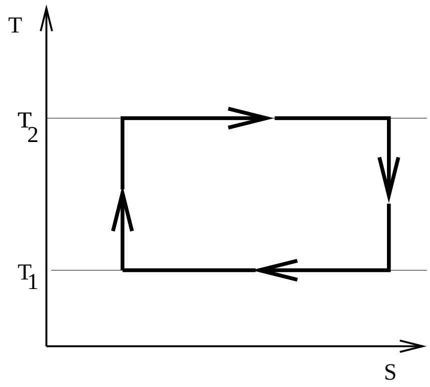
Statistical & thermal physics revision questions

1. [June 2007] Explain what is meant by a *function of state*. Give an expression for the change in entropy when a body is heated between temperatures T_1 and T_2. [5]

2. [June 2007] Consider the Carnot cycle for an ideal gas operating between temperatures T_1 and T_2.



The figure shows the cycle's entropy–temperature diagram. Show that the efficiency of the cycle is

$$\eta = 1 - \frac{T_1}{T_2}.$$

Hence estimate the maximum efficiency possible for a practical steam engine when the steam is heated to 800 K. [7]

3. [June 2008] Write down expressions for the thermodynamic Helmholtz and Gibbs functions and their differentials in terms of the functions of state U, p, V, T and S. Use these to derive two of the Maxwell relations. A Joule-Kelvin expansion is a process in which a gas is allowed to expand isenthalpically. How can this be achieved in practice? [7]

Such an expansion can lead to a change in temperature of the gas by an amount related to the Joule–Kelvin coefficient, defined by

$$\mu_{\text{JK}} = \left(\frac{\partial T}{\partial p}\right)_H .$$

Obtain an expression for μ_{JK} in terms of T, V, the heat capacity and the expansion coefficient at constant pressure. Show that no heating or cooling will occur for a perfect gas. [7]

A gas is found to obey the equation of state

$$p(V - b) = RT - \frac{ap}{RT}.$$

Obtain an expression for μ_{JK} and explain the significance of this result. [6]

[Turn over]

4. [June 2007] Show that the relation between pressure p and temperature T when two phases of a single substance are in equilibrium is given by the Clausius–Clapeyron equation,

$$\frac{\mathrm{d}p}{\mathrm{d}T} = \frac{\Delta S}{\Delta V} = \frac{L}{T\Delta V},$$

where ΔS is the change in specific entropy for a given change in specific volume ΔV when the substance changes phase, and where L is the specific latent heat. [8]

A pressurised vessel containing liquid water and water vapour only is heated until the pressure reaches twice the value of atmospheric pressure. Obtain an expression for the specific volume of the vapour by assuming that it behaves as a perfect gas. Hence calculate the temperature at which this pressure will be reached. You may assume the latent heat to be independent of temperature, and that the specific volume of the water is negligible compared with that of the vapour. [10]

Explain briefly why $\mathrm{d}p/\mathrm{d}T$ is negative when liquid water and water ice are in equilibrium. [2]

[The specific latent heat of vaporisation of water is 2.272×10^6 J kg^{-1}, the mean molecular mass of water is 18.0 a.m.u., and atmospheric pressure may be taken to be 10^5 Pa.]

5. [June 2001] The energy density U of radiation from a black body at temperature T is given by

$$U = aT^4 \quad \text{where} \quad a = \frac{\pi^2 k_\mathrm{B}^4}{15\hbar^3 c^3}.$$

Derive Stefan's law for the power radiated per unit area by a black body, and obtain an expression for the Stefan–Boltzmann constant which appears in this law. [5]

Obtain an expression for the entropy density of black-body radiation. [5]

Black-body radiation, initially occupying a volume V at a temperature T_i, expands adiabatically to a final volume αV where α is a constant. What is its final temperature? [5]

[June 2003] The surface temperature of the Sun is 5700 K, and the spectrum of radiation which it emits has a maximum at a wavelength of 510 nm. Estimate the surface temperature of the North Star, for which the corresponding maximum is at 350 nm. [5]

6. In the classical limit, the partition function for N molecules of an ideal gas can be conveniently written $Z = (Vz)^N/N!$ where z depends only on the temperature T, some internal properties of a molecule and fundamental constants. Find an expression for the chemical potential in terms of $k_\mathrm{B}T$, N/V and z. [5]

[Turn over]

7. Define the partition function Z for a system in terms of the energies E_j of its quantum states j and the inverse temperature $\beta = 1/k_\mathrm{B}T$. Write down the probability p_j that the system is in a given state j. [2]

Show that the system's internal energy U, entropy $S = -k_\mathrm{B}\sum_j p_j \ln p_j$ and Helmholtz free energy F are given by

$$U = -\frac{\partial \ln Z}{\partial \beta}, \quad S = \frac{U}{T} + k_\mathrm{B} \ln Z \text{ and } F = -k_\mathrm{B}T \ln Z.$$ [4]

In a simplified model of a crystal, each molecule is a point mass that is attached to its site by a force, so that at each site there is a three-dimensional quantised harmonic oscillator with natural angular frequency ω. Show that in this approximation the Helmholtz free energy of a crystal of N sites is

$$F = \frac{3}{2}N\hbar\omega + 3Nk_\mathrm{B}T \ln\left(1 - \mathrm{e}^{-\hbar\omega/k_\mathrm{B}T}\right).$$ [7]

Show that the crystal's heat capacity (a) tends to $3Nk_\mathrm{B}$ in the limit of high temperature, $T \to \infty$, and (b) vanishes in the limit of low temperature, $T \to 0$. [7]

$$\left[\sum_{n=0}^{n=\infty} r^n = \frac{1}{1-r}, \text{ for } |r| < 1.\right]$$

Andrew Boothroyd
Trinity 2014

[**LAST PAGE**]