Statistical and Thermal Physics

Second year physics course A1

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Problem Set 1

(A. T. Boothroyd)

PROBLEM SET 1: Basic Thermodynamics

Problem set 1 can be covered in one tutorial or class held during Week 5 of Michaelmas Term

Functions of two variables

1.1 In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

(a)The definition of x implies that

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r}. \tag{1}$$

But we also have $x^2 + y^2 = r^2$, so differentiating with respect to r gives

$$2x\frac{\partial x}{\partial r} = 2r \qquad \Longrightarrow \qquad \frac{\partial x}{\partial r} = \frac{r}{x}. \tag{2}$$

But equations (1) and (2) imply that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$. What's gone wrong?

(b) Show that

$$\left(\frac{\partial x}{\partial r}\right)_y \left(\frac{\partial r}{\partial y}\right)_x \left(\frac{\partial y}{\partial x}\right)_r = -1$$

Expansions, cycles and heat engines

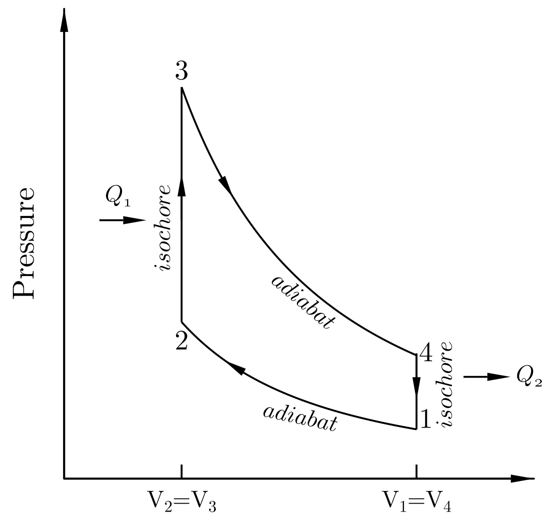
1.2 Two thermally insulated cylinders, A and B, of equal volume, both equipped with pistons, are connected by a valve. When open, the valve allows unrestricted flow. Initially A has its piston fully withdrawn and contains a perfect monatomic gas at temperature $T_{\rm i}$, and B has its piston fully inserted, and the valve is closed. The thermal capacity of the cylinders is to be ignored. The valve is fully opened and the gas slowly drawn into B by pulling out the piston B; piston A remains stationary. Show that the final temperature of the gas is $T_{\rm f} = T_{\rm i}/2^{2/3}$.

[For a monatomic ideal gas, $\gamma = 5/3$.]

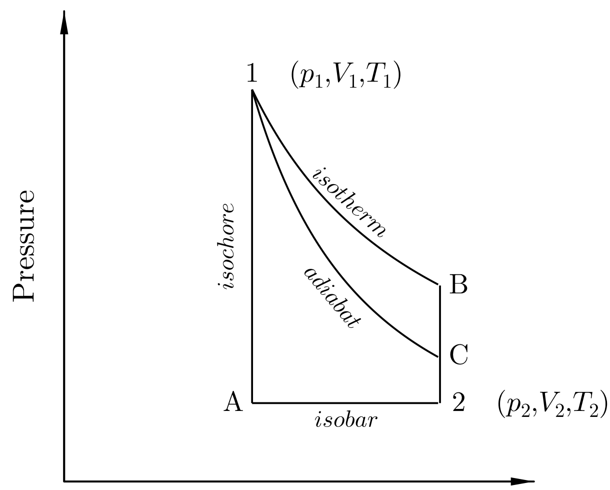
1.3 A possible ideal-gas cycle operates as follows:
(i) From an initial state (p_1, V_1) the gas is cooled at constant pressure to (p_1, V_2);
(ii) The gas is heated at constant volume to (p_2, V_2);
(iii) The gas expands adiabatically back to (p_1, V_1).
Assuming constant heat capacities, show that the thermal efficiency is

$$1 - \gamma\frac{(V_1/V_2) - 1}{(p_2/p_1) - 1}.$$

(You may quote the fact that in an adiabatic change of an ideal gas, pV^γ stays constant, where $\gamma = C_p/C_V$.)



1.4 Show that the efficiency of the standard Otto cycle (shown above) is $1 - r^{1-\gamma}$, where $r = V_1/V_2$ is the compression ratio.



1.5 An ideal gas is changed from an initial state (p_1, V_1, T_1) to a final state (p_2, V_2, T_2) by the following quasi-static processes shown in the figure: (i) 1A2 (ii) 1B2 and (iii) 1C2. For each process, obtain the work that must be done on the system and the heat that must be added in terms of the initial and final state variables, and hence show that $\Delta U = C_V(T_2 - T_1)$ independent of path. (Assume that the heat capacity C_V is constant.)

1.6 A building is maintained at a temperature T by means of an ideal heat pump which uses a river at temperature T_0 as a source of heat. The heat pump consumes power W, and the building loses heat to its surroundings at a rate $\alpha(T - T_0)$. Show that T is given by

$$T = T_0 + \frac{W}{2\alpha}\left(1 + \sqrt{1 + 4\alpha T_0/W}\right).$$