, are mounted coaxially with their adjacent surfaces 1 mm apart. They are in a chamber containing Ar gas at S.T.P. (viscosity $2.1\times10^{-5}\ \mathrm{N\,s\,m^{-2}}$) and are free to rotate about their common axis. One of them rotates with an angular velocity of 10 rad s^{-1}. Find the couple which must be applied to the other to keep it stationary.

4.9 Measurements of the viscosity η of argon gas (^{40}Ar) over a range of pressures yield the following results at two temperatures:

$$\begin{array}{ll} \text{at 500 K} & \eta \approx 3.5 \times 10^{-5}\ \mathrm{kg\ m^{-1}\ s^{-1}} \\ \text{at 2000 K} & \eta \approx 8.0 \times 10^{-5}\ \mathrm{kg\ m^{-1}\ s^{-1}} \end{array}$$

The viscosity is found to be approximately independent of pressure. Discuss the extent to which these data are consistent with (i) simple kinetic theory, and (ii) the diameter of the argon atom (0.34 nm) deduced from the density of solid argon at low temperatures.

4.10 a) Argue qualitatively or show from elementary kinetic theory that the coefficient of self-diffusion D, the thermal conductivity κ and the viscosity η of a gas are related via

$$D \sim \frac{\kappa}{c_V} \sim \frac{\eta}{\rho},$$

where c_V is the heat capacity per unit volume ($3nk_B/2$ for ideal monatomic gas) and ρ is the mass density of the gas.

b*) Starting from the kinetic equation for the distribution function $F^*(t, \mathbf{r}, \mathbf{v})$ of some labelled particle admixture in a gas, derive the self-diffusion equation

$$\frac{\partial n^*}{\partial t} = D \frac{\partial^2 n^*}{\partial z^2}$$

for the number density $n^*(t, z) = \int d^3\mathbf{v} F^*(t, z, \mathbf{v})$ of the labelled particles (which we assume to change only in the z direction). Derive also the expression for the self-diffusion coefficient D, given that

—the molecular mass of the labelled particles is m^*,

—the temperature of the unlabelled ambient gas is T (assume it is uniform),

—collision frequency of the labelled particles with the unlablelled ones is ν_c^*.

Assume that the ambient gas is static (no mean flows), that the density of the labelled particles is so low that they only collide with the unlabelled particles (and not each other) and that the frequency of these collisions is much larger than the rate of change of any mean quantities. Use the Krook collision operator, assuming that collisions relax the distribution of the labelled particles to a Maxwellian F_M^* with density n^* and the same velocity (zero) and temperature (T) as the ambient unlabelled gas.

Hint. Is the momentum of the labelled particles conserved? You should discover that self-diffusion is related to the mean velocity u_z^* of the labelled particles (you can assume $u_z^* \ll v_{\mathrm{th}}$). You can calculate this velocity either directly from $\delta F^* = F^* - F_M^*$ or from the momentum equation for the labelled particles.

c*) Derive the momentum equation for the mean flow of the labelled particles and obtain the result you have known since school: friction force (collisional drag exerted on labelled particles by the ambient population) is proportional to the mean velocity (of the lablelled particles). What is the proportionality coefficient? This, by the way, is the "Aristotelian equation of motion" — Aristotle thought force was generally proportional to velocity. It took a while for another man to figure out the more general formula.

Show from the momentum equation you have derived that the flux of labelled particles is proportional to their pressure gradient: $n^* u_z^* = -(1/m^* \nu_c^*) \partial p^* / \partial z$.

Heat Diffusion Equation

4.11 a) A cylindrical wire of thermal conductivity κ, radius a and resistivity ρ uniformly carries a current I. The temperature of its surface is fixed at T_0 using water cooling. Show that the temperature $T(r)$ inside the wire at radius r is given by

$$T(r) = T_0 + \frac{\rho I^2}{4\pi^2 a^4 \kappa}(a^2 - r^2).$$

b) The wire is now placed in air at temperature T_{air} and the wire loses heat from its surface according to Newton's law of cooling (so that the heat flux from the surface of the wire is given by $\alpha(T(a) - T_{\mathrm{air}})$ where α is a constant. Find the temperature $T(r)$.

4.12 A microprocessor has an array of metal fins attached to it, whose purpose is to remove heat generated within the processor. Each fin may be represented by a long thin cylindrical copper rod with one end attached to the processor; heat received by the rod through this end is lost to the surroundings through its sides.

The internal energy density ε of the rod is related to its temperature T via $\varepsilon = \rho c_m T$, where ρ is mass density, c_m the specific (i.e., per unit mass) heat capacity of the metal (not $3k_B/2m$; you will learn what it is later in the course). Show that the temperature $T(x,t)$ at location x along the rod at time t obeys the equation

$$\rho c_m \frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2}{a} R(T),$$

where a is the radius of the rod, and $R(T)$ is the rate of heat loss per unit area of surface at temperature T.

The surroundings of the rod are at temperature T_0. Assume that $R(T)$ has the form (Newton's law of cooling)

$$R(T) = A(T - T_0).$$

In the steady state:

(a) obtain an expression for T as a function of x for the case of an infinitely long rod whose hot end has temperature T_{m};

(b) show that the heat that can be transported away by a long rod of radius a is proportional to $a^{3/2}$, provided that A is independent of a.

In practice the rod is not infinitely long. What length does it need to have for the results above to be approximately valid? The radius of the rod is 1.5 mm.

[The thermal conductivity of copper is $380\,\mathrm{W\,m^{-1}\,K^{-1}}$. The cooling constant $A = 250\,\mathrm{W\,m^{-2}\,K^{-1}}$.]

4.13 One face of a thick uniform layer is subject to a sinusoidal temperature variation of angular frequency ω. Show that damped sinusoidal temperature oscillations propagate into the layer and give an expression for the decay length of the oscillation amplitude.

A cellar is built underground covered by a ceiling which is 3 m thick made of limestone. The outside temperature is subject to daily fluctuations of amplitude 10°C and annual fluctuations of 20°C. Estimate the magnitude of the daily and annual temperature variations within the cellar. Assuming that January is the coldest month of the year, when will the cellar's temperature be at its lowest?

[The thermal conductivity of limestone is 1.6 $\mathrm{Wm^{-1}K^{-1}}$, and the heat capacity of limestone is $2.5 \times 10^6 \mathrm{JK^{-1}m^{-3}}$.]

Pressure, Energy, Effusion *(revision)*

4.14 Consider an insulated cylindrical vessel filled with monatomic ideal gas, closed on one side and plugged by a piston on the other side. The piston is very slowly pulled out (its velocity u is much smaller than the thermal velocity of the gas molecules). Show *using kinetic theory, not thermodynamics,* that during this process the pressure p of the gas inside the vessel and its volume V are related by $pV^{5/3}$ =const.

Hint. Consider how the energy of a gas particle changes after each collision with the piston and hence calculate the rate of change of the internal energy of the gas inside the vessel.

4.15 Consider two chambers of equal volume separated by an insulating wall and containing an ideal gas maintained at two distinct temperatures $T_1 = 225$ K and $T_2 = 400$ K. Initially the two chambers are connected by a long tube whose diameter is much larger than the mean free path in either chamber and equilibrium is established (while maintaining T_1 and T_2). Then the tube is removed, the chambers are sealed, but a small hole is opened in the insulating wall, with diameter $d \ll \lambda_{\mathrm{mfp}}$ (for either gas).

a) In what direction will the gas flow through the hole: $1 \to 2$ or $2 \to 1$?

b) If the total mass of the gas in both chambers is M, what is the mass ΔM of the gas that will be transferred through the hole from one chamber to the other before a new equilibrium is established?

Answer. You should find that

$$\Delta M = \frac{\sqrt{T_1 T_2}}{T_1 + T_2} \frac{\sqrt{T_2} - \sqrt{T_1}}{\sqrt{T_1} + \sqrt{T_2}} M.$$